Invent. math. 95, 421–435 (1989)

Inventiones mathematicae

Rational points of bounded height on Fano varieties

Dedicated to A.I. Kostrikin on the occasion of his 60th birthday

Jens Franke [1], Yuri I. Manin [2], and Yuri Tschinkel [3]

[1] Universität Jena, Sekt. Math., Universitäts-Hochhaus 17.OG, DDR-6900 Jena and Dept. of Math., Moscow University

[2] Steklov Math. Institute, 42 Vavilova, 117966 GSP-1, Moscow, USSR

[3] Dept. of Math., Moscow University, 117234, Moscow, USSR

Introduction

Let F be a finite extension of $\mathbb{Q}$ endowed with a system of norms $|\cdot|_v$ such that the product formula $\prod_v |a|_v = 1$ is valid for all $a \in F$, and $\prod_{v/p} |p|_v = p^{-1}$ for a prime $p \in \mathbb{Z}$. Let V be an algebraic variety defined over F and $\mathbb{L}$ a metrized line bundle on V, i.e., a system $(L, |\cdot|_v)$ consisting of a line bundle L and a family of Banach v-adic metrics on $L \otimes F_v$ for all v's satisfying well-known conditions (cf. [1], Chap. VI). Then we can define a height function h_L: $X(F) \to \mathbb{R}$ by the formula $h_{\mathbb{L}}(x) = \prod_v |s(x)|_v^{-1}$, where s is a local section of L non-vanishing at x. Our valuations are normed in such a way that h_L remains invariant with respect to base changes $F' \supset F$.

Assume that L is ample and put

$$N(V, \mathbb{L}, H) = \mathrm{card}\{x \in V(F) \,|\, h_{\mathbb{L}}(x) \leqq H\}.$$

The asymptotic behavior of $N(V, \mathbb{L}, H)$ as $H \to \infty$ is a crude but essential characteristics of arithmetical properties of V. It is therefore desirable to understand its connections with algebro-geometric properties of (V, L).

Projective algebraic varieties V are divided into three large classes with respect to the behaviour of their canonical bundle ω.

(I) General type varieties, i.e., varieties with ample ω. A bold generalization of Mordell's conjecture states that the F-points on a general type variety lie on a proper Zariski-closed subset (cf. [1], p. 349, where this conjecture is deduced from a more precise Vojta's conjecture). Therefore, if $V(F)$ is infinite, there should be a proper subvariety of V which contains all F-points and whose irreducible components are not of general type. Thus the investigation of $V(F)$ may be subdivided into two steps.

a) Determine (irreducible components of) $W=\overline{V(F)}$ (Zariski closure).

b) Investigate $W(F)$.

It is now known that the moduli space M_g of stable curves of genus g is of general type for $g \geqq 24$ ([2]). A tantalizing problem arises to describe $\overline{M_g(\mathbb{Q})}$ and its part corresponding to smooth curves. A similar problem can be posed about the moduli space of principally polarized abelian varieties A_g.

We are not aware of any conjectures about values of $\dim(\overline{M_g(\mathbb{Q})})$, $\dim(\overline{A_g(\mathbb{Q})})$, and asymptotics of $N(M_g, \omega, H)$ and $N(A_g, \omega, H)$ (ω carries a natural metric investigated by Faltings).

(II) Intermediate type varieties, for which neither ω, nor ω^{-1} is ample. Arithmetics of this class is virtually unexplored, with one very important exception, that of abelian varieties. From the Mordell-Weil theorem and a refined Néron-Tate height theory one deduces that

$$N(A, \mathbb{L}, H) \sim c(\log H)^{r/2}, \tag{0.1}$$

where $r=rk(A(F))$ and c is a constant that can be expressed through $\mathrm{card}(A(F)_{\mathrm{tors}})$ and the volume of the ellipsoid $\hat{h}_L \leqq 1$ in $\mathbb{R} \otimes A(F)$. The famous Birch-Swinnerton-Dyer conjecture connects (0.1) with the behaviour of the L-function of A at the center of the critical strip.

(III) Fano varieties, i.e., varieties with ample ω^{-1}. The simplest Fano variety is $\mathbb{P}_F^n$. Schanuel [3] proved that

$$N(\mathbb{P}_F^n, O(1), H) \sim c' H^{n+1}.$$

Since $\omega_{\mathbb{P}^n}^{-1} = O(n+1)$, it follows that $N(\mathbb{P}_F^n, \omega^{-1}, H) \sim cH$. Prompted by this observation and some numerical data about the distribution of rational points on $x_0^3 + 2x_1^3 + 3x_2^3 + 4x_3^3 = 0$ (cf. Appendix) Yu.I. Manin conjectured that the following asymptotic behaviour

$$N(V, \omega_V^{-1}, H) \sim cH(\log H)^t \tag{0.2}$$

should be typical for Fano manifolds V with dense $V(F)$, excluding some degenerate cases (cf. below). Including a power of logarithm is necessary since it already occurs for products of projective spaces. The available information is compatible with the further conjecture that $t = rk(\mathrm{Pic}\, V) - 1$.

This article is devoted to the proof of the following facts, giving some support to these conjectures.

a) Asymptotic (0.2) with error term $O(H(\log H)^{t-1})$ is stable with respect to the direct product.

b) Asymptotic (0.2) is consistent with the predictions of the Hardy-Littlewood method for complete intersections in $\mathbb{P}^n$.

c) Asymptotic (0.2) is valid for generalized flag manifolds $P \backslash G$ where G is a semisimple algebraic group and P is a parabolic subgroup.

We prove a) and b) in § 1. The most technical part c) is due to J. Franke who made the crucial remark that the Dirichlet series

$$Z(s) = \sum_{x \in (P \backslash G)(F)} h_{\omega^{-1}}(x)^{-s}$$

can be identified with one of the Langlands-Eisenstein series [4]. The use of the heavy machinery of [4] is perhaps reminiscent of shooting sparrows with cannons. However, it allows one to borrow heavily from Langlands work, who established most of the analytical properties of $Z(s)$ needed to deduce (0.2) and in fact a considerably sharper result (cf. Sect. 2, Corollary to Theorem 5).

