Invent. math. 95, 629–666 (1989)

Inventiones mathematicae

Eta invariants of Dirac operators on locally symmetric manifolds

Henri Moscovici* and Robert J. Stanton**
Department of Mathematics, The Ohio State University, Columbus, OH 43210-1174, USA

Introduction

The η-invariant of a self-adjoint elliptic differential operator on a compact manifold X was introduced by Atiyah, Patodi and Singer [A-P-S], in connection with the index theorem for manifolds with boundary. It is a spectral invariant which measures the asymmetry of the spectrum Spec(A) of such an operator A. To define it, one starts by setting, for $\operatorname{Re}(s) \gg 0$,

$$\eta(s, A) = \sum_{\lambda \in \operatorname{Spec}(A) - \{0\}} \frac{\operatorname{sgn} \lambda}{|\lambda|^s} = \operatorname{Tr}\left(A (A^2)^{-\frac{s+1}{2}}\right). \tag{0.1}$$

This is a holomorphic function which can be meromorphically continued to $\mathbb{C}$. Indeed, from the identity

$$\eta(s, A) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty t^{\frac{s-1}{2}} \operatorname{Tr}(A e^{-tA^2})\, dt \tag{0.2}$$

and the asymptotic behaviour of the heat operator at $t=0$, it follows that $\eta(s, A)$ admits a meromorphic extension to the whole s-plane, with at most simple poles at $s = \frac{\dim X - k}{\operatorname{ord} A}$, $(k = 0, 1, 2, \ldots)$ and locally computable residues. The remarkable, and considerably more difficult to establish, fact is that $s=0$ is not a pole, and this makes it possible to define the η-invariant of A by setting

$$\eta(A) = \eta(0, A). \tag{0.3}$$

In particular one can attach an η-invariant to any operator of Dirac type on a compact Riemannian manifold of odd dimension. (On even dimensional manifolds, Dirac operators have symmetric spectrum and, therefore, trivial η-

* Supported in part by NSF Grants DMS-8503357 and DMS-8803072
** Supported in part by NSF Grant DMS-8803072

invariants.) An important case of such an operator is the (even part of the) tangential signature operator, B, acting on the even forms of M; its η-invariant

(0.4) $$\eta_X = \eta(B)$$

is called the η-invariant of X.

Besides the essential role played in the index theorem for manifolds with boundary, where they contribute the non-local boundary correction terms, η-invariants of Dirac operators are closely related to several important invariants from differential topology (see [A-P-S], [D2], [K-S]). They have also been related to global anomalies in gauge theories (see [Wi], [B-F]).

For X a compact oriented $(4n-1)$-dimensional Riemannian manifold of constant negative curvature, Millson [M] has proved a remarkable formula relating η_X to the closed geodesics on X. Specifically, Millson defines a Selberg type zeta function by the formula

(0.5) $$\log Z(s) = \sum_{[\gamma] \neq 1} \frac{\operatorname{Tr}\tau_\gamma^+ - \operatorname{Tr}\tau_\gamma^-}{|\det(I - P_h(\gamma))|^{1/2}} \frac{e^{-s l(\gamma)}}{m(\gamma)},$$

where $[\gamma]$ runs over the nontrivial conjugacy classes in $\Gamma = \pi_1(X)$, $l(\gamma)$ is the length of the (unique) closed geodesic c_γ in the free homotopy class corresponding to $[\gamma]$, $m(\gamma)$ is the multiplicity of c_γ, $P_h(\gamma)$ is the restriction of the linear Poincaré map $P(\gamma) = d\Phi_1$ at $(c_\gamma, \dot{c}_\gamma) \in TX$ to the directions normal to the geodesic flow Φ_t and $\tau_\gamma^\pm$ is the parallel translation around c_γ on $\Lambda_\gamma^\pm = \pm i$ eigenspace of $\sigma_B(\dot{c}_\gamma)$, with σ_B denoting the principal symbol of B. He then proves that

(0.6) *$Z(s)$ admits a meromorphic continuation to the entire complex plane;*

(0.7) $$\log Z(0) = \pi i \eta_X;$$

and

(0.8) *$Z(s)$ satisfies the functional equation $Z(s)Z(-s) = e^{2\pi i \eta_X}$.*

The appropriate class of Riemannian manifolds for which a result of this type can be expected is that of non-positively curved locally symmetric manifolds, while the class of self-adjoint operators whose eta invariants are interesting to compute is that of Dirac-type operators, eventually with additional coefficients in locally flat bundles. It is the purpose of this paper to formulate and prove such an extension of Millson's formula.

We shall now present our main results. Let X denote a compact oriented odd-dimensional locally symmetric manifold, whose simply connected cover $\tilde{X}$ is a symmetric space of noncompact type. Let D denote a generalized Dirac operator associated to a locally homogeneous Clifford bundle over X. The fixed point set of the geodesic flow, acting on the unit sphere bundle T^1X, is a disjoint union of submanifolds X_γ, parametrized by the nontrivial conjugacy classes $[\gamma] \neq 1$ in $\Gamma = \pi_1(X)$. Each X_γ is itself a (possibly flat) locally symmetric manifold of nonpositive sectional curvature. We denote by $\mathscr{E}_1(\Gamma)$ the set of those conjugacy classes $[\gamma]$ for which X_γ has the property that the Euclidean de Rham factor of $\tilde{X}_\gamma$ is 1-dimensional. Thus, for $[\gamma]\in\mathscr{E}_1(\Gamma)$, $\tilde{X}_\gamma\cong\mathbb{R}\times\tilde{X}'_\gamma$ and the lines $\mathbb{R}\times\{x'\}$, $x'\in\tilde{X}'_\gamma$, are the axes of γ. Projected down to X_γ, they become closed geodesics, c_γ, which foliate X_γ. The space of leaves $\hat{X}_\gamma$ turns out to be an orbifold. The eigenvalues of absolute value 1 of the linear Poincaré map $P(\gamma)$ determine a bundle $C\hat{X}_\gamma$ over $\hat{X}_\gamma$ (the "center" bundle), and the parallel translation around the leaves c_γ gives rise to an orthogonal transformation $\hat{\tau}_\gamma$ of $C\hat{X}_\gamma$. $C\hat{X}_\gamma$ contains the tangent bundle $T\hat{X}_\gamma$ and we let $N\hat{X}_\gamma$ denote the orthogonal complement of $T\hat{X}_\gamma$ in $C\hat{X}_\gamma$. Since $T\hat{X}_\gamma$ corresponds to the eigenvalue 1 of $\hat{\tau}_\gamma$, $N\hat{X}_\gamma$ decomposes as

$$N\hat{X}_\gamma=N\hat{X}_\gamma(-1)\oplus\sum_{0<\theta<\pi}N\hat{X}_\gamma(\theta),$$

according to the other eigenvalues -1, $e^{\pm i\theta}(0<\theta<\pi)$.

The restriction to X_γ of the vector bundle $\mathbb{E}$, can be pushed down to a vector bundle $\hat{\mathbb{E}}_\gamma$ over $\hat{X}_\gamma$, which splits into subbundles $\hat{\mathbb{E}}_\gamma^\pm$ corresponding to the eigenvalue $\pm i$ of the symbol of D. One thus obtains a $\hat{\tau}_\gamma$-equivariant complex $\hat{\sigma}_\gamma^D\colon\hat{\mathbb{E}}_\gamma^+\to\hat{\mathbb{E}}_\gamma^-$ over $T\hat{X}_\gamma$ and, therefore, a class $[\hat{\sigma}_\gamma^D]\in K^0_{\hat{\tau}_\gamma}(T\hat{X}_\gamma)$, the $\hat{\tau}_\gamma$-equivariant K-theory group (with compact supports) of $T\hat{X}_\gamma$. As in [A-S; § 3], we can then form the cohomology class $ch\,\hat{\sigma}_\gamma^D(\hat{\tau}_\gamma)\in H^{ev}(T\hat{X}_\gamma;\mathbb{C})$. By analogy with the Lefschetz formula of Atiyah-Singer [A-S; Thm. (3.9)], and using the stable characteristic classes $\mathscr{R}$, $\mathscr{S}^\theta$ and $\mathscr{T}$ defined therein, we set:

$$(0.9)\quad L(\gamma,D)=\left\{\frac{ch\,\hat{\sigma}_\gamma^D(\hat{\tau}_\gamma)\,\mathscr{R}(N\hat{X}_\gamma(-1))\prod_{0<\theta<\pi}\mathscr{S}^\theta(N\hat{X}_\gamma(\theta))\,\mathscr{T}(\hat{X}_\gamma)}{\det(I-\hat{\tau}_\gamma|N\hat{X}_\gamma)}\right\}[T\hat{X}_\gamma].$$

For $[\gamma]\neq 1$, the closed geodesics c_γ in the free homotopy class associated to $[\gamma]\neq 1$ have the same length l_γ. If $[\gamma]\in\mathscr{E}_1(\Gamma)$, then $q=\frac{1}{2}\dim N\hat{X}_\gamma$ is integer and independent of γ. Also, for $[\gamma]\in\mathscr{E}_1(\Gamma)$, $\Gamma_\gamma^{\#}=\Gamma_\gamma\cap C_\gamma$, where C_γ is the connected center of G_γ, is infinite cyclic; we let $m_\gamma=[\Gamma_\gamma^{\#}:Z_\gamma]$, where Z_γ is the group generated by γ in Γ. Again for $[\gamma]\in\mathscr{E}_1(\Gamma)$, we denote by $P_h(\gamma)$ the hyperbolic part of the linear Poincaré map $P(\gamma)$, i.e., the restriction of $P(\gamma)$ to the subbundle of $TT^1X|TX_\gamma$ determined by the eigenvalues of absolute value <1 (stable) and >1 (unstable); this notation is consistent with that employed in (0.5).

Our main result establishes that a zeta function can be defined, initially for $\mathrm{Re}(s^2)\gg 0$, by the formula

$$(0.10)\qquad \log Z(s,D)=(-1)^q\sum_{[\gamma]\in\mathscr{E}_1(\Gamma)}\frac{L(\gamma,D)}{|\det(I-P_h(\gamma))|^{1/2}}\,\frac{e^{-sl_\gamma}}{m_\gamma},$$

and furthermore that:

(0.11) *$Z(s,D)$ has a meromorphic extension to the entire complex plane;*

$$(0.12)\qquad \eta(D)=\frac{1}{\pi i}\log Z(0,D);$$

and,

(0.13) *$Z(s, D)$ satisfies the functional equation*, $Z(s, D)\,Z(-s, D) = e^{2\pi i \eta(D)}$.

Besides the intricate way the geometric dependence of $\eta(D)$ is encoded in the Lefschetz-type coefficients, the new and surprising feature of formula (0.12) is the appearance of only rank one geodesics. An immediate consequence is the vanishing of all the eta invariants when G has no factors locally isomorphic to $SO(p, q)$, pq odd, or $SL(3, \mathbb{R})$. As mentioned before, the result can be extended to η-invariants with coefficients in flat bundles. In particular, we obtain zeta function formulae for the diffeomorphism (as opposed to metric) invariants defined by taking the signature with coefficients in a locally flat bundle of virtual dimension zero.

A few comments on the proof are now in order. Like Millson's, it is based on the use of the Selberg trace formula. We shall, therefore, highlight only the way in which the difficulties, not merely technical, posed by the handling of the arbitrary split-rank case are overcome. One starts by expanding $\mathrm{Tr}(De^{-tD^2})$ as a series of orbital integrals associated to the conjugacy classes $[\gamma]$ in Γ. Each such integral, over a necessarily semisimple orbit, can be in turn expressed in terms of the "noncommutative" Fourier transform of the odd heat kernel, along the tempered unitary dual of G, the group of isometries of the symmetric space $\tilde{X}$. One of the key results in this paper is the explicit calculation of $\mathrm{Tr}\,\pi(D)$ for $\pi = \pi_{P,\xi,\nu}$ a principal series representation induced off a parabolic $P = MAN$, which implies, in particular, that $\mathrm{Tr}\,\pi(D) = 0$, unless A is the split part of a fundamental Cartan subgroup and $\dim A = 1$. This explains the occurence of only one type of conjugacy classes, namely $\mathscr{E}_1(\Gamma)$, in (0.10). More importantly, it makes it possible to bring the expression for $\mathrm{Tr}(De^{-tD^2})$ to a manageable, albeit still group-theoretical, form. The transition to the geometric form, specifically the expression (0.9) of the "Lefschetz numbers" $L(\gamma, D)$ requires some additional work, analogous to computations in [H-P] and [Sc]. Finally, the meromorphic continuation as well as the functional equation for the zeta function $Z(s, D)$ are proved by identifying $Z(s, D)$ as an infinite determinant (defined by the "high temperature" regularization) of the Cayley transform $\dfrac{D - is}{D + is}$.

Contents

§ 1. Dirac bundles

To establish our notation, we recall in this section some standard material on Dirac bundles; for details we refer the reader to [L-M].

Let X denote a compact Riemannian manifold and let $T(X)$ denote the tangent bundle of X and $\mathbb{C}l(X)$ the complexified Clifford bundle. Let $\mathbb{E}$ be a complex vector bundle over X and suppose that there is a bundle map from $\mathbb{C}l(X)$ to $\operatorname{End}\mathbb{E}$ that is an algebra homomorphism on each fiber and covers the identity

$$\begin{array}{ccc} \mathbb{C}l(X) & \longrightarrow & \operatorname{End}\mathbb{E} \\ & \searrow \quad \swarrow & \\ & X & \end{array} \tag{1.1}$$

Given such a structure there always exists an inner product on each fiber $\mathbb{E}_x$ for which unit vectors in $T(X)_x \subseteq \mathbb{C}l(X)_x$ act by unitary transformations. A bundle $\mathbb{E}$ together with such a $\mathbb{C}l(X)$ action and smoothly varying inner product will be called a Clifford module bundle.

Since X is Riemannian, there is a canonical connection on $T(X)$ and hence on $\mathbb{C}l(X)$. We denote that connection by ∇^R. A Clifford module bundle is called a Dirac bundle if it has a connection ∇ satisfying the compatibility condition

$$\nabla_Z(v\cdot s) = (\nabla_Z^R v)\cdot s + v\cdot(\nabla_Z s) \tag{1.2}$$

where s is a local section of $\mathbb{E}$, v is a local section of $\mathbb{C}l(X)$, Z a vector field and $\cdot$ denotes the module multiplication. On a Dirac bundle one then has a Dirac operator defined by

$$Ds = \sum_i e_i \cdot (\nabla_{e_i} s)$$

where $\{e_i\}$ is any local orthonormal frame for X.

Our concern, starting in § 3, will be with bundles that satisfy one further condition, namely local homogeneity. To define this we let $\tilde{X}$ be the simply connected cover of X and for any vector bundle $\mathbb{E}$ over X let $\tilde{\mathbb{E}}$ denote the pull-back to $\tilde{X}$. Let G be a group that acts on $\tilde{X}$ by isometries.

Definition. A vector bundle $\mathbb{E}$ over X is G-locally homogeneous if there is a smooth action of G on $\tilde{\mathbb{E}}$ which is linear on the fibers and covers the action of G on $\tilde{X}$.

Notice that for any such G, $T(X)$ is G-locally homogeneous in a natural way via the differential. Hence so is any bundle obtained from $T(X)$ by tensor products. Since G acts by isometries on $\tilde{X}$, it follows that there is a smooth action of G on $\mathbb{C}l(\tilde{X})$; thus $\mathbb{C}l(X)$ is G-locally homogeneous. Likewise, other standard constructions from linear algebra applied to any G-locally homogeneous $\mathbb{E}$ will give in a natural way corresponding G-locally homogeneous vector bundles. In particular, $\operatorname{End}\mathbb{E} \simeq \mathbb{E}^* \otimes \mathbb{E}$ is G-locally homogeneous whenever $\mathbb{E}$ is.

When we work with G-locally homogeneous bundles we shall require all constructions to be G-equivariant. For example, if $\mathbb{E}$ is a Clifford module bundle which is also G-locally homogeneous, then we shall require the natural action on $\mathbb{C}l(\tilde{X})$ and $\operatorname{End}(\tilde{\mathbb{E}})$ to be equivariant with the module action, that is, for each g in G we have the commutative diagram

$$\begin{array}{ccc} \mathbb{C}l(\tilde{X}) & \longrightarrow & \operatorname{End}\tilde{\mathbb{E}} \\ \downarrow g & & \downarrow g \\ \mathbb{C}l(\tilde{X}) & \longrightarrow & \operatorname{End}\tilde{\mathbb{E}} \end{array} \quad . \tag{1.3}$$

Similarly, if $\mathbb{E}$ is a Dirac bundle which is G-locally homogeneous we shall require G-equivariance for $\tilde{\nabla}$ the lift of ∇ to $\tilde{\mathbb{E}}$. Thus the corresponding Dirac operator $\tilde{D}$ is then G-equivariant, i.e., D is G-locally homogeneous.

When G is $I_+(\tilde{X})$, the full connected group of orientation preserving isometries of $\tilde{X}$, we shall refer to G-locally homogeneous bundles as locally homogeneous. This agrees with the usual terminology when $\tilde{X}$ is a homogeneous space G/K with $G=I_+(\tilde{X})$.

Returning to the general situation, one knows that a Dirac operator is elliptic and is essentially self-adjoint. We denote its closure acting in the Hilbert space of square integrable sections of $\mathbb{E}$ also by D.

§ 2. The Cayley transform determinant

To motivate our infinite determinant construction we consider first a self-adjoint operator on a finite dimensional Hilbert space. The Cayley transform of such an operator D is the unitary operator

$$C=\frac{D-i}{D+i}.$$

More generally, for $s\in\mathbb{C}$, consider the family of operators

$$C(s)=\frac{D-is}{D+is}.$$

This family is meromorphic, with poles at $s\in i\operatorname{Spec}'(D)(\operatorname{Spec}(D)-\{0\})$, all of which are simple, and having residue

$$\operatorname{res}_{-i\lambda} C(s)=2i\lambda P_\lambda,$$

where P_λ is projection onto the $i\lambda$ eigenspace. For $\lambda\in\operatorname{Spec}(D)$ let $m(\lambda)$ denote the multiplicity. One has

$$\det C(s)=(-1)^{m(0)}\det' C(s)$$

where

$$\det' C(s)=\prod_{\lambda\in\operatorname{Spec}'(D)}\left(\frac{\lambda-is}{\lambda+is}\right)^{m(\lambda)}.$$

Set $\lambda_0 = \min_{\lambda \in \mathrm{Spec}'(D)} |\lambda|$ and let log be the principal branch of the logarithm. If s is in the plane cut from $\pm i\lambda_0$ to $\pm i\infty$, one has

$$\log \det{}' C(s) = \sum_{\lambda \in \mathrm{Spec}'(D)} m(\lambda) \log\left(\frac{\lambda - is}{\lambda + is}\right)$$

and

$$\frac{d}{ds} \log \det{}' C(s) = -\sum_{\lambda \in \mathrm{Spec}'(D)} m(\lambda)\left(\frac{i}{\lambda + is} + \frac{i}{\lambda - is}\right) = -2i \sum_{\lambda \in \mathrm{Spec}'(D)} m(\lambda) \frac{\lambda}{\lambda^2 + s^2}$$

or

$$\frac{d}{ds} \log \det{}' C(s) = -2i \operatorname{Tr} \frac{D}{D^2 + s^2} \tag{2.1}$$

Thus we obtain the following characterization:

(2.2) $\det{}' C(s)$ is the unique meromorphic function whose logarithmic derivative satisfies (2.1) and normalized by $\det{}' C(0) = 1$.

