Invent. math. 73, 71–106 (1983)

Inventiones mathematicae
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Einstein Metrics and Complex Structures

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0. Main Results

The main purpose of this paper is to point out some similarities between Einstein metrics and complex structures. The results may be regarded as generalizations of some facts which hold on 2-dimensional manifolds. First we consider an Einstein metric and a complex structure. Recall a classical result of Newlander and Nirenberg.

0.1. **Proposition.** *Any integrable almost complex structure (i.e., an almost complex structure whose Nijenhuis torsion tensor vanishes) defines a complex structure.*

Of course, we cannot say that an Einstein metric is "holomorphic". The following result is the best we can expect in general.

0.2. **Proposition** ([12, Theorem 5.2]). *Any C^r-Einstein metric ($r \geqq 2$) is real analytic with respect to some real analytic structure compatible with the original C^r-differentiable structure.*

On a two or three dimensional manifold, this fact is obvious because in these cases Einstein metrics have constant sectional curvature. Moreover, we know the following

0.3. **Fact.** *Any (Einstein) metric on an orientable 2-dimensional manifold is Kählerian, that is, there exists a complex structure such that the metric becomes a compatible Kähler metric.*

And conversely,

0.4. **Fact.** *Any complex structure on a 2-dimensional manifold admits a compatible Kähler-Einstein metric.*

★ This research was done when the author was supported by the Sakkokai Foundation and invited to Ecole Polytechnique

Fact 0.4 was generalized to higher dimensions for some compact manifolds by Calabi, Aubin and Yau.

0.5. **Proposition** ([3], [4, Théorème 4], [31, p. 364 Theorem 2]). *Let J be a complex structure with negative or vanishing first Chern class. If J admits a compatible Kähler metric, then it also admits a compatible Kähler-Einstein metric.*

This result suggests some strong relationship between Einstein metrics and complex structures. But it seems that Fact 0.3 is regarded as that which holds characteristically in the 2-dimensional case. In fact, on an odd dimensional manifold, we can expect nothing more than Proposition 0.2. However, Hitchin obtained the following

0.6. **Proposition** ([18, Remark 2.2]). *Any Einstein metric on the $K3$-surface is Kählerian.*

Here, *the $K3$-surface* is a 4-dimensional C^∞-manifold which is defined by the equation: $\sum_{i=1}^{4} (z^i)^4 = 0$ in $P^3(\mathbf{C})$. We will give a weak generalization of this result.

0.7. **Theorem** (Theorem 10.5). *Let (J, g) be a Kähler-Einstein structure on a compact manifold M. Assume that the first Chern class is non-positive and that the local deformation space of the complex structure J coincides with an open set of the cohomology group $H^1(M, \Theta)$ with coefficient in the sheaf Θ of germs of holomorphic vector fields. Then any Einstein metric g_1 on M sufficiently close to the metric g is Kählerian.*

It is an interesting problem: *Are the assumptions for the local deformation space and that g_1 is close to g really necessary?* We may say that Theorem 0.7 is a kind of converse to Proposition 0.5. Next, we consider families of structures.

0.8. **Proposition** ([25]). *The space of all complex structures on a manifold locally forms a complex analytic set.*

For the meaning of this proposition, see Definition 2.7 of the local pre-moduli space of Einstein metrics. We will give a corresponding result for families of Einstein metrics.

0.9. **Theorem** (Theorem 3.1). *The local pre-moduli space of Einstein metrics forms a real analytic set.*

As a corollary, we will get

0.10. **Theorem** (Theorem 3.2). *Let g be an Einstein metric. The three notions: to be non-deformable, to be formally non-deformable and to be rigid are equivalent.*

By analogy with the construction of complex analytic structure on the space of complex structures in Proposition 0.8, we can construct a canonical riemannian metric on a family of riemannian metrics. Combining Propositions 0.5, 0.8 and Theorem 0.7, we have families of Kähler-Einstein structures which has a complex structure and a riemannian metric.

0.11. **Theorem** (Theorem 12.3). *Let* (J_t, g_t) *be a normal and stable family* (Definition 12.1) *of Kähler-Einstein structures. Then the canonical riemannian metric is a Kähler metric compatible with the complex structure introduced in Proposition* 0.8.

In the 2-dimensional case, the family is called *the Teichmüller space* and the canonical riemannian metric is known as *the Peterson-Weil metric.* We will also see that the canonical riemannian metric on a local pre-moduli space of Einstein metrics is real analytic with respect to the real analytic structure introduced in Theorem 0.9. So we can expect that this riemannian metric satisfies some elliptic equation. For example, we may ask, "*Is the canonical riemannian metric an Einstein metric?*" This problem is rather difficult except on a family of flat riemannian metrics. Even in the 2-dimensional case, it remains open.

An important difference between the theory of deformations of Einstein metrics and that of complex structures lies in the obstruction for integrability of infinitesimal deformations.

0.12. **Proposition** ([19, p. 452 Theorem]). *Let* J *be a complex structure on* M. *If* $H^2(M, \Theta)=0$, *then for any infinitesimal deformation* I *of* J, *there exists an actual deformation of* J *whose infinitesimal deformation coincides with* I.

For the theory of complex deformations, we may say that the space $H^2(M, \Theta)$ is *the obstruction space.* For the theory of Einstein deformations, we will give a similar but negative result (Theorem 0.13). It seems to be impossible to say that if some space defined solely in terms of a given Einstein metric g vanishes, then, for any infinitesimal deformation h of g, there exists an Einstein deformation of g whose infinitesimal deformation coincides with h.

0.13. **Theorem** (Proposition 5.4). *The obstruction space for the space* EEID *of all essential infinitesimal Einstein deformations is the space* EEID *itself.*

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The author would like to express his sincere gratitude to Professors M. Berger, A.L. Besse, J.-P. Bourguignon, J.L. Kazdan, J. Lafontaine, D. Meyer, and P. Pansu for many useful discussions and their encouragements.

1. Deformations and Infinitesimal Deformations of an Einstein Metric

First, we introduce some fundamental definitions and facts due to [8]. Throughout this paper, M *denotes* a compact connected n-dimensional C^∞-manifold without boundary. All objects are assumed to belong to the real C^∞-category, unless otherwise stated. By *a family of geometric structures on* M we mean a family of structures which depends smoothly on t, where t runs through an open set of a Euclidean space $\mathbf{R}^k$ or more generally a manifold. This open set is called *the parameter space.*

1.1. *Definition.* Let g be an Einstein metric on M with volume 1. A family g_t of Einstein metrics on M with volume 1 such that $g_0 = g$ is called *an Einstein deformation of* g.

Note that any family g_t of Einstein metrics on M can be reduced to an Einstein deformation by multiplying it by some function of t.

1.2. *Definition.* Let g_t be an Einstein deformation of an Einstein metric g on M. If there is a family γ_t of diffeomorphisms of M such that $g_t = \gamma_t^* g$, then g_t is said to be *trivial.*

Let g_t be an Einstein deformation of g with parameter space P. Then each g_t satisfies *the Einstein equation:*

(1.2.1) $$E(g_t) \equiv r_t - \Big(\int_{g_t} u_t/n\Big)\, g_t = 0,$$

where r_t is *the Ricci tensor,* u_t *the scalar curvature* and *the operator* $\int_g$ is defined by

(1.2.2) $$\int_g f = \int_M f v_g,$$

v_g being *the volume element defined by* g. If we differentiate the Eq. (1.2.1) with respect to t, then we get a second order linear differential equation:

(1.2.3) $$E'_g(h) \equiv (1/2)(\bar{\Delta} + 2L - 2\delta^*\delta - \operatorname{Hess} \operatorname{tr})\, h = 0$$

for $h = v[g_t]$, $v \in T_0 P$, where the operators are defined by

(1.2.4) $$(\bar{\Delta}\psi)_{ij} = -D^m D_m \psi_{ij},$$

(1.2.5) $$(L\psi)_{ij} = R_i{}^k{}_j{}^m \psi_{km},$$

(1.2.6) $$(\delta\psi)_i = -D^m \psi_{mi}$$

for bilinear forms ψ (which need not be symmetric), and

(1.2.7) $$(\delta^* \xi)_{ij} = (1/2)(D_i \xi_j + D_j \xi_i)$$

for 1-forms ξ, D being *the covariant derivative* and the sign convention of the curvature tensor R is taken so that $R_{ijij} \leqq 0$ for the standard sphere.

1.3. *Definition.* Let g be an Einstein metric on M with volume 1. A symmetric bilinear form h is called *an infinitesimal Einstein deformation of* g if h satisfies the Eq. (1.2.3) and the following equation (to preserve the volume):

(1.3.1) $$\int_g \operatorname{tr} h = 0.$$

The space of all infinitesimal Einstein deformations of g is *denoted by* EID(g) or simply EID.

Remark that the equation

(1.3.2) $$L_X g = 2\delta_g^* X,$$

where L_X denotes the Lie derivative, holds under the canonical identification between 1-forms and vector fields given by the metric g.

1.4. *Definition.* Let g be an Einstein metric on M with volume 1. An infinitesimal Einstein deformation of g of the form $L_X g$ is said to be *trivial.* The space of all trivial infinitesimal Einstein deformations is *denoted by* ETID(g) or simply ETID. An infinitesimal Einstein deformation h of g is said to be *essential* if h is orthogonal to the space ETID with respect to the global inner product defined by g. The space of all essential infinitesimal Einstein deformations of g is *denoted by* EEID(g) or simply EEID. (See Definition 2.3.)

The three equations in the following Lemma may be regarded as defining the space EEID.

1.5. **Lemma** ([8, Lemma 7.1, (7.1)]). *Let g be an Einstein metric on M with volume* 1. *A symmetric bilinear form h is an element of* EEID(g) *if and only if h satisfies the following equations.*

(1.5.1) $$(\bar{\Delta} + 2L) h = 0,$$

(1.5.2) $$\delta h = 0,$$

(1.5.3) $$\operatorname{tr} h = 0.$$

In particular, the space EEID *is finite dimensional.*

1.6. *Remark.* By definition, the isometry group $I(g)$ of g acts on the space EEID(g). This action induces an action of the space $K(g)$ of all Killing vector fields.

1.7. *Example.* Let (T^n, g) be a flat torus. Then the equations in Lemma 1.5 reduce to $Dh = 0$ and $\operatorname{tr} h = 0$. Thus $\dim \mathrm{EEID}(g) = n(n+1)/2 - 1$. For any infinitesimal Einstein deformation h, there is an Einstein deformation g_t such that $g'_0 = h$. All Einstein deformations g_t of g are families of flat metrics ([8, Addition 8.1], [7, Proposition 3.2]).

1.8. *Example.* Let g be an Einstein metric on M whose sectional curvature ranges in $(3n/(7n-4), 1]$ ($(1/4, 1]$ for $n = 4$). Then g has constant sectional curvature ([6, Théorème 2], [15, Theorem 1]). It implies that if g has positive constant sectional curvature, then there is no non-trivial Einstein deformation of g. On the other hand, EEID(g)=0 ([8, Corollary 7.3, Lemma 7.4]).

1.9. *Example.* Let (M, g) be a locally symmetric Einstein manifold of noncompact type without 2-dimensional factor. Then EEID(g)=0 and there is no non-trivial Einstein deformation of g ([23, Corollary 3.5]).

1.10. *Example.* Let (M, g) be a simply connected irreducible symmetric space of compact type. Then the space EEID(g) vanishes except for the following types: $SU(k+1)$ $(k\geqq 2)$, $SU(k)/SO(k)$ $(k\geqq 2)$, $SU(2k)/Sp(k)$ $(k\geqq 3)$, $U(p+q)/U(p)\times U(q)$ $(p\geqq q\geqq 2)$ and E_6/F_4 ([24, Theorem 5.7]).

2. Moduli Spaces of Einstein Metrics

We recall Ebin's slice theorem. In this section we set $N=[n/2]+1$. Remark that by Sobolev's embedding theorem (c.f. [16]) H^s-differentiability implies C^{s-N}-differentiability for $s\geqq N$. So, for $s\geqq N$, the space of all H^s-riemannian metrics on M makes sense, which we denote by $\mathcal{M}^s$. And the group of all H^{s+1}-diffeomorphisms of M also makes sense, which we denote by $\mathcal{D}^{s+1}$. They become Hilbert manifolds ([29]). For $g\in\mathcal{M}^\infty$, we denote by $I(g)$ the isometry group of (M, g).