We note finally that (0.2) cannot be true without some further non-degeneracy assumptions. For example, consider the case of a smooth cubic surface $V \subset \mathbb{P}^3_F$. If one of the 27 lines on it is F-rational, we have by Schanuel's theorem

$$N(V, \omega_V^{-1}, H) \geqq N(\mathbb{P}^1, \omega_V^{-1}|_{\mathbb{P}^1}, H) \geqq c H^2$$

since $\omega_V^{-1}|_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$.

Generalizing this counter-example consider the following situation. Let $W \subset V$ be Fano manifolds. Suppose that for some $a > 1$ we have $(\omega_V^{-1}|_W)^{\otimes a} \simeq \omega_W^{-1}$. Then (0.2) cannot be true for V and W simultaneously.

Would it suffice to exclude such "pathology" in order to ensure (0.2) or is it necessary also to assume that $\deg_{\omega^{-1}} V$ is sufficiently large?

Unfortunately, we do not know answers even for cubic or more general del Pezzo surfaces.

§ 1. Products and complete intersections

1. Products. Since $\omega_{V \times W} = \omega_V \boxtimes \omega_W$ and $h_{L_1 \boxtimes L_2}(x, y) = h_{L_1}(x)\, h_{L_2}(y)$, the stability of the asymptotic behaviour (0.2) follows at once from the following elementary statement, which is probably well-known. We sketch a proof for the sake of completeness.

Let λ: $0 < \lambda_1 \leqq \lambda_2 \leqq \ldots$ and μ: $0 < \mu_1 \leqq \mu_2 \leqq \ldots$ be two real increasing sequences. Put $\lambda\mu = \{\lambda_i \mu_j$, ordered increasingly$\}$. Put $N_\lambda(H) = \text{card}\{i \mid \lambda_i < H\}$ and similarly for $N_\lambda(H)$, $N_{\lambda\mu}(H)$.

2. Proposition. *Assume that*

$$N_\lambda(H) = c_\lambda H \log^s H + O(H \log^{s-1} H),$$

$$N_\mu(H) = c_\mu H \log^r H + O(H \log^{r-1} H).$$

Then

$$N_{\lambda\mu}(H) = B(r+1, s+1)\, c_\lambda c_\mu H \log^{r+s+1} H + O(H \log^{r+s} H),$$

where B is the beta-function.

Proof. We have

$$N_{\lambda\mu}(H)=\sum_{i=1}^{N_\lambda(H/\mu_1)} N_\mu(H/\lambda_i)=c\sum_{i=1}^{N_\lambda(H/\mu_1)}\frac{H}{\lambda_i}\log^r\left(\frac{H}{\lambda_i}\right)$$
$$+O\left(\sum_{i=1}^{N_\lambda(H/\mu_1)}\frac{H}{\lambda_i}\log^{r-1}\left(\frac{H}{\lambda_i}\right)\right). \tag{1.1}$$

Here the error term is of the same structure as the leading one with the logarithm power diminished by one. The same effect occurs repeatedly in the subsequent estimates. Therefore we shall write out explicitely only leading terms. Put

$$a(j)=\operatorname{card}\{i\,|\,\lambda_1+j\leqq\lambda_i<\lambda_1+j+1\},\qquad N=[H/\mu_1-\lambda_1]+1.$$

Then

$$\sum_{i=1}^{N_\lambda(H/\mu_1)}\frac{H}{\lambda_i}\log^r\left(\frac{H}{\lambda_i}\right)\sim\sum_{j=1}^{N}a(j)\frac{H}{j}\log^r(H/j). \tag{1.2}$$

In order to calculate (1.2), we use the Abel summation trick and estimate

$$\sum_{j=0}^{M}a(j)=N_\lambda(M+\lambda_1+1)=c_\lambda M\log^s M+O(M\log^{s-1}M).$$

In this way we obtain for the leading term of (1.1) the asymptotics

$$c_\lambda c_\mu H\sum_{j=1}^{N} j\log^r j\int_j^{j+1}x^{-2}\log^s\left(\frac{H}{x}\right)dx.$$

Finally, using a mean value theorem we can replace the sum by the integral

$$\int_1^{N+1}\log^r x(\log H-\log x)^s\,d(\log x)$$
$$\sim B(r+1,s+1)(\log H)^{r+s+1}+O((\log H)^{r+s}).$$

3. *Hardy-Littlewood method.* The adelic version of the classical Hardy-Littlewood method is explained in a very clear and conceptual way in [9] following important previous work [6–8]. We shall use here its barest rudiments, namely, a formal description of the "singular series" which in favorable cases can be proved to dominate a weighted sum of delta-functions supported by F-points of a Fano manifold V (or, rather, a cone over V).

Denote by A_F the adèle ring of F. Let W be an affine variety over F. The general Hardy-Littlewood formula is an identity

$$\sum_{x\in W(F)}\varphi(x)=\int_{W(A_F)}\varphi(x)\,\omega_{HL}(x)+R(\varphi)=P(\varphi)+R(\varphi), \tag{1.3}$$

where $\omega_{HL}(x)$ is a measure on $W(A_F)$ which will be explicitely described below in certain cases and φ belongs to a certain space of summable functions on

$W(A_F)$. The integral $P(\varphi)$ at the r.h.s. is the singular series, and getting good estimates of the error term $R(\varphi)$ is a real job for a number-theorist. We have nothing new to say about it.

Here is our small observation. Let W be a complete intersection in F^n, given by forms of degree

$$d_1, \ldots, d_m: f_1(x_1, \ldots, x_n) = \ldots = f_m(x_1, \ldots, x_n) = 0.$$

This means, in particular, that $W_0 = W \setminus \{0\}$ is a pointed cone over the projective complete intersection $V \subset \mathbb{P}_F^{n-1}$ defined by the same equations in homogeneous coordinates. We shall assume that V is a smooth projective manifold.