Let now D be a Dirac operator as in § 1. As in the finite dimensional case, the family of operators

$$C(s) = \frac{D - is}{D + is}$$

is meromorphic with simple poles at $s \in i\,\mathrm{Spec}'(D)$. We shall show that there is a unique determinant function; $\det{}' C(s)$, as in (2.2). Since $D(D^2 + s^2)^{-1}$ is not a trace class operator, first we shall describe the high temperature regularization of the trace.

Theorem 2.1

(a)
$$\operatorname{Tr}^0\left(\frac{D}{D^2 + s^2}\right) = \lim_{\varepsilon \downarrow 0} \operatorname{Tr}\left(\frac{D}{D^2 + s^2} e^{-\varepsilon(D^2 + s^2)}\right)$$

is a meromorphic function with simple poles $\{\pm i\lambda \mid \lambda \in \mathrm{Spec}'(D)\}$ *and residues*

$$\operatorname{res}_{i\lambda} \operatorname{Tr}^0\left(\frac{D}{D^2 + s^2}\right) = \frac{1}{2i}(m(\lambda) - m(-\lambda)).$$

(b) *For* $\operatorname{Re} s^2 > -\lambda_0^2$ *one has*

$$\operatorname{Tr}^0\left(\frac{D}{D^2 + s^2}\right) = \int_0^\infty e^{-ts^2} \operatorname{Tr}(D e^{-tD^2})\, dt.$$

(c) *For any* $\varepsilon > 0$ *one has*

$$\operatorname{Tr}^0\left(\frac{D}{D^2 + s^2}\right) = \int_0^\varepsilon e^{-ts^2} \operatorname{Tr}(D e^{-tD^2})\, dt + \operatorname{Tr}\left(\frac{D}{D^2 + s^2} e^{-\varepsilon(D^2 + s^2)}\right).$$

Proof. We need estimates on $\mathrm{Tr}(De^{-tD^2})$ for t small and t large. The first is obtained from [B–F]: $\mathrm{Tr}(De^{-tD^2})=O(t^{1/2})$, $t\downarrow 0$. The other estimate is elementary. Fix $t_0>0$. Then for $t\geqq t_0$

$$\begin{aligned}|\mathrm{Tr}(De^{-tD^2})|\leqq \mathrm{Tr}(|D|e^{-tD^2}) &= \sum_{v\in\mathrm{Spec}'(|D|)} m(v)\,v\,e^{-tv^2}\\ &= e^{-t\lambda_0^2}\sum_{v\in\mathrm{Spec}'(|D|)} m(v)\,v\,e^{-t(v^2-\lambda_0^2)}\\ &\leq e^{-t\lambda_0^2}\sum_{v\in\mathrm{Spec}'(|D|)} m(v)\,v\,e^{-t_0(v^2-\lambda_0^2)}\\ &= e^{-(t-t_0)\lambda_0^2}\sum_{v\in\mathrm{Spec}'(|D|)} m(v)\,v\,e^{-t_0v^2}\\ &= c\,e^{-t\lambda_0^2}.\end{aligned}$$

These two estimates allow us to conclude that the function

$$\Psi(s)=\int_0^\infty e^{-ts^2}\,\mathrm{Tr}(De^{-tD^2})\,dt \tag{2.3}$$

is analytic for $\mathrm{Re}\,s^2>-\lambda_0^2$. Actually, from the estimates we can conclude more. Fix $\varepsilon>0$ and write

$$\Psi(s)=\int_0^\varepsilon e^{-ts^2}\,\mathrm{Tr}((De^{-tD^2})\,dt+\int_\varepsilon^\infty e^{-ts^2}\,\mathrm{Tr}(De^{-tD^2})\,dt. \tag{2.4}$$

It is obvious that the first integral is entire, and (using [B-F]) that it has limit zero as $\varepsilon\downarrow 0$, uniformly on compact subsets. For the second integral, if $\mathrm{Re}\,s^2>-\lambda_0^2$ we may use Fubini to get

$$\int_\varepsilon^\infty e^{-ts^2}\,\mathrm{Tr}(De^{-tD^2})\,dt=\mathrm{Tr}\int_\varepsilon^\infty De^{-t(s^2+D^2)}\,dt=\mathrm{Tr}\left(\frac{D}{D^2+s^2}\,e^{-\varepsilon(D^2+s^2)}\right)$$

Now $(D^2+s^2)^{-1}$ is a meromorphic operator-valued function and for each $\varepsilon>0$, $\mathrm{Tr}\left(\frac{D}{D^2+s^2}\,e^{-\varepsilon(D^2+s^2)}\right)$ is a meromorphic function with poles at $\{\pm i\lambda\,|\,\lambda\in\mathrm{Spec}'(D)\}$ and

$$\mathrm{res}_{i\lambda}\,\mathrm{Tr}\left(\frac{D}{D^2+s^2}\,e^{-\varepsilon(D^2+s^2)}\right)=\frac{1}{2i}\,[m(\lambda)-m(-\lambda)].$$

Hence, for each $\varepsilon>0$, the right hand side of (2.4) defines a meromorphic continuation of $\Psi(s)$, which must be unique. Denote it by $\mathrm{Tr}^0\left(\frac{D}{D^2+s^2}\right)$. Then this meromorphic function has all the properties stated in the theorem. □

We are now in the position to make the definition:

(2.5) $\det'\left(\frac{D-is}{D+is}\right)$ is the unique meromorphic function whose logarithmic derivative is $\frac{2}{i}\operatorname{Tr}^0\left(\frac{D}{D^2+s^2}\right)$ and whose value at $s=0$ is 1.

Recall now the definition of the η-invariant, (0.1)–(0.3). From (0.2) and the estimate in [B-F], $\operatorname{Tr}(De^{-tD^2})=O(t^{1/2})$, it follows that the integral converges and defines $\eta(s)$ on $\operatorname{Re} s>-2$; in particular

$$\eta(D)=\frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}\operatorname{Tr}(De^{-tD^2})\,dt.$$

Proposition 2.2. $\lim_{x\to+\infty}\det'\frac{D-ix}{D+ix}=e^{-\pi i\eta(D)}$.

Proof. Since

$$\int_0^\infty dt\int_0^\infty e^{-ts^2}|\operatorname{Tr}(De^{-tD^2})|\,ds=\frac{\sqrt{\pi}}{2}\int_0^\infty t^{-1/2}|\operatorname{Tr}(De^{-tD^2})|\,dt<\infty$$

one can use Fubini's theorem to obtain

$$\begin{aligned}\int_0^\infty \Psi(s)\,ds&=\int_0^\infty ds\int_0^\infty e^{-ts^2}\operatorname{Tr}(De^{-tD^2})\,dt\\&=\int_0^\infty dt\int_0^\infty e^{-ts^2}\operatorname{Tr}(De^{-tD^2})\,ds\\&=\frac{\sqrt{\pi}}{2}\int_0^\infty t^{-1/2}\operatorname{Tr}(De^{-tD^2})\,dt\\&=\frac{\pi}{2}\eta(D).\end{aligned}$$

Thus

$$\lim_{x\to\infty}\log\det'\frac{D-ix}{D+ix}=\frac{2}{i}\int_0^\infty\Psi(s)\,ds=-\pi i\eta(D).\quad\square$$

We note that since $\operatorname{Tr}\left(\frac{D}{D^2+s^2}e^{-\varepsilon(D^2+s^2)}\right)$ is invariant under $s\to -s$, our determinant satisfies the functional identity

(2.6) $$\det'\frac{D+is}{D-is}\det'\frac{D-is}{D+is}=1.$$

Remark. Set $\varepsilon = 1/x$. If one replaces the metric g by $g_\varepsilon = g/\varepsilon^2$ then Proposition 2.2 says that the adiabatic limit of the determinant of the Cayley transform of D_ε is $e^{-\pi i \eta(D)}$.

§ 3. Dirac operators on locally symmetric spaces

For the remainder of the paper we shall require X to be locally symmetric. We will then use harmonic analysis to study the kernel of the odd heat operator directly, rather than through its spectral decomposition as in § 2. More precisely we follow the familiar Selberg approach and evaluate the trace of the odd heat operator, De^{-tD^2}, by means of orbital integrals. The success, in this instance, of this approach ultimately rests on the computation of the Fourier transform of the Dirac operator in Proposition 3.6.

Let $\tilde{X}$ be a globally symmetric space of noncompact type and dimension $2n+1$, and let G denote the connected component of the group of orientation-preserving isometries of $\tilde{X}$. Then G is a connected semisimple Lie group, and if K is a fixed maximal compact subgroup, then $\tilde{X}$ is naturally isometric to G/K.

Let $\mathfrak{p}$ denote the tangent space to $\tilde{X}$ at eK and denote by Spin$(\mathfrak{p})$ the usual $\mathbb{Z}_2$ cover of $SO(\mathfrak{p})$ contained in the Clifford algebra $\mathbb{C}l(\mathfrak{p})$. Since the dimension of $\mathfrak{p}$ is odd, $\mathbb{C}l(\mathfrak{p})$ has exactly two distinct simple modules $(c_\pm, L_\pm)$; these modules, however, when restricted to Spin$(\mathfrak{p})$ are equivalent. Passing to a covering group if necessary, we may suppose K maps into Spin$(\mathfrak{p})$. Let (σ, S) denote the representation of K obtained from either of these modules. We shall refer to (σ, S) as the spin representation of K.

Lemma 3.1. *Let $\tilde{X}$ be an odd dimensional homogeneous space G/K and $\mathbb{E}$ a G-homogeneous Clifford module bundle over $\tilde{X}$. Then $\mathbb{E}$ is associated to a representation of K of the form $(\sigma \otimes \tau, S \otimes V)$.*

Proof. Let E be the vector space $\mathbb{E}_{eK}$ and let $c(\cdot)$ denote the action of $\mathbb{C}l(\mathfrak{p})$ on E. Since $\mathbb{E}$ is homogeneous, there is a representation (ρ, E) of K on $\mathbb{E}_{eK}$, with $\mathbb{E}$ associated to (ρ, E).

Now E is also a module for $\mathbb{C}l(\mathfrak{p})$, with $\mathfrak{p}$ odd dimensional, and so as $\mathbb{C}l(\mathfrak{p})$ module

$$E \simeq L_+ \otimes V_+ \oplus L_- \otimes V_- .$$

Here $L_\pm \otimes V_\pm$ are the ± 1 eigenspaces of $c(\omega^{\mathbb{C}})$, $\omega^{\mathbb{C}} \in \mathbb{C}l(\mathfrak{p})$ the complex volume element ($\omega^{\mathbb{C}} = i^{l+1} e_1 \dots e_{2n+1}$, $n \equiv l(2)$). We shall show that each of $L_\pm \otimes V_\pm$ are of the form $S \otimes V$ as K-modules.

Restricting $c(\cdot)$ to K, one gets

$$(3.1) \qquad (c(\cdot)|_K, E) \simeq (\sigma \otimes \mathbb{1} \oplus \sigma \otimes \mathbb{1}, S \otimes V_+ \oplus S \otimes V_-)$$

Using the G-equivariance (1.3), in particular K-equivariance, gives

$$(3.2) \qquad \rho(k)\, c(v)\, \rho(k^{-1}) = c(kvk^{-1}),$$

where K is viewed as a subgroup of $\mathrm{Spin}(\mathfrak{p}) \subseteq \mathbb{C}l(\mathfrak{p})$. Since $\omega^{\mathbb{C}}$ is central, from (3.2) it follows that $\rho(k)$ acts on each of $L_+ \otimes V_+$ and $L_- \otimes V_-$. From (3.1) we get $c_\pm(kvk^{-1}) = \sigma\otimes\mathbb{1}(k)\, c_\pm(v)\, \sigma\otimes\mathbb{1}(k^{-1})$, thus $\sigma\otimes\mathbb{1}(k^{-1})\,\rho(k)$ intertwines the action of $\mathbb{C}l(\mathfrak{p})$. Hence for each $k\in K$ there are $\tau_\pm \in \mathrm{End}(V_\pm)$ with

$$\sigma\otimes\mathbb{1}(k^{-1})\,\rho(k)|_{L_\pm\otimes V_\pm} = \mathbb{1}\otimes\tau_\pm(k).$$

If suffices to show $\tau_\pm$ is a homomorphism.

$$\begin{aligned}\mathbb{1}\otimes\tau_\pm(k_1k_2) &= \sigma\otimes\mathbb{1}(k_1k_2)^{-1}\,\rho(k_1k_2)|_{L_\pm\otimes V_\pm}\\ &= \sigma\otimes\mathbb{1}(k_2^{-1})\,\sigma\otimes\mathbb{1}(k_1)^{-1}\,\rho(k_1)\,\rho(k_2)|_{L_\pm\otimes V_\pm}\\ &= \sigma\otimes\mathbb{1}(k_2^{-1})\,\mathbb{1}\otimes\tau_\pm(k_1)\,\rho(k_2)|_{L_\pm\otimes V_\pm}\\ &= \mathbb{1}\otimes\tau_\pm(k_1)\,\mathbb{1}\otimes\tau_\pm(k_2). \quad \square\end{aligned}$$

We identify $\Gamma(\tilde{\mathbb{E}})$, the space of smooth sections of $\tilde{\mathbb{E}}$, with $[C^\infty(G)\otimes S\otimes V]^K$, where K acts on $C^\infty(G)$ via the right regular representation $R(G)$. On $\Gamma(\tilde{\mathbb{E}})$ there is a natural connection $\nabla\colon \Gamma(\tilde{\mathbb{E}}) \to \Gamma(\tilde{\mathbb{E}}\otimes T^*(\tilde{X}))$, given by

$$\nabla f = \Sigma(R(X_i)\otimes\mathbb{1})f\otimes X_i^*.$$

Here $\{X_i\}$ is a basis of $\mathfrak{p}$ and $\{X_i^*\}$ the dual basis. Clearly ∇ commutes with the natural action of G on $\Gamma(\tilde{\mathbb{E}})$, but it also anti-commutes with the action of the Cartan involution θ on sections. Indeed, if $X\in\mathfrak{p}$,

$$\begin{aligned}R(X)f^\theta(g) &= \frac{d}{dt} f^\theta(g\exp tX)|_{t=0}\\ &= \frac{d}{dt} f(\theta(g)\exp - tX)|_{t=0}\\ &= -(R(X)f)^\theta(g).\end{aligned}$$

Lemma 3.2. *Let $\tilde{\mathbb{E}}$ be a homogeneous vector bundle over $\tilde{X}$. Then there is a unique connection on $\Gamma(\tilde{\mathbb{E}})$ that is G-homogeneous and anti-commutes with the Cartan involution, θ.*

Proof. Let ∇ be the natural connection and ∇' any other connection as in the Lemma. Then $\nabla' - \nabla$ is of order zero, i.e., $\nabla' = \nabla + \Sigma L_i\otimes X_i^*$ where $L_i\in\mathrm{End}(E)$ and such that $L = \Sigma L_i\otimes X_i^*$ in $\mathrm{Hom}(E, E\otimes\mathfrak{p}^*)$ is K-equivariant. Since ∇' and ∇ anti-commute with θ, so must L. But L is of order zero, so $(Lf)(x) = L(f(x))$ and hence must commute with θ. $\square$

Corollary 3.3. *Let $\tilde{X}$ be an odd dimensional symmetric space and $\tilde{\mathbb{E}}$ a G-homogeneous Clifford module bundle over $\tilde{X}$. Then on $\Gamma(\tilde{\mathbb{E}})$ there exists an essentially unique Dirac operator which is G-homogeneous and anti-commutes with the Cartan involution.*

Proof. Let ∇ be the unique connection on $\Gamma(\mathbb{E})$ given by Lemma 3.2. Let $s \in \Gamma(\mathbb{E}) \simeq [C^\infty(G) \otimes S \otimes V]^K$ and $v \in \Gamma(\mathbb{C}l(\tilde{X})) \simeq [C^\infty(G) \otimes \mathbb{C}l(\mathfrak{p})]^K$. Then

$$\begin{aligned}\nabla(v \cdot s)(g) &= \Sigma(R(X_i) \otimes \mathbb{1})(v(g) \cdot s(g)) \otimes X_i^* \\ &= \Sigma[R(X_i)\, v(g)] \cdot s(g) \otimes X_i^* \\ &\quad + \Sigma v(g) \cdot R(X_i)\, s(g) \otimes X_i^* \\ &= (\nabla v) \cdot s(g) + (v \cdot \nabla s)(g).\end{aligned}$$

Hence $\mathbb{E}$ is a Dirac bundle and thus has a Dirac operator. It follows from the properties of ∇ that this Dirac operator is G-homogeneous and anti-commutes with θ. On the other hand, suppose $\mathbb{E}$ is a Dirac bundle with a homogeneous Dirac operator. Since the module structure on $\mathbb{E}$ is G-homogeneous as is the Dirac operator, it follows that the connection must be. Hence (Lemma 3.2) it is natural connection, and the Dirac operator is the one described previously. □

Henceforth, we fix $\mathbb{E}$, a G-homogeneous Clifford module bundle on $\tilde{X}$. We shall use the Dirac operator

$$\tilde{D} = \sum_i R(X_i) \otimes c(X_i)\, c(\omega^{\mathbb{C}}),$$

here $\{X_i\}$ is an oriented orthonormal basis of $\mathfrak{p}$ and $c(\cdot)$ denotes Clifford multiplication on E. The twist with the volume element enables us to handle the general case when both simple modules $L_\pm$ occur. When only one occurs, this operator is a scalar times the usual Dirac operator. This invariant operator $\tilde{D}$ is known to be elliptic and formally self-adjoint. More generally, if (π, H_π) is any unitary representation of G with smooth vectors H_π^∞, define an operator on $[H_\pi^\infty \otimes S \otimes V]^K$ by

$$\tilde{D}_\pi = \sum_i \pi(X_i) \otimes c(X_i)\, c(\omega^{\mathbb{C}}).$$

For computations, it is convenient to identify $c(v)$ with $c_+(v) \otimes I_+ \oplus c_-(v) \otimes I_-$, $v \in \mathbb{C}l(\mathfrak{p})$, and $c(x)$ with $s(x) \otimes I$, $x \in \mathrm{Spin}(\mathfrak{p})$. We shall use this identification freely in this section.

Combining the computations in [B-W] p. 68 for the non-equirank case without coefficients, with those in [A-Sc] p. 54 for the equirank case but with coefficients, one gets a formula for $\tilde{D}_\pi^2$ on $[H_\pi^\infty \otimes S \otimes V]^K$,

$$\begin{aligned}\tilde{D}_\pi^2 = &-\pi(\Omega) \otimes I \otimes I - I \otimes \sigma(\Omega_K) \otimes I \\ &+ I \otimes I \otimes \tau(\Omega_K).\end{aligned}$$

Here Ω is the Casimir operator of G and Ω_K the Casimir operator of K, constructed using the Killing form of $\mathfrak{g}$. When (τ_μ, V) is irreducible, the formula simplifies to

$$\tilde{D}_\pi^2 = -\pi(\Omega) \otimes I \otimes I + (\|\mu + \rho_k\|^2 - \|\rho\|^2) I, \tag{3.3}$$

where μ, ρ_k and ρ are as in [B-W].