2.1. **Lemma** ([13, Theorem 8.20]). *Let $g\in\mathcal{M}^\infty$. If $s\geqq N+2$, then there exist a submanifold $\mathcal{S}_g^s$ of $\mathcal{M}^s$ and a local cross section $\chi^{s+1}: I(g)\backslash\mathcal{D}^{s+1}\to\mathcal{D}^{s+1}$ defined on an open neighbourhood $\mathcal{U}^{s+1}$ of the coset $I(g)$ with the following properties.*

(2.1.1) *If $\gamma\in I(g)$, then $\gamma^*(\mathcal{S}_g^s)=\mathcal{S}_g^s$.*

(2.1.2) *Let $\gamma\in\mathcal{D}^{s+1}$. If $\gamma^*(\mathcal{S}_g^s)\cap\mathcal{S}_g^s\neq\phi$, then $\gamma\in I(g)$.*

(2.1.3) *The map $F^s: \mathcal{S}_g^s\times\mathcal{U}^{s+1}\to\mathcal{M}^s$ defined by $F^s(g_1, u)=\chi^{s+1}(u)^* g_1$ is a homeomorphism onto an open neighbourhood $\mathcal{V}^s$ of g in $\mathcal{M}^s$.*

By the property (2.1.3), any $g_1\in\mathcal{V}^s$ is isometric with some $g_2\in\mathcal{S}_g^s$. And by the property (2.1.2), two H^s-riemannian metrics g_1 and g_2 in $\mathcal{S}_g^s$ are H^{s+1}-isometric if and only if they are isometric under an isometry $\gamma\in I(g)$. This means that the quotient space $\mathcal{M}^s/\mathcal{D}^{s+1}$ is locally identified with the quotient space $\mathcal{S}_g^s$/I(g). The following Lemma is the infinitesimal version of slice theorem.

2.2. **Lemma** ([13, Proposition 8.8]). *Let $g\in\mathcal{M}^\infty$. Then the following orthogonal decomposition holds.*

$$H^s(S^2M)=\delta_g^*(H^{s+1}(S^1M))\oplus\operatorname{Ker}\delta_g\cap H^s(S^2M), \tag{2.2.1}$$

where S^pM denotes the symmetric p-tensor bundle over M. The spaces $H^s(S^2M)$, $\delta_g^*(H^{s+1}(S^1M))$ and Ker δ_g *are the tangent spaces at g respectively of $\mathcal{M}^s$, $(\mathcal{D}^{s+1})^*g$ and $\mathcal{S}_g^s$.*

2.3. *Definition.* Let $g\in\mathcal{M}^\infty$. Let ψ be a symmetric bilinear form on M. We decompose ψ into $\delta_g^*\xi+h$; $\delta_g h=0$. The symmetric bilinear form h (resp. $\delta_g^*\xi$) is called *the essential part* (resp. *the trivial part*) of ψ.

Lemma 2.1 is adapted to the ILH-category (c.f. [28]).

2.4. **Lemma** ([23, Theorem 2.2]). *In Lemma 2.1, the spaces $\mathscr{S}^s_g$, $\mathscr{U}^{s+1}$, $\mathscr{V}^s$ and the map χ^{s+1} can be taken so that $\mathscr{S}^s_g=\mathscr{S}^{N+2}_g\cap\mathscr{M}^s$, $\mathscr{U}^{s+1}=\mathscr{U}^{N+3}\cap(I(g)\backslash\mathscr{D}^{s+1})$, $\mathscr{V}^s=\mathscr{V}^{N+2}\cap\mathscr{M}^s$ and that $\chi^{s+1}=\chi^{N+3}|\mathscr{U}^{s+1}$ for $s\geqq N+2$. For any integers $s\geqq N+2$ and $k\geqq 0$, the mappings*

$$F^{s+k}_s\equiv F^{s+k}\colon \mathscr{S}^{s+k}_g\times\mathscr{U}^{s+k+1}\to\mathscr{V}^s \tag{2.4.1}$$

$$p^{s+k}_s\times q^{s+k}_s\equiv(F^{s+k})^{-1}\colon \mathscr{V}^{s+k}\to\mathscr{S}^s_g\times\mathscr{U}^{s+1} \tag{2.4.2}$$

are C^k-differentiable.

This means that, if we treat only C^∞-riemannian metrics, then the map F may be regarded as a C^∞-diffeomorphism.

In the following, we treat two C^∞-structures on M. We denote by C_∞ the original C^∞-structure on M and by C_r the C^r-structure induced by C_∞. If we denote by C'_∞ another C^∞-structure on M, then C'_r will denote the C^r-structure induced by C'_∞.

2.5 **Lemma.** *Let $s\geqq N+2$ and $g\in\mathscr{M}^\infty$. If $g_1\in\mathscr{S}^s_g$ is C^∞-differentiable with respect to some C^∞-structure C'_∞ on M such that $C'_{N+1}=C_{N+1}$, then g_1 is C^∞-differentiable with respect to the original C^∞-structure C_∞, i.e., $g_1\in\mathscr{S}^\infty_g$.*

Proof. It is known by a theorem of Whitney (c.f. [17, p. 51 Theorem 2.9]) that if $C'_{N+1}=C_{N+1}$, then there exists a C^{N+1}-diffeomorphism γ of M such that $\gamma^*C'_\infty=C_\infty$. It means that $\gamma^*g_1\in\mathscr{M}^\infty$. On the other hand, γ^*g_1 is an H^s-riemannian metric with respect to γ^*C_∞. Let $\{x^i\}$ (resp. $\{\bar{x}^i\}$) be a local C^∞-coordinate with respect to C_∞ (resp. γ^*C_∞) and $\{\Gamma^i{}_{jk}\}$ (resp. $\{\bar{\Gamma}^i{}_{jk}\}$) the Christoffel symbols of γ^*g_1 with respect to $\{x^i\}$ (resp. $\{\bar{x}^i\}$). Then the functions $\bar{x}^i$ are C^{N+1}-differentiable and $\bar{\Gamma}^i{}_{jk}$ are H^s-differentiable. By the transformation rule for the Christoffel symbols:

$$\frac{\partial^2\bar{x}^k}{\partial x^i\partial x^j}=\Gamma^m{}_{ij}\frac{\partial\bar{x}^k}{\partial x^m}-\bar{\Gamma}^k{}_{lm}\frac{\partial\bar{x}^l}{\partial x^i}\frac{\partial\bar{x}^m}{\partial x^j}, \tag{2.5.1}$$

we see the following. If $\bar{x}^i$ are H^t-differentiable for $N+1\leqq t\leqq s$, then the right hand side is H^{t-1}-differentiable by the composition law ([29, Theorem 11.3]) and so $\bar{x}^i$ are H^{t+1}-differentiable. Thus, by induction, we see that $\bar{x}^i$ are H^{s+1}-differentiable, which means that $\gamma\in\mathscr{D}^{s+1}$.

Now we approximate $\gamma\in\mathscr{D}^{s+1}$ by a sequence $\{\gamma_i\}$ of C^∞-diffeomorphisms. Then the sequence $\{\gamma_i^{-1*}\gamma^*g_1\}$ of C^∞-riemannian metrics converges to g_1 in $\mathscr{M}^s$, which implies that there is a C^∞-riemannian metric in $\mathscr{V}^s$ (defined in Lemma 2.1) which is H^{s+1}-isometric with g_1. Applying Lemma 2.4, we obtain a riemannian metric $g_2\in\mathscr{S}^\infty_g$ which is also H^{s+1}-isometric with g_1. Denote by $\eta\in\mathscr{D}^{s+1}$ this isometry, i.e., $\eta^*g_2=g_1$. Then the property (2.1.2) implies that $\eta\in I(g)$, in particular η is in $\mathscr{D}^\infty$ and so $g_1\in\mathscr{M}^\infty$. Q.E.D.

2.6. **Lemma.** *Let $s\geqq n+2$ and $g\in\mathscr{M}^\infty$. Then all H^s-Einstein metrics in $\mathscr{S}^s_g$ are in $\mathscr{S}^\infty_g$.*

Proof. Let $g_1 \in \mathscr{S}_g^s$ be an H^s-Einstein metric. By Sobolev's embedding theorem, g_1 is C^{s-N}-differentiable. Then the result follows from Proposition 0.2 and Lemma 2.5. Q.E.D.

2.7. *Notation. Denote by* $\mathscr{M}_1^s$ the space of all H^s-riemannian metrics on M with volume 1.

2.8. *Definition.* Let $g \in \mathscr{M}_1^\infty$ be an Einstein metric. The space of all Einstein metrics in $\mathscr{S}_g^\infty \cap \mathscr{M}_1^\infty$ is called *the local pre-moduli space of Einstein metrics around g on M*, and denoted by ELPM(g) or simply ELPM.

2.9. *Remark.* By definition, the isometry group $I(g)$ acts on the space ELPM(g). This action induces an action of the space $K(g)$ on ELPM(g). See Remark 1.6.

3. Real Analyticity of the Local Pre-Moduli Space of Einstein Metrics

We know that "the local pre-moduli space of complex structures" forms a complex analytic set (Proposition 0.8). Concerning the moduli space of Einstein metrics, we have the following

3.1. **Theorem.** *Let $g \in \mathscr{M}_1^\infty$ be an Einstein metric. If $s \geqq n+2$, then there is an open neighbourhood $\mathscr{U}^s$ of g in $\mathscr{S}_g^s \cap \mathscr{M}_1^s$ such that the space* ELPM(g)$\cap \mathscr{U}^s$ *forms a real analytic set in a finite dimensional real analytic submanifold Z^s of $\mathscr{U}^s$ whose tangent space $T_g Z^s$ at g coincides with the space* EEID(g).

This theorem is proved by using the real analytic implicit function theorem in Banach space. See Appendix. As a corollary, we get the following

3.2. **Theorem.** *Let $g \in \mathscr{M}_1^\infty$ be an Einstein metric. The following four conditions are equivalent.*

(3.2.1) *The Einstein metric g is deformable, that is, there is a non-trivial Einstein deformation of g.*

(3.2.2) *There is a continuous one-parameter family g_t of Einstein metrics on M such that $g_0 = g$ and that g_t are not homothetic with g for all $t \neq 0$.*

(3.2.3) *The Einstein metric g is not rigid, that is, for any open neighbourhood $\mathscr{V}$ of g in $\mathscr{M}^s$, there is an Einstein metric $g_1 \in \mathscr{V}$ which is not homothetic with g.*

(3.2.4) *The Einstein metric g is formally deformable, that is, there is a formal power series*

$$g(t) = \sum_{i=0}^{\infty} \frac{1}{i!} h^{(i)} t^i, \quad h^{(0)} = g \tag{3.2.5}$$

with coefficients in $C^\infty(S^2 M)$ which satisfies the formal equation $E(g(t)) = 0$ but is not in the formal orbit space $(\mathscr{D}^{s+1})^* g$.

Proof. Assume that condition (3.2.1) holds. We may assume that the parameter space of g_t is one-dimensional and that g_1 is not homothetic with g. [13, Theorem 8.10] says that the orbit space $(\mathscr{D}^{s+1})^* g$ is closed in $\mathscr{M}^s$. Therefore there is a maximal real number $0 \leqq t_0 < 1$ such that g_t are isometric with g for all $0 \leqq t \leqq t_0$. Let $\gamma \in \mathscr{D}^{s+1}$ be an isometry between g_{t_0} and g, i.e., $\gamma^* g_{t_0} = g$. Since the metrics g_{t_0} and g are C^∞, so is γ. Thus we obtain a new family $\bar{g}_t = \gamma^* g_{t-t_0}$ such that $\bar{g}_0 = g$ and that for all sufficiently small $t > 0$ $\bar{g}_t$ is not isometric with g, i.e., condition (3.2.2) holds. Obviously, condition (3.2.2) implies condition (3.2.3). Assume that g is not rigid. By Lemmas 2.1 and 2.6, it means that the point g is not isolated in the space ELMP(g). Then by Theorem 3.1, there is a non-trivial real analytic curve in the local pre-moduli space, which implies that g is formally deformable. Finally, assume that g is formally deformable and let the formal power series (3.2.5) is a non-trivial formal deformation. Set

$$g_{(k)t} = \sum_{i=0}^{k} \frac{1}{i!} h^{(i)} t^i. \tag{3.2.6}$$

Then each $g_{(k)t}$ is a C^∞-curve in $\mathscr{M}^\infty$, so we obtain a curve $\tilde{g}_{(k)t}$ in the space $\mathscr{S}_g^s$ by Lemmas 2.1 and 2.4. Let $h_{(k)}^{(i)}$ be the i-th derivative of $\tilde{g}_{(k)t}$ at $t=0$. Then we see that

$$h_{(k)}^{(i)} = h_{(j)}^{(i)} \quad \text{for } i \leqq \min\{k, j\}, \tag{3.2.7}$$

and so the formal curve $\tilde{g}(t)$ defined by

$$\tilde{g}(t) = \sum_{i=0}^{\infty} \frac{1}{i!} h_{(i)}^{(i)} t^i \tag{3.2.8}$$

is a non-trivial curve in the space ELPM(g). But [2, Theorem 1.2] says that any formal curve

$$c(t) = \sum_{i=0}^{\infty} \frac{1}{i!} c^{(i)} t^i \tag{3.2.9}$$

in an analytic set can be approximated by convergent curves, that is, for any positive integer k, there exists a convergent curve

$$c_{(k)t} = \sum_{i=0}^{\infty} \frac{1}{i!} c_{(k)}^{(i)} t^i \tag{3.2.10}$$

in the real analytic set such that $c_{(k)}^{(i)} = c^{(i)}$ for $i \leqq k$. Therefore we obtain a non-trivial convergent curve in the space ELPM(g), i.e., g is deformable. Q.E.D.