4. Proposition. a) $\omega_V^{-1} = O(n - \sum d_i)$. *In particular, V is a Fano manifold if and only if $n > \sum d_i$.*

b) *For a real $\tau > 0$ put $\tilde{\tau} = (\tilde{\tau}_v) \in A_F$, where $\tilde{\tau}_v = \tau^{[F_v:Q_v]/[F:Q]}$ if $v|\infty$, $\tilde{\tau}_v = 1$ otherwise. Let φ be a function on $W(A_F)$ and $\varphi_\tau(x_1, \ldots, x_n) = \varphi(\tilde{\tau}^{-1} x_1, \ldots, \tilde{\tau}^{-1} x_n)$. Assume that $P(\varphi_\tau)$ makes sense for all τ. Then*

$$P(\varphi_\tau) = \tau^{n - \Sigma d_i} P(\varphi_1). \tag{1.4}$$

Proof. Part a) is well known. To prove b), note that

$$\int_{W_0(A_F)} \varphi_\tau(x)\, \omega_{HL}(x) = \int_{W_0(A_F)} \varphi_1(x)\, \omega_{HL}(\tilde{\tau} x).$$

Hence it suffices to check that

$$\omega_{HL}(\tilde{\tau} x) = \tau^{n - \Sigma d_i} \omega_{HL}(x).$$

But this follows from the following description of ω_{HL} (cf. [9]). Consider $f = (f_1, \ldots, f_m)$ as a map $f: A^n \to A^m$ of affine spaces over F. Denote the standard volume forms by $\omega_n = dx_1 \wedge \ldots \wedge dx_n$, $\omega_m = dy_1 \wedge \ldots \wedge dy_m$. Define locally on W_0 a form Ω_{HL} by the following prescription:

$$\omega_n = \theta \wedge f^*(\omega_m), \qquad \Omega_{HL} = \theta|_{W_0}.$$

Then ω_{HL} is a measure constructed in the standard way from Ω_{HL} and the Tate-Tamagawa measure on A_F. (cf. [6, 8]). Now we have

$$\omega_n(\tilde{\tau} x) = \tau^n \omega_n(x), \qquad f^*(\omega_m)(\tilde{\tau} x) = \tau^{\Sigma d_i} f^*(\omega_m)(x)$$

since $f^*(\omega_m) = df_1 \wedge \ldots \wedge df_m$. This proves (1.4).

Finally, we must explain in which sense (1.4) conforms with conjecture (0.2). Assume for simplicity that φ_1 is the restriction to $W(A_F)$ of the characteristic function of the product of balls at infinite places and v-adic integers at finite places. Then $\sum_{x \in W_0(F)} \varphi_\tau(x)$ calculates the number of F-points of W_0 with integer coordinates whose conjugates lie in the expanded archimedean balls. For $F = \mathbb{Q}$ this is clearly equivalent to calculating the number of F-points on V with $O(1)$-height $\leqq \tau$, with a correcting factor $(1 + o(1))/2\zeta(m)$ (since one really needs only points of $W_0(\mathbb{Q})$ with co-prime coordinates). Hopefully, one can use the same trick for general F. Then (1.4) will show that the number of points with $O(1)$-height $\leqq \tau$ on V grows as $c\tau^{n-\Sigma d_i}$ if the singular series $P(\varphi)$ is an asymptotic approximation to $\sum_x \varphi_\tau(x)$. Finally, part a) of the Proposition 4 allows one to conclude that $N(V, \omega_V^{-1}, H)$ grows linearly with H.

§ 2. Flag manifolds

1. Notation. Let G be a semisimple linear algebraic group over an algebraic number field F. We fix once and for all a minimal F-rational parabolic subgroup $P_0 \subset G$. Let P be a standard (i.e., containing P_0) parabolic subgroup. We denote by $V = P \backslash G$ the generalized flag manifold and by $\pi: G \to V$ the canonical projection.

By X^* (resp. X_*) we shall denote the group of characters (resp. cocharacters) defined over F. Any element $\chi \in X^*(P)$ defines a line bundle L_χ on V, whose sections on an open subset $U \subset V$ are

$$\Gamma(U, L_\chi) = \{f \in \Gamma(\pi^{-1}(U), O_G) \mid f(pg) = \chi(p) f(g) \text{ for all } p \in P, g \in G\}. \tag{2.1}$$

The correspondence $\chi \to L_\chi$ defines an embedding (because of $X^*(G) = 0$ and Rosenlicht's theorem) of finite index (since $\operatorname{Pic}(G)$ is finite) $X^*(P) \to \operatorname{Pic} V$, hence $rk \operatorname{Pic} V = rk\, X^*(P)$.

Let A_F be the adele ring of F. We choose a maximal compact subgroup $K = \prod_v K_v \subset G(A_F)$ in such a way that $G(A_F) = P_0(A_F) K$. The bundles $L_\chi \otimes F_v$ on $V \otimes F_v$ then carry natural K_v-invariant v-adic metrics defined by the following rule. Let s be a section of L_χ on a neighbourhood of $x \in V(F_v)$ corresponding to a function f as in (1). We choose $k \in K_v$ with $\pi(k) = x$ and put

$$|s|_{x,v} = |f(k)|_v.$$

Then a height function on $V(F)$ is defined by

$$h_\chi(x) = h_{L_\chi}(x) = \prod_v |s|_{x,v}^{-1},$$

where s is a section of L_χ in a neighbourhood of x which is F-rational and non-vanishing at x. We see that a choice of K determines a metrization of L_χ.