We denote by $e^{-t\tilde{D}^2}$ the heat operator for the non-negative operator $\tilde{D}^2$, and summarize its main properties.

(3.4) The Schwartz kernel of $e^{-t\tilde{D}^2}$ can be identified with a section, $\tilde{h}_t$, in $[C^\infty(G)\otimes \mathrm{End}(S\otimes V)]^{K\times K}$ that acts by convolution on $[C^\infty(G)\otimes S\otimes V]^K$.

Let $\mathscr{C}^p(G)$ denote the Harish-Chandra p-integrable Schwartz space and set $\mathscr{S}(G)=\bigcap_{p>0}\mathscr{C}^p(G)$.

(3.5) For each $t>0$, $\tilde{h}_t$ is in $[\mathscr{S}(G)\otimes \mathrm{End}(S\otimes V)]^{K\times K}$.

This is proved in [B-M] for even dimensional $\tilde{X}$ but the argument is valid for odd dimensions as well using (3.3).

(3.6) If (π, H_π) is an irreducible unitary representation of G and (τ, V) is irreducible, then on the finite dimensional space $[H_\pi^\infty\otimes S\otimes V]^K$ we have

$$\pi(\tilde{h}_t)=e^{-t\tilde{D}_\pi^2}=e^{t(\|\Lambda+\rho\|^2-\|\mu+\rho_k\|^2)}I$$

where Ω acts on H_π by the scalar $\|\Lambda+\rho\|^2-\|\rho\|^2$.

(Again see [B-M]).

(3.7) For each $t>0$, the odd heat operator $\tilde{D}e^{-t\tilde{D}^2}$ has kernel

$$\tilde{k}_t\in[\mathscr{S}(G)\otimes \mathrm{End}(S\otimes V)]^{K\times K}.$$

Indeed, this follows from (3.5) and the fact that $\mathscr{S}(G)$ is invariant under R.

We shall need these constructs on locally symmetric spaces. So let Γ be a discrete, torsion free subgroup of G with $X=\Gamma\backslash\tilde{X}$ compact. Since $\tilde{\mathbb{E}}$ is homogeneous we may form $\mathbb{E}=\Gamma\backslash\tilde{\mathbb{E}}$. Smooth sections of $\mathbb{E}$ may be identified with $[C^\infty(\Gamma\backslash G)\otimes S\otimes V]^K$. We let D denote the differential operator induced on sections of $\mathbb{E}$ by $\tilde{D}$. Then D is elliptic, formally self-adjoint, with finite dimensional kernel. The corresponding heat operator e^{-tD^2} defines a trace class operator on $[L^2(\Gamma\backslash G)\otimes S\otimes V]^K$ and has kernel h_t equal to a smooth $\mathrm{End}(S\otimes V)$-valued function on $\Gamma\backslash G\times\Gamma\backslash G$ with $\mathrm{Tr}\, e^{-tD^2}=\int_{\Gamma\backslash G}\mathrm{tr}\, h_t(\dot{x},\dot{x})\, d\mu(\dot{x})$. The kernels h_t and $\tilde{h}_t$ are related by

$$h_t(p,q)=\sum_{\Gamma}\tilde{h}_t(y^{-1}\gamma x), \tag{3.8}$$

where $p=\Gamma x$ and $q=\Gamma y$, x, y in G.

To compute $\mathrm{Tr}\, De^{-tD^2}$ it will be enough to evaluate the Fourier transform of the odd heat kernel on the tempered unitary dual. For this we need a more explicit formula for $\tilde{D}_\pi$, where π is a representation induced from a parabolic subgroup.

Consider the Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$, and let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. Extend $\mathfrak{a}$ to a Cartan subalgebra $\mathfrak{h}_\mathbb{C}=\mathfrak{t}_\mathbb{C}\oplus\mathfrak{a}_\mathbb{C}$ and let Δ be the roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. For the normalization of root vectors and similar facts, we refer to [He]. Set $\Delta_{\mathfrak{p}} = \{\alpha \in \Delta \mid \alpha \neq \alpha^{\theta}\}$. For each $\alpha \in \Delta_{\mathfrak{p}}$ we choose root vectors E_α with the following properties:

(i) $E_\alpha = Y_\alpha + X_\alpha$ with $Y_\alpha \in \mathfrak{k}_{\mathbb{C}}$, $X_\alpha \in \mathfrak{p}_{\mathbb{C}}$ and $\theta E_\alpha = -E_{-\alpha}$.

(ii) Denoting the Killing form by (,), we have:

$$(E_\alpha, E_\alpha) = 0; \quad (E_\alpha, \theta E_\alpha) = 1; \quad (X_\alpha, X_\alpha) = \tfrac{1}{2};$$

$$(X_\alpha, Y_\beta) = 0; \quad (X_\alpha, X_\beta) = 0 = (Y_\alpha, Y_\beta) \quad \alpha \neq \beta.$$

(iii) Set $A_\alpha = -[E_\alpha, \theta E_\alpha] = 2[Y_\alpha, X_\alpha]$ in $\mathfrak{a}_{\mathbb{C}}$. Then if H is in $\mathfrak{a}_{\mathbb{C}}$, $(H, A_\alpha) = \alpha(H)$, and $(A_\alpha, X_\beta) = 0 = (A_\alpha, Y_\beta)$, for all $\beta \in \Delta_{\mathfrak{p}}$.

(iv) Define $N_{\alpha,\beta}$, α, β in $\Delta_{\mathfrak{p}}$ by $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$ if $\alpha+\beta \in \Delta$ and zero otherwise. Then $N_{\alpha,\beta} = -N_{\beta,\alpha}$, and if α, β, γ are in Δ with $\alpha+\beta+\gamma = 0$ we have $N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}$. Moreover, from (i) we get $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$.

For U, V in $\mathfrak{p}$ we denote by $U \wedge V \in \operatorname{End}(\mathfrak{p})$ the map $U \wedge V(X) = (U, X)V - (V, X)U$. Also if γ is in $\Delta_{\mathfrak{p}}$, by $|\gamma|$ we mean the positive root proportional to γ.

Lemma 3.4. *Let $\alpha \in \Delta_{\mathfrak{p}}^{+}$ with $E_\alpha = Y_\alpha + X_\alpha$, Y_α in $\mathfrak{k}_{\mathbb{C}}$ and X_α in $\mathfrak{p}_{\mathbb{C}}$. Then*

$$\operatorname{ad}|\mathfrak{p}_{\mathbb{C}} Y_\alpha = X_\alpha \wedge A_\alpha + \tfrac{1}{2} \sum_{\substack{\beta \in \Delta_{\mathfrak{p}}^{+} \\ N_{\alpha\beta} \neq 0}} N_{\alpha,\beta} X_\beta \wedge X_{\alpha+\beta}$$

$$+ \tfrac{1}{2} \sum_{\substack{\beta \in \Delta_{\mathfrak{p}}^{+} \\ N_{\alpha,-\beta} \neq 0}} N_{\alpha,-\beta} X_\beta \wedge X_{|\alpha-\beta|}.$$

Proof. Let $\{A_i\}$ be any orthonormal basis of $\mathfrak{a}$. Then $\{A_i, X_\alpha, \alpha \in \Delta_{\mathfrak{p}}^{+}\}$ is an orthogonal basis of $\mathfrak{p}_{\mathbb{C}}$. We shall show both sides agree on this basis. We shall use (i)–(iv) repeatedly.

For A_i, $\operatorname{ad}|\mathfrak{p}_{\mathbb{C}} Y_\alpha(A_i) = -\alpha(A_i) X_\alpha$, while $X_\alpha \wedge A_\alpha(A_i) = -\alpha(A_i) X_\alpha$ and $X_\beta \wedge X_\gamma(A_i) = 0$. For $\gamma \in \Delta_{\mathfrak{p}}^{+} \setminus \{\alpha\}$ with $\alpha \pm \gamma \notin \Delta$,

$$[Y_\alpha, X_\gamma] = \tfrac{1}{4}\{[E_\alpha, E_\gamma] - \theta[E_\alpha, E_\gamma] - [E_\alpha, \theta E_\gamma] + \theta[E_\alpha, \theta E_\gamma]\}$$
$$= 0.$$

On the other hand, for such γ, $X_\alpha \wedge A_\alpha(X_\gamma) = 0$, $X_\beta \wedge X_{\alpha+\beta}(X_\gamma) = 0$ for otherwise $\gamma = \beta$ and hence $\alpha+\gamma$ is a root or $\gamma = \alpha+\beta$ and then $\gamma - \alpha$ is a root, and similarly $X_\beta \wedge X_{|\alpha-\beta|}(X_\gamma) = 0$. For X_α we have $[Y_\alpha, X_\alpha] = \tfrac{1}{2} A_\alpha$, $X_\alpha \wedge A_\alpha(X_\alpha) = (X_\alpha, X_\alpha) A_\alpha = \tfrac{1}{2} A_\alpha$, and $X_\beta \wedge X_{\alpha+\beta}(X_\alpha) = 0 = X_\beta \wedge X_{|\alpha-\beta|}(X_\alpha)$.

Hence it suffices to examine X_γ where at least one of $\alpha \pm \gamma$ is a root. Now $[Y_\alpha, X_\alpha] = \tfrac{1}{2} N_{\alpha,\gamma} X_{\alpha+\gamma} + \tfrac{1}{2} N_{\alpha,-\gamma} X_{|\alpha-\gamma|}$, and $X_\alpha \wedge A_\alpha(X_\gamma) = 0$. The only terms in the sums that might be non-zero on X_γ are: $X_\gamma \wedge X_{\alpha+\beta}$, $X_\gamma \wedge X_{|\alpha-\gamma|}$ when $\beta = \gamma$;

$X_{\gamma-\alpha}\wedge X_\gamma$ when $\gamma=\alpha+\beta$; and when $\gamma=|\alpha-\beta|$ either $X_{\alpha-\gamma}\wedge X_\gamma$ or $X_{\alpha+\gamma}\wedge X_\gamma$ according to $\gamma=\alpha-\beta$ or $\gamma=\beta'-\alpha$. If $\alpha-\gamma>0$ we get

$$\begin{aligned}&\tfrac{1}{2}[N_{\alpha,\gamma}X_\gamma\wedge X_{\alpha+\gamma}+N_{\alpha,-\gamma}X_\gamma\wedge X_{\alpha-\gamma}+N_{\alpha,-(\alpha-\gamma)}X_{\alpha-\gamma}\wedge X_\gamma\\&\quad+N_{\alpha,-(\alpha+\gamma)}X_{\alpha+\gamma}\wedge X_\gamma]=\tfrac{1}{2}[N_{\alpha,\gamma}-N_{\alpha,-(\alpha+\gamma)}]X_\gamma\wedge X_{\alpha+\gamma}\\&\quad+\tfrac{1}{2}[N_{\alpha,-\gamma}-N_{\alpha,-(\alpha-\gamma)}]X_\gamma\wedge X_{\alpha-\gamma}.\end{aligned}$$

Using (iv) and recalling that $(X_\gamma, X_\gamma)=\frac{1}{2}$, we find that this evaluated on X_γ gives $\frac{1}{2}N_{\alpha,\gamma}X_{\alpha+\gamma}+\frac{1}{2}N_{\alpha,-\gamma}X_{\alpha-\gamma}$. The remaining case $\gamma-\alpha>0$ is done similarly. □

Let $\mathfrak{q}$ be a standard cuspidal parabolic subalgebra which we may assume can be expressed as $\mathfrak{q}=\mathfrak{m}_\mathfrak{q}\oplus\mathfrak{a}_\mathfrak{q}\oplus\mathfrak{n}_\mathfrak{q}$ with $\mathfrak{a}_\mathfrak{q}\subseteq\mathfrak{a}$, and $\mathfrak{m}_\mathfrak{q}=\mathfrak{m}_\mathfrak{q}\cap\mathfrak{k}\oplus\mathfrak{m}_\mathfrak{q}\cap\mathfrak{p}$. Let Q be the normalizer of $\mathfrak{q}$ in G; Q has Langlands decomposition $Q=M_QA_QN_Q$. Let (ξ, W_ξ) be an irreducible unitary representation of M_Q and e^ν a quasi-character of A_Q. Set $\pi_{\xi,\nu}=\mathrm{Ind}_Q^G\,\xi\otimes e^\nu\otimes I$, acting by the left regular representation on

$$H_{\xi,\nu}=\{f\colon G\to W_\xi \mid f(gman)=e^{-(\nu+\rho_Q)\log a}\,\xi(m)^{-1}f(g)\},$$

with norm squared $\int_K |f(k)|^2_{W_\xi}\,dk$. Let us note that unitary induction corresponds to ν imaginary valued. For technical reasons that will become clear later we take M to be a subgroup of M_Q such that $\exp(\mathfrak{m}\cap\mathfrak{p})\subseteq M\subseteq M_Q$. Let now (ξ, W_ξ) be an irreducible unitary representation of M, e^ν a quasi-character of A_Q and form $\pi_{\xi,\nu}=\mathrm{Ind}^G_{MA_QN_Q}\,\xi\otimes e^\nu\otimes I$ as before.

To compute $\tilde{D}_{\pi_{\xi,\nu}}$ on $[H_{\xi,\nu}\otimes S\otimes V]^K$, we first observe that $[H_{\xi,\nu}\otimes S\otimes V]^K$ is naturally isomorphic to $[W_\xi\otimes S\otimes V]^{K\cap M}$ via the map $f\mapsto f(e)$ (see [B-M] p. 178 for a proof for minimal parabolics that extends easily to the case at hand). Let $\{A_i\}$ (resp. $\{X_j\}$) be any orthonormal basis of $\mathfrak{a}_\mathfrak{q}$ (resp. $\mathfrak{m}_\mathfrak{q}\cap\mathfrak{p}$) and X_α as before, for $E_\alpha\in\mathfrak{n}_{\mathfrak{q},\mathbb{C}}$. Then $\{A_i, X_j, \sqrt{2}X_\alpha\}$ is an orthonormal basis of $\mathfrak{p}_\mathbb{C}$. If λ is any linear functional on $\mathfrak{a}_{\mathfrak{q},\mathbb{C}}$ we denote by $A_\lambda\in\mathfrak{a}_{\mathfrak{q},\mathbb{C}}$ the vector with $\lambda(H)=(A_\lambda, H)$.

Proposition 3.5. *On* $[W_\xi\otimes S\otimes V]^{K\cap M}$, $\tilde{D}_{\pi_{\xi,\nu}}$ *is given by:*

$$\tilde{D}_{\pi_{\xi,\nu}}=I\otimes c(A_\nu)\,c(\omega^\mathbb{C})+\sum_{\alpha\in\Delta(\mathfrak{n}_\mathfrak{q})} I\otimes c(X_\alpha)\,c(\omega^\mathbb{C})\cdot I\otimes I\otimes\tau(Y_\alpha)$$
$$-\tfrac{1}{2}\sum_{\alpha,\beta}N_{\alpha,\beta}\,I\otimes c(X_\alpha)\,c(X_\beta)\,c(X_{\alpha+\beta})\,c(\omega^\mathbb{C})+\sum\xi(X_j)\otimes c(X_j)\,c(\omega^\mathbb{C}). \tag{3.9}$$

Proof. $\tilde{D}_{\pi_{\xi,\nu}}$ is a sum of terms involving A_i, X_α, X_j; we shall compute the contribution of each. For any f in $H_{\xi,\nu}\otimes S\otimes V$, $(\pi_{\xi,\nu}(A_i)\otimes c(A_i))f(e)=(\nu+\rho_Q)(A_i)(I\otimes c(A_i))f(e)$. Similarly $(\pi_{\xi,\nu}(X_j)\otimes c(X_j))f(e)=(\xi(X_j)\otimes c(X_j))f(e)$. And writing $X_\alpha=E_\alpha-Y_\alpha$, for such f we have

$$(\pi_{\xi,\nu}(X_\alpha)\otimes c(X_\alpha))f(e)=-(\pi_{\xi,\nu}(Y_\alpha)\otimes c(X_\alpha))f(e).$$

Now assuming that f is in $[H_{\xi,\nu}\otimes S\otimes V]^K$ we get that

$$(\pi_{\xi,\nu}(X_\alpha)\otimes c(X_\alpha)f(e)=(I\otimes c(X_\alpha)\cdot I\otimes\sigma(Y_\alpha)\otimes I)f(e)$$
$$+(I\otimes c(X_\alpha)\cdot I\otimes I\otimes\tau(Y_\alpha))f(e)$$

The second term is kept; for the first term we use Lemma 3.4 and $s(X_\beta\wedge X_\gamma)=\frac{1}{2}c(X_\beta)c(X_\gamma)$ $(\beta\neq\gamma)$ getting on $f(e)$

$$(I\otimes c(X_\alpha)\cdot I\otimes\sigma(Y_\alpha)\otimes I)=+\tfrac{1}{2}I\otimes c(X_\alpha)^2c(A_\alpha)$$
$$-\tfrac{1}{4}\sum_{\substack{\beta\\ N_{\alpha,\beta}\neq 0}} N_{\alpha,\beta}I\otimes c(X_\alpha)c(X_\beta)c(X_{\alpha+\beta})$$
$$-\tfrac{1}{4}\sum_{\substack{\beta\\ N_{\alpha,-\beta}\neq 0}} N_{\alpha,-\beta}I\otimes c(X_\alpha)c(X_\beta)c(X_{|\alpha-\beta|}).$$

Summing over $\alpha\in\Delta(\mathfrak{n}_\mathfrak{q})$, the first term becomes $-\frac{1}{2}\sum_\alpha I\otimes c(A_\alpha)=-I\otimes c(A_{\rho_Q})$, and the remaining terms may be combined pairwise to give

$$-\tfrac{1}{2}\sum_\alpha\sum_{\substack{\beta\\ N_{\alpha,\beta}\neq 0}} N_{\alpha,\beta}I\otimes c(X_\alpha)c(X_\beta)c(X_{\alpha+\beta}).$$

Indeed the terms in these two sums are in bijective correspondence $(\alpha,\beta)\mapsto(\alpha+\beta,\beta)$ and $(\gamma,\lambda)\to$ either $(\gamma-\lambda,\lambda)$ or $(\lambda-\gamma,\gamma)$ according to $|\gamma-\lambda|$, also $N_{\alpha+\beta,-\beta}=N_{-\alpha,-\beta}=N_{\alpha,\beta}$ as is seen from (iv). Hence the $\mathfrak{n}_\mathfrak{q}$ contribution is

$$-I\otimes c(A_{\rho_Q})-\tfrac{1}{2}\sum_\alpha\sum_{\substack{\beta\\ N_{\alpha,\beta}\neq 0}} N_{\alpha,\beta}I\otimes c(X_\alpha)c(X_\beta)c(X_{\alpha+\beta}).$$

For the $\mathfrak{a}_\mathfrak{q}$ contribution we get

$$\sum_i(\nu+\rho_Q)(A_i)I\otimes c(A_i)=I\otimes c(A_\nu)+I\otimes c(A_{\rho_Q}).$$

Combining the $\mathfrak{n}_\mathfrak{q}$, $\mathfrak{a}_\mathfrak{q}$ and $\mathfrak{m}_\mathfrak{q}\cap\mathfrak{p}$ contributions and taking into account $\omega^{\mathbb{C}}$, we get the result. □

Fix a unit vector Y in $\mathfrak{a}_\mathfrak{q}$, and let $\mathfrak{p}^Y$ be the orthogonal complement of $\mathbb{R}Y$ in $\mathfrak{p}$. Since $\mathfrak{p}$ is odd dimensional, the spinor representation (s,S) when restricted to $\mathrm{Spin}(\mathfrak{p}^Y)\subseteq\mathrm{Spin}(\mathfrak{p})$ breaks up into two irreducible summands. We label these $S_\pm$ according to whether they are the $\pm i$ eigenspace of $c(Y)c(\omega^{\mathbb{C}})$. This is possible because $c(Y)^2=-1$, $c(\omega^{\mathbb{C}})^2=1$ and $\mathrm{Spin}(\mathfrak{p}^Y)$ centralizes Y within $\mathbb{C}l(\mathfrak{p})$. We note that $S_\pm$ depend on Y but we shall disregard this in the notation. The group $K\cap M_Q$ centralizes $\mathfrak{a}_\mathfrak{q}$ and is easily seen to map into $\mathrm{Spin}(\mathfrak{p}^Y)$ by σ; thus $S_\pm$ are also $K\cap M_Q$ (hence $K\cap M$) invariant.