3.3. *Remark.* The fact that the formal deformability implies the deformability may be used to construct some new Einstein metrics. But up to now we have no application of this relation.

3.4. *Remark.* In the category of deformations of complex structures, the rigidity does not imply the non-deformability. For counter examples, see [27, pp. 23–26].

3.5. **Corollary.** *Let $g \in \mathscr{M}_1^\infty$ be an Einstein metric. If each essential infinitesimal Einstein deformation h of g is integrable, that is, if there exists an Einstein deformation g_t of g such that $g_0' = h$, then the space* ELPM(g) *forms a submanifold of $\mathscr{M}^s$ around* g *whose tangent space T_g(ELPM(g)) coincides with the space* EEID(g).

Proof. It is sufficient to prove that if f is a real analytic function defined on an open neighbourhood of the origin 0 in a Euclidean space $\mathbf{R}^k$ such that $f(0)=0$ and that for any $v\in\mathbf{R}^k$ there exists a curve $c_v(t)$ in $\mathbf{R}^k$ which satisfies $c_v(0)=0$, $c_v'(0)=v$ and $f(c_v(t))=0$, then f is identically zero. We prove it by induction. First, we have

$$0=(f\circ c_v)_0'=f_0'(c_v'(0))=f_0'(v). \tag{3.5.1}$$

Therefore $f_0'=0$. Assume that $f_0^{(i)}=0$ for $i\leqq r$. Then we see

$$0=\left(\frac{d}{dt}\right)_0^{r+1}(f\circ c_v)(t)=f_0^{(r+1)}(c_v'(0),\ldots,c_v'(0)), \tag{3.5.2}$$

and so $f_0^{(r+1)}=0$. Q.E.D.

3.6. *Example.* If EEID$(g)=0$, then ELPM(g) reduces to one point. See Examples in Sect. 1.

3.7. *Example.* Let g be the symmetric Einstein metric on $P^{2m}(\mathbf{C})\times S^2$. Then EEID$(g)\neq 0$ but ELPM(g) reduces to one point ([24, Theorem 5.7]).

4. The Canonical Riemannian Metric on the Space ELPM(g)

If the local pre-moduli space ELPM of Einstein metrics around an Einstein metric g forms a submanifold of $\mathscr{M}^s$, then it has the induced riemannian metric. But this metric may depend on the origin g (c.f. Lemma 4.7). So we define another riemannian metric.

4.1. *Definition.* A family g_t of riemannian metrics on M with parameter space P is said to be *effectively parametrized* if we have

$$v[g_t]\notin \operatorname{Im}\delta_{g_t}^* \tag{4.1.1}$$

for all $t\in P$ and non-zero $v\in T_tP$. A family g_t is said to be *normal* if the dimension of the space $K(g_t)$ of all Killing vector fields is constant for $t\in P$.

Let g_t be a family of riemannian metrics on M with parameter space P. For each $t\in P$ and $v\in T_tP$, we have the decomposition (2.2.1):

$$v[g_t]=h+L_Xg_t;\qquad \delta_{g_t}h=0. \tag{4.1.2}$$

We set

$$v^H=v-X, \tag{4.1.3}$$

which is regarded as a vector field along the map: $M\to M\times P$. If we denote by $[v^H,g_t]$ the symmetric bilinear form $v[g_t]-L_Xg_t$, then

$$h=[v^H,g_t]\in\operatorname{Ker}\delta_{g_t}. \tag{4.1.4}$$

We define a positive semi-definite inner product (,) on T_tP by setting

$$(v,w)=\langle[v^H,g_t],[w^H,g_t]\rangle_{g_t}, \tag{4.1.5}$$

where $\langle\ ,\ \rangle$ denotes the global inner product. Note that if $K(g_t) \neq 0$ then the vector field X which satisfies Eq. (4.1.2) is not unique, but $[v^H, g_t]$ is well-defined and so inner product (4.1.5) is well-defined.

4.2. **Lemma.** *Let g_t be an effectively parametrized normal family of riemannian metrics on M with parameter space P. Then the inner product* (4.1.5) *defines a riemannian metric on P.*

Proof. Since inner product (4.1.5) is positive definite on each T_tP by assumption, it is sufficient to show that the essential part h in decomposition (4.1.2) depends C^∞-ly on t and v. This follows directly from the next lemma. We will also see that the vector v^H can be taken so that it depends C^∞-ly on t and v. Q.E.D.

4.3. **Lemma.** *Let ω_t be a family of volume elements on M, E_t, F_t families of vector bundles over M with fiber metrics g_t^E, g_t^F and Q_t: $C^\infty(E_t) \to C^\infty(F_t)$ a family of differential operators of order k with injective symbol. Assume that ω_t, E_t, F_t, g_t^E, g_t^F and Q_t depends C^∞-ly (resp. real analytically) on t. That is, there are bundle isomorphisms e_t: $E_0 \to E_t$ and f_t: $F_0 \to F_t$ such that the coefficients of $e_t^* g_t^E$, $f_t^* g_t^F$ and $(f_t^{-1})_* \circ Q_t \circ (e_t)_*$ depend C^∞-ly (resp. real analytically) on t. Then the dimension of the space* $\operatorname{Ker} Q_t$ *is upper semicontinuous. If the dimension of the space* $\operatorname{Ker} Q_t$ *is constant, then the decompositions*

(4.3.1) $$H^s(E_t) = Q_t^*(H^{s+k}(F_t)) \oplus \operatorname{Ker} Q_t,$$

(4.3.2) $$H^s(F_t) = Q_t(H^{s+k}(E_t)) \oplus \operatorname{Ker} Q_t^*$$

depend C^∞-ly (resp. real analytically) on t, where Q_t^* is the formal adjoint operator of Q_t with respect to g_t^E, g_t^F and ω_t. Moreover the isomorphisms

(4.3.3) $$Q_t^* + 1 : Q_t(H^{s+2k}(E_t)) \oplus \operatorname{Ker} Q_t \to H^s(E_t),$$

(4.3.4) $$Q_t + 1 : Q_t^*(H^{s+2k}(F_t)) \oplus \operatorname{Ker} Q_t^* \to H^s(F_t)$$

also depend C^∞-ly (resp. real analytically) on t.

Proof. We may assume that the vector bundles E_t and F_t do not depend on t. The decomposition (4.3.2) for each t is due to [13, Theorem 8.5]. If we remark that $Q_t^* Q_t$ is an elliptic operator for each t, then the other isomorphisms for each t are easy to check. By Remark 13.8, the families of operators Q_t etc. are C^∞-(resp. real analytic) curves in the Banach spaces $L(H^{s+k}(E), H^s(F))$ etc. of all continuous linear operators for sufficiently large s. First we consider the map

(4.3.5) $$\text{projection} \circ Q_t : Q_0^*(H^{s+k}(F)) \oplus \operatorname{Ker} Q_0 \to Q_0(H^s(F)).$$

For $t=0$, the restriction of this map on $Q_0^*(H^{s+k}(F))$ is an isomorphism. Hence by the implicit function theorem, there is a unique homomorphism ψ_t: $\operatorname{Ker} Q_0 \to Q_0^*(H^{s+k}(F))$ which depends C^∞-ly (resp. real analytically) on t such that

(4.3.6) $$Q_t(\psi_t(z) + z) \in \operatorname{Ker} Q_0^* \quad \text{for } z \in \operatorname{Ker} Q_0.$$

Let $x \in \operatorname{Ker} Q_t$ and decompose it as

(4.3.7) $$x = u_x + z_x; \quad u_x \in \operatorname{Im} Q_0^*, \quad z_x \in \operatorname{Ker} Q_0.$$

Then we see that

(4.3.8) $$Q_t(u_x + z_x) = 0.$$

Since such ψ_t is unique, we get $\psi_t(z_x) = u_x$. In particular, if $z_x = 0$, then $x = 0$. Thus we have an injection: $\operatorname{Ker} Q_t \to \operatorname{Ker} Q_0$, from which the upper semicontinuity follows. Assume that the dimension of $\operatorname{Ker} Q_t$ is constant. Then we see that for any $z \in \operatorname{Ker} Q_0$, there exists $x \in \operatorname{Ker} Q_t$ such that $z = z_x$. But then

(4.3.9) $$Q_t(\psi_t(z) + z) = Q_t(x) = 0.$$

Thus if we set $a_t = 1 + \psi_t$, then a_t gives an isomorphism: $\operatorname{Ker} Q_0 \to \operatorname{Ker} Q_t$.

Next consider the map

(4.3.10) $$Q_t^* + a_t \colon H^s(F) \oplus \operatorname{Ker} Q_0 = \operatorname{Ker} Q_0^* \oplus Q_0(H^{s+k}(E)) \oplus \operatorname{Ker} Q_0 \\ \to H^{s-k}(E) = Q_t^*(H^s(F)) \oplus \operatorname{Ker} Q_t.$$

For $t = 0$, the restriction of this map to $Q_0(H^{s+k}(E)) \oplus \operatorname{Ker} Q_0$ is an isomorphism. Therefore by the implicit function theorem there exists a homomorphism

(4.3.11) $$\psi_t = \psi_t^I + \psi_t^K \colon \operatorname{Ker} Q_0^* \to Q_0(H^{s+k}(E)) \oplus \operatorname{Ker} Q_0$$

which depends C^∞-ly (resp. real analytically) on t such that

(4.3.12) $$Q_t^*(x + \psi_t^I(x)) + a_t \psi_t^K(x) = 0 \quad \text{for } x \in \operatorname{Ker} Q_0^*.$$

But this implies that $Q_t^*(x + \psi_t^I(x)) = a_t \psi_t^K(x) = 0$. Thus if we set $b_t = 1 + \psi_t^I$, this gives an isomorphism: $\operatorname{Ker} Q_0^* \to \operatorname{Ker} Q_t^*$.

Finally we consider the maps

(4.3.13) $$Q_t^* + a_t \colon Q_0(H^{s+2k}(E)) \oplus \operatorname{Ker} Q_0 \to H^s(E),$$

(4.3.14) $$Q_t + b_t \colon Q_0^*(H^{s+2k}(F)) \oplus \operatorname{Ker} Q_0^* \to H^s(F).$$

They are isomorphisms for $t = 0$, so we have the inverse maps ψ_t and ϕ_t, i.e.,

(4.3.15) $$(Q_t^* + a_t)\psi_t = \mathrm{id}_{H^s(E)},$$

(4.3.16) $$(Q_t + b_t)\phi_t = \mathrm{id}_{H^s(F)},$$

which depend C^∞-ly (resp. real analytically) on t. Then the map $Q_t^* \psi_t + a_t \psi_t$ gives the decomposition (4.3.1) and the map $Q_t \phi_t + b_t \phi_t$ gives the decomposition (4.3.2). Then the spaces $Q_t(H^{s+k}(E))$ and $Q_t^*(H^{s+k}(F))$ depend C^∞-ly (resp. real analytically) on t, thus also depend isomorphisms (4.3.3) and (4.3.4). Q.E.D.

4.4. *Remark*. This Lemma in the C^∞-category is essentially done in [22, Theorem 5]. In their proof they used potential theory.

4.5. *Definition.* Let g_t be an effectively parametrized normal family with parameter space P. The riemannian metric on P defined in Lemma 4.2 is called *the canonical riemannian metric on* P and *denoted by* g^P.

4.6. *Definition.* Let g_t (resp. $\bar{g}_s$) be a family of riemannian metrics on M with parameter space P (resp. $\bar{P}$). They are said to be *equivalent* if there are a diffeomorphism $\psi: P \to \bar{P}$ and a family γ_t of diffeomorphisms of M with parameter space P such that

$$(4.6.1) \qquad \gamma_t^* \bar{g}_{\psi(t)} = g_t.$$

4.7. **Lemma.** *Let g_t be an effectively parametrized normal family of riemannian metrics on M with parameter space P and $\bar{g}_t$ an equivalent family. Then the family $\bar{g}_t$ is also effectively parametrized and normal, and ψ becomes an isometry.*

Proof. Obviously, $\bar{g}_t$ is normal. We may assume that the parameter space of $\bar{g}_t$ is also P. Then if we differentiate Eq. (4.6.1), we get

$$(4.7.1) \qquad [v, g_t] = \gamma_t^* [[v, \gamma_t] \circ \gamma_t^{-1}, \bar{g}_t] + \gamma_t^* [v, \bar{g}_t].$$

Therefore if h is the essential part of $[v, g_t]$ with respect to g_t, then $\gamma_t^{-1*} h$ is the essential part of $[v, \bar{g}_t]$ with respect to $\gamma_t^{-1*} g_t = \bar{g}_t$. This implies that $\bar{g}_t$ is effectively parametrized, and we see that

$$(4.7.2) \qquad \begin{aligned} (v_1, v_2) &= \langle h_1, h_2 \rangle_{g_t} \\ &= \langle \gamma_t^{-1*} h_1, \gamma_t^{-1*} h_2 \rangle_{\gamma_t^{-1*} g_t}, \end{aligned}$$

where h_i is the essential part of $[v_i, g_t]$. Q.E.D.