We recall now how to describe the anticanonical bundle on V in group-theoretical terms. Let $M \supseteq M_0$ be Levi factors of P and P_0, and let $A \subseteq A_0$ be their split components. Then $X^*(M) = X^*(P)$ is a sublattice of finite index in $X^*(A)$. Let $\mathfrak{a} = X_*(A) \otimes_Z \mathbb{R}$, $\check{\mathfrak{a}} = X^*(A) \otimes_Z \mathbb{R}$. Denote by $\mathfrak{e}$ the Lie algebra of the radical of P. Let $\rho = \rho_P$ be one half the sum of the roots of A in $\mathfrak{e}$ counted with multiplicity equal to the dimension of their eigenspaces. Then $L_{-2\rho}$ is the anticanonical bundle of V.

Our main tool will be the zeta-function

$$Z_\chi(s)=\sum_{x\in V(F)} h_\chi(x)^{-s} \tag{2.2}$$

corresponding to $\chi=-2\rho$. In order to establish its analytical properties we shall express it through Eisenstein series [4].

2. *Eisenstein series.* For every v, we define a function $H_{P,K_v}(g)\in\mathfrak{a}$ of $g\in G(F_v)$. Consider the Iwasawa decomposition $g=pk$, $p\in P_0(F_v)$, $k\in K_v$ and put for any $\chi\in X^*(P)$:

$$\exp(\langle H_{P,K_v}(g),\chi\rangle)=|\chi(p)|_v.$$

The corresponding global function is then defined by

$$H_P(g)=\sum_v H_{P,K_v}(g_v),\qquad g=(g_v)\in G(A_F).$$

We note that the sum is finite. If confusions are impossible, the subscript K is omitted.

We now put for $\lambda\in\check{\mathfrak{a}}\otimes\mathbb{C}$, $g\in G(A)$

$$E_P^G(\lambda,g)=\sum_{\gamma\in P(F)\backslash G(F)}\exp(\langle\lambda+\rho_P,H_P(\gamma g)\rangle). \tag{2.3}$$

By the remarks following the proof of Lemma 4.1 in [4], this series absolutely converges if $\operatorname{Re}\lambda\in\rho_P+\check{\mathfrak{a}}^+$, where $\check{\mathfrak{a}}^+$ is defined in the following way.

Let $\mathfrak{a}_0=X_*(A_0)\otimes_Z\mathbb{R}$, $\check{\mathfrak{a}}_0=X^*(A_0)\otimes\mathbb{R}$. Restriction of characters defines a natural embedding $\check{\mathfrak{a}}\to\check{\mathfrak{a}}_0$ and a natural projection $\mathfrak{a}_0\to\mathfrak{a}$. Furthermore, since $A\subset A_0$, there is a canonical embedding $\mathfrak{a}\to\mathfrak{a}_0$ which is a section of the projection defined above. Hence we get direct sum decompositions $\mathfrak{a}_0=\mathfrak{a}\oplus\mathfrak{a}_0^P$, $\check{\mathfrak{a}}_0=\check{\mathfrak{a}}\oplus\check{\mathfrak{a}}_0^P$. Let Φ_0 be the set of roots of A_0 in G, and let $\Delta_0\subset\Phi_0$ be the set of simple positive roots in the cone defined by P_0. It is known that Φ_0 is a root system. For every $\alpha\in\Phi_0$, we denote by $\check{\alpha}\in\mathfrak{a}_0$ the corresponding coroot. Put $\check{\mathfrak{a}}_0^+=\{x\in\check{\mathfrak{a}}_0\mid\langle x,\check{\alpha}\rangle>0$ for every $\alpha\in\Delta_0\}$. Finally, let

$$\mathfrak{a}^+=\text{interior of}\,(\check{\mathfrak{a}}\cap\text{closure of}\,\check{\mathfrak{a}}_0^+).$$

By the results of Langlands, the Eisenstein series (2.3) has a meromorphic continuation whose properties we shall recall, but first we establish how (2.2) is related to (2.3).

3. Proposition. *The zeta-function* (2.2) *absolutely converges for* $\chi\in X^*(P)\cap(-\check{\mathfrak{a}}^+)$ *(corresponding to ample bundles) and* $-(\operatorname{Re}s)\chi\in\check{\mathfrak{a}}^++2\rho$ *and is equal to* $E_P^G(-(s\chi+\rho_P),e)$, *where* $e\in G(A_F)$ *is the identity element.*

Proof. Let $g\in G(F)\subset G(A_F)$. Let us check that

$$h_\chi(\pi(g))=\exp(\langle H_P(g),\chi\rangle).$$

In fact, if $g=pk$, $p\in P_0(A_F)$, $k\in K$ and f is a function as in (2.1) defining a F-rational section of L_χ in a neighbourhood of $x=\pi(g)$ with $f(g)\neq 0$ we have

$$|s|_{v,\pi(g)}=|f(k)|_v=|\chi(p)|_v^{-1}\,|f(g)|_v.$$

Therefore, using the product formula, we get

$$h_\chi(\pi(g))=\prod_v |s|_{v,\pi(g)}^{-1}=\prod_v \exp(\langle H_{P,K_v}(g),\chi\rangle)\,|f(g)|_v^{-1}$$
$$=\exp(\langle H_P(g),\chi\rangle)$$

since $f(g)\in F^*$. Comparing this with (2.3) we get our result since $\pi\colon G\to V$ induces a surjection on F-rational points.

4. Remarks. a) Applying this proposition to the anticanonical class $L_{-2\rho}$, we get

$$Z_{-2\rho}(s)=\sum_{x\in V(F)} h_{-2\rho}(x)^{-s}=E_P^G((2s-1)\rho_P, e).$$

b) We can change K by putting $K'=pKp^{-1}$ for $p\in P_0(A_F)$. Indicating by a prime the corresponding zeta-function we get

$$Z'_\chi(s)=\exp(-s\langle\chi, H_P(p)\rangle)\,E_P^G(-(s\chi+\rho), p).$$

c) The constant term of E_P^G (which is comparatively easy to compute) represents a mean value of a Z-function over several choices of K.