Proposition 3.6. *Let* $Y \in \mathfrak{a}_\mathfrak{q}$ *be a unit vector and* $S_\pm$ *the irreducible* spin *representations of* $\mathrm{Spin}(\mathfrak{p}^Y)$. *Then*

$$\mathrm{Tr}\,\tilde{D}_{\pi_{\xi,\nu}} = \nu(Y)\dim[W_\xi \otimes (S_+ - S_-) \otimes V]^{K\cap M}. \tag{3.10}$$

Moreover,

$$\mathrm{Tr}\,\tilde{D}_{\pi_{\xi,\nu}} = 0 \quad \text{if } \dim \mathfrak{a}_\mathfrak{q} \geq 2. \tag{3.11}$$

Proof. $S_\pm$ are the $\pm i$ eigenspaces of $c(\omega^{\mathbb{C}} Y)$. Now any odd element in $\mathbb{C}l(\mathfrak{p})$ generated by $\mathfrak{p}^Y$ anti-commutes with $\omega^{\mathbb{C}} Y$, hence maps $S_\pm$ to $S_\mp$. From this it follows that all the terms in (3.9) but $I \otimes c(A_\nu)\, c(\omega^{\mathbb{C}})$ have trace zero. We write $A_\nu = \nu(Y)\, Y + Z$ where $Z \in \mathfrak{a}_\mathfrak{q} \cap \mathfrak{p}^Y$. Again we have $\mathrm{Tr}\, I \otimes c(Z)\, c(\omega^{\mathbb{C}}) = 0$ for parity reasons, leaving formula (3.10) for $\mathrm{Tr}\,\tilde{D}_{\pi_{\xi,\nu}}$.

To prove (3.11) let $Z \in \mathfrak{a}_\mathfrak{q} \cap \mathfrak{p}^Y$ and non-zero. Then $c(Z)^2$ a scalar implies that $c(Z)$ is invertible. But $Z \in \mathfrak{a}_\mathfrak{q}$ together with (3.2) gives that $c(Z)$ intertwines the $K \cap M$ action; while $Z \in \mathfrak{p}^Y$, non-zero, implies that $c(Z)$ interchanges $S_\pm$. Hence S_+ is equivalent to S_- as $K \cap M$ module and thus (3.11) follows.

Remark. Assume that $V = V_\mu$ is irreducible, then

$$\dim[W_\xi \otimes (S_+ - S_-) \otimes V]^{K\cap M} = 0 \qquad \text{if } \dim \mathfrak{a}_\mathfrak{q} = 1 \tag{3.12}$$
$$\text{and} \quad \|\Lambda_\xi\|^2 - \|\mu + \rho_k\|^2 \neq 0.$$

Indeed, recall (3.3) which says $(\tilde{D}_{\pi_{\xi,0}})^2$ acts as a scalar on $[H_{\xi,0} \otimes S \otimes V]^K$, easily computed to be $-\|\Lambda_\xi\|^2 + \|\mu + \rho_k\|^2$ where Λ_ξ is the infinitesimal character of W_ξ and $V = V_\mu$ is irreducible. Hence $\tilde{D}_{\pi_{\xi,0}}$ is an isomorphism: $[W_\xi \otimes S_\pm \otimes V]^{K\cap M} \to [W_\xi \otimes S_\mp \otimes V]^{K\cap M}$ provided $\|\Lambda_\xi\|^2 - \|\mu + \rho_k\|^2 \neq 0$. Since, from (3.9), $\tilde{D}_{\pi_{\xi,\nu}} = I \otimes c(A_\nu)\, c(\omega^{\mathbb{C}}) + \tilde{D}_{\pi_{\xi,0}}$ we get (3.12). □

§4. The trace of the odd heat operator

We are now in a position to compute orbital integrals of the odd heat kernel $\tilde{D} e^{-t\tilde{D}^2}$. For this purpose we shall follow closely the notation used in [H-C I] and [H-C S] and, for brevity, refer the reader to these papers for notation, normalization of measures, etc. not explained herein.

A brief summary of choices of Haar measures is in order. The Killing form, via B_θ, induces a Euclidean structure on $\mathfrak{g}$ and any subspaces. Normalize Lebesgue measure on any subspace so that the volume of the unit cube is one. Any Lie subgroup L of G has the Haar measure, denoted dL, implemented by a differential form, near the identity, with pull-back via exp the chosen Lebesgue measure. On a compact subgroup L, denote by $dl = v_L(L)^{-1}\, dL$ the Haar measure with total mass one. Any parabolic subgroup P with Langlands decomposition MAN fixes a "standard" Haar measure dm on M. Finally, measures on quotient spaces are chosen so that the Fubini theorem holds.

Let $\mathfrak{h}=\mathfrak{a}_I \oplus \mathfrak{a}_R$ be a standard Cartan subalgebra and $A=A_I A_R$ the corresponding Cartan subgroup. Set $\mathfrak{m}^1$ = centralizer of $\mathfrak{a}_R$; $\mathfrak{m}^1=\mathfrak{m}\oplus\mathfrak{a}_R$. For any choice of compatible orders on $\Delta_{\mathfrak{m}}$ and $\Delta_{\mathfrak{g}}$ define two functions on A by

$$\Delta_+(a)=|\det(I-\operatorname{Ad}a^{-1})|_{\mathfrak{g}/\mathfrak{m}^1}|^{1/2}$$
$$'\Delta_I(a)=\prod_{\alpha\in\Delta_{\mathfrak{m}}^+}[1-\xi_{-\alpha}(a)].$$

Let A' denote the regular elements in A and let $h=a_I a_R$ be in A'. Denote the projection $G\to G/A_R$ by $x\mapsto x^*$, and for f in $C_c^\infty(G)$ set

$$'F_f^A(h)={}'\Delta_I(h)\Delta_+(h)\int_{G/A_R} f(h^{x^*})\,dx^* \tag{4.1}$$

here $h^{x^*}=xhx^{-1}$ and dx^* is normalized so that $dx=dx^*\,dA_R$.

Let G_h = centralizer of h in G and G_h^0 the component of the identity in G_h. Then for the normalization of measures $dx=d\bar{x}\,dG_h^0$, define the orbital integral

$$O_f(h)=\int_{G/G_h^0} f(h^{\bar{x}})\,d\bar{x}.$$

Since h is regular, $G_h^0=A_I^0 A_R$, and for the measures as chosen, one has

$$O_f(h)=\int_{G/A_R} f(h^{x^*})\,dx^*.$$

For h regular, Harish-Chandra has shown that the distribution $f\mapsto O_f(h)$ extends to $\mathscr{C}^2(G)$, in particular can be evaluated on the odd heat kernel (see (3.5)).

Let $\succ$ denote the partial order on the standard Cartan subgroups: $A\succ B$ if a certain finite group $\mathfrak{w}(\mathfrak{a}_R|\mathfrak{b}_R)\neq 0$. If $A=A_I A_R$ is any Cartan subgroup let A^*, A_I^*, A_R^* denote the set of irreducible unitary characters of A, A_I, A_R. In [S] Harish-Chandra stated that $'F_f^B(h)$ is supported on those A^* where $A\succ B$, here $f\in\mathscr{C}^2(G)$. While it is unclear whether Harish-Chandra stated Theorem 15 only for equirank G, it is clear from Herb's work [Hb] ($h\in C_c^\infty(G)$) that the result is valid without the equirank condition. Since we need only symmetry and support features of orbital integrals but not the very detailed inversion formula of Herb, we shall follow the notation in [H-C]. With this caveat in mind, as a special case of his Theorem 15, we have

Proposition 4.1. *Let $b\in B'$ and $f\in\mathscr{C}^2(G)$. Assume*

$$\hat{f}_A\equiv 0 \quad \text{if } A\succ B.$$

Then

$$'F_f^B(b)=\int_{B^*}[W(G/B)]^{-1}\sum_{s\in W(G/B)}\varepsilon_I(s)\langle s\cdot b^*,b\rangle\hat{f}_B(b^*)\,db^*. \tag{4.2}$$

Definition. For B the fundamental Cartan subgroup, the functions with $\hat{f}_A\equiv 0$, $A\succ B$, shall be called pseudo-cusp forms.

We now take a closer look at (4.2). Let $b^*\in(B^*)'$, $b^*=(a_I^*,\nu)$ with $\nu\in\mathfrak{a}_R^*$. The regular element a_I^* in A_I^* gives rise to a discrete series representation of

M in the following way. Let M^0 be the component of the identity in M and let $C = \ker \mathrm{Ad}|_M$; set $M^+ = M^0 C$. The unitary character a_I^* gives a regular integral element of $\mathfrak{a}_I^*$ together with a compatible character on C. Let $(\pi_{\omega(a_I^*)}, H(a_I^*))$ be the discrete series representation of M^+ associated to the $W(M^+, A_I)$ orbit of a_I^*. Set $\pi_{\xi(a_I^*)} = \mathrm{Ind}_{M^+}^M \pi_{\omega(a_I^*)}$ and let $H(\xi)$ be the Hilbert space for π_ξ. The representation $(\pi_\xi, H(\xi))$ is a discrete series representation of M; this construction exhausts $\mathscr{E}_2(M)$, the set of equivalence classes of discrete series representations of M and two are equivalent if the M^+ parameters are in the same $W(M, A_I)$ orbit.

Form $\pi_{\xi,\nu} = \mathrm{Ind}_{MA_RN}^G \pi_\xi \otimes e^\nu \otimes 1$, and let $\Theta_{\pi_{\xi,\nu}}$ denote the character. Then $\Theta_{\pi_{\xi,\nu}}(f) = \varepsilon_I(s) \hat{f}_B(b^*)$ where s in $W(G/B)$ sends b^* to the (ξ, ν) data and $\varepsilon_I(s) = \pm 1$. The functions $\hat{f}_B$ are skew relative to $W(G/B)$, i.e., $\hat{f}_B(s \cdot b^*) = \hat{f}_B(s \cdot a_I^*, s \cdot \nu)$ $= \varepsilon_I(s) \hat{f}_B(a_I^*, \nu) = \varepsilon_I(s) \hat{f}_B(b^*)$. In particular if w is in $W(M, A_I)$, $w \cdot \nu = \nu$ so $\hat{f}_B(w \cdot b^*) = \hat{f}_B(w \cdot a_I^*, \nu) = \varepsilon_I(w) \hat{f}_B(A_I^*, \nu)$.

The distributional character of discrete series of M^+ and M are given on A_I' by locally summable functions $\Theta_{\omega(a_I^*)}$ and $\Theta_{\xi(a_I^*)}$. Set $'\Phi_\omega(a_I) = {}'\Delta_M(a_I)\Theta_\omega(a_I)$ $= \sum_{W(M^+, a_I)} \varepsilon_I(s) \langle s \cdot a_I^*, a_I \rangle$, $'\Phi_\xi(a_I) = {}'\Delta_M(a_I) \Theta_\xi(a_I)$. Here $'\Delta_M = \prod_{\alpha \in \Delta_{\mathfrak{m}}^+} [1 - \xi_{-\alpha}]$ and $'\Delta_M = {}'\Delta_{M,c} {}'\Delta_{M,n}$ with c and n denoting the compact and non-compact roots.

Since M is cuspidal, $\mathfrak{m} \cap \mathfrak{p}$ is even dimensional. Let $\sigma_\pm$ denote the spin representations of $K \cap M^0$ and $\chi_{\sigma_\pm}$ their characters. If B is fundamental, then B is connected and $M^+ = M^0$; and thus $\sigma_\pm$ are representations of $K \cap M^+$. It is known that $(\chi_{\sigma_+} - \chi_{\sigma_-})|_{A_I'} = \xi'_{\rho_{M,n}} \Delta_{M,n}$, here $\xi_{\rho_{M,n}}$ is defined on A_I', being the highest weight of σ_+. Let W be any finite dimensional virtual representation of $K \cap M^+$ and suppose $(\sigma_+ - \sigma_-)$ divides W, denoted $W \in (\sigma_+ - \sigma_-)$. In this case $[{}'\Delta_{M,n}]^{-1} \chi_W$ is an analytic function on A_I.

Let $b^* \in A_I^*$ and $\log b^*$ in $\mathfrak{a}_I^*$. The character b^* is said to be regular if $\prod_{\alpha \in \Delta_{\mathfrak{m}}^+} (\log b^* + \rho_M, \alpha) \neq 0$. The discrete series, $\mathscr{E}_2(M^+)$, are in one-to-one correspondence with $W(M^+, A_I)$ orbits of the regular characters. If b^* is singular, Harish-Chandra has also constructed an invariant distribution also denoted Θ_{b^*}. While this is not necessarily the character of an irreducible representation of M^+, it is known from [H-S] that it is a virtual character with constituents irreducible representations induced off parabolic subgroups of M^+ associated to a non-fundamental Cartan subgroup of M^+. For b^* singular let $W(b^*) \subseteq W_{\mathbb{C}}(M^+, A_I)$ be the isotropy subgroup. One has $\Theta_{b^*} = \sum_{W(b^*)} \varepsilon(w) \Theta'_{w \cdot b^*}$ with $\Theta'_{w \cdot b^*}$ the character of the induced representation acting on a Hilbert space $H(w \cdot b^*)$. Let $\mathscr{E}_2^s(M^+)$ denote the $W(M^+, A_I)$ orbits of the singular characters.

Lemma 4.2. *Let B a fundamental Cartan subgroup. Let W be a finite dimensional virtual representation of $K \cap M^+$ and suppose that $W \in (\sigma_+ - \sigma_-)$. Then on A_I*

$$\text{(4.3)} \quad \frac{\chi_W \overline{{}'\Delta_{M,c}}}{{}'\Delta_{M,n}} = \sum_{\omega \in \mathscr{E}_2(M^+)} \dim [H(\omega) \otimes W]^{K \cap M^+} \overline{{}'\Phi}_\omega + \sum_{\omega \in \mathscr{E}_2^s(M^+)} \sum_{W(\omega)} \varepsilon(\omega) \dim [H(w \cdot \omega) \otimes W]^{k \cap M^+} \overline{{}'\Delta_M} \overline{\Theta'}_{w \cdot \omega}.$$

Proof. One knows that only finitely many terms on the right side of (4.3) have $\dim[H(\omega)\otimes W]^{K\cap M^+}\neq 0$. From the Weyl character formula and the Harish-Chandra character formula it follows that both sides of (4.3) are finite sums of characters of A_I. Using Harish-Chandra's orthogonality relations, it suffices to evaluate the integrals (da_I normalized Haar measure)

$$\int_{A_I} \frac{\chi_W \overline{'\Delta_{M,c}}}{'\Delta_{M,n}}\, '\Phi_\omega\, da_I$$

and

$$\int_{A_I} \frac{\chi_W \overline{'\Delta_{M,c}}}{'\Delta_{M,n}}\, '\Delta_M \Theta'_{w\cdot\omega}\, da_I.$$

Notice that since $W\in(\sigma_+ - \sigma_-)$ the left side of (4.3) is skew under the action of $W(M^+, A_I)$, and the right side of (4.3) involves a spanning set of skew Fourier series on A_I.

We evaluate the first integral; the second is done similarly. As the integrand is analytic, we need integrate only over A'_I. Hence

$$\int_{A'_I} \frac{\chi_W \overline{'\Delta_{M,c}}}{'\Delta_{M,n}}\, '\Phi_\omega\, da_I = \int_{A'_I} \frac{\chi_W \overline{'\Delta_{M,c}}}{'\Delta_{M,n}}\, '\Delta_M \Theta_\omega\, da_I$$
$$= \int_{A'_I} \chi_W \Theta_\omega |'\Delta_{M,c}|^2\, da_I.$$

On the connected group M^+ the character Θ_ω and the K-character τ_ω agree on $(K\cap M^+)'$ ([A-Sc] p. 16). Thus we get the integral is

$$\int_{A'_I} \chi_W \tau_\omega |'\Delta_{m,c}|^2\, da_I = |W(M^+, A_I)| \dim[H(\omega): \hat{W}]^{K\cap M^+}$$
$$= |W(M^+, A_I)| \dim[H(\omega)\otimes W]^{K\cap M^+}. \qquad \square$$

Lemma 4.3. *Let $f\in\mathscr{C}^2(G)$ be a pseudo-cusp form and suppose that*

$$\hat{f}_B(a_I^*, v) = \tilde{\phi}(v) \dim[H(\omega(a_I^*)) \otimes W]^{K\cap M^+}.$$

Then

$$'F_f^B(h_I \cap h_R) = \frac{'\Delta_{M,c}(h_I)}{\overline{'\Delta_{M,n}(h_I)}} |W(G/B)|^{-1} \sum_{W(G/B)} \phi(s\cdot h_R)\, \bar{\chi}_W(s\cdot h_I) \tag{4.4}$$

where ϕ is the $\mathfrak{a}_R$-Fourier transform of $\tilde{\phi}$.