4.8. *Remark.* This Lemma means that the canonical riemannian metric is a well-defined notion on a family of riemannian metrics as a fiber structure: $M \times P \to P$ (c.f. [21, Definition 1.1]).

If the space ELPM becomes a submanifold of $\mathcal{S}_g^s$, it may be regarded as a family of riemannian metrics on M with parameter space ELPM.

4.9. *Definition.* Let $g \in \mathcal{M}_1^\infty$ be an Einstein metric. The space ELPM(g) is said to be *normal* (resp. *effectively parametrized*) if it forms a submanifold of $\mathcal{M}^s$ around g and if it is normal (resp. effectively parametrized) as a family of riemannian metrics.

4.10. **Lemma.** *Let $g_0 \in \mathcal{M}_1^\infty$ be an Einstein metric. Assume that the space* ELPM(g_0) *becomes a submanifold of $\mathcal{M}^s$ around g_0. Then the following conditions are equivalent.*

(4.10.1) ELPM(g_0) *is effectively parametrized.*

(4.10.2) ELPM(g_0) *is normal.*

(4.10.3) $K(g) = K(g_0)$ *for* $g \in$ ELPM(g_0).

(4.10.4) $K(g_0)$ *acts trivially on* ELPM(g_0).

(4.10.5) $K(g_0)$ *acts trivially on* T_{g_0}(ELPM(g_0)), *where $K(g)$ denotes the space of all Killing vector fields on (M, g).*

Proof. We will show the implications: $1\to 3$, $2\to 3\to 4$ and $5\to 4\to 1$. Combining the obvious implications: $3\to 2$ and $4\to 5$, we get the equivalence.

$1\to 3$: Assume that the space ELPM is effectively parametrized. Let $X\in K(g_0)$. Then, since $K(g_0)$ acts on ELPM, we have $L_X g\in T_g(\mathrm{ELPM})$, therefore $L_X g=0$, i.e., $X\in K(g)$. But here by the property (2.1.2), $I(g)\subset I(g_0)$ for $g\in\mathscr{S}^s_{g_0}$. Thus $K(g)=K(g_0)$.

$2\to 3$: Assume that the space ELPM is normal. Then the inclusion $I(g)\subset I(g_0)$ implies that $K(g)=K(g_0)$.

$3\to 4$: Assume that $K(g)=K(g_0)$ for $g\in\mathrm{ELPM}$. Let $X\in K(g_0)$. Then $L_X g=0$, which implies that $K(g_0)$ acts trivially on ELPM.

$5\to 4$: Assume that $K(g_0)$ acts trivially on $T_{g_0}(\mathrm{ELPM})$. Let $X\in K(g_0)$ and $\gamma(t)$ the one-parameter group of diffeomorphisms generated by X. Then $\gamma(t)$ acts trivially on $T_{g_0}(\mathrm{ELPM})$. But here, the manifold ELPM is an $I(g_0)$-invariant submanifold of $\mathscr{M}^s$, so the induced riemannian metric on ELPM is $I(g_0)$-invariant. Thus the triviality of the action of $\gamma(t)$ on $T_{g_0}(\mathrm{ELPM})$ extends to that on the space ELPM, which implies that $K(g_0)$ acts trivially on ELPM.

$4\to 1$: Assume that $K(g_0)$ acts trivially on the space ELPM. Let $g\in\mathrm{ELPM}$ and X a vector field on M such that $L_X g\in T_g(\mathrm{ELPM})$. Denote by $\gamma(t)$ the one-parameter group of diffeomorphisms generated by X. Then in the situation of Lemmas 2.1 and 2.4, we have

(4.10.6) $$\gamma(t)^* g=F^{s-1}_{s-2}(p^s_{s-1}(\gamma(t)^* g), q^s_{s-1}(\gamma(t)^* g)).$$

Therefore, by the property (2.1.2) we see that

(4.10.7) $$\chi^s(q^s_{s-1}(\gamma(t)^* g))\circ\gamma(t)^{-1}\in I(g_0),$$

which implies that

(4.10.8) $$(\chi^s)'\circ(q^s_{s-1})'_g(L_X g)-X\in K(g_0).$$

But here, since $L_X g\in T_g\mathscr{S}^s_{g_0}$, we have

(4.10.9) $$(q^s_{s-1})'_g(L_X g)=0.$$

So $X\in K(g_0)$. Therefore, by assumption we see that $L_X g=0$. Q.E.D.

4.11. **Corollary.** *Let g be an Einstein metric on M with non-positive Ricci curvature and with volume* 1. *If (M,g) has no local flat factor and if the space* ELPM(g) *forms a submanifold of $\mathscr{M}^s$, then the canonical riemannian metric on the space* ELPM(g) *is well defined.*

Proof. By a well-known theorem of Bochner, the assumption implies that $K(g)$ vanishes. Q.E.D.

4.12. **Corollary.** *Let g be a flat riemannian metric on M with volume* 1. *Then the canonical riemannian metric on the space* ELPM(g) *is well defined.*

Proof. Let $h\in\mathrm{EEID}$. Then h is parallel and so we can easily construct an Einstein deformation g_t such that $g_0'=h$. Therefore, by Corollary 3.5, the space ELPM becomes a submanifold of $\mathscr{M}^s$ whose tangent space at g coincides with

EEID. Moreover, if X is a Killing vector field of (M, g), then X is parallel and so $L_X h=0$. Q.E.D.

4.13. **Theorem.** *Let* $g_0 \in \mathcal{M}_1^\infty$ *be an Einstein metric. If the space* ELPM(g_0) *is normal (or equivalently if* ELPM(g_0) *forms a manifold and if one of the conditions in* Lemma 4.10 *holds), then the canonical riemannian metric on* ELPM(g_0) *is real analytic.*

Proof. It is sufficient to prove that the essential part h of ψ depends real analytically on g and $\psi \in T_g$(ELPM). This follows from Theorem 3.1 and Lemma 4.3. Q.E.D.

4.14. *Example.* Let (T^n, g_0) be a flat torus of volume 1. Let EPM be the set of all T^n-invariant riemannian metrics on T^n with volume 1. Then the space ELPM(g_0) is an open neighbourhood of g_0 in EPM. In this case, the canonical riemannian metric on ELPM(g_0) coincides with the induced metric from $\mathcal{M}^s$. The space ELPM(g_0) is then isometric with an open set of the irreducible symmetric space $SL(n, \mathbf{R})/SO(n, \mathbf{R})$ (=EPM).

5. The Obstruction Space for the Space EEID

In this section, we treat only deformations of riemannian metrics. But to understand what we do, it would be better to recall Proposition 0.12, which says that the space $H^2(M, \Theta)$ is the "obstruction" for the integrability (see Corollary 3.5) of infinitesimal complex deformations (see Definition 6.2). Bourguignon posed the following

5.1. *Question. Is there a space which plays the role of an obstruction space for the integrability of infinitesimal Einstein deformation?*

We will give a negative answer to this question. Consider an equation $F(g)=0$ for riemannian metrics. (For example, Einstein's equation $E(g)=0$.) Assume that there is a linear operator B_g such that the equation

(5.1.1) $$B_g(F(g))=0$$

becomes an identity. For example, if $F=E$, we have such an operator, namely the Bianchi identity operator:

(5.1.2) $$B_g(\psi)=\delta_g \psi+(1/2)\, d\,\mathrm{tr}_g \psi.$$

Let g_t be a 1-parameter family of riemannian metrics on M, and set $h_i = \left(\frac{d}{dt}\right)_0^i g_t$. Then we have the expression

(5.1.3) $$\left(\frac{d}{dt}\right)_0^r F(g_t)=P_{g_0}^r(h_1, \ldots, h_{r-1})+F'_{g_0}(h_r).$$

If we take the r-th derivative of identity (5.1.1), we get the identity

(5.1.4) $$\sum_{i=1}^{r} \binom{r}{i} \left[\left(\frac{d}{dt}\right)_0^i B_{g_t}\right]\left[\left(\frac{d}{dt}\right)_0^{r-i} F(g_t)\right]=0.$$

Now, we assume that

$$(5.1.5)\qquad \left(\frac{d}{dt}\right)_0^i F(g_t)=0 \quad \text{for } 0\leqq i\leqq r-1.$$

Then equality 5.1.4 becomes

$$(5.1.6)\qquad B_{g_0}(P^r_{g_0}(h_1,\dots,h_{r-1})+F'_{g_0}(h_r))=0.$$

Remark that this equality holds for any h_r. So we have, since B_{g_0} is linear,

$$(5.1.7)\qquad B_{g_0}\circ F'_{g_0}=0,$$

$$(5.1.8)\qquad B_{g_0}(P^r_{g_0}(h_1,\dots,h_{r-1}))=0.$$

We want to solve the equation

$$(5.1.9)\qquad F'_{g_0}(h_r)=-P^r_{g_0}(h_1,\dots,h_{r-1})$$

for h_r. There is a solution h_r if and only if $P^r_{g_0}(h_1,\dots,h_{r-1})\in \operatorname{Im} F'_{g_0}$. But here we have Eq. (5.1.8), so if $\operatorname{Ker} B_{g_0}\subset \operatorname{Im} F'_{g_0}$, then there exists a solution of (5.1.9). In general, by equality (5.1.7), we have

$$(5.1.10)\qquad \operatorname{Ker} B_{g_0}\supset \operatorname{Im} F'_{g_0}.$$

5.2. *Definition.* Let g be a riemannian metric on M such that $F(g)=0$. The space $\operatorname{Ker} B_g/\operatorname{Im} F'_g$ is called *the obstruction space.* If $\operatorname{Im} F'_g$ is closed, then the space $\operatorname{Ker} B_g\cap(\operatorname{Im} F'_g)^\perp$ also is called *the obstruction space.*

In fact, we saw that if the obstruction space vanishes, then all infinitesimal deformations are formally integrable (see Corollary 3.5). In the case of Einstein's equation $E(g)=0$, it implies the following

5.3. **Lemma.** *Let g be an Einstein metric on M with volume* 1. *If the obstruction space* $\operatorname{Ker} B_g\cap(\operatorname{Im} E'_g)^\perp$ *vanishes, then the space* ELPM *forms a submanifold of M^s whose tangent space T_g*(ELPM) *coincides with the space* EEID.

Proof. By Corollary 3.5, it is sufficient to prove that for any $h\in$EEID, there is an Einstein deformation g_t of g such that $g'_0=h$. But it is already shown in the above discussion that there exists a formal power series

$$(5.3.1)\qquad g(t)=\sum_{i=0}^{\infty}\frac{1}{i!}h_i t^i$$

which satisfies the formal equation $E(g(t))=0$ such that $h_0=g$ and $h_1=h$. Then the result follows by the same argument as in the proof of Theorem 3.2. Q.E.D.

But unfortunately, we have the next proposition which says that we cannot apply this lemma.

5.4. **Proposition** ([24, Proposition 3.2]). *Let g be an Einstein metric on M with volume* 1. *Then the obstruction space* $\operatorname{Ker} B_g\cap(\operatorname{Im} E'_g)^\perp$ *coincides with the space* EEID *itself.*

5.5. *Remark.* We see that the definition of the obstruction space depends on the choice of the identity. But it seems to the author that for Einstein's equation, there is no other effective identity than the Bianchi identity. On the other hand, if an equation $F(g)=0$ is sufficiently nice, for example if the space of all "essential infinitesimal deformations" is finite dimensional, Lemma 5.3 remains true for this equation. If a solution g of the equation has "an essential infinitesimal deformation" and if "the obstruction space" vanishes, then we get a manifold as "the local pre-moduli space" for the equation. To analyse the real analytic set ELPM as a subset of such a manifold, we pose the following

5.6. *Problem. Find such an equation $F(g)=0$. We claim also that Einstein metrics g satisfy $F(g)=0$.*

6. Deformations and Infinitesimal Deformations of a Complex Structure

We introduce the notion of complex deformations, which was developed by [21, 22]. We introduce it in an exactly similar way as in Sect. 1 to analyse deformations of Kähler-Einstein structures.

6.1. *Definition.* Let J be a complex structure on M. A family J_t of complex structures on M such that $J_0=J$ is called *a complex deformation of* J. A complex deformation J_t of J is said to be *trivial* if the complex structure J_t is isomorphic with J for each t.

A tensor field J of type (1, 1) is a complex structure if and only if the following equations are satisfied.