5. Theorem. *The anticanonical zeta-function $Z_{-2\rho}(s)$ has the following properties:*

a) *It admits a meromorphic continuation to the whole complex s-plane.*

b) *It is holomorphic in the domain* $\mathrm{Re}(s)>1$, *has a pole of order* $t=rk(P)=rk(\mathrm{Pic}\,V)$ *at* $s=1$, *and no other poles with* $\mathrm{Re}(s)=1$.

c) *It has no singularities on the line* $\mathrm{Re}(s)=1/2$.

Corollary. *There is a polynomial $p(x)$ of degree t such that*

$$N(V, L_{-2\rho}, H)=\mathrm{card}\{x\in V(F)\,|\,h_{L_{-2\rho}}(x)\leqq H\}=H\,p(\log H)+o(H).$$

This follows directly from the theorem by a standard Tauberian argument. It is possible to obtain for the error-term an estimate of the type $H^{1-\varepsilon}$. Comparing with the Schanuel theorem [3], *one may expect that the best value of ε is* $[F:Q]^{-1}\dim V^{-1}$.

The leading coefficient of p can be computed in terms of residua of scattering operators (cf. (2.11) below). In the case of Chevalley groups, the result can be made quite explicit (cf. (2.12)).

6. Proof. Part a) of the theorem follows from Proposition 3 and the general theory of Eisenstein series (cf. [4], Appendix II). In our situation, the meromorphic continuation of E_P^G is fairly easy since it is clear how to express E_P^G as a residue of $E_{P_0}^G$. The first part of b) also follows from Proposition 3. In order to prove the second part of b) we need some results of the theory of cuspidal Eisenstein series.

Let $w \in W$, the Weyl group of Φ_0. It has a F-rational representative $w' \in N_G(A_0)$ (the normalizer of A_0 in G). We put

$$c(w, \lambda) = \int_{w' N_0(A_F) w'^{-1} \cap N_0(A_F) \backslash N_0(A_F)} \exp(\langle H_{P_0}(w'^{-1} n), \lambda + \rho_{P_0} \rangle)\, dn,$$

where N_0 is the radical of P_0. The Haar measures are normalized by

$$\int_{N_0(F) \backslash N_0(A_F)} dn = 1$$

and

$$\int_{N_0(F) \cap w' N_0(F) w'^{-1} \backslash N_0(A_F) \cap w' N_0(A_F) w'^{-1}} dn = 1.$$

Then

$$\int_{N_0(F) \backslash N_0(A_F)} E_{P_0}(\lambda, n g)\, dn = \sum_{w \in W} c(w, \lambda) \exp(\langle H_{P_0}(g), w\lambda + \rho_{P_0} \rangle).$$

By results of Langlands ([4], § 6), the functions $c(w, \cdot)$ have meromorphic continuations. They satisfy functional equations

$$c(w, \lambda)\, E^G_{P_0}(w\lambda) = E^G_{P_0}(\lambda),$$

$$c(wv, \lambda) = c(w, v\lambda)\, c(v, \lambda).$$

If we consider the partial Eisenstein series

$$E^G_{P_0}(\lambda, g) = \sum_{\gamma \in P_0(F) \backslash P(F)} \exp(\langle H_{P_0}(\gamma g), \lambda + \rho_{P_0} \rangle),$$

then

$$\int_{N_0(F) \backslash N_0(A_F)} E^P_{P_0}(\lambda, n g)\, dn = \sum_{w \in W^A} c(w, \lambda) \exp(\langle H_{P_0}(g), w\lambda + \rho_{P_0} \rangle),$$

where $W^A = \{w \in W \mid w \text{ identically acts on } \mathfrak{a}\}$. Furthermore,

$$c(w, \lambda + \vartheta) = c(w, \lambda) \tag{2.5}$$

if $w \in W^A$ and $\vartheta \in \check{\mathfrak{a}}$.

The following lemma seems to be well-known to the experts, but for the convenience of the reader we add a proof.

7. Lemma. *The singular hyperplanes of $c(w, \cdot)$ containing ρ_{P_0} are precisely the hyperplanes*

$$\langle \check{\alpha}, \lambda - \rho_{P_0} \rangle = 0, \tag{2.6}$$

where α is a simple positive root such that $w\alpha$ is a negative root. Their multiplicity is one. No other singular divisor intersects $\rho_{P_0} + i\check{\mathfrak{a}}_0$.

Proof of the lemma. First we consider the case $w = s_\alpha$ (the reflection with respect to α) with $\alpha \in \Delta_0$. By (2.5), (2.6) is the only possible singular hyperplane containing ρ_{P_0}. To prove that it is indeed singular, we may by (2.5) assume that α is the only simple root, i.e., that the F-rank of G is one. Since the constant function is square integrable on $G(F)\backslash G(A_F)$ and orthogonal to the space of cusp forms, the process of moving the contour of integration described in [4], § 7 proves that (2.6) must be singular. (In our situation, the difficult theory of [4, § 7] is superfluous and the process is quite similar to the well-known case $G=SL_2$). The fact that the pole (2.6) must be simple has been proved in [5], Lemma 98. The fact that $c(s_\alpha, \lambda)$ has no more poles on $\rho_{P_0}+i\check{\mathfrak{a}}_0$ has been proved in [4], p. 128, or [5], Theorem 7.1.

In the general case, we fix a minimal representation

$$w=s_{\alpha_1} \ldots s_{\alpha_k}, \qquad \alpha_i \in \Delta_0. \tag{2.7}$$

We put $w_j=s_{\alpha_{j+1}} \ldots s_{\alpha_k}$. Then

$$c(w, \lambda)=c(s_{\alpha_1}, w_1 \lambda) \ldots c(s_{\alpha_k}, \lambda).$$

We need the following sublemma:

8. Sublemma. Let v be the dimension of $\mathfrak{e}_\beta$, the eigenspace of the root β. If $\alpha \in \Phi_0$ is a positive root with $\alpha/2 \notin \Phi_0$, then

$$\langle \check{\alpha}, \rho_{P_0} \rangle \geqq v_\alpha + 2 v_{2\alpha} \tag{2.8}$$

and the equality occurs if and only if $\alpha \in \Delta_0$.