Proof. From (4.2) we have

$$'F_f^B(h_I h_R) = \int_{B^*} [W(G/B)]^{-1} \sum_{s\in W(G/B)} \varepsilon_I(s) \langle s\cdot b^*, b\rangle \hat{f}_B(b^*)\, db^*$$
$$= [W(G/B)]^{-1} \int_{A_I^*} \int_{\mathfrak{a}_R^*} \sum_{s\in W(G/B)} \varepsilon_I(s) \langle s\cdot a_I^*, h_I\rangle\, s\cdot v(h_R) \hat{f}_B(a_I^*, v)\, dv$$
$$= [W(G/B)]^{-1} \int_{A_I^*/W(M^+, A_I)} \sum_{W(M^+, A_I)}$$
$$\cdot \int_{\mathfrak{a}_R} \sum_{W(G/B)} \varepsilon_I(s) \langle sw\cdot a_I^*, h_I\rangle\, s\cdot v(h_R) \hat{f}_B(w\cdot a_I^*, v)\, dv.$$

We compute first the contribution of $(A_I^*)'$. Identify $(A_I^*)/W(M^+, A_I)$ with $\mathscr{E}_2(M^+)$ and use $\hat{f}(w \cdot a_I^*, v) = \varepsilon_I(\omega)\hat{f}(a_I^*, v)$ to get

$$\sum_{\mathscr{E}_2(M^+)} \sum_{W(M^+, A_I)} \varepsilon_I(w) \sum_{W(G/B)} \varepsilon_I(s)\langle sw \cdot a_I^*, h_I\rangle \int_{\mathfrak{a}_R^*} s \cdot v(h_R)\hat{f}_B(a_I^*, v)\, dv$$

$$= \sum_{\mathscr{E}_2(M^+)} \sum_{W(M^+, A_I)} \varepsilon_I(w) \sum_{W(G/B)} \varepsilon_I(s)\langle sw \cdot a_I, h_I\rangle$$

$$\cdot \phi(s^{-1} h_R) \dim[H(\omega(a_I^*)) \otimes W]^{K \cap M^+}$$

$$= \sum_{W(G/B)} \phi(s^{-1} h_R) \sum_{\mathscr{E}_2(M^+)} \sum_{W(M^+, A_I)} \varepsilon_I(w)\, \varepsilon_I(s)\, \xi_{s \cdot \rho_M - \rho_M}(h_I)$$

$$\cdot \langle w \cdot a_I^*, s^{-1} \cdot h_I\rangle \dim[H(\omega(a_I^*)) \otimes W]^{K \cap M^+}$$

$$= \sum_{W(G/B)} \varepsilon_I(s)\, \phi(s^{-1} h_R) \sum_{\mathscr{E}_2(M^+)} \dim[H(\omega(a_I^*)) \otimes W]^{K \cap M^+}\, '\Phi_\omega(s^{-1} h_I).$$

The contribution from the singular characters is done similarly except that one replaces $\sum_{W(M^+, A_I)} \varepsilon_I(w)\langle w \cdot a_I^*, s^{-1} h_I\rangle$ with $\sum_{W(\omega)} \varepsilon_I(w)' \Delta_M(s^{-1} h_I)\, \Theta'_{w \cdot \omega}(s^{-1} h_I)$.

Then from Lemma 4.2 we obtain

$$'F_f^B(h_I \cap h_R) = [W(G/B)]^{-1} \sum_{W(G/B)} \varepsilon_I(s)\, \phi(s^{-1} h_R)\, \xi_{s \cdot \rho_M - \rho_M}(h_I)$$

$$\cdot \frac{'\Delta_{M,c}(s^{-1} h_I)}{\overline{'\Delta_{M,n}(s^{-1} h_I)}}\, \bar{\chi}_W(s^{-1} h_I).$$

Write $\frac{'\Delta_{M,c}}{\overline{'\Delta_{M,n}}} = \frac{'\Delta_M}{|'\Delta_{M,n}|^2}$, and use Lemma 27.1 from [H-C, I] together with the invariance of $|'\Delta_{M,n}|^2$ to get the expression

$$'F_f^B(h_I h_R) = \frac{'\Delta_M(h_I)}{|'\Delta_{M,n}(h_I)|^2} \frac{1}{|W(G/B)|} \sum_{W(G/B)} \phi(s^{-1} h_R)\, \overline{\chi_W}(s^{-1} h_I). \quad \square$$

Let us point out that Lemma 4.3 applies to the odd heat kernel, $\tilde{k}_t$ or rather its local trace $\operatorname{tr} \tilde{k}_t$ (defined after trivializing the bundle). First we observe that the odd heat kernel is a pseudo-cusp form. Indeed recall from (3.5) that it is in $\mathscr{S}(G)$. If we decompose V into irreducible K-modules, on each of them $(\tilde{D}_{\pi_{\xi,v}})^2$ is a scalar operator, and consequently (Prop. 3.6) $\operatorname{Tr}(\tilde{D}_{\pi_{\xi,v}} e^{-t\tilde{D}^2_{\pi_{\xi,v}}}) \equiv 0$ if $\dim \mathfrak{a}_q \geq 2$. Thus we get the stronger statement.

Lemma 4.4. *If A is any standard Cartan subgroup with $\mathbb{R}$-rank $A > 1$, then* $(\operatorname{tr} \tilde{k}_t)^\wedge_A \equiv 0$. *In particular* $\operatorname{tr} \tilde{k}_t$ *is a pseudo-cusp form.*

Remark. Taking a brief look at the classification of simple non-compact Lie groups, one finds that the only ones with an $\mathbb{R}$-rank one fundamental Cartan subgroup are, up to local isomorphism, $SL(3, \mathbb{R})$ and $SO_e(p, q)$, pq odd.

From Proposition 3.6 we also get that if B is $\mathbb{R}$-rank 1 and $(a_I^*, v) \in B^*$

$$(4.5) \qquad (tr\, \tilde{k}_t)^\wedge_B\, (a_I^*, v) = v\, e^{-tv^2} \dim[H(\omega(a_I^*)) \otimes (S_+ - S_-) \otimes V_\mu]^{K \cap M_Q^0}$$

with $\dim[H(\omega(\mathfrak{a}_I^*))\otimes(S_+ - S_-)\otimes V_\mu]^{K\cap M_Q^0}=0$ if $\|\Lambda_\omega\|^2 - \|\mu+\rho_k\|^2 \neq 0$.

In the notation of Lemma 3.10, $(S_+ - S_-)\otimes V$ is W. The next result shows that $W\in(\sigma_+ - \sigma_-)$.

Lemma 4.5. Let $\tilde{V}$, V_1, V_2 be even dimensional complex vector spaces and suppose $\tilde{V}=V_1\oplus V_2$ orthogonal sum. Let $S_\pm(\cdot)$ be the Spin modules for Spin$(\cdot)$. Then as Spin(V_1) virtual modules,

$$S_+(\tilde{V})-S_-(\tilde{V})=[S_+(V_1)-S_-(V_1)]\otimes[S_+(V_2)-S_-(V_2)].$$

Proof. $\mathbb{C}l(\tilde{V})\simeq\mathbb{C}l(V_1)\hat{\otimes}\mathbb{C}l(V_2)$, (graded tensor product of algebras) and $S(\tilde{V})\simeq S(V_1)\hat{\otimes}S(V_2)$ the graded tensor product of modules, hence we have

$$S_+(\tilde{V})\simeq S_+(V_1)\otimes S_+(V_2)\oplus S_-(V_1)\otimes S_-(V_2)$$
$$S_-(\tilde{V})\simeq S_+(V_1)\otimes S_-(V_2)\oplus S_-(V_1)\otimes S_+(V_2)$$

and

$$S_+(\tilde{V})-S_-(\tilde{V})\simeq[S_+(V_1)-S_-(V_1)]\otimes[S_+(V_2)-S_-(V_2)]. \quad \square$$

Returning to the odd heat kernel, let us fix B $\mathbb{R}$-rank 1, $A_\mathfrak{p}\in\mathfrak{a}_R$ a unit vector, and MA_RN the standard cuspidal parabolic associated to B. Then set $\tilde{V}=\mathfrak{p}\ominus\mathfrak{a}_R$, $V_1=\mathfrak{m}\cap\mathfrak{p}$ and $V=V_1\oplus V_2$, and recall that $K\cap M^0\subseteq\text{Spin}(V_1)$. We get from Lemma 4.4, $\sigma_+ - \sigma_- = S_+(V_1)-S_-(V_1)$ is a factor in $S_+ - S_- = S_+(\tilde{V})-S_-(\tilde{V})$; hence $(\sigma_+ - \sigma_-)$ divides W.

Corollary 4.6. *Let $f=\operatorname{tr}\tilde{k}_t$ and B an $\mathbb{R}$-rank 1 Cartan subgroup. Let $h_k h_\mathfrak{p} = h_k\exp(r_h A_\mathfrak{p})$ be in B'. Then*

$$'F_f^B(h_k h_\mathfrak{p})=i\,\frac{2\pi r_h}{(4\pi t)^{3/2}}\,e^{-r_h^2/4t}\,\frac{'\Delta_{M,c}(h_k)}{'\Delta_{M,n}(h_k)}\,\bar{\chi}_W(h_k)$$

where $W=(S_+ - S_-)\otimes V$.

Proof. Given the preceding results, it suffices to see that

$$\frac{1}{|W(G/B)|}\sum_{W(G/B)}\varphi(s\cdot r_h A_\mathfrak{p})\,\bar{\chi}_W(s\cdot h_k)=\varphi(r_h A_\mathfrak{p})\,\bar{\chi}_W(h_k).$$

But from [Hi] we have $W(G/B)|_{\mathfrak{a}_R}=W(\mathfrak{a}_R)\simeq\mathbb{Z}_2$ or trivial since B has $\mathbb{R}$-rank 1, and $W(G/B)$ can be represented by elements of K. Since V is a representation of K, $\chi_V(s\cdot h_k)=\chi_V(h_k)$ follows.

If $s\in W(M,A_I)$, then $s\cdot h_\mathfrak{p}=h_\mathfrak{p}$. But $S_\pm$ are the $\pm i$ eigenspaces of $c(A_\mathfrak{p})\,c(\omega^{\mathbb{C}})$ and s is represented by conjugation by $k_s\in K$, thus from (3.2) k_s: $S_\pm\to S_\pm$. Hence $\varphi(s\cdot r_h A_\mathfrak{p})=\varphi(r_h A_\mathfrak{p})$ and $\chi_W(s\cdot h_k)=\chi_W(h_k)$.

If $s\in W(G/B)$ represents the non-trivial element of $W(\mathfrak{a}_R)$, then $s\cdot A_\mathfrak{p}=-A_\mathfrak{p}$. Now in this case k_s: $S_\pm\to S_\mp$. Hence $\chi_W(s\cdot h_k)=-\chi_W(h_k)$ and $\varphi(s\cdot r_h A_\mathfrak{p})=-\varphi(r_h A_\mathfrak{p})$. $\square$

From the relationship between orbital integrals and F_f we get.

Corollary 4.7. *Again let $f=\operatorname{tr}\tilde{k}_t$ and $h=h_k h_\mathfrak{p}\in A'$. Then*

(i) $$\int_{G/G_h^0} f(h^{\bar{x}})\,d\bar{x}=i\,2\pi\,\frac{r_h e^{-r_h^2/4t}}{(4\pi t)^{3/2}}\,\frac{\bar{\chi}_W(h_k)}{\Delta_+(h)\,|\Delta_{M,n}(h_k)|^2},\qquad h\in B',$$

(ii) $$\int_{G/G_h^0} f(h^{\bar{x}})=0,\qquad h\in A'\neq B'.$$

Next we evaluate orbital integrals of $\operatorname{tr}\tilde{k}_t$ for singular h. That we need to consider only B fundamental and $\mathbb{R}$-rank 1 will be especially helpful.

Indeed as B is fundamental the roots are either imaginary or complex. If $\gamma\in\Gamma$ is conjugate to $h\in B$, and for some complex root α, $\xi_\alpha(h)=1$, then since B has $\mathbb{R}$-rank one we must have $h\in A_I$. But then γ is a torsion element of Γ (Γ is discrete) contrary to the hypotheses on Γ. Hence $\gamma\in\Gamma$ is singular precisely when there is an imaginary root, thus $\alpha\in\Delta_M$, with $\xi_\alpha(h)=1$. We fix $\gamma\in\Gamma$ and suppose, without loss of generality, that γ is in B.

We shall follow Harish-Chandra [DS II, p. 32–37] with some minor notational differences. Let $\mathfrak{g}_\gamma$ be the centralizer of γ in $\mathfrak{g}_{\mathbb{C}}$. Then $\mathfrak{b}=\mathfrak{a}_I\oplus\mathfrak{a}_R$ is contained in $\mathfrak{g}_\gamma$ and is fundamental. Order the roots compatibly on $\mathfrak{a}_R$ and $\mathfrak{b}$. Let P_γ be the positive roots of $(\mathfrak{g}_\gamma, \mathfrak{b})$ and P_γ^c the remaining positive roots. Set $\tilde{\omega}_\gamma=\prod_{\alpha\in P_\gamma} H_\alpha$. Then for $f\in\mathscr{C}^2(G)$ one has

$$(4.6)\qquad \int_{G/G_\gamma} f(\gamma^{\bar{x}})=\frac{N^{-1}c_0^{-1}}{\xi_\rho(\gamma)\prod_{\alpha\in P_\gamma^c}[1-\xi_\alpha(\gamma)^{-1}]}\,\tilde{\omega}_\gamma F_f^B(\gamma),$$

here N is the order of the finite group G_γ/ZG_γ^0 which in this case is G_γ/G_γ^0 or $W(G_\gamma, B)/W(G_\gamma^0, B)$, and $c_0=c_\gamma$ is computed by Harish-Chandra ([HC I] Theorem 37.1) to be $(-1)^q(2\pi)^q 2^{v/2}\tilde{\omega}_k(\rho_k)\,|W(G_\gamma^0, B)|$. Notice that if γ is regular, then $c_\gamma=1$.

Recall that we use $'F_f^B$ and, since there are no real roots, $F_f^B=\xi_{\rho_M}{}'F_f^B$. Similarly $(\chi_{\sigma_+}-\chi_{\sigma_-})|_{A_I}=\xi_{\rho_{M,n}}{}'\Delta_{M,n}$, and the Weyl group action $\xi_{w\cdot\lambda}=\xi_{w\lambda}\xi_{w\rho_{M,c}-\rho_{M,c}}$. Now W is divisible by $\sigma_+-\sigma_-$; say $W=(\sigma_+-\sigma_-)\otimes\sum a_i W_{\lambda_i}$ with λ_i the $(K\cap M^0;\, A_I)$ highest weights and a_i integers.

Corollary 4.8. Let $f=\operatorname{tr}\tilde{k}_t$ and B an $\mathbb{R}$-rank one fundamental Cartan subgroup. Let $\gamma=\gamma_k\gamma_p=\gamma_k\exp l_\gamma A_p$ be in B. Then

$$(4.7)\qquad \int_{G/G_\gamma^0} f(\gamma^{\bar{x}})\,d\bar{x}=c_\gamma^{-1}(i2\pi)\frac{1}{\xi_\rho(\gamma)\prod_{\alpha\in P_\gamma^c}[1-\xi_\alpha(\gamma^{-1})]}\frac{l_\gamma e^{-l_\gamma^2/4t}}{(4\pi t)^{3/2}}$$
$$\cdot\sum a_i\sum_{W(M^0,A_I)}\varepsilon(w)\,\tilde{\omega}_\gamma(w(\lambda_i+\rho_{M,c}))\,\bar{\xi}_{w\cdot\lambda_i}(\gamma_k).$$

Proof. From Corollary 4.6 we have for regular $h=h_k\exp rA_p$,

$$'F_f^B(h)=(i2\pi)\frac{r e^{-r^2/4t}}{(4\pi t)^{3/2}}\frac{'\Delta_{M,c}(h_k)}{'\Delta_{M,n}(h_k)}\bar{\chi}_W(h_k)$$
$$=(i2\pi)\frac{r e^{-r^2/4t}}{(4\pi t)^{3/2}}\bar{\xi}_{\rho_{M,n}}(h_k)'\Delta_{M,c}(h_k)\sum a_i\bar{\chi}_{W_{\lambda_i}}(h_k).$$

Hence

$$F_f^B(h)=(i2\pi)\frac{r e^{-r^2/4t}}{(4\pi t)^{3/2}}\xi_{\rho_{M,n}}(h_k)'\Delta_{M,c}(h_k)\sum a_i\bar{\chi}_{W_{\lambda_i}}(h_k).$$

As $\tilde{\omega}_\gamma$ involves only imaginary roots, the differentiation is easily done (after using the Weyl character formula) giving the result. □

We are finally in a position to compute $\operatorname{Tr} D e^{-tD^2}$ in group theoretical terms. Let $\mathscr{E}_1(\Gamma)$ denote the set of Γ-conjugacy classes of non-trivial elements in Γ contained in a fundamental, $\mathbb{R}$-rank one Cartan subgroup of G. It follows from [Ms] that there are infinitely many such Γ-conjugacy classes; also if G is of $\mathbb{R}$-rank one and non-equirank with K then $\mathscr{E}_1(\Gamma)$ consists of all non-trivial Γ-conjugacy classes.

Theorem 4.9. *Let X be a compact locally symmetric manifold of odd dimension and $\mathbb{E}$ a locally homogeneous Dirac bundle over X with Dirac operator D. Then*

$$\begin{aligned}(4.8)\qquad \operatorname{Tr}(D e^{-tD^2}) = \sum_{[\gamma]\in\mathscr{E}_1(\Gamma)} & [G_\gamma : G_\gamma^0]^{-1} \operatorname{vol}(G_\gamma/\Gamma_\gamma)(2\pi i) c_\gamma^{-1} \\ & \cdot \frac{l_\gamma e^{-l_\gamma^2/4t}}{(4\pi t)^{3/2}} \frac{1}{\xi_\rho(h_\gamma) \prod_{\alpha\in P_\gamma^c} [1-\xi_{-\alpha}(h_\gamma)} \\ & \cdot \sum a_i \sum_{W(M^0, A_I)} \varepsilon(w)\tilde{\omega}_\gamma(w\cdot\lambda_i)\,\xi_{w\cdot\lambda_i}(h_{\gamma,k})\end{aligned}$$

here $h_\gamma \in B$ is G-conjugate to γ.

Proof. It suffices to recall that

$$\operatorname{Tr}(D e^{-tD^2}) = \sum_{[\gamma]} \operatorname{vol}(G_\gamma/\Gamma_\gamma)\, O_{\operatorname{tr}\tilde{k}_t}(\gamma)$$

and the relationship between orbital integrals and $'F_f$ (4.1). That $\operatorname{tr}\tilde{k}_t \in \mathscr{S}(G)$ is admissible is well-known (e.g. [Mo], (4.4)).

§ 5. Cohomological interpretation

The goal of this section is an expression for $\operatorname{Tr}(D e^{-tD^2})$ in geometric terms. A formula for the zeta function and one for the eta function will then follow from appropriate integral transforms and some additional analysis.