(6.1.1) $$J^2=-\mathrm{id}_{TM},$$

(6.1.2) $$N(J)=0,$$

where N denotes *the Nijenhuis torsion tensor*, defined by

(6.1.3) $$N(J)(X,Y)=[X,Y]-[JX,JY]+J[JX,Y]+J[X,JY].$$

6.2. *Definition.* Let J be a complex structure on M. A tensor field I of type (1, 1) is called *an infinitesimal complex deformation of* J if the following linear equations are satisfied.

(6.2.1) $$IJ+JI=0,$$

(6.2.2) $$N'_J(I)=0.$$

The space of all infinitesimal complex deformations of J is *denoted by* CID(J) or simply by CID.

6.3. *Definition.* Let J be a complex structure and g a riemannian metric on M. An infinitesimal complex deformation of the form $L_X J$ is said to be *trivial.* The space of all trivial infinitesimal complex deformations of J is *denoted by* CTID(J) or simply by CTID. An infinitesimal complex deformation I of J is said to be *essential with respect to* g if I is orthogonal to the space CTID(J)

with respect to the global inner product defined by g. The space of all essential infinitesimal complex deformations of J with respect to g is denoted by CEID(J, g) or simply CEID(J), CEID.

6.4. **Lemma.** *Let J be a complex structure on M. A (real) tensor field I of type $(1, 1)$ is an infinitesimal complex deformation of J if and only if I satisfies the following two equations*

$$I^{\alpha}{}_{\beta}=0, \tag{6.4.1}$$

$$\partial_{\alpha} I^{\bar{\gamma}}{}_{\beta}-\partial_{\beta} I^{\bar{\gamma}}{}_{\alpha}=0. \tag{6.4.2}$$

If g is a Kähler metric on M, then Eq. (6.4.2) becomes

$$D_{\alpha} I^{\bar{\gamma}}{}_{\beta}-D_{\beta} I^{\bar{\gamma}}{}_{\alpha}=0, \tag{6.4.3}$$

and an element $I \in$CID is in the space CEID(J, g) if and only if I satisfies the equation

$$D^{\gamma} I^{\bar{\alpha}}{}_{\gamma}=0. \tag{6.4.4}$$

Proof. Equations (6.4.1), (6.4.2) and (6.4.3) are given by a straightforward tensor computations. Equation (6.4.4) follows from the equation

$$(L_X J)(\partial_{\alpha})=-2\sqrt{-1}(D_{\alpha} X^{\bar{\gamma}})\,\bar{\partial}_{\gamma} \text{ for vector fields } X. \tag{6.4.5}$$

Q.E.D.

6.5. **Lemma.** *Let J be a complex structure and g a riemannian metric on M. Then the space* CID(J) *admits an orthogonal decomposition*

$$\mathrm{CID}(J)=\mathrm{CEID}(J, g) \oplus \mathrm{CTID}(J) \tag{6.5.1}$$

with respect to the global inner product defined by g.

Proof. It suffices to show that the space CTID is a closed subspace, which is a direct consequence of decomposition (4.3.2) (for $t=0$). Q.E.D.

Also for the space CEID, a similar result as Lemma 1.5 holds. See Lemma 8.1. We defined the space CEID in analogy with the space EEID. But we have another space of infinitesimal deformations of complex structures, defined in [20, Sect. 5]. This space is the cohomology group $H^1(M, \Theta)$ with coefficients in the sheaf Θ of germs of holomorphic vector fields.

6.6. **Lemma.** *Let J be a complex structure and g a riemannian metric on M. Then the space* CEID(J, g) *and $H^1(M, \Theta)$ are canonically isomorphic.*

Proof. We use the ordinary notations for chain complexes and denote by X the sheaf of germs of smooth vector fields. First we construct a linear map: CID $\rightarrow H^1(M, \Theta)$. Let $I \in$CID. By Eq. 6.4.2 and by Dolbeault's lemma, there is a local complex vector field η such that $I^{\bar{\gamma}}{}_{\beta}=\partial_{\beta} \eta^{\bar{\gamma}}$. Taking the real part ξ of the anti-holomorphic part of η, we see that $L_{\xi} J=I$ by formula 6.4.5, i.e., there exists $\{\xi_{\alpha}\} \in C^0(M, X)$ such that $L_{\xi_{\alpha}} J=I$. If we define $\{\phi_{\alpha\beta}\} \in Z^1(M, X)$ by $\phi_{\alpha\beta}=\xi_{\alpha}-\xi_{\beta}$, then we see that $\{\phi_{\alpha\beta}\} \in Z^1(M, \Theta)$, and so it defines an element in

$H^1(M, \Theta)$. We easily see that this correspondence: $\mathrm{CID} \to H^1(M, \Theta)$ is well-defined, and its kernel coincides with the space CTID. So this map defines an injection: $\mathrm{CEID} \to H^1(M, \Theta)$. Next we show that this map is surjective. Let $\{\phi_{\alpha\beta}\} \in Z^1(M, \Theta)$. Since $Z^1(M, \Theta) \subset Z^1(M, X) = B^1(M, X)$, there exists $\{\eta_\alpha\} \in C^0(M, X)$ such that $\phi_{\alpha\beta} = \eta_\alpha - \eta_\beta$. Then $L_{\eta_\alpha} J$ defines an infinitesimal complex deformation. It is easy to see that I is mapped to $\{\phi_{\alpha\beta}\} \in H^1(M, \Theta)$ by the above correspondence: $\mathrm{CID} \to H^1(M, \Theta)$. Q.E.D.

Let J be a complex structure on M. If $I \in \mathrm{CID}$, then by Lemma 2.4, we see that $JI \in \mathrm{CID}$, i.e., the space CID becomes a complex vector space. If $\{\phi_{\alpha\beta}\} \in Z^1(M, \Theta)$, then $\{J\phi_{\alpha\beta}\} \in Z^1(M, \Theta)$. Using the operator J on $Z^1(M, \Theta)$, the space $H^1(M, \Theta)$ also becomes a complex vector space. Moreover, if g is a compatible Kähler metric on M, then by Eq. (6.4.3) and (6.4.4), the space $\mathrm{CEID}(J, g)$ becomes a subspace of the complex vector space $\mathrm{CID}(J)$. We easily see the following

6.7. **Lemma.** *Let (J, g) be a Kähler structure on M. Then the isomorphism defined in* Lemma 6.6 *is an isomorphism of complex vector spaces.*

7. The Space EEID on a Kähler-Einstein Manifold

First, we prepare the following

7.1. **Lemma.** *Let (J, g) be a Kähler-Einstein structure on M. Then the following formulae hold.*

(7.1.1) $D^\gamma D_\gamma \psi_{\alpha_1 \ldots \alpha_k \bar\beta_1 \ldots \bar\beta_l} = D^{\bar\gamma} D_{\bar\gamma} \psi_{\alpha_1 \ldots \alpha_k \bar\beta_1 \ldots \bar\beta_l} + (k-l)\, e\, \psi_{\alpha_1 \ldots \alpha_k \bar\beta_1 \ldots \bar\beta_l}$,

(7.1.2) $-D^\alpha (D_\alpha \psi_{\beta\gamma} - D_\beta \psi_{\alpha\gamma}) = \tfrac{1}{2}(\bar\Delta + 2L)\psi_{\beta\gamma} + D_\beta D^\alpha \psi_{\alpha\gamma}$,

(7.1.3) $-D^\alpha (D_\alpha \psi_{\beta\bar\gamma} - D_\beta \psi_{\alpha\bar\gamma}) = \tfrac{1}{2}(\bar\Delta + 2L + 2e)\psi_{\beta\bar\gamma} + D_\beta D^\alpha \psi_{\alpha\bar\gamma}$,

(7.1.4) $\langle D_\alpha \psi_{\beta\gamma} - D_\beta \psi_{\alpha\gamma}, D_{\bar\alpha} \psi_{\bar\beta\bar\gamma} - D_{\bar\beta} \psi_{\bar\alpha\bar\gamma} \rangle$
$= \langle (\bar\Delta + 2L)\psi_{\beta\gamma}, \psi_{\bar\beta\bar\gamma} \rangle - 2\langle D^\alpha \psi_{\alpha\gamma}, D^{\bar\beta} \psi_{\bar\beta\bar\gamma} \rangle$,

(7.1.5) $\langle D_\alpha \psi_{\beta\bar\gamma} - D_\beta \psi_{\alpha\bar\gamma}, D_{\bar\alpha} \psi_{\bar\beta\gamma} - D_{\bar\beta} \psi_{\bar\alpha\gamma} \rangle$
$= \langle (\bar\Delta + 2L + 2e)\psi_{\beta\bar\gamma}, \psi_{\bar\beta\gamma} \rangle - 2\langle D^\alpha \psi_{\alpha\bar\gamma}, D^{\bar\beta} \psi_{\bar\beta\gamma} \rangle$,

where e is the constant Ricci curvature and $\langle\ ,\ \rangle$ denotes the global inner product defined by g. If ϕ is an anti-symmetric 2-tensor, then

(7.1.6) $$L\phi_{\alpha\beta} = 0.$$

Proof. Straightforward tensor calculation. C.f. [11, Sect. 6]. Q.E.D.

Now let (J, g) be a Kähler-Einstein structure on M with volume 1 and let $h \in \mathrm{EEID}$. We decompose h into its hermitian part h_H and its anti-hermitian part h_A:

(7.1.7) $$h_H(JX, JY) = h_H(X, Y),$$

(7.1.8) $$h_A(JX, JY) = -h_A(X, Y).$$

Then from equation 1.5.1: $(\bar{\Delta}+2L)h=0$ and equality (7.1.4), h_A satisfies Eq. (1.5.2): $\delta h_A=0$. Obviously h_A satisfies Eqs. (1.5.1) and (1.5.3): $\mathrm{tr}\, h_A=0$, therefore $h_A \in \mathrm{EEID}$ and so $h_H \in \mathrm{EEID}$. Q.E.D.

7.2. *Notation.* Let (J, g) be a Kähler-Einstein structure on M with volume 1. The space of all hermitian (resp. anti-hermitian) essential infinitesimal Einstein deformations is *denoted by* EEID_H (resp. EEID_A).

7.3. **Proposition.** *Let* (J, g) *be a Kähler-Einstein structure on* M *with volume* 1. *Then the decomposition:*

(7.3.1) $$\mathrm{EEID}=\mathrm{EEID}_H \oplus \mathrm{EEID}_A$$

holds. An anti-hermitian symmetric 2-tensor field h *is an element of* EEID_A *if and only if the following equations are satisfied.*

(7.3.2) $$D_\alpha h_{\beta\gamma} - D_\beta h_{\alpha\gamma}=0,$$

(7.3.3) $$D^\alpha h_{\alpha\beta}=0.$$

Moreover, if $e<0$, *then* $\mathrm{EEID}_H=0$. *If* $e=0$, *then a hermitian symmetric 2-tensor field* h *is an element of* EEID_H *if and only if the following equations are satisfied.*

(7.3.4) $$D_\alpha h_{\beta\bar{\gamma}} - D_\beta h_{\alpha\bar{\gamma}}=0,$$

(7.3.5) $$h_\alpha^{\ \alpha}=0.$$

Proof. Let $h \in \mathrm{EEID}_A$. Then from equality (7.1.4) and Eq. (1.5.1): $(\bar{\Delta}+2L)h=0$, Eqs. (7.3.2) and (7.3.3) follow. Conversely, if an anti-hermitian symmetric 2-tensor field h satisfies Eqs. (7.3.2) and (7.3.3), then by equality (7.1.2), Eq. (1.5.1) holds. This implies that $h \in \mathrm{EEID}$.

Assume that $e \leqq 0$ and let $h \in \mathrm{EEID}_H$. Then, from equality (7.1.5) and Eqs. (1.5.1) and (1.5.2): $\delta h=0$ it follows that, if $e<0$, $h=0$ and, if $e=0$, h satisfies Eq. (7.3.4). Equation (7.3.5) is equivalent to Eq. (1.5.3): $\mathrm{tr}\, h=0$. Conversely, assume that $e=0$ and that a hermitian symmetric 2-tensor field h satisfies Eqs. (7.3.4) and (7.3.5). Then

(7.3.6) $$D^{\bar{\alpha}} h_{\bar{\alpha}\gamma} = D^{\bar{\alpha}} h_{\gamma\bar{\alpha}} = D_\gamma h^{\bar{\alpha}}{}_{\bar{\alpha}}=0,$$

i.e., Eq. (1.5.2) holds. Then from equality (7.1.3) Eq. (1.5.1) follows, which implies that $h \in \mathrm{EEID}$. Q.E.D.