Proof of the sublemma. We compare the expressions

$$s_\alpha \rho_{P_0} = \rho_{P_0} - \alpha \langle \check{\alpha}, \rho_{P_0} \rangle$$

and

$$s_\alpha \rho_{P_0} = \rho_{P_0} - \sum_{\beta \in \Phi_0^+, s_\alpha \beta \notin \Phi_0^+} v_\beta \beta,$$

where Φ_0^+ is the set of positive roots. The inequality (2.8) follows at once. The equality occurs if and only if α and 2α are the only positive roots β with $s_\alpha \beta \in -\Phi_0^+$, i.e., if the length of s_α is one. In view of $\alpha/2 \notin \Phi_0$, this is equivalent to $\alpha \in \Delta_0$. The proof of the sublemma is complete.

By the minimality of the expression (2.7), we have $w_j^{-1} \alpha_j \in \Phi_0^+$. The sublemma implies

$$\langle \check{\alpha}_j, w_j \rho_{P_0} \rangle \geqq \langle \check{\alpha}_j, \rho_{P_0} \rangle,$$

and equality occurs if and only if $w_j^{-1} \alpha_j \in \Delta_0$. From (2.5) and the convergence assertion in [4, Lemma 4.1] or [5, Lemma 23] it follows that the factor $c(s_{\alpha_j}, w_j \lambda)$ is regular at $i\check{\mathfrak{a}}_0+\rho_{P_0}$ unless $w_j^{-1} \alpha_j \in \Delta_0$, in which case it has a simple pole along $\langle w_j^{-1} \alpha_j, \lambda - \rho_{P_0} \rangle = 0$, but no more singularities meeting $\rho_{P_0}+i\check{\mathfrak{a}}_0$. Since the roots $w_j^{-1} \alpha_j$ are precisely the positive roots β with $\beta/2 \notin \Phi_0$ and $w\beta < 0$, and each of them occurs only once, the Lemma 7 follows.

9. The end of the proof. Let $\Delta_0^P = \{\alpha \in \Delta_0 \mid \alpha \text{ vanishes on } \mathfrak{a}\}$. We put

$$c_P = \lim_{\lambda \to \rho_{P_0}} \Big(\prod_{\alpha \in \Delta_0^P} \langle \check{\alpha}, \lambda - \rho_{P_0} \rangle \Big) c(w_0^A, \lambda),$$

where w_0^A is the longest element of W^A. By a lemma of Langlands ([4, Lemma 3.6] or [5, Theorem 4]), our previous remarks imply

$$\lim_{\substack{\lambda\to 0\\ \lambda\in\check{\mathfrak{a}}_0^P}} \Big(\prod_{\alpha\in\Delta_0^P}\langle\check{\alpha},\lambda\rangle\Big) E_{P_0}^P(\lambda+\vartheta+\rho_{P_0}, g)=c_P\exp(\langle H_P(g),\vartheta+2\rho_P\rangle) \tag{2.9}$$

for $\vartheta\in\check{\mathfrak{a}}$. In fact, we can use (2.4) and Lemma 7 to compare the constant terms of both sides along P_0.

Using (2.9) and the obvious relation

$$E_{P_0}^G(\lambda, g)=\sum_{\gamma\in P(F)\backslash G(F)} E_{P_0}^P(\lambda,\gamma g),\qquad \lambda\in\rho_{P_0}+\check{\mathfrak{a}}_0^+$$

we get

$$\lim_{\substack{\lambda\to 0\\ \lambda\in\check{\mathfrak{a}}_0^P}} \Big(\prod_{\alpha\in\Delta_0^P}\langle\check{\alpha},\lambda\rangle\Big) E_{P_0}^G(\lambda+\vartheta+\rho_{P_0}, g)=c_P\, E_P^G(\vartheta+\rho_P, g). \tag{2.10}$$

Applying (2.10) once more with P replaced by G, we get

$$\lim_{\vartheta\to 0}\Big(\prod_{\alpha\in\Delta_0-\Delta_0^P}\langle\check{\alpha},\vartheta\rangle\Big) E_P^G(\vartheta+\rho_P, g)=c_G/c_P.$$

If we put $\vartheta=2(s-1)\rho_P$, we get the final result of our computations:

$$\lim_{s\to 1}(s-1)^{rk(P)}Z_{-2\rho}(s)=c_G/\Big(c_P\cdot\prod_{\alpha\in\Delta_0-\Delta_0^P}(2\langle\check{\alpha},\rho_P\rangle)\Big). \tag{2.11}$$

(It should perhaps be mentioned that in computing the right side of (2.11), one has to view ρ_P as an element of $\check{\mathfrak{a}}_0$, for in general $\check{\alpha}$ does not belong to $\mathfrak{a}$.)

By the well-known relation between the singularities of $E_{P_0}^G$ and of the functions $c(w,\cdot)$, Lemma 7 and (2.10) imply that there are at most $t=rk(P)$ singular hyperplanes of E_P^G containing ρ_F and no other singular hyperplanes meeting $\rho_{P_0}+i\check{\mathfrak{a}}_0$. It follows that $Z_{-2\rho}(s)$ has a pole of order $\leqq t$ at $s=1$, and that the t-th residue, given by (11), does not vanish. The proof of Theorem 5.b) is complete. Theorem 5.c) follows from [4, Corollary to Lemma 7.6].