We recall a few facts about the geometry of the geodesic flow on T^1X. The conjugacy classes in Γ (the fundamental group of X) are in $1-1$ correspondence with the set of free homotopy classes of closed curves in X. For each conjugacy class $[\gamma] \neq 1$ consider the periodic geodesics (of period one) in the corresponding free homotopy class. Take a horizontal lift of each of these geodesics to T^1X and call the resulting set of curves (in T^1X) X_γ. Concerning X_γ, one knows that it is a smooth connected manifold canonically diffeomorphic to $\Gamma_{\gamma'}\backslash G_{\gamma'}/U_{\gamma'}$ ($U_{\gamma'}$ maximal compact in $G_{\gamma'}$) for any $\gamma' \in [\gamma]$; that distinct conjugacy classes give disjoint submanifolds of T^1X; and that the fixed point set of the geodesic flow (at $t=1$) consists of the union of the X_γ ([DKV] § 5). The locally symmetric spaces $\Gamma_{\gamma'}\backslash G_{\gamma'}/U_{\gamma'}$ are also isometric; more generally, for any x in the G-conjugacy class of γ the spaces $\Gamma_\gamma^x\backslash G_x/U_x$ ($\Gamma_\gamma^x = x\Gamma_\gamma x^{-1}$) and $\Gamma_\gamma\backslash G_\gamma/U_\gamma$ are isometric. As Γ is co-compact, any $\gamma\in\Gamma$ is a semisimple element in G; thus γ is G-conjugate to an element in standard position (relative to our choice earlier of Cartan involution). Henceforth, we assume γ is in standard position, then we write $\gamma = \gamma_I \exp Y$. We endow X_γ with the metric from $\Gamma_\gamma \backslash G_\gamma / K_\gamma$, and then identify $X_\gamma \subseteq T^1 X$ with $\Gamma_\gamma \backslash G_\gamma / K_\gamma$.

As X_γ has non-positive sectional curvature, it has a foliation by the Euclidean local de Rham factor. For $\gamma \in \mathscr{E}_1(\Gamma)$, this local factor is one dimensional and the foliation is easily described; viz., through each point in X_γ there passes a unique closed geodesic-take these geodesics to be the leaves of the foliation. Let $\mathbb{L}X_\gamma$ be the line bundle generated by the tangent to the closed geodesic and ${}^0TX_\gamma$ the normal bundle; then TX_γ is the orthogonal sum of ${}^0TX_\gamma$ and $\mathbb{L}X_\gamma$.

We shall now explicitly describe the parallel transport around a closed geodesic associated to γ.

Lemma 5.1. *Let $V = \Gamma_\gamma \backslash G_\gamma \times_{K_\gamma} V$ be a locally homogeneous vector bundle with invariant connection, over X_γ, associated to a representation ρ of K_γ on V. Given $p = \Gamma_\gamma g K_\gamma$, let c_p be the unique closed geodesic passing through p, which is the projection of an axis of γ. If $\tau(c_p)$: $V_p \to V_p$ denotes the parallel transport map around c_p then*

$$\tau(c_p)[\Gamma_\gamma g, v] = [\Gamma_\gamma g, \rho(\gamma_I)^{-1} v], \qquad v \in V.$$

Proof. Let $q = g K_\gamma \in \tilde{X}_\gamma$ and let c_q be the axis passing through q, i.e., $c_q(t) = g \exp t Y K_\gamma$. Consider now a section σ of $\tilde{V} = G_\gamma \times_{K_\gamma} V$ over c_q which is parallel along c_q. Then $\sigma(x K_\gamma) = [x, f(x)]$ where f: $G_\gamma \to V$ has the property $f(xk) = \rho(k)^{-1} f(x)$, $k \in K_\gamma$. The parallelism condition $\nabla_{\dot{c}_q(t)} \sigma(c_q(t)) = 0$ is equivalent to $\frac{d}{ds} f(g \exp t Y \exp s Y)|_{s=0} = 0$; whence $\frac{d}{dt} f(g \exp t Y) = 0$, for any $t \in \mathbb{R}$, i.e., $f(g \exp t Y) = f(g)$. Therefore,

$$\begin{aligned} f(\gamma g) &= f(g\gamma) = f(g \exp Y \gamma_I) = \rho(\gamma_I)^{-1} f(g \exp Y) \\ &= \rho(\gamma_I)^{-1} f(g), \quad \text{i.e.,} \\ \sigma(\gamma g) &= [\gamma g, f(\gamma g)] = \gamma \cdot [g, \rho(\gamma_I)^{-1} f(g)]. \end{aligned}$$

This shows that the parallel transport $\tilde{\tau}_{q,\gamma q}(\gamma)$ from q to γq along c_q is given by

$$\tilde{\tau}_{q,\gamma q}(\gamma)[g, v] = \gamma [g, \rho(\gamma_I)^{-1} v],$$

which, when projected down to X_γ, proves the claim. □

We denote by $\Pi(\gamma)$ the compact, topologically cyclic group generated by the parallel transport τ in $\Gamma_\gamma \backslash G_\gamma \times_{K_\gamma} \mathfrak{p}$ around the geodesic c_p. Recall that $\mathbb{L}X_\gamma$ is the line bundle generated by the vector field $\dot{c}_p(0)$, $p \in X_\gamma$, and let $\mathbb{L}^\perp X_\gamma$ be its full orthocomplement, i.e.,

$$\mathbb{L}^\perp X_\gamma = \Gamma_\gamma \backslash G_\gamma \times_{K_\gamma} \mathfrak{p}^Y, \qquad \mathfrak{p} = \mathbb{R} Y \oplus \mathfrak{p}^Y.$$

Notice that from Lemma 5.1 $\Pi(\gamma)$ acts on $\mathbb{L}^\perp X_\gamma$, and trivially on ${}^0TX_\gamma$, viewed as a subbundle via the natural inclusion ${}^0i_\gamma$: ${}^0TX_\gamma \to \mathbb{L}^\perp X_\gamma$.

Recall that $X_\gamma \subseteq T^1 X$ and that $TX_\gamma \subseteq T^{\mathrm{hor}}(TX)$, the horizontal bundle over TX. At each $u \in X_\gamma$ consider the linear Poincaré map $P(\gamma)_u$, i.e., the differential of the geodesic flow $(t=1)$ at the fixed point u. In [DKV] § 5.4, it is shown that for each generalized eigenvalue λ of $P(\gamma)_u$ with $|\lambda|=1$ there is an eigenspace in $T_u^{\mathrm{hor}}(TX) \otimes \mathbb{C}$. Let $C_u(TX)$ be the subspace of $T_u^{\mathrm{hor}}(TX)$ whose complexification consists of eigenvectors of $P(\gamma)_u$ of modulus one, and let $C(TX)$ be the resulting bundle over X_γ, the "center bundle". Notice that TX_γ is certainly contained in $C(TX)$. We let NX_γ denote the subbundle of $C(TX)$ orthogonal to TX_γ. Then NX_γ can be viewed as the bundle of "twists" ([K]). The parallel transport group $\Pi(\gamma)$ acts on NX_γ and hence NX_γ decomposes into eigenbundles

$$NX_\gamma = NX_\gamma(-1) \oplus \sum_{0<\theta<\pi} NX_\gamma(\theta)$$

according to the eigenvalues $-1, e^{\pm i\theta}\, 0<\theta<\pi$ of τ; that the eigenvalue 1 does not occur, follows from [DKV, Prop. 5.8]. As in ([A-S III], § 3) we attach to each eigenbundle the stable characteristic classes $\mathcal{R}(-1)$ and $\mathcal{S}(\theta)$. The hyperbolic directions, i.e., the subspace of $T_u(TX)$ for the generalized eigenvalues λ, $|\lambda| \neq 1$, are invariant by $P(\gamma)_u$; we denote by $P_h(\gamma)$ the restriction of $P(\gamma)_u$ to this space.

In order to define our local Lefschetz number we must first associate an equivariant K-theory class to our Dirac operator. Recall that $\mathbb{E} = \Gamma \backslash G \times_K E$ is the original Clifford module bundle over X; let $\mathbb{E}_\gamma = \Gamma_\gamma \backslash G_\gamma \times_{K_\gamma} E$ be its restriction to X_γ. For each $p \in X_\gamma$ the involution $c(\omega^{\mathbb{C}} \dot{c}_p(0))$ splits $E_{\gamma,p}$ into the $\pm i$ eigenspaces $E^\pm_{\gamma,p}$. This determines a splitting of the bundle $\mathbb{E}_\gamma$ as a direct sum of two subbundles $\mathbb{E}^\pm_\gamma = \Gamma_\gamma \backslash G_\gamma \times_{K_\gamma} E^\pm$. If $Z \in \mathfrak{p}^Y$, $c(Z)$ anti-commutes with $c(Y)$ and thus exchanges E^+ and E^-. Moreover, the Clifford multiplication gives a K_γ module homomorphism

$$\mathfrak{p}^Y \to \mathrm{Hom}(E^+, E^-).$$

Indeed from (3.2) we have with $\rho^\pm = \rho|_{E^\pm}$

$$c(\mathrm{Ad}\, kZ) = \rho^-(k)\, c(Z)\, \rho^+(k), \qquad k \in K_\gamma. \tag{5.1}$$

Let $j_\gamma\colon \mathbb{L}^\perp X_\gamma \to X_\gamma$ be the projection map and consider the pull-back bundles

$$j_\gamma^* \mathbb{E}^\pm_\gamma = (\Gamma_\gamma \backslash G_\gamma \times \mathfrak{p}^Y) \times_{K_\gamma} E^\pm.$$

The Clifford multiplication induces a homomorphism of vector bundles (over $\mathbb{L}^\perp X_\gamma$)

$$\sigma^D_\gamma\colon j^*_\gamma \mathbb{E}^+_\gamma \to j^*_\gamma \mathbb{E}^-_\gamma,$$

which is an isomorphism outside the zero-section. Moreover, in view of Lemma 5.1 and of identity (5.1) σ^D_γ commutes with the parallel transport around the geodesic c_p, $p \in X_\gamma$. One concludes that σ^D_γ defines a class in $K_{\Pi(\gamma)}(\mathbb{L}^\perp X_\gamma)$, the $\Pi(\gamma)$-equivariant K-theory with compact support of $\mathbb{L}^\perp X_\gamma$.

Recall ${}^0i_\gamma\colon {}^0TX_\gamma \to \mathbb{L}^\perp X_\gamma$ is the natural inclusion and set

$${}^0\sigma_\gamma^D = {}^0i_\gamma^* \sigma_\gamma^D \in K_{\Pi(\gamma)}({}^0TX_\gamma);$$

this is the local symbol of D. Since $\Pi(\gamma)$ acts trivially on ${}^0TX_\gamma$, $K_{\Pi(\gamma)}({}^0TX_\gamma) \cong K({}^0TX_\gamma) \otimes R(\Pi(\gamma))$. One can, therefore, define, as in [A-S III, § 3], the cohomology class

$$ch\,{}^0\sigma_\gamma^D(\tau(\gamma)) \in H^{ev}({}^0TX_\gamma; \mathbb{C}).$$

Finally we let ${}^0\mathcal{T}(X_\gamma)$ denote the Todd class of ${}^0TX_\gamma$. Putting all this together, we arrive at the definition of the local Lefschetz number

(5.2)
$$L(\gamma, D) = \left\{ \frac{ch\,{}^0\sigma_\gamma^D(\tau(\gamma))\,\mathcal{R}(NX_\gamma(-1)) \prod_{0<\theta<\pi} \mathcal{S}(NX_\gamma(\theta))\,{}^0\mathcal{T}(X_\gamma)}{|\det(I - P(\gamma)|_{NX_\gamma})|^{1/2}} \right\} ([{}^0TX_\gamma] \cap [\theta])$$

here θ is the 1-form on X_γ dual to the unit vector field $\dot{c}_p(0)$.

Recall from § 3, that to any vector in $\mathfrak{p}$ there is a splitting of the Spin module S into $S_+ \oplus S_-$. Take $[\gamma] \in \mathcal{E}_1(\Gamma)$, assumed to be in standard position, $\gamma = \gamma_I \exp l_\gamma Y$ with Y a unit vector in $\mathfrak{a}$ and $l_\gamma > 0$, and set $S = S_+ \oplus S_-$ relative to Y. Let $(S_+ - S_-) \otimes V = (\sigma_+ - \sigma_-) \otimes \sum a_i W_{\lambda_i}$ be the decomposition as $K \cap M^0$ modules, where V is as in Lemma 3.1.

Proposition 5.2.

(5.3)
$$L(\gamma, D) = (-1)^{\# P_{I,n}} \frac{[G_\gamma : G_\gamma^0]^{-1} \operatorname{vol}(G_\gamma/\Gamma_\gamma)\, c_\gamma^{-1}}{\xi_{\rho_I}(\gamma) \prod_{\alpha \in P_{\mathfrak{k},I}^c} [1 - \xi_{-\alpha}(\gamma)]} \cdot \sum a_i \sum_{W(M^0, A_I)} \varepsilon(w)\, \tilde{\omega}_\gamma(w \cdot \lambda_i)\, \xi_{w \cdot \lambda_i}(\gamma_I).$$

Remark. This Lefschetz number agrees with the one described in the introduction, due to the normalization of $[\theta]$.

Proof. The proof is obtained through several Lemmas, technical in nature, which handle problems stemming from the disconnectedness of G_γ, and then lead by universality properties of characteristic classes together with fiber integration, to a computation done in [H-P]. We emphasize that we do not reduce our problem to the "global" situation in [H-P] because in general we do not have a co-compact subgroup of an equirank group, but rather we reduce the computation to the "local" situation in [H-P], and hence ultimately to the computation in [Sc].

First we want to replace X_γ by its orientable cover $\bar{X}_\gamma \simeq \Gamma_\gamma \backslash G_\gamma / K_\gamma^0$, here K_γ^0 is the identity component of K_γ. Lifting everything to $\bar{X}_\gamma$ has the effect of multiplying the expression by the order $[K_\gamma : K_\gamma^0]$ of the covering. The next step is the Thom isomorphism. As explained in ([A−S III] § 2), one can replace the evaluation on $[{}^0T\bar{X}_\gamma] \cap [\bar{\theta}]$, via the Thom isomorphism, by evaluation on $[\bar{X}_\gamma] \cap [\bar{\theta}]$. For this one must replace $ch\,{}^0\sigma_\gamma^D(\tau(\gamma))$ by

$\left(\frac{ch E^+(\tau(\gamma)) - ch E^-(\tau(\gamma))}{{}^0e_\gamma}\right)(\Gamma_\gamma \backslash G_\gamma) \in H^{ev}(\bar{X}_\gamma; \mathbb{C})$ where ${}^0e_\gamma \in H^*(BK_\gamma^0)$ is the restriction of the Euler class of $H^*(BSO({}^0\mathfrak{p}_\gamma))$ via the representation Ad: $K_\gamma^0 \to SO({}^0\mathfrak{p}_\gamma)$. We note that ${}^0e_\gamma \neq 0$ since K_γ^0 has no trivial weight space in ${}^0\mathfrak{p}_\gamma$. Thus we have

$$(5.4) \qquad L(\gamma, D) = \frac{(-1)^{n_\gamma}[K_\gamma; K_\gamma^0]}{|\det(I - P(\gamma)|_{NX_\gamma})|^{1/2}} \left(\frac{ch E^+(\tau(\gamma)) - ch E^-(\tau(\gamma))}{{}^0e_\gamma}\right)(\Gamma_\gamma \backslash G_\gamma)$$
$$\cdot \mathscr{R}(N\bar{X}_\gamma(-1)) \prod_{0<\theta<\pi} \mathscr{S}(N\bar{X}_\gamma(\theta))([\bar{X}_\gamma] \cap [\theta])$$

where $n_\gamma = \frac{1}{2}\dim {}^0\mathfrak{p}_\gamma$.

To evaluate these classes we shall separate the contributions from the split component A of G_γ and the remaining reductive factor $G'_\gamma C_{\gamma_I}$. Here C_{γ_I} is a torus, with $C_\gamma = C_{\gamma_I} A$ the connected center of G_γ and G'_γ a semisimple (frequently disconnected) group with finite center.

Lemma 5.3. *Let* $[\gamma] \in \mathscr{E}_1(\Gamma)$, $\gamma = \gamma_I \exp l_\gamma Y$. *Then*

(i) $Z(\Gamma_\gamma)$, *the center of* Γ_γ, *is free abelian of rank* 1.

(ii) $\Gamma_\gamma \cap C_\gamma$ *is free abelian of rank* 1.

Proof. (i) Since the dimension of the Euclidean local factor of $\Gamma_\gamma \backslash G_\gamma / K_\gamma$ is one, it is well known that $Z(\Gamma_\gamma)$ then is free of rank one.

(ii) $\Gamma_\gamma \cap C_\gamma$ is finitely generated ([W]) and torsion free. Suppose $\gamma_1 = t_1 \exp H$ and $\gamma_2 = t_2 \exp \alpha H$ are two generators, here $t_i \in C_{\gamma_I}$ and $H \in \mathfrak{a}$. If $\alpha = \frac{p}{q}$ is rational, then $\gamma_1^p \gamma_2^{-q} \in \Gamma_\gamma \cap C_{\gamma_I}$ hence is torsion. So suppose α is irrational. Using Dirichlet's theorem, for any n there are integers p_n, q_n with $|q_n \alpha - p_n| < \frac{1}{n}$. Let $t \in C_{\gamma_I}$ be a limit point of $\{t_2^{q_n} t_1^{-p_n}\}$; then t is a limit point of $\{\gamma_2^{q_n} \gamma_1^{-p_n}\}$. But $\Gamma_\gamma \cap C_\gamma$ is discrete and closed; hence there is a neighborhood of t, N_t, with $N_t \cap \Gamma_\gamma \cap C_\gamma$ containing at most t. Thus rank $\Gamma_\gamma \cap C_\gamma \leqq 1$. Now γ is in the center of G_γ and the center of G'_γ is finite, so for some $N \geqq 1$, $\gamma^N \in C_\gamma$; hence rank $\Gamma_\gamma \cap C_\gamma = 1$. □

We take a generator γ^* for $\Gamma_\gamma \cap C_\gamma$ with $\gamma = (\gamma^*)^{m_\gamma}$, $m_\gamma \geqq 1$. The integer m_γ is the algebraic multiplicity of the geodesics in X_γ. We write $\gamma^* = \gamma_I^* \exp \|\gamma^*\| Y$.

Lemma 5.4. $\Gamma_\gamma \cap C_\gamma \backslash C_\gamma / C_{\gamma_I}$ *is isometric to* S^1 *via* $t \mapsto \Gamma_\gamma \cap C_\gamma(\exp t \|\gamma^*\| Y) C_{\gamma_I}$, $t \in [0, 1]$.

Proof. The map is clearly surjective. Suppose that $\exp t \|\gamma^*\| Y \in \Gamma_\gamma \cap C_\gamma \cdot C_{\gamma_I}$. Then $\exp t \|\gamma^*\| Y = (\gamma^*)^n k = (\gamma_I^*)^n k \exp n \|\gamma^*\| Y$, thus $t = n$, so $t = 0$ or 1. □

Lemma 5.5. *Set* $\Gamma'_\gamma = G'_\gamma \cap \Gamma_\gamma C_\gamma$. *Then* Γ'_γ *is a discrete, co-compact subgroup of* G'_γ.