7.4. **Proposition.** *Let* (J, g) *be a Kähler-Einstein structure on* M *with* $e=0$ *and with volume* 1. *Then*

(7.4.1) $$\mathrm{EEID}_H \cong H^{1,1}(M, \mathbf{R})/\mathbf{R}\cdot\omega,$$

where ω *denotes the Kähler form.*

Proof. For $h \in \mathrm{EEID}_H$, we define a real 2-form ψ by $\psi = hJ$, i.e.,

(7.4.2) $$\psi_{\alpha\bar{\beta}} = -\sqrt{-1}\, h_{\alpha\bar{\beta}}.$$

Then from Eq. (1.5.2), $\delta\psi=0$ follows, and from Eq. (7.3.4) it follows that $d\psi=0$. It is easy to see that this correspondence gives the isomorphism (7.4.1). Q.E.D.

8. The Space CEID on a Kähler-Einstein Manifold

Let (J, g) be a Kähler structure on M and let $I \in$CEID. A tensor of type $(1, 1)$ is identified with a tensor of type $(0, 2)$ via the usual correspondence defined by the metric tensor. Thus, by Eq. (6.4.1), the tensor field I is identified with an anti-hermitian 2-tensor field, which is denoted by $I_{\alpha\beta}$ or $I_{\bar{\alpha}\bar{\beta}}$.

8.1. **Lemma.** *Let (J, g) be a Kähler-Einstein structure on M. An anti-hermitian 2-tensor field I is an element of* CEID *if and only if it satisfies the following equation*

(8.1.1) $$(\bar{\Delta}+2L)I=0.$$

Proof. If $I \in$CEID, then from Eq. (6.4.3) and (6.4.4):

$$D_\alpha I^{\bar{\gamma}}{}_\beta - D_\beta I^{\bar{\gamma}}{}_\alpha=0, \qquad D^\gamma I^{\bar{\alpha}}{}_\gamma=0,$$

and formula (7.1.2), Eq. (8.1.1) follows. If Eq. (8.1.1) holds, then from formula (7.1.4), we see that Eqs. (6.4.3) and (6.4.4) follow. Q.E.D.

Let I_S (resp. I_A) be the symmetric part (resp. antisymmetric part) of $I \in$CEID. Then by Lemma 8.1, we see that I_S and $I_A \in$CEID. *Denote by* CEID_S (resp. CEID_A) the space of all symmetric (resp. antisymmetric) elements of CEID.

8.2. **Proposition.** *Let (J, g) be a Kähler-Einstein structure on M. Then the decomposition*

(8.2.1) $$\mathrm{CEID}=\mathrm{CEID}_S \oplus \mathrm{CEID}_A$$

holds. An antisymmetric anti-hermitian 2-tensor field I belongs to CEID_A *if and only if I is parallel, i.e.,*

(8.2.2) $$DI=0.$$

In particular, we have the isomorphism:

(8.2.3) $$\mathrm{CEID}_A \cong H^{2,0}(M, \mathbf{C}).$$

Proof. Let I be an antisymmetric anti-hermitian 2-tensor field. Then, by formula (7.1.6), Eq. (8.1.1) is equivalent to the equation $\bar{\Delta}I=0$, which is also equivalent to Eq. (8.2.2). The last isomorphism then follows from the equivalence between the properties of being parallel and being harmonic. Q.E.D.

8.3. **Proposition.** *Let (J, g) be a Kähler-Einstein structure on M with $e \neq 0$. Then* $\mathrm{CEID}_A=0$.

Proof. If $e \neq 0$, then by formula (7.1.1), there is no non-zero parallel holomorphic 2-tensor field. So using isomorphism (8.2.3), $\mathrm{CEID}_A = 0$.

9. Kähler Relation Between the Space EID and CID

Let (J, g) be a Kähler structure on M and (J_t, g_t) a one-parameter family of Kähler structure such that $(J_0, g_0) = (J, g)$. Then the following equations are satisfied.

(9.1.1) $$(g_t(X, J_t Y) + g_t(Y, J_t X))' = 0,$$

(9.1.2) $$d\omega_t' = 0,$$

where ω_t is the Kähler form defined by $\omega_t(X, Y) = g_t(X, J_t Y)$. Set $g_0' = h$, $J_0' = I$ and $\omega_0' = \phi$. Then we see that ϕ is given by the following equations

(9.1.3) $$\phi_{\alpha\beta} = \sqrt{-1}\, h_{\alpha\beta} + I_{\alpha\beta},$$

(9.1.4) $$\phi_{\alpha\bar\beta} = -\sqrt{-1}\, h_{\alpha\bar\beta} + I_{\alpha\bar\beta}.$$

From Eq. (9.1.1), we see that

(9.1.5) $$2\sqrt{-1}\, h_{\alpha\beta} + (I_{\alpha\beta} + I_{\beta\alpha}) = 0.$$

Combining with Eq. (9.1.3), we get

(9.1.6) $$\phi_{\alpha\beta} = (1/2)\,(I_{\alpha\beta} - I_{\beta\alpha}).$$

Therefore the following relation always holds.

(9.1.7) $$\begin{aligned} 2(d\phi)_{\alpha\beta\gamma} &= D_\alpha (I_{\beta\gamma} - I_{\gamma\beta}) + \text{alternating terms} \\ &= (D_\alpha I_{\beta\gamma} - D_\gamma I_{\beta\alpha}) + \text{alternating terms} \\ &= 0, \end{aligned}$$

where the last equality follows from Eq. (6.4.3): $D_\alpha I^{\bar\gamma}{}_\beta - D_\beta I^{\bar\gamma}{}_\alpha = 0$. From Eqs. (6.4.1): $I^\alpha{}_\beta = 0$ and (9.4.1), we see that

(9.1.8) $$\phi_{\alpha\bar\beta} = -\sqrt{-1}\, h_{\alpha\bar\beta}.$$

Combining with Eq. (9.1.6), we get

(9.1.9) $$(d\phi)_{\alpha\beta\bar\gamma} = -\sqrt{-1}\, D_\alpha h_{\beta\bar\gamma} + \sqrt{-1}\, D_\beta h_{\alpha\bar\gamma} + (1/2)\, D_{\bar\gamma}(I_{\alpha\beta} - I_{\beta\alpha}).$$

Thus we may set the following

9.2. *Definition.* Let (J, g) be a Kähler structure on M. A symmetric 2-tensor field h and an infinitesimal complex deformation I are said to be *Kähler related* if they satisfy the following equations

(9.2.1) $$2\sqrt{-1}\, h_{\alpha\beta} + (I_{\alpha\beta} + I_{\beta\alpha}) = 0,$$

(9.2.2) $$2\sqrt{-1}\,(D_\alpha h_{\beta\bar\gamma} - D_\beta h_{\alpha\bar\gamma}) = D_{\bar\gamma}(I_{\alpha\beta} - I_{\beta\alpha}).$$

9.3. **Lemma.** *Let* (J, g) *be a Kähler-Einstein structure on* M *with volume* 1. *Then the space* EEID_A *and the space* CEID_S *are isomorphic under a canonical correspondence:* $I \to h$ *defined by*

$$h_{\alpha\beta} = \sqrt{-1}\, I_{\alpha\beta}. \tag{9.3.1}$$

Moreover, this isomorphism is equivalent with the Kähler relation.

Proof. The first half is obvious by Lemma 6.4 and Proposition 7.3. Under the assumption that h is anti-hermitian and that I is symmetric, Eq. (9.2.2) always holds and Eq. (9.3.1) is equivalent with Eq. (9.2.1). Q.E.D.

9.4. **Corollary.** *Let* (J, g) *be a Kähler-Einstein structure on* M *with volume* 1. *Then,*

(9.4.1) *if* $e<0$, $\dim \mathrm{EEID} = 2 \dim_{\mathbf{C}} H^1(M, \Theta)$,

(9.4.2) *if* $e=0$, $\dim \mathrm{EEID} = (\dim H^{1,1}(M, \mathbf{R}) - 1) + 2(\dim_{\mathbf{C}} H^1(M, \Theta) - \dim_{\mathbf{C}} H^{2,0}(M, \mathbf{C}))$,

(9.4.3) *if* $e>0$, $\dim \mathrm{EEID} \geqq 2 \dim_{\mathbf{C}} H^1(M, \Theta)$.

Proof. Combination of Lemma 6.6, Propositions 7.3, 7.4, 8.2 and Lemma 9.3. Q.E.D.

9.5. *Remark.* Formula (9.4.2) on the $K3$-surface is obtained in [10, p. 174 Theorem].

9.6. *Remark.* In general, we cannot replace the inequality sign in (9.4.3) by an equality sign. For example, on $P^1(\mathbf{C}) \times P^{2m}(\mathbf{C})$, $H^1(M, \Theta) = 0$ ([9, Theorem VII, Corollary]) but $\mathrm{EEID} \neq 0$ (Example 3.17).

9.7. **Lemma.** *Let* (J, g) *be a Kähler-Einstein structure on* M *with volume* 1. *If* $e>0$, *then for any non-zero element* $h \in \mathrm{EEID}_H$, *there is no element* $I \in \mathrm{CID}$ *which is Kähler related with* h.

Proof. Assume that I is Kähler related with h. Then by Eq. (9.2.1), I is anti-symmetric, and so from Eq. (9.2.2) it follows that

$$\sqrt{-1}(D_\alpha h_{\beta\bar\gamma} - D_\beta h_{\alpha\bar\gamma}) = D_{\bar\gamma} I_{\alpha\beta}. \tag{9.7.1}$$

But on the other hand, by Eq. (1.5.2): $\delta h = 0$, we know that

$$D^{\bar\gamma} D_\alpha h_{\beta\bar\gamma} = D_\alpha D^{\bar\gamma} h_{\beta\bar\gamma} = 0. \tag{9.7.2}$$

Therefore $D^{\bar\gamma} D_{\bar\gamma} I_{\alpha\beta} = 0$, which implies that

$$D_{\bar\gamma} I_{\alpha\beta} = 0. \tag{9.7.3}$$

Combining Eqs. (9.7.1), (1.5.1): $(\bar\Delta + 2L) h = 0$, (1.5.2): $\delta h = 0$ and (7.1.3), we see that $e h = 0$. Q.E.D.

9.8. **Lemma.** *Let* (J, g) *be a Kähler-Einstein structure on* M *with volume* 1. *The space of all infinitesimal complex deformations* $I \in \mathrm{CID}$ *which are Kähler related* *with* $0 \in \mathrm{EID}$ *coincides with the space* CEID_A. *The space of all infinitesimal Einstein deformations* $h \in \mathrm{EID}$ *which are Kähler related with* $0 \in \mathrm{CID}$ *vanishes if* $e<0$, *coincides with the space* EEID_H *if* $e=0$ *and coincides with the space* $\{L_X g;$ *X is a holomorphic vector field*$\}$ *if* $e>0$.

Proof. If $I \in \mathrm{CEID}_A$, then I is parallel by Proposition 8.2 and so I is Kähler related with $0 \in \mathrm{EID}$. Conversely if $I \in \mathrm{CID}$ is Kähler related with $0 \in \mathrm{EID}$, then I is antisymmetric and $D_{\bar{\gamma}} I_{\alpha\beta}=0$, which implies that Eq. (6.4.4) holds. Therefore $I \in \mathrm{CEID}_A$.

If $e=0$ and if $h \in \mathrm{EEID}_H$, then by Eq. (7.3.4), h is Kähler related with $0 \in \mathrm{CID}$. If $e>0$ and if X is a holomorphic vector field on M, then it is easy to see that the infinitesimal Einstein deformation $L_X g$ is Kähler related with $L_X J =0 \in \mathrm{CID}$. Conversely, assume that $h \in \mathrm{EID}$ is Kähler related with $0 \in \mathrm{CID}$. From decomposition (2.2.1), there exist an element $\psi \in \mathrm{EEID}$ and a vector field X on M such that $h=\psi+L_X g$. Then ψ is Kähler related with $-L_X J \in \mathrm{CID}$. By Lemma 9.3, there is an element $I_1 \in \mathrm{CEID}_S$ which is Kähler related with the anti-hermitian part ψ_A of ψ, so the hermitian part ψ_H of ψ is Kähler related with $-(I_1+L_X J)$. If $e<0$, then $\psi_H=0$ by Proposition 7.3. If $e>0$, then $\psi_H=0$ by Lemma 9.7. If $e=0$, then we have seen that ψ_H is Kähler related with $0 \in \mathrm{CID}$. Thus, in any case, $I_1+L_X J$ is Kähler related with $0 \in \mathrm{EID}$. But we have seen that then $I_1+L_X J \in \mathrm{CEID}_A$, which implies that $L_X J=0$ and $I_1=0$ by decompositions (6.5.1) and (7.3.1). Therefore $\psi_A=0$. If $e<0$, then we have seen that $\psi=\psi_H=0$ and so $h=0$ since there is no non-zero holomorphic vector field. If $e=0$, then $h=\psi \in \mathrm{EEID}_H$ since all holomorphic vector fields are Killing vector fields. If $e>0$, then we have seen that $\psi=\psi_H=0$ and so $h=L_X g$ where X is a holomorphic vector field. Q.E.D.