10. Remarks. a) Suppose that G is of adjoint type and splits over F. We choose a Chevalley basis $\{X_\alpha\}_{\alpha\in\Phi_0}$ of the Lie algebra $\mathfrak{g}$, where Φ_0 is the root system of A_0. In the case of a non-archimedean place v, choose K_v to be the stabilizer of the corresponding Chevalley lattice in $\mathfrak{g}\otimes_F F_v$. If v is a real (resp. complex) place, we assume that the Cartan involution defining K_v is the unique automorphism (resp. anti-linear automorphism) of $\mathfrak{g}\otimes_F F_v$ which sends X_α to $X_{-\alpha}$. The functions $c(w,\lambda)$ are then given by

$$c(\omega,\lambda)=\prod_{\substack{\alpha>0\\ w\alpha<0}}\frac{\xi(\langle\check{\alpha},\lambda\rangle)}{\xi(\langle\check{\alpha},\lambda\rangle+1)}. \tag{2.12}$$

The product is over all positive roots α with $w\alpha<0$, and ξ is given by

$$\xi(s)=D^{s/2}(\pi^{-s/2}\Gamma(s/2))^{r_1}((2\pi)^{-s}\Gamma(s))^{r_2}\zeta_F(s),$$

where D is the discriminant of F and ζ_F is Dedekind's zeta function. Formula (2.12) is standard (cf. for instance [4, p. 285]). With the help of the functional equations for $c(w, \lambda)$, it is reduced to the case $G = PGL_2$, in which it is proved by a straightforward computation. Now, (2.12) implies a formula for the leading term of the asymptotics

$$\lim_{s \to 1} (s-1)^{rk(p)} Z_{-2\rho}(s)$$

$$= \prod_{\alpha \in \Phi_P - \Delta_0} \frac{\xi(\langle \check{\alpha}, \rho_P \rangle)}{\xi(\langle \check{\alpha}, \rho_P \rangle + 1)} \prod_{\alpha \in \Delta_0 - \Delta_0^P} \frac{\mathrm{Res}_{s=1} \xi}{\xi(2) \, 2 \langle \check{\alpha}, \rho_P \rangle},$$

where Φ_P is the set of roots of A_0 which occur in the radical of P.

Furthermore, the functional equations of the Eisenstein series imply a functional equation relating $Z_{-2\rho}(s)$ and $Z_{-2\rho}(1-s)$. In the case $V = \mathbb{P}^n(\mathbb{Q})$, this functional equation coincides with the functional equation derived by expressing $Z(s)$ as a quotient of the Epstein and the Riemann Zeta function.

b) Suppose that for every place v we are given a K_v-finite function φ_v on $(P \cap K_v) \backslash K_v$ such that $\varphi_v = 1$ for almost every v. If one wants to investigate the series

$$Z_\varphi(s) = \sum_{x \in V(F)} h_{-2\rho}(x)^{-s} \prod_v \varphi_v(x),$$

one can do so by means of Eisenstein series. For every $\lambda \in \check{\mathfrak{a}}_{\mathbb{C}}$ and every v, φ_v defines an element of $\mathrm{Ind}_{P(F_v)}^{G(F_v)}(\rho_\lambda)$, where $\rho_\lambda \colon P(F_v) \to \mathbb{C}$ is the character defined by λ. One has an Eisenstein series $E_P^G(\varphi, \lambda, g)$, $\lambda \in \check{\mathfrak{a}}_{\mathbb{C}}$, $g \in G(A_F)$ which meromorphically depends on λ, and which satisfies the equality

$$Z_\varphi(s) = E_P^G(\varphi, (2s-1)\rho, e).$$

The role of the numbers $c(w, \lambda)$ is now played by the intertwining operators $C(w, \lambda)$. The singular hyperplanes of the operator $C(w, \lambda)$ are given by (2.6) (the proof is the same as above), however the residue of $C(w, \lambda)$ along one of these hyperplanes may annihilate a part of $\mathrm{Ind}_{P(A_F)}^{G(A_F)}(\rho_\lambda)$. Parts a) and c) of 5. remain true, in b) the result is that the order of $Z(s)$ at $s=1$ is at most $rk(P)$.

Some remarks concerning the dependence of the higher residue

$$r = \lim_{s \to 1} (s-1)^{rk(P)} Z_\varphi(s) \tag{2.14}$$

on φ are perhaps in order. It is clear that r vanishes unless all hyperplanes (2.6) are indeed singular hyperplanes of $C(w_0, \lambda)$, where w_0 is the longest element of W. The operator C is a product of local intertwining operators

$$C(w_0, \lambda) = \prod_v C_v(w_0, \lambda)$$

over all places v. Let s be the finite set of places with $\varphi_v \neq 1$. For $v \notin S$, $C_v(w_0, \lambda)(\varphi_v) = c_v(w_0, \lambda)\, \varphi_v$, and $c(w_0, \lambda)$ is the product of $c_v(w_0, \lambda)$ over all places v. At $\lambda = \rho_{P_0}$, each $C_v(w_0, \lambda)$ is regular and different from zero. Lemma 7 implies that the singular hyperplanes of the product $\prod_{v \notin S} c_v(w_0, \lambda)$ are given by (2.6), their multiplicity is one. For each v, $C_v(w_0, \lambda)$ is regular at $\lambda = \rho_{P_0}$. It follows that r vanishes if and only if $C_v(w_0, \rho_{P_0})(\varphi_v) = 0$ for some $v \in S$.

One can prove that this happens if and only if $\int_{K_v} \varphi_v(k)\,dk=0$. If P_0 remains minimal over F_v, then our claim follows from the Langlands classification theorem. $C_v(w_0,\cdot)$ is proportional to the intertwining operator used in the proof of Langlands' theorem (cf. [10, Corollary 4.6 in Chapter IV and Proposition 2.6 in Chapter XI]), and the Langlands quotient of $\mathrm{Ind}_{P_0(F_v)}^{G(F_v)}(2\rho_{P_0})$ is the trivial one-dimensional representation. It follows that the projection of φ_v to this quotient vanishes iff $\int_{K_v} \varphi_v(k)\,dk=0$.

The same fact concerning the vanishing of $C_v(w_0,\rho_{P_0})(\varphi_v)$ can be proved by a direct computation using the integration formula which played a crucial role in the proof of Langlands' theorem. This computation goes through in the general case, i.e., without the assumption that P_0 is F_v-minimal.