Proof. This result is a variation of Lemma 3.3 in [W]. To see that Γ'_γ is discrete, let $\gamma'_i \to 1$ in Γ'_γ, with $\gamma'_i = \gamma_i c_i \in \Gamma_\gamma C_\gamma$. Then for any $\gamma \in \Gamma_\gamma$, the commutators $[\gamma_i, \delta] = [\gamma'_i, \delta] \to 1$ in Γ_γ. Since Γ_γ is discrete, it follows that $[\gamma_i, \delta] = 1$ for $i \geqq i(\delta)$. But

Γ_γ is finitely generated, so γ_i must be in $Z(\Gamma_\gamma)$ for i large. Now $\Gamma_\gamma \cap G_\gamma^0$ is uniform in G_γ^0 so by the Selberg density property γ_i (and hence γ_i') centralizes G_γ^0, for i large. As $[\gamma] \in \mathscr{E}_1(\Gamma)$, the fundamental Cartan subgroup B is contained in G_γ^0. But then γ_i' centralizes B, hence $\gamma_i' \in B$. Since γ_i' must then be in the center of G_γ^0 but also $\{\gamma_i'\} \to 1$ we have for large i, $\gamma_i' = 1$. The proof that Γ_γ' is co-compact is the same as in [W]. □

To handle possible torsion in Γ_γ' we take ${}^0\Gamma_\gamma'$ a normal subgroup of Γ_γ' with $|\Gamma_\gamma' : {}^0\Gamma_\gamma'| < \infty$ and torsion free. Since G is not equirank we may assume that it is linear and then the existence of ${}^0\Gamma_\gamma'$ follows from [B]. We set ${}^0\Gamma_\gamma = \Gamma_\gamma \cap {}^0\Gamma_\gamma' C_\gamma$.

Lemma 5.6. *${}^0\Gamma_\gamma$ is normal in Γ_γ and $|\Gamma_\gamma : {}^0\Gamma_\gamma| < \infty$.*

Proof. Let $\beta \in \Gamma_\gamma$ and $\alpha \in {}^0\Gamma_\gamma$, and write $\beta = \beta' c_\beta \in \Gamma_\gamma' C_\gamma$ (resp. $\alpha = \alpha' c_\alpha$). Then $\beta \alpha \beta^{-1} = \beta \alpha' \beta^{-1} c_\alpha = \beta' \alpha' (\beta')^{-1} c_\alpha \in \Gamma_\gamma' C_\gamma \cap \Gamma_\gamma = {}^0\Gamma_\gamma$. Next let $\alpha \in \Gamma_\gamma$, $\alpha = \alpha' c$, and let α_j', $1 \leqq j \leqq |\Gamma_\gamma' : {}^0\Gamma_\gamma'|$, be representatives for $\Gamma_\gamma' / {}^0\Gamma_\gamma'$. Then for some j, $\alpha' = \alpha_j' \beta'$ with $\beta' \in {}^0\Gamma_\gamma'$, and hence $\alpha = \alpha_j' \beta' c$. Since $\alpha_j' \in \Gamma_\gamma'$, there are $\alpha_j \in \Gamma_\gamma$ and $c_j \in C_\gamma$ with $\alpha_j = \alpha_j' c_j$. Then $\alpha = \alpha_j \beta' c c_j^{-1}$ and so $\alpha_j^{-1} \cdot \alpha \in {}^0\Gamma_\gamma$, i.e., $\alpha \in \alpha_j^0 \Gamma_\gamma$. Thus $|\Gamma_\gamma : {}^0\Gamma_\gamma| \leqq |\Gamma_\gamma' : {}^0\Gamma_\gamma'|$. □

Lemma 5.7. *S^1 acts freely on ${}^0\Gamma_\gamma \backslash G_\gamma / K_\gamma$ with quotient ${}^0\Gamma_\gamma' \backslash G_\gamma' / K_\gamma'$.*

Proof. Assume on the contrary that there is $g \in G_\gamma$ and t with $g \exp t \|\gamma^*\| Y \in {}^0\Gamma_\gamma g K_\gamma$, i.e., $g \exp t \|\gamma^*\| Y = \alpha' c g k$. Writing $g = g' c^*$, we get $g' \exp t \|\gamma^*\| Y = \alpha' c g' k = \alpha' g' k' a$ where $a \in A$. Now $x \in G_\gamma$ is uniquely expressible in the form $x' a$, $x' \in G_\gamma' C_{\gamma_I}$, $a \in A$. Hence $g' = \alpha' g' k'$ or $g'^{-1} \alpha' g' \in K_\gamma \cap G_\gamma' = K' \gamma$. Thus $\alpha' \in g' K' g'^{-1} \cap {}^0\Gamma_\gamma'$ and hence is torsion; so $\alpha' = 1$. But then $c \in \Gamma_\gamma \cap C_\gamma$ which is generated by $\gamma^* = \gamma_I^* \exp \|\gamma^*\| Y$. It follows that $t = 1$ or 0 and the action is free.

Now let π: $G_\gamma / K_\gamma \to G_\gamma' / K_\gamma'$ be the obvious projection: $\pi(g K_\gamma) = g' K_\gamma'$, where $g = g' c$. Then π induces a map π: ${}^0\Gamma_\gamma \backslash G_\gamma / K_\gamma \to {}^0\Gamma_\gamma' \backslash G_\gamma' / K_\gamma'$ that is clearly surjective. It suffices to determine the fiber over a base point, ${}^0\Gamma_\gamma' K_\gamma'$. Now $\pi({}^0\Gamma_\gamma g K_\gamma) = {}^0\Gamma_\gamma' g' K_\gamma' = {}^0\Gamma_\gamma' K_\gamma'$ means that $g' \in {}^0\Gamma_\gamma' K_\gamma'$. Write $g' = (\alpha c_1^{-1}) k'$, or $g' c_1 = \alpha k'$. Multiplying by c we get $g c_1 = \alpha k' c$, or writing $c c_1^{-1} = c_I \exp t \|\gamma^*\| Y$, $\alpha^{-1} g = \exp t \|\gamma^*\| Y c_I k'$. Thus ${}^0\Gamma_\gamma g K_\gamma = {}^0\Gamma_\gamma \exp t \|\gamma^*\| Y K_\gamma$. □

Returning to the proof of Proposition 5.2, we set $\bar{X}_\gamma = \Gamma_\gamma \backslash G_\gamma / K_\gamma^0$, ${}^0X_\gamma = {}^0\Gamma_\gamma \backslash G_\gamma / K_\gamma$, ${}^0\bar{X}_\gamma = {}^0\Gamma_\gamma \backslash G_\gamma / K_\gamma^0$, ${}^0X_\gamma' = {}^0\Gamma_\gamma' \backslash G_\gamma' / K_\gamma'$ and ${}^0\bar{X}_\gamma' = {}^0\Gamma_\gamma' \backslash G_\gamma' / K_\gamma'^0$. Then ${}^0\bar{X}_\gamma$ is a finite $(|\Gamma_\gamma : {}^0\Gamma_\gamma|)$ cover of $\bar{X}_\gamma$ and so it suffices to evaluate the classes on ${}^0\bar{X}_\gamma$. Also ${}^0X_\gamma$ is an S^1-fibration over ${}^0X_\gamma'$ and the classes under consideration are the pull-back to ${}^0X_\gamma$ of the corresponding classes on ${}^0X_\gamma'$. The next result, fiber integration, reduces the problem to the evaluation on ${}^0X_\gamma'$ (or the oriented cover ${}^0\bar{X}_\gamma'$) of these classes.

Lemma 5.8. *Let $[\omega] \in H^*({}^0\bar{X}_\gamma'; \mathbb{C})$, ω a top degree form. Let ${}^0\bar{\pi}$: ${}^0\bar{X}_\gamma \to {}^0\bar{X}_\gamma'$ be the projection. Then*

$$\langle {}^0\bar{\pi}^*[\omega], [{}^0\bar{X}_\gamma] \cap [\bar{\theta}_\gamma] \rangle = \mathrm{vol}(C_\gamma / C_\gamma \cap {}^0\Gamma_\gamma) \langle [\omega], [{}^0\bar{X}_\gamma'] \rangle.$$

Proof. One has

$$\begin{aligned} \langle {}^0\bar{\pi}^*[\omega], [{}^0\bar{X}_\gamma] \cap [\bar{\theta}_\gamma] \rangle &= \int_{{}^0\bar{X}_\gamma} {}^0\bar{\pi}^*[\omega] \wedge \bar{\theta}_\gamma \\ &= \int_{{}^0\bar{X}_\gamma'} \omega \wedge {}^0\bar{\pi}_*(\bar{\theta}_\gamma) \end{aligned}$$

where ${}^0\bar{\pi}_*$ is integration over the fiber. But since $\bar{\theta}_\gamma$ is induced by the volume form of $C_\gamma/C_\gamma \cap {}^0\Gamma_\gamma$, we get ${}^0\bar{\pi}_*(\bar{\theta}_\gamma) = \mathrm{vol}(C_\gamma/C_\gamma \cap {}^0\Gamma_\gamma)$. □

Lemma 5.9

$$\text{(i)}\quad {}^0\Gamma_\gamma \cap C_\gamma = \Gamma_\gamma \cap C_\gamma.$$

$$\text{(ii)}\quad \mathrm{vol}(C_\gamma/C_\gamma \cap {}^0\Gamma_\gamma) = \|\gamma^*\|.$$

Proof. (i) ${}^0\Gamma_\gamma < \Gamma_\gamma$, thus ${}^0\Gamma_\gamma \cap C_\gamma \subseteq \Gamma_\gamma \cap C_\gamma$. But ${}^0\Gamma_\gamma = \Gamma_\gamma \cap {}^0\Gamma'_\gamma C_\gamma$, thus $\Gamma_\gamma \cap C_\gamma \subseteq {}^0\Gamma_\gamma$.

(ii) In Lemma 5.7 we saw that the fiber is given by ${}^0\Gamma_\gamma g \exp t \|\gamma^*\| YK_\gamma$, $t \in [0, 1]$ and Y a unit vector. Hence $\mathrm{vol}(C_\gamma/C_\gamma \cap {}^0\Gamma_\gamma) = \|\gamma^*\|$. □

We recall that γ is in standard position and $[\gamma] \in \mathscr{E}_1(\Gamma)$, i.e., $\gamma \in B = A_I A$, the fundamental Cartan subgroup of G. Let P, as before, be the associated cuspidal parabolic subgroup, and, for an appropriate order, let $P = MAN$ be a Langlands decomposition. Then $\gamma = \gamma_I \exp l_\gamma Y$ with $\gamma_I \in M^{\,0}$ an elliptic element. Notice that $M^0_{\gamma_I} = (G'_\gamma)^0 C_{\gamma_I}$ and $M_{\gamma_I} = G'_\gamma C_{\gamma_I}$ (as follows from ([DKV] Lemma 4.1)). We let $K^M_{\gamma_I}$ (resp. $K^{M^0}_{\gamma_I}$) denote the maximal compact subgroup of M_{γ_I} (resp. $M^0_{\gamma_I}$) relative to the restriction to M of the Cartan involution. Then $M_{\gamma_I}/K^M_{\gamma_I} \simeq M^0_{\gamma_I}/K^{M^0}_{\gamma_I}$, and ${}^0\Gamma'_\gamma \backslash M_{\gamma_I}/K^M_{\gamma_I} \simeq {}^0X'_\gamma$ is a finite cover (of order $|{}^0\Gamma'_\gamma : {}^0\Gamma'_\gamma \cap (G'_\gamma)^0|$) of ${}^0\Gamma'_\gamma \cap M^0_{\gamma_I} \backslash M^0_{\gamma_I}/K^{M^0}_{\gamma_I}$. Although there need not be a discrete, co-compact subgroup of M^0, nevertheless, because of ${}^0\Gamma'_\gamma$, we are in the same "local" setting as in [H-P]. Recall that $(S_+ - S_-) \otimes V = \sum a_i(\sigma_+ - \sigma_-) \otimes W_{\lambda_i}$ with W_{λ_i} modules for K^{M^0}. Then for each λ_i we define the Lefschetz numbers $L(\gamma_I, \lambda_i - \rho_{M,n})$ as in [H-P]

$$(5.5)\qquad L(\gamma_I, \lambda_i - \rho_{M,n}) = \left\{ \frac{ch\, i^*_\gamma\, \sigma(\mathbb{E}_{\lambda_i - \rho_{M,n}})(f)\, \mathscr{R}(N^\gamma(-1)) \prod\limits_{0<\theta<\pi} \mathscr{S}(N^\gamma(\theta))\, \mathscr{T}(X^\gamma)}{\det(I - f|_{N_\gamma})} \right\} [TX^\gamma].$$

The characteristic classes in $L(\gamma_I, \lambda_i - \rho_{M,n})$ are given by universality properties of the structure group K_γ and the tangent bundle; hence are the same as the classes in $L(\gamma, D)$ (5.2). The only difference is in Chern of the symbol class, but here $ch\, {}^0\sigma^D_\gamma(\tau(\gamma)) = \sum a_i\, ch\, i^*_\gamma\, \sigma(\mathbb{E}_{\lambda_i - \rho_{M,n}})(\tau(\gamma))$. Hence we evaluate $L(\gamma, D)$ using the calculation in [H-P], and ultimately the one in [Sc] upon which it depends. For convenience we state the formula for $L(\gamma_I, \lambda_i - \rho_{M,n})$:

$$(5.6)\qquad L(\gamma_I, \lambda_i - \rho_{M,n}) = (-1)^{n+n_\gamma} [K_{\gamma_I} : K^0_{\gamma_I}]^{-1} |W_{\gamma_I}|^{-1} |W^{\mathbb{C}}_{\gamma_I}|^{-1} \cdot \frac{\prod\limits_{\alpha \in \rho_{\gamma_I}} \langle \rho_{\gamma_I}, \alpha \rangle^{-1} v({}^0\Gamma'_\gamma \cap M^0_{\gamma_I} \backslash M^0_{\gamma_I})}{\xi_{\rho_I}(\gamma_I) \prod\limits_{\alpha \in P \backslash P_{\gamma_I}} [1 - \xi_{-\alpha}(\gamma_I)]} \sum_{w \in W(M_0, A_I)} \varepsilon(w)\, \tilde{\omega}(w \cdot \lambda_i)\, \xi_{w \cdot \lambda_i}(\gamma_I).$$

Except for some obvious notational differences (and some unexplained notation for which we refer to [H-P]) this will give (5.3) once we reconcile the differences in choice of Haar measure between [H-P] and us. Actually it is easier to relate ours to the one in [Sc] and then [Sc] and [H-P] ([H-P] p. 217);

$$(5.7)\qquad |W^{\mathbb{C}}_{\gamma_I}|^{-1}\prod_{\alpha\in\rho_{\gamma_I}}\langle\rho_{\gamma_I},\alpha\rangle^{-1}v({}^0\Gamma'_\gamma\cap M^0_{\gamma_I}\backslash M^0_{\gamma_I}) =(2\pi)^{-\#P_{\gamma_I}}(-1)^{n_\gamma}v'({}^0\Gamma'_\gamma\cap M^0_{\gamma_I}\backslash M^0_{\gamma_I}).$$

Let us recall that $c_\gamma=(-1)^q(2\pi)^q2^{\nu/2}\tilde\omega_k(\rho_k)|W(G^0_\gamma,B)|$ and $\tilde\omega_k(\rho_k)=(2\pi)^{\gamma_k}v_{G_\gamma}(A_I)v_{G_\gamma}(K_\gamma)^{-1}$ ([H-C III], Lemma 37.4). Then we get the following relationships after an examination of the normalization of measures and the notation: $[G_\gamma:G^0_\gamma]=[K_{\gamma_I}:K^0_{\gamma_I}]$; $|W(G^0_\gamma,B)|=|W_{\gamma_I}|$; $q+\gamma_k=\#P_\gamma$; $2^{-\nu/2}v_{G_\gamma}(K_\gamma)\operatorname{vol}(G_\gamma/\Gamma_\gamma)=v_{G_\gamma}(A_I)\operatorname{vol}(C_\gamma/C_\gamma\cap\Gamma_\gamma)|\Gamma_\gamma:{}^0\Gamma_\gamma|^{-1}v'({}^0\Gamma'\cap M^0_{\gamma_I}\backslash M^0_{\gamma_I})$. Finally the sign results in $(-1)^{\#P_{I,n}}$. Thus we prove Proposition 5.2. □

We conclude this section with a reformulation in geometric terms of Theorem 4.9 for $\operatorname{Tr}(De^{-tD^2})$.

Theorem 5.10

$$(5.8)\qquad \operatorname{Tr}(De^{-tD^2})=(-1)^q(2\pi i)\sum_{[\gamma]\in\mathscr{E}_1(\Gamma)}\frac{\|\gamma^*\|\,L(\gamma,D)}{|\det(I-P_h(\gamma))|^{1/2}}\,\frac{l_\gamma e^{-l_\gamma^2/4t}}{(4\pi t)^{3/2}}$$

here γ is conjugate to $\gamma_I\exp l_\gamma Y\in A_IA$, and $q=\#P_{I,n}$ is $\frac{1}{2}$ the dimension of the space of leaves.

§ 6. The zeta function formula

In this section we define a geometric zeta function of Selberg type (actually its logarithmic derivative) with the aid of Theorem 5.10. Our approach consists of the use of functional calculus and estimates on the heat kernel and the spectral analysis from § 2 bypassing the usual Paley-Wiener technique.

Proposition 6.1. *Let* $\operatorname{Re}s^2\gg 0$. *Then*

$$\int_0^\infty e^{-s^2t}\operatorname{Tr}(De^{-tD^2})\,dt=(-1)^q(i/2)\sum_{[\gamma]\in\mathscr{E}_1(\Gamma)}\frac{L(\gamma,D)}{|\det I-P_h(\gamma)|^{1/2}}\,l_{\gamma*}\,e^{-sl_\gamma}.$$

Proof. The result follows from the identity

$$(6.1)\qquad \int_0^\infty s\notin(-\infty,0],\, e^{-s^2t}\frac{e^{-l_\gamma^2/4t}}{(4\pi t)^{3/2}}\,dt=\frac{e^{-sl_\gamma}}{4\pi l_\gamma},$$

together with (5.8) and an interchange of integrals.

To justify the interchange we fix T, $0<T<\infty$ and set $I_T=\int_0^T e^{-s^2t}\operatorname{Tr}(De^{-tD^2})\,dt$ and $I_\infty=\int_T^\infty e^{-s^2t}\operatorname{Tr}(De^{-tD^2})\,dt$. From Theorem 2.1(c) we have $I_\infty=\operatorname{Tr}(D(D^2+s^2)^{-1}e^{-T(D^2+s^2)})$. We claim that this trace can be computed from the trace formula applied to the Schwartz kernel of $\operatorname{tr}(\tilde{D}(\tilde{D}^2+s^2)^{-1} e^{-T(\tilde{D}^2+s^2)})$ and also that the result is the absolutely convergent series

(6.2) $$(-1)^q (i/2) \sum_{[\gamma]\in\mathscr{E}_1(\Gamma)} \frac{L(\gamma, D)}{|\det I - P_h(\gamma)|^{1/2}} l_{\gamma *}\left[\int_{-\infty}^{\infty} \frac{v}{v^2+s^2} e^{-T(v^2+s^2)} e^{ivl_\gamma} dv\right].$$

Indeed, choose ε, $0<\varepsilon<T$, and consider $[\tilde{D}(\tilde{D}^2+s^2)^{-1} e^{-\varepsilon(\tilde{D}+s^2)}] e^{-(T-\varepsilon)(\tilde{D}^2+s^2)}$. The first of these factors is a smoothing operator (since $\tilde{D}^2 \geqq 0$), while the second is in $[\mathscr{S}(G)\otimes \operatorname{End}(S\otimes(V))]^{K\times K}$. The convolution of the kernels of these two operators remains in the Schwartz space and so the kernel is admissible. It follows that the series of orbital integrals is absolutely convergent. To evaluate the orbital integrals recall from §3 that the Schwartz kernel of $\tilde{D} e^{-\varepsilon\tilde{D}^2}$ is a pseudo-cusp form hence, $\tilde{D}_\pi^2+s^2$ being a diagonal operator, that the Schwartz kernel of $\tilde{D}(\tilde{D}^2+s^2)^{-1} e^{-\varepsilon(\tilde{D}^2+s^2)}$ is a pseudo-cusp form. The claim now follows from the observation that Proposition 3.6 applies.