9.9. **Proposition.** *Let* (J, g) *be a Kähler-Einstein structure on* M *with volume* 1. *Let* $h_1 \in \mathrm{EID}$ *and* $I_1 \in \mathrm{CID}$. *We decompose them by decompositions* (2.2.1), (6.5.1), (7.3.1) *and* (8.2.1) *as*

$$h_1=h+L_X g, \quad h=h_A+h_H; \quad h_A \in \mathrm{EEID}_A,\ h_H \in \mathrm{EEID}_H, \tag{9.9.1}$$

$$I_1=I+L_Y J, \quad I=I_S+I_A; \quad I_S \in \mathrm{CEID}_S,\ I_A \in \mathrm{CEID}_A. \tag{9.9.2}$$

Then h_1 *and* I_1 *are Kähler related if and only if condition* (9.9.3), *or equivalently one of conditions* (9.9.4), (9.9.5) *and* (9.9.6), *holds.*

(9.9.3) *$X-Y$ is a holomorphic vector field,* h_H *is Kähler related with* $0 \in \mathrm{CID}$ *and* h_A *is Kähler related with* I_S.

(9.9.4) $e<0$, $X=Y$ *and* $h_{\alpha\beta}=\sqrt{-1}\, I_{\alpha\beta}$.

(9.9.5) $e=0$, *$X-Y$ is a Killing vector field and* $2h_{\alpha\beta}=\sqrt{-1}\,(I_{\alpha\beta}+I_{\beta\alpha})$.

(9.9.6) $e>0$, *$X-Y$ is a holomorphic vector field,* $h_{\alpha\bar{\beta}}=0$ *and* $h_{\alpha\beta}=\sqrt{-1}\, I_{\alpha\beta}$.

Proof. Assume condition (9.6.3). Then $L_X g$ and $L_Y J$ are Kähler related and h is Kähler related with I_S. If $e \neq 0$, then $I_A=0$ by Proposition 8.3. Even if $e=0$, I_A is Kähler related with $0 \in \mathrm{EID}$ by Lemma 9.8. Therefore h_1 is Kähler related with I_1. Conversely, assume that h_1 and I_1 are Kähler related. Then since $L_X g$ is Kähler related with $L_X J$, h is Kähler related with $I + L_{(Y-X)}J$. By Lemma 9.3, there is an element $I_2 \in \mathrm{CEID}_S$ which is Kähler related with h_A. Then h_H is Kähler related with $I - I_2 + L_{(Y-X)}J$. But here, if $e<0$ then $h_H=0$ by Lemma 7.3, if $e=0$ then h_H is Kähler related with $0 \in \mathrm{CID}$ by Lemma 9.8, and if $e>0$ then $h_H=0$ by Lemma 9.7. Thus in any case h_H is Kähler related with $0 \in \mathrm{CID}$ and so $I - I_2 + L_{(Y-X)}J$ is Kähler related with $0 \in \mathrm{EID}$. Then by Lemma 9.8, $I - I_2 + L_{(Y-X)}J \in \mathrm{CEID}_A$. This implies that $I - I_2 \in \mathrm{CEID}_A$ and $L_{(Y-X)}J = 0$, i.e., $Y - X$ is a holomorphic vector field and $I_2 = I_S$. But by definition of I_2, I_2 is Kähler related with h_A.

To prove the equivalence between conditions (9.9.3) and (9.9.4), (9.9.5), (9.9.6), it is sufficient to see the following. If $e<0$, then there is no non-zero holomorphic vector field. If $e=0$, then all holomorphic vector fields are Killing vector fields. If $e \leqq 0$, then h_H is always Kähler related with $0 \in \mathrm{CID}$ by Lemmas 7.3 and 9.8. If $e>0$, then h_H is Kähler related with $0 \in \mathrm{CID}$ only if $h_H=0$ by Lemma 9.7. If $e \neq 0$, then $I_S = I$ by Proposition 8.3. Combining these informations with Lemma 9.3 gives the equivalence. Q.E.D.

10. Einstein Metrics and Complex Structures

We expect that in the situation of Proposition 0.5, if we deform the complex structure J, then the Einstein metric g depends C^∞-ly on J. The following result justifies this observation also valid for the case of positive Chern class.

10.1. **Proposition.** *Let (J, g) be a Kähler-Einstein structure on M with volume* 1. *If the constant Ricci curvature $e>0$, then we assume that there is no non-zero holomorphic vector field. Let J_t be a one-parameter complex deformation of J. Then there exists an Einstein deformation g_t (defined for small t) of g such that each metric g_t is a Kähler metric compatible with J_t. Moreover, if we have an infinitesimal Einstein deformation $h \in \mathrm{EID}$ which is Kähler related with $I = J'_0$, then we can choose g_t so that $g'_0 = h$.*

Proof. Recall the formula

$$\rho = \sqrt{-1}\,\partial\bar{\partial}\log|g| \tag{10.1.1}$$

for a Kähler structure, where ρ is the Ricci form defined by $\rho_{ij} = r_{ik}J^k{}_j$, ∂ and $\bar{\partial}$ the ordinary differential operators defined by J and $|g| = \det(g_{\alpha\bar{\beta}})$ for a complex coordinate system. By [22, Theorem 15], there is a one-parameter family $\hat{g}_t$ of riemannian metrics on M such that each $\hat{g}_t$ is a Kähler metric compatible with J_t. First we assume that $e \neq 0$. Set

$$\bar{g}_t = e^{-1}\hat{r}_t, \tag{10.1.2}$$

where $\hat{r}_t$ is the Ricci tensor of $\hat{g}_t$. Then the function f_t defined by

$$f_t = \log(|\bar{g}_t|\,|\hat{g}_t|^{-1}) \tag{10.1.3}$$

is well-defined and, by formula (10.1.1), satisfies the equation

$$(10.1.4)\qquad \bar{\rho}_t - e\,\bar{\omega}_t = \sqrt{-1}\,\partial_t \bar{\partial}_t f_t,$$

where $\bar{\omega}_t$ is the Kähler form of $(J_t, \bar{g}_t)$. We consider the equation

$$(10.1.5)\qquad \log(|\bar{\omega}_t + \sqrt{-1}\,\partial_t \bar{\partial}_t \psi|_t\, |\bar{\omega}_t|_t^{-1}) + f_t - e\,\psi = 0$$

for a function ψ. By formula (10.1.1), we can check that the Kähler metric defined by the Kähler form

$$(10.1.6)\qquad \omega_t = \bar{\omega}_t + \sqrt{-1}\,\partial_t \bar{\partial}_t \psi$$

is an Einstein metric (c.f. [5, Eq. I, $\mathrm{II}^{\pm}$]), Consider the map: $\mathbf{R} \times H^{s+2}(M) \to H^s(M)$ $(s \geqq N+2 = [n/2]+3)$ defined by

$$(10.1.7)\qquad (t, \psi) \mapsto \log(|\bar{\omega}_t + \sqrt{-1}\,\partial_t \bar{\partial}_t \psi|_t\, |\bar{\omega}_t|_t^{-1}) + f_t - e\,\psi.$$

Then the derivative of this map with respect to ψ at $(0,0)$ is given by

$$(10.1.8)\qquad \psi' \mapsto (1/2)\,\Delta\psi' - e\,\psi',$$

which is an isomorphism from $H^{s+2}(M)$ onto $H^s(M)$. In fact if $2e$ is an eigenvalue of Δ with ψ' as a corresponding eigenfunction, then $J d\psi'$ becomes a holomorphic vector field ([26, pp. 134–136, 147]). Thus the implicit function theorem implies that a solution $\psi_t \in H^s(M)$ of (10.1.5) exists and depends C^∞-ly on t. By changing constant factor if necessary, we obtain an H^s-Einstein deformation of g defined by Eq. (10.1.6). Then $g'_0 \in \mathrm{EID}$ is Kähler related with I, so $g'_0 - h$ is Kähler related with $0 \in \mathrm{CID}$. This implies that $g'_0 = h$ by Lemma 9.8.

If $e = 0$, the proof is less simple. By assumption, $\hat{g}'_0$ and h are Kähler related with the same I, so $\hat{g}'_0 - h$ is Kähler related with $0 \in \mathrm{CID}$. Then from Eqs. (9.2.1) and (9.2.2), it follows that the tensor field $\hat{\psi} = (\hat{g}'_0 - h)J$ is a closed hermitian form. [20, Theorem 4.2] says that the dimension of the space $H^{1,1}_t(M, \mathbf{R})$ defined by $(J_t, \hat{g}_t)$ is constant for t. Hence, by Lemma 4.3, we obtain a one-parameter family $\hat{\psi}_{Ht}$ of 2-forms such that $\hat{\psi}_{H0} = \hat{\psi}$ and that each $\hat{\psi}_{Ht}$ is in $H^{1,1}_t(M, \mathbf{R})$. We set

$$(10.1.9)\qquad \bar{g}_t = \hat{g}_t + t\,\phi_t,$$

where ϕ_t is defined by $\phi_t = \hat{\psi}_{Ht} J_t$. Then we see that $\bar{g}_t$ is a Kähler metric compatible with J_t and that $\bar{\omega}'_0 - hJ$ is cohomologous to 0. Now we define a function f_t by

$$(10.1.10)\qquad \bar{\rho}_t = \sqrt{-1}\,\partial_t \bar{\partial}_t f_t.$$

Since $\bar{\rho}_t$ is cohomologous to 0, such an f_t exists and is unique up to constant for each t. Then, by Lemma 4.3, we see that such a function f_t can be taken to depend C^∞-ly on t and so that $f_0 = 0$ (which is obvious when $e \neq 0$). We replace the map (10.1.7) by the map: $\mathbf{R} \times \mathrm{Ker}(\int_g | H^{s+2}(M)) \times \mathbf{R} \to H^s(M)$ defined by

$$(10.1.11)\qquad (t, \psi, c) \mapsto \log(|\bar{\omega}_t + \sqrt{-1}\,\partial_t \bar{\partial}_t \psi|_t\, |\bar{\omega}_t|_t^{-1}) + f_t + c.$$

The derivative of this map with respect to ψ and c at $(0, 0, 0)$ is given by

(10.1.12) $$(\psi', c') \mapsto (1/2)\,\Delta\psi' + c',$$

which is an isomorphism from $\mathrm{Ker}\int_g \times \mathbf{R}$ onto $H^s(M)$. Thus the implicit function theorem implies that the solution $\psi_t \in \mathrm{Ker}\int_g$ exists, depends C^∞-ly on t and gives an Einstein metric defined by Eq. (10.1.6). Moreover, $\omega'_0 - \bar{\omega}'_0$ and $\bar{\omega}'_0 - hJ$ are cohomologous to 0, and so is $\omega'_0 - hJ$. But, by Lemma 9.8, we see that $g'_0 - h \in \mathrm{EEID}_H$ and so $\omega'_0 - hJ$ is harmonic (see Proposition 7.4). Thus $g'_0 = h$. By changing the constant factor if necessary, we obtain an H^s-Einstein deformation of g.

Finally, we show the smoothness. Since each g_t is a C^2-Kähler-Einstein metric, g_t is C^∞ by [12, Theorem 6.1] (c.f. Proposition 0.2). Since the solution g_t is uniquely constructed in the above proof, we can repeat this proof, i.e., we can apply the implicit function theorem, for any $r \geqq s$ at each t. This means that g_t is a C^∞-curve in $\mathcal{M}^r$ for all $r \geqq s$. Q.E.D.

10.2. *Definition.* A complex structure J on M is said to *belong to a non-singular complete family of complex structures* if there is a family J_t of complex structures on M with parameter space P such that $J_0 = J$ and that the map: $T_0 P \to \mathrm{CID}$ defined by

(10.2.1) $$v \mapsto [v, J_t]$$

is an isomorphism onto CEID.

10.3. *Remark.* By Lemma 6.6, it is obvious that the above definition does not depend on the choice of the riemannian metric with respect to which the space CEID is defined.

10.4. *Remark.* Recall Proposition 0.10. That is, if $H^2(M, \Theta)$ vanishes, then J belongs to a non-singular complete family of complex structures.

10.5. **Theorem.** *Let (J, g) be a Kähler-Einstein structure on M with volume* 1. *Assume that the complex structure J belongs to a non-singular complete family of complex structures. Moreover, if $e > 0$, we assume that $\mathrm{EEID}_H(J, g)$ vanishes and that there is no non-zero holomorphic vector field. Then any Einstein metric g_1 on M sufficiently close to g is Kählerian, that is, there is a complex structure J_1 on M such that g_1 becomes a Kähler metric compatible with J_1. Moreover, such J_1 can be taken to depend C^∞-ly on g_1.*

10.6. *Remark.* If the manifold M is of dimension 2 or the $K3$-surface, then the condition that g_1 is close to g is not necessary. See Proposition 0.3. Remark also that in these two cases the assumption for the original Kähler-Einstein structure (J, g) is satisfied.