Our remarks prove that (2.14) is proportional to $\int_K \varphi(k)\,dk$.

c) In a large number of cases, Lemma 7.2 in [4] can be used to prove that the poles of $Z_X(s)$ in the strip $1/2<\mathrm{Re}(s)\leqq 1$ are real. A non-real pole of $Z(s)$ in this strip gives rise to a singular hyperplane σ of $E_P^G(\cdot)$ which meets $\check{\mathfrak{a}}^+ + i\check{\mathfrak{a}}$ but for which $X(\sigma)$, the point of smallest norm in σ, is not real. For cuspidal Eisenstein series (i.e., $P=P_0$) this is impossible. In the general case one has to check that the image σ' of σ in $\check{\mathfrak{a}}+\rho_{P_0}-\rho_P\subset\check{\mathfrak{a}}_0$ belongs to the collection of hyperplanes constructed in [4, Theorem 7.7]. The main difficulty is to prove that σ' does not fall victim to a cancellation effect which is possible when residua along σ' several times occur in the process of moving the contour of integration. We don't know how to do this in general.

Appendix

Consider a primitive integer solution $x=(x_i)$ of the equation

$$x_0^3+2x_1^3+3x_2^3+4x_3^3=0$$

as a $\mathbb{Q}$-point on a minimal cubic surface V in $\mathbb{P}^3$. Put $h(x)=\sum_{i=0}^{3}|x_i|$. It is a height of x with respect to the obvious metrization of ω_V^{-1}. Denote by $x\circ y$ the third intersection point of V with the line passing through x, $y\in V(\mathbb{Q})$. Call x decomposable if for some $z\in V(Q)$ with $h(z)<h(x)$ we have $h(x\circ z)<h(x)$.

In the following table the numbers

$$N(H)=\mathrm{card}\{x\in V(\mathbb{Q})\,|\,h(x)\leqq H\},$$

$$N_i(H)=\mathrm{card}\{x\in V(\mathbb{Q})\,|\,h(x)\leqq H,\ x \text{ is indecomposable}\}$$

are given as a function of H. The table and the accompanying graph suggest the linear growth of both functions.

A complete table of solutions with $h(x)\leqq 200$ and their decompositions into solutions of lower height was first compiled by Yu.I. Manin. Similar lists for $h(x)\leqq 1100$ were calculated by Don Zagier to whom we are thankful for his permission to publish excepts from them.

Height ≤	Number of solutions	Number of indecomposable solutions
50	16	3
100	32	4
150	49	8
200	63	10
250	79	12
300	94	14
350	109	17
400	127	19
450	139	21
500	159	21
550	181	24
600	193	24
650	207	25
700	223	26
750	238	28
800	264	33
850	277	34
900	296	35
950	320	36
1000	340	39
1050	363	40
1100	379	42

Applying a similar search of points to a smooth cubic curve we would find that $N_i(H)$ stabilizes for H sufficiently large. This is equivalent to the Mordell-Weil theorem. Since for our cubic surface $N_i(H)$ does not show tendency to stabilize we hardly can hope that this fact generalizes. However, there still remains a possibility that all points with sufficiently large height can be generated by drawing secants and tangents starting from a finite set.

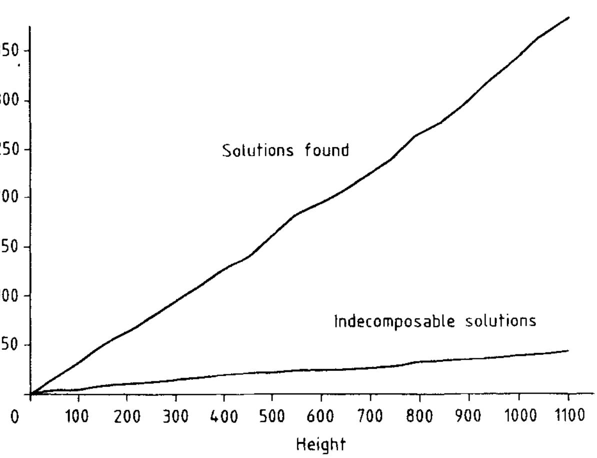

Fig. 1

References

1. Artin, M., Chai, C.-L. et al.: Arithmetic geometry. Cornell, G., Silverman, J.H. (eds.). Berlin-Heidelberg-New York: Springer 1986
2. Eisenbud, D., Harris, J.: The Kodaire dimension of the moduli space of curves of genus >23. Invent. Math. **90**, 359–388 (1987)
3. Schanuel, S.: Heights in number-fields. Bull. Soc. Math. France **107**, 433–449 (1979)
4. Langlands, R.P.: On the functional equations satisfied by Eisenstein series. (Lecture Notes in Math., Vol. 544). Berlin-Heidelberg-New York: Springer 1976
5. Harish-Chandra.: Automorphic forms on semisimple Lie groups. (Lecture notes in Math., Vol. 62). Berlin-Heidelberg-New York: Springer 1968
6. Weil, A.: Sur la formule de Siegel dans la théorie des groupes classiques. Acta Math. **113**, 1–87 (1965) [=Oe. Sci. [1965]]
7. Mars, J.G.M.: Sur l'approximation du nombre de solutions de certaines équations diophantiennes. Ann. Sci. EC. Norm. Super., IU. Ser. **6**, 357–387 (1973)
8. Igusa, J.-I.: Lectures on forms of higher degree. Tata Institute of Fund. Research. Berlin-Heidelberg-New York: Springer 1978
9. Patterson, S.J.: The Hardy-Littlewood method and Diophantine analysis in the light of Igusa's work. Math. Gött. Schriftenreihe Geom. Anal. **11**, 1–45 (1985)
10. Borel, A., Wallach, N.: Continuous cohomology, discrete subgroups, and representations of real reductive groups. (Annals of Math. Studies, Vol. 94). Princeton Univ. Press. Princeton 1980

Oblatum 14-I-1988 & 28-VI-1988