The finite time case, I_T, is handled by the dominated convergence theorem. Recall that $\tilde{h}_t$ is the heat kernel on $\tilde{X}$. Define $\sigma: G\to\mathbb{R}$ by $\sigma(x)=\sigma(\exp X\, k)=\|X\|$. Then the left invariant distance on $\tilde{X}=G/K$ is given by $d(gK, hK)=\sigma(g^{-1}h)$. In [D] one finds uniform, finite time estimates for the scalar heat kernel on manifolds admitting a properly discontinuous group of isometries with compact quotient. In a standard way, as in see e.g., [R-S], these results extend to the vector valued case. Then one has the estimates: $0<t\leqq T$

(6.3) $$\|\tilde{h}_t(gK, hK)\| \leqq C t^{-\frac{n}{2}} \exp\left(-\frac{\sigma^2(g^{-1}h)}{4t}\right)$$

$$\|D_g^i D_h^j \tilde{h}_t(gK, hK)\| \leqq C t^{-\frac{n}{2}-i-j} \exp\left(-\frac{\sigma^2(g^{-1}h)}{4t}\right),$$

here D_g, D_h are first order differential operators. From these one gets an estimate on the odd heat kernel, $k_t(x,x)$, on $X=\Gamma\backslash\tilde{X}$ for $0<t\leqq T$:

(6.4) $$|\operatorname{tr} k_t(x,x)| \leqq C t^{-\frac{n}{2}-1} \sum_{\gamma\in\Gamma} \exp\left(-\frac{\sigma^2(x^{-1}\gamma x)}{4t}\right).$$

Then we have

$$|I_T| = \left|\int_0^T \operatorname{Tr}(D e^{-tD^2}) e^{-s^2 t} dt\right|$$

$$\leqq \int_0^T e^{-\operatorname{Re} s^2 t} \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} |\operatorname{tr} \tilde{k}_t(x^{-1}\gamma x)|\, d\dot{x}\, dt$$

$$\leqq \int_0^T e^{-\operatorname{Re} s^2 t} \int_{\Gamma\backslash G} C \sum_{\gamma\in\Gamma} t^{-\frac{n}{2}-1} \exp\left(-\frac{\sigma^2(x^{-1}\gamma x)}{4t}\right) d\dot{x}\, dt,$$

(6.5) $$|I_T| \leqq \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} C'(\operatorname{Re} s^2)^{n/2} \exp(-(\operatorname{Re} s^2)^{1/2} \sigma(x^{-1}\gamma x))\, p(s, \sigma(x^{-1}\gamma x)).$$

In the last line we use the generalization of (6.1) obtained by integration by parts

$$(6.1)' \qquad \int_0^\infty e^{-t} e^{-a^2/t} \frac{dt}{t^{\frac{n}{2}+1}} = C \frac{e^{-a}}{a} \times \text{polynomial}\left(\frac{1}{a}\right).$$

Now it is well known that $e^{-\sigma(y)}$ is dominated by $\varphi_0(y)^b$, $b>0$ where φ_0 is the basic zonal spherical function. Hence $e^{-u\sigma(y)} \leqq \varphi_0(y)^{bu}$ $(u>0)$ and so $\exp(-(\operatorname{Re} s^2)^{1/2}\sigma(y))\, p(s,\sigma(y))$ is dominated by a positive power of $\varphi_0(y)^{(\operatorname{Re} s^2)^{1/2}}$. For $\operatorname{Re} s^2 \gg 0$, one knows that $\varphi_0(y)^{(\operatorname{Re} s^2)^{1/2}}$ is admissible, thus $\sum_{\gamma\in\Gamma} \varphi_0(x^{-1}\gamma x)^{(\operatorname{Re} s^2)^{1/2} b}$ is absolutely uniformly convergent on compact subsets of G. Hence the integrand in (6.4) is continuous and, as $\Gamma \backslash G$ is compact, we get that the right-hand side of (6.4) is finite. Hence we can interchange the integrals in I, getting

$$(6.6) \qquad I_T = (-1)^q (i/2) \sum_{[\gamma]\in\mathscr{E}_1(\Gamma)} \frac{L(\gamma, D)}{|\det I - P_h(\gamma)|^{1/2}}\, l_{\gamma*} \int_0^T e^{-s^2 t} \frac{e^{-l_\gamma^2/4t}}{(4\pi t)^{3/2}}\, dt.$$

Adding (6.6) to (6.2) and undoing the Fourier transform in (6.2) we obtain the Proposition. □

Remark. The number q has a geometric formulation. It is onehalf the dimension of the fiber of the center bundle $C(TX)$ over X_γ.

Definition. Let $\operatorname{Re} s^2 \gg 0$ and define $\log Z(s, D)$ by

$$(6.7) \qquad \log Z(s, D) = \sum_{[\gamma]\in\mathscr{E}_1(\Gamma)} (-1)^q \frac{L(\gamma, D)}{|\det I - P_h(\gamma)|^{1/2}} \frac{e^{-s l_\gamma}}{m_\gamma}$$

here m_γ is the algebraic multiplicity defined in § 5.

This series converges absolutely and uniformly on compact subsets of $\operatorname{Re} s^2 \gg 0$ as is seen by writing $m_\gamma = l_\gamma / l_{\gamma*}$, noticing that $\{l_\gamma \mid [\gamma]\in\mathscr{E}_1(\Gamma)\}$ is bounded from below, and dominating by the series in Proposition 6.1. One also has $\lim_{s\to+\infty} \log Z(s, D) = 0$. Indeed the absolute convergence for a fixed s_0 allows the application of the dominated convergence theorem. Summarizing, we have

Proposition 6.2. *The series*

$$\sum_{[\gamma]\in\mathscr{E}_1(\Gamma)} (-1)^q \frac{L(\gamma, D)}{|\det I - P_h(\gamma)|^{1/2}} \frac{e^{-s l_\gamma}}{m_\gamma}$$

defines a holomorphic function in $\operatorname{Re} s^2 \gg 0$, *denoted by* $\log Z(s, D)$. *Moreover* $\lim_{s\to+\infty} \log Z(s, D) = 0$.

On the other hand, in § 2 we saw that $\log \det'\left(\frac{D-is}{D+is}\right)$ is a meromorphic function and a comparison of Proposition 6.1 with (2.5) gives $\log Z(s, D)$

$=\log\det'\left(\frac{D-is}{D+is}\right)+\pi i\eta_D$ for $\operatorname{Re}s^2\gg 0$. Thus $\log Z(s,D)$ has a meromorphic continuation, and the identity $\det'\left(\frac{D-is}{D+is}\right)\det'\left(\frac{D+is}{D-is}\right)=1$ yields the functional equation

$$Z(s,D)\,Z(-s,D)=e^{2\pi i\eta_D}. \tag{6.8}$$

Hence we have

Theorem 6.3. *Set*

$$\log Z(s,D)=\sum_{[\gamma]\in\mathscr{E}_1(\Gamma)}(-1)^q\frac{L(\gamma,D)}{|\det I-P_h(\gamma)|^{1/2}}\frac{e^{-sl_\gamma}}{m_\gamma},$$

for $\operatorname{Re}s^2\gg 0$. *Then* $\log Z(s,D)$ *has a meromorphic continuation to* $\mathbb{C}$ *given by the identity*

$$\log Z(s,D)=\log\det'\left(\frac{D-is}{D+is}\right)+\pi i\eta_D.$$

Moreover, $Z(s,D)$ *satisfies the functional equation*

$$Z(s,D)\,Z(-s,D)=e^{2\pi i\eta_D}.$$

§ 7. Twisted eta invariants and applications

In this final section we extend the zeta function approach to the computation of the reduced η-invariants of Atiyah-Patodi-Singer [A-P-S].

Let $\varphi\colon\Gamma\to U(F)$ be a unitary representation of Γ on F. The associated Hermitian vector bundle $\mathbb{F}=\tilde{X}\times_\Gamma F$ over X inherits a flat connection from the trivial connection on $\tilde{X}\times F$. If $L\colon C^\infty(X,V)$ is a differential operator acting on the sections of the vector bundle V, then L extends canonically to a differential operator $L_\varphi\colon C^\infty(X,V\otimes\mathbb{F})\to C^\infty(X,V\otimes\mathbb{F})$, uniquely characterized by the property that L_φ is locally isomorphic to $L\oplus\ldots\oplus L$(dim F times). Explicitly, L_φ can be obtained as follows. First, lift L to a Γ-periodic differential operator $\tilde{L}\colon C^\infty(\tilde{X},\tilde{V})\to C^\infty(\tilde{X},\tilde{V})$, where $\tilde{V}$ is the pull-back of V. Since $\tilde{\mathbb{F}}$ is the trivial bundle $\tilde{X}\times F$, $C^\infty(\tilde{X},\tilde{V}\otimes\tilde{\mathbb{F}})\cong C^\infty(\tilde{X},\tilde{V})\otimes F$ and thus $\tilde{L}\otimes I_F$ defines a differential operator. This operator is obviously Γ-periodic and, therefore, drop down to give a differential operator acting on $C^\infty(X,V\otimes\mathbb{F})$, which clearly satisfies the required property.

Consider now a locally homogeneous Dirac bundle $\mathbb{E}$ over X and the corresponding Dirac operator $D: C^\infty(X,\mathbb{E})\to C^\infty(X,\mathbb{E})$. We recall that, via the identification of $C^\infty(X,\mathbb{E})$ with $(C^\infty(\Gamma\backslash G)\otimes E)^K$, one has

$$D=\sum_i R_\Gamma(X_i)\otimes c(X_i)\,c(\omega^{\mathbb{C}})$$

where $\{X_i\}$ is an orthonormal basis of $\mathfrak{p}$.

Lemma 7.1. *The space of L^2-sections $L^2(X,\mathbb{E}\otimes\mathbb{F})$ can be identified with $(L^2(\Gamma\backslash G;\varphi)\otimes E)^K$, where $L^2(\Gamma\backslash G;\varphi)$ is the Hilbert space of the induced representation $R_{\Gamma,\varphi}=\mathrm{ind}_\Gamma^G\varphi$. Moreover, via this identification, the extension of D by φ becomes*

$$D_\varphi=\sum_i R_{\Gamma,\varphi}(X_i)\otimes c(X_i)\,c(\omega^{\mathbb{C}}).$$

Proof. The first assertion follows from the fact that $\mathbb{E}=\Gamma\backslash G\times_K E\cong G\times_{\Gamma\times K}E$ (where Γ acts trivially on E), $\mathbb{F}=G/K\times_\Gamma F\cong G\times_{\Gamma\times K}F$ (where K acts trivially on F), and therefore, $\mathbb{E}\otimes\mathbb{F}\cong G\times_{\Gamma\times K}E\otimes F$. Thus,

$$C^\infty(X,\mathbb{E}\otimes\mathbb{F})\cong(C^\infty(G)\otimes F\otimes E)^{\Gamma\times K}\cong(C^\infty(G,F)^\Gamma\otimes E)^K,$$

which, by completion with respect to the appropriate L^2-norm, gives

$$L^2(X,\mathbb{E}\otimes\mathbb{F})\cong(L^2(\Gamma\backslash G;\varphi)\otimes E)^K.$$

Now let $D'_\varphi=\sum_i R_{\Gamma,\varphi}(X_i)\otimes c(X_i)c(\omega^{\mathbb{C}})$. Its lift $\tilde{D}'_\varphi$ to $C_c^\infty(\tilde{X},\tilde{\mathbb{E}}\otimes\tilde{\mathbb{F}})$ $\cong(C_c^\infty(G)\otimes F\otimes E)^K=(C_c^\infty(G)\otimes E)^K\otimes F$ is given by the formula

$$\tilde{D}'_\varphi=\sum_i R(X_i)\otimes c(X_i)\,c(\omega^{\mathbb{C}})\otimes I_F,$$

which implies that D_φ coincides with D'_φ. □

Let us now recall the definition of the reduced η-invariants [A-P-S]. One starts with a self-adjoint elliptic operator L: $C^\infty(X,V)\to C^\infty(X,V)$ and a unitary representation φ: $\Gamma\to U(F)$. One then forms the twisted operator L_φ: $C^\infty(X,V\otimes\mathbb{F})\to C^\infty(X,V\otimes\mathbb{F})$, which clearly remains elliptic and, since φ is unitary, self-adjoint. One can, therefore, consider its η-function $\eta(L_\varphi,s)$. The difference

$$\tilde{\eta}(L,\varphi,s)=\eta(s,L_\varphi)-\dim F\cdot\eta(s,L) \tag{7.1}$$

is the reduced η-function of L with respect to the representation φ and

$$\tilde{\eta}_\varphi(L)=\eta(L,\varphi,0) \tag{7.2}$$

is the reduced η-invariant corresponding to L and φ. Its reduction mod $\mathbb{Z}$, is a homotopy invariant of L, more precisely depends only of the stable homotopy class of the leading symbol $[\sigma(L)]\in K_c^1(TX)$ [A-P-S, Part III].

We specialize now to the case of locally homogeneous Dirac operators. Let $D\colon C^\infty(X, \mathbb{E}) \to C^\infty(X, \mathbb{E})$ be such an operator and let (φ, F) be a unitary representation of Γ. From Lemma 6.1 it follows that

$$D_\varphi e^{-tD_\varphi^2} = R_{\Gamma,\varphi}(k_t).$$

Applying the trace formula corresponding to $R_{\Gamma,\varphi}$, one obtains

$$\operatorname{Tr}(D_\varphi e^{-tD_\varphi^2}) = \sum_{[\gamma]\neq 1} \operatorname{Tr}\varphi(\gamma)\operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma\backslash G} k_t(x^{-1}\gamma x)\, d\dot{x}.$$

It is now obvious that we can repeat the arguments of the preceding sections to construct a "twisted" zeta function $Z(s, D_\varphi)$, meromorphic on $\mathbb{C}$, given for $\operatorname{Re}(s^2) \gg 0$ by the formula

$$\log Z(s, D_\varphi) = (-1)^q \sum_{[\gamma]\in\mathscr{E}_1(\Gamma)} \operatorname{Tr}\varphi(\gamma) \frac{L(\gamma, D)}{|\det I - P_h(\gamma)|^{1/2}} \frac{e^{-sl_\gamma}}{m_\gamma},$$

and that one has

$$\eta(D_\varphi) = \frac{1}{\pi i} \log Z(0, D_\varphi).$$

Passing to the reduced η-invariant, one obtains the following result.

Theorem 7.2. *With the above notation one has*

$$\tilde{\eta}_\varphi(D) = \frac{1}{\pi i} \log \tilde{Z}_\varphi(0, D), \tag{7.3}$$

where $\tilde{Z}_\varphi(s, D)$ is meromorphic on $\mathbb{C}$, given on $\operatorname{Re}(s^2) \gg 0$ by the formula

$$\log \tilde{Z}_\varphi(s, D) = (-1)^q \sum_{[\gamma]\in\mathscr{E}_1(\Gamma)} (\operatorname{Tr}\varphi(\gamma) - \dim F) \frac{L(\gamma, D)}{|\det I - P_h(\gamma)|^{1/2}} \frac{e^{-sl_\gamma}}{m_\gamma} \tag{7.4}$$

and satisfies, the functional equation

$$\tilde{Z}_\varphi(s, D)\, \tilde{Z}_\varphi(-s, D) = e^{2\pi i \tilde{\eta}_\varphi(D)}. \tag{7.5}$$

We close with a few applications of the theorem which help clarify its meaning.

Consider an arbitrary Riemannian metric g on X. Let $B_g\colon C^\infty(X, \Lambda^{ev} T_{\mathbb{C}}^* X) \to C^\infty(X, \Lambda^{ev} T_{\mathbb{C}}^* X)$ be the corresponding tangential signature operator, i.e.,

$$B_g | C^\infty(X, \Lambda^{2p} T_{\mathbb{C}}^* X) = i^{(\dim X + 1)/2} (-1)^{p+1} (*_g d - d *_g). \tag{7.7}$$

It is well-known that B_g is a Dirac-type operator. More precisely, under the canonical isomorphism $\Lambda^{ev} T^*_{\mathbb{C}} X \cong \mathrm{Cliff}^0(TX, g)$, one has

$$B_g \cong \sum_i c(e_i)\, c(\omega^{\mathbb{C}})\, \nabla_{e_i}, \tag{7.8}$$

for any local orthonormal frame $\{e_i\}$ on X. When g is the canonical locally symmetric metric we drop the subscript g from the notation.

Corollary 7.3. *For any Riemannian metric g on X, one has*

$$\tilde{\eta}_\varphi(B_g) = \frac{(-1)^q}{\pi i} \sum_{[\gamma]\in\mathscr{E}_1(\Gamma)} (\operatorname{Tr}\varphi(\gamma) - \dim F) \frac{L(\gamma, B)}{|\det I - P_h(\gamma)|^{1/2}} \frac{e^{-s l_\gamma}}{m_\gamma}\bigg|_{s=0}. \tag{7.9}$$

Proof. By [A-P-S, Part III, Thm. 2.4],

$$\tilde{\eta}_\varphi(B_g) = \tilde{\eta}_\varphi(B)$$

and thus (7.9) follows from (7.3). □

In view of this corollary we denote by $\tilde{\eta}_{\varphi, X}$ the number $\tilde{\eta}_\varphi(B_g)$ which is independent of the metric g.

Corollary 7.4. *Assume that $\tilde{\eta}_{\varphi,X} \neq 0$ for some unitary representation φ of $\pi_1(X)$. Then G contains factors locally isomorphic to $SL(3, \mathbb{R})$ or $SO(p, q)$, pq odd.*

Proof. This follows from the remark following Lemma 4.4. □

In another direction we have the following result concerning $\log \tilde{Z}_\varphi(s, D)$.

Corollary 7.5. *Assume that $X = \partial Y$, that $\mathbb{E}$ extends to a Clifford bundle on Y and that $\varphi: \pi_1(X) \to U(F)$ extends to a representation of $\pi_1(Y)$. Then $\tilde{Z}_\varphi(0, D) = \pm 1$.*

Proof. Set $\tilde{\xi}_\varphi(D) = \frac{1}{2}(\tilde{\eta}_\varphi(D) - \dim F \dim \ker D + \dim \ker D_\varphi)$. Then $\tilde{\xi}_\varphi(D) \in \mathbb{Z}$, as follows from [A-P-S II] (Th. 3.3). Hence

$$\tilde{Z}_\varphi(0, D) = \frac{Z(0, D_\varphi)}{Z(0, D)^{\dim F}} = (-1)^{\dim F \dim \ker D - \dim \ker D_\varphi}.$$

In particular $\tilde{Z}_\varphi(0, D) = \pm 1$. □

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Oblatum 23-V-1988 & 12-IX-1988