Proof. In the following, we omit the suffix s which means H^s since all objects are C^∞ and the mappings p, q and χ in Lemma 2.4 may be considered to be C^∞. By Proposition 10.1, we have a C^∞-map ε: $P \times \mathrm{EEID}_H \to \mathcal{M}_1$ such that each $(t, h) \in P \times \mathrm{EEID}_H$ corresponds to a Kähler-Einstein metric compatible with J_t. Moreover, by Proposition 7.3 and Lemma 6.8, the image of the differential ε'_0 at the origin coincides with the space EEID. We take the composition $p\circ\varepsilon$. Then the image of the differential $(p\circ\varepsilon)'_0$ at the origin coincides with the space EEID. Owing to Theorem 3.1, it means that the local pre-moduli space ELPM(g) locally becomes a submanifold of $\mathcal{S}_g$ whose tangent space at g coincides with EEID and that $p\circ\varepsilon$ is a local submersion from $P\times \mathrm{EEID}_H$ onto ELPM defined on an open neighbourhood of the origin. Therefore there is a cross section ψ: ELPM$\to P\times\mathrm{EEID}_H$. Now we may assume that the metric g_1 has volume 1. Since $p(g_1)\in$ELPM, we can define $\psi p(g_1)$ and the equation $p\varepsilon\psi p(g_1)=p(g_1)$ holds. Therefore we see that

$$\varepsilon\psi p(g_1)=(\chi q(\varepsilon\psi p(g_1)))^* p(g_1). \tag{10.6.1}$$

Here, $\varepsilon\psi p(g_1)$ is a Kähler metric compatible with the complex structure $\psi p(g_1)$. Thus $p(g_1)$ is a Kähler metric compatible with the complex structure $(\chi q(\varepsilon\psi p(g_1)))^{-1*}(\psi p(g_1))$ and so $g_1=(\chi q(g_1))^* p(g_1)$ is a Kähler metric compatible with

$$(\chi q(g_1))^*(\chi q(\varepsilon\psi p(g_1)))^{-1*}(\psi p(g_1)).\quad \text{Q.E.D.}$$

10.7. **Corollary.** *Let (J,g) be a Kähler-Einstein structure on M with volume* 1 *whose complex structure J is in a non-singular complete family of complex structures. Assume that $c_1<0$ or that $c_1=0$ and the second Betti number $b_2=1$. Then there is a local one-to-one correspondence between complex structures J_1 on M and compatible Kähler-Einstein metrics g_1. In particular, the space* ELPM(g) *may be regarded as a family of complex structures on M. Moreover the spaces* $\mathrm{EEID}_H(J_1,g_1)$ *and* $\mathrm{CEID}_A(J_1,g_1)$ *vanish for all pairs (J_1,g_1).*

Proof. By Remark 10.4, we can apply Theorem 10.5. If $\mathrm{EEID}_H(J_1,g_1)=0$ and $\mathrm{CEID}_A(J_1,g_1)=0$, then by Proposition 9.9, we see that the correspondence between complex structures and Einstein metrics becomes a local diffeomorphism around J_1. The vanishings of the spaces EEID_H and CEID_A follow from Proposition 7.4 and 8.2. Q.E.D.

10.8. *Example.* As a particular case, we can apply Theorem 10.5 to complex hypersurfaces of a complex projective space. Let $V_{m,d}$ be the set of all homogeneous polynomials f on $\mathbf{C}^{m+2}$ such that f defines a non-singular irreducible hypersurface in $P^{m+1}(\mathbf{C})$. The complex automorphism group $SL(m+2,\mathbf{C})$ of $P^{m+1}(\mathbf{C})$ acts canonically on $V_{m,d}$. Let $H_{m,d}$ be the quotient space $V_{m,d}/SL(m+2,\mathbf{C})$. The space $H_{m,d}$ may be regarded as a set of complex hypersurfaces of $P^{m+1}(\mathbf{C})$. Let $(M,J)\in H_{m,d}$. If $d\geqq m+2$, $m\geqq 2$ and $(m,d)\neq(2,4)$, then the first Chern class c_1 of (M,J) is negative or vanishes. Therefore we can apply Proposition 0.5. Let $MH_{m,d}$ be the set of all Kähler-Einstein structures obtained by Proposition 0.5. [21, Theorem 14.1] says that, under the same assumption for m and d, for any $(M,J)\in H_{m,d}$ the complex structure J is in a non-singular complete family of complex structures, and this family may be regarded as an open neighbourhood of J in $H_{m,d}$. Thus we can apply Theorem 10.5. Let $(M,J,g)\in MH_{m,d}$. If g_1 is an Einstein metric on M which is sufficiently close to g, then there exists a complex structure J_1 on M such that (M,J_1,g_1) is isomorphic to certain $(M,J_2,g_2)\in MH_{m,d}$ as Kähler manifolds. If

$d>m+2$, then the first Chern class of J is negative. Hence we see, by Corollary 9.4, that

$$\dim \mathrm{EEID}(g)=2\left\{\binom{m+d+1}{d}-(m+2)^2\right\}. \tag{10.8.1}$$

11. The Complex Structure on a Family of Complex Structures

In this section, we recall some results obtained by Kodaira and Spencer. The idea of our proofs is similar with that of [21] except for notations.

11.1. *Definition.* A family J_t of complex structures on M with parameter space P is said to be *normal* if the dimension of the space $H^0(M, \Theta(J_t))$ is constant, and said to be *stable* if the linear map: $T_tP \to H^1(M, \Theta(J_t))$ defined by

$$v \mapsto [v, J_t] \tag{11.1.1}$$

is an isomorphism onto a complex subspace of $H^1(M, \Theta(J_t))$ for each $t \in P$.

Let (J_t, g_t) be a family of Kähler structures on M with parameter space P. For $t \in P$ and $v \in T_tP$, we can define an element $I \in \mathrm{CEID}(J_t, g_t)$ by

$$[v, J_t]=I+[X, J_t] \tag{11.1.2}$$

for some vector field X on M. We set

$$v^H=v-X. \tag{11.1.3}$$

If the family J_t is stable, then there is a unique vector $w \in T_tP$ such that $[w^H, J_t] = J_t[v^H, J_t]$. We define a complex structure J^P on T_tP by

$$[(J^P v)^H, J_t]=J_t[v^H, J_t]. \tag{11.1.4}$$

11.2. **Proposition** (c.f. [21, Proposition 11.1]). *Let (J_t, g_t) be a family of Kähler structures on M. Assume that the family J_t is normal and stable. Then the operator J^P depends C^∞-ly on $t \in P$ and becomes a complex structure on P.*

Proof. Since the family J_t is normal, we can apply Lemma 4.3 and see that X can be taken to depend C^∞-ly on t and $v \in T_tP$. Therefore, if v is a vector field on P, then we may assume that v^H is also a C^∞-vector field, which implies that J^P is a C^∞-tensor field on P. Now, we show that this almost complex structure J^P on P is integrable. For v, $w \in T_tP$, we set

$$A(v, w)=[v^H, w^H]-[v, w]^H. \tag{11.2.1}$$

It is a well-defined vector field on M for each pair v, $w \in T_tP$. I.e., $A(v, w)$ does not depend on the extension of v and w. Moreover we set

$$N^A(v, w)=A(v, w)-A(J^P v, J^P w)+J_tA(J^P v, w)+J_tA(v, J^P w) \tag{11.2.2}$$

and denote by N^P the Nijenhuis torsion tensor of J^P. We see that

$$
\begin{aligned}
(11.2.3)\quad & [N^P(v,w)^H, J_t] \\
&= [[v,w]^H, J_t] - [[J^P v, J^P w]^H, J_t] + [J^P[J^P v, w]^H, J_t] + [J^P[v, J^P w]^H, J_t] \\
&= [[v^H, w^H], J_t] - [A(v,w), J_t] - [[J^P v^H, J^P w^H], J_t] + [A(J^P v, J^P w), J_t] \\
&\quad + J_t[[J^P v^H, w^H], J_t] - J_t[A(J^P v, w), J_t] + J_t[[v^H, J^P w^H], J_t] - J_t[A(v, J^P w), J_t] \\
&= -[N^A(v,w), J_t] \\
&\quad + [v^H, [w^H, J_t]] - [w^H, [v^H, J_t]] - [J^P v^H, [J^P w^H, J_t]] + [J^P w^H, [J^P v^H, J_t]] \\
&\quad + J_t[J^P v^H, [w^H, J_t]] - J_t[w^H, [J^P v^H, J_t]] + J_t[v^H, [J^P w^H, J_t]] - J_t[J^P w^H, [v^H, J_t]] \\
&= -[N^A(v,w), J_t] + [v^H, [w^H, J_t]] - [w^H, [v^H, J_t]] \\
&\quad - [J^P v^H, J_t][w^H, J_t] - J_t[J^P v^H, [w^H, J_t]] + [J^P w^H, J_t][v^H, J_t] + J_t[J^P w^H, [v^H, J_t]] \\
&\quad + J_t[J^P v^H, [w^H, J_t]] - J_t[w^H, J_t][v^H, J_t] + [w^H, [v^H, J_t]] \\
&\quad + J_t[v^H, J_t][w^H, J_t] - [v^H, [w^H, J_t]] - J_t[J^P w^H, [v^H, J_t]] \\
&= -[N^A(v,w), J_t].
\end{aligned}
$$

Since the family J_t is stable, it means that $N^P(v,w)=0$. Q.E.D.

We have seen also the following

11.3. **Corollary.** *In the situation of* Proposition 11.2, *$N^A(v,w)$ is a holomorphic vector field for J_t for each $v, w \in T_t P$.*

Next, if $H^0(M, \Theta(J_t))=0$, then we can define an almost complex structure J^T on $T=M\times P$ by

$$(11.3.1)\qquad J^T v^H = (J^P v)^H \quad \text{for } v\in TP,$$

$$(11.3.2)\qquad J^T X = J_t X \quad \text{for } X\in TM,$$

since the vector field introduced in Eq. (11.1.2) is unique, and so the vector v^H is well-defined for each $v\in T_t P$.

11.4. **Proposition** (c.f. [21, Proposition 18.3, Theorem 18.4]). *Let (J_t, g_t) be a family of Kähler structures on M with parameter space P. If the family J_t is normal and stable and if $H^0(M, \Theta(J_0))=0$, then the almost complex structure J^T on T defined above is integrable.*

Proof. Denote by N^T the Nijenhuis tensor of J^T. Then $N^T(X,Y)=0$ for X, $Y\in TM$. For $v\in TP$ and $X\in TM$, we see

$$
\begin{aligned}
(11.4.1)\quad & \\
N^T(v^H, X) &= [v^H, X] - [J^P v^H, J_t X] + J_t[J^P v^H, X] + J_t[v^H, J_t X] \\
&= [v^H, X] - [J^P v^H, J_t]X - J_t[J^P v^H, X] + J_t[J^P v^H, X] + J_t[v^H, J_t]X - [v^H, X] \\
&= 0.
\end{aligned}
$$

For v, $w\in TP$, we see

$$(11.4.2)\qquad \begin{aligned} N^T(v^H, w^H) &= [v^H, w^H] - [J^P v^H, J^P w^H] + J^T[J^P v^H, w^H] + J^T[v^H, J^P w^H] \\ &= A(v, w) + [v, w]^H - A(J^P v, J^P w) - [J^P v, J^P w]^H \\ &\quad + J_t A(J^P v, w) + J^P[J^P v, w]^H + J_t A(v, J^P w) + J^P[v, J^P w]^H \\ &= N^A(v, w) + N^P(v, w)^H. \end{aligned}$$

But here $N^P(v, w) = 0$ by Proposition 11.2 and $N^A(v, w) = 0$ by assumption and Corollary 11.3. Q.E.D.

12. The Canonical Riemannian Metric on a Family of Kähler-Einstein Structures

Let (J_t, g_t) be a family of Kähler-Einstein structures on M. If the family J_t is normal and stable and if the family g_t is normal and effectively parametrized, then the parameter space P can be endowed with the canonical complex structure and the canonical riemannian metric.

12.1. *Definition.* A family (J_t, g_t) of Kähler-Einstein structures on M with volume 1 is said to be *normal* if the family J_t is normal, and said to be *stable* if the family J_t is stable and if the spaces $\mathrm{CEID}_A(J_t, g_t)$ and $\mathrm{EEID}_H(J_t, g_t)$ vanish for all $t \in P$.

12.2. **Lemma.** *If a family (J_t, g_t) of Kähler-Einstein structures on M is normal and stable, then the family g_t is normal and effectively parametrized. In particular, the parameter space P can be endowed with the canonical riemannian metric.*

Proof. Let (J, g) be a Kähler-Einstein structure on M. We know that if $e<0$ then $\operatorname{Ker}\delta_g^* = 0$, if $e = 0$ then $\operatorname{Ker}\delta_g^* = H^0(M, \Theta)$ and if $e>0$ then $\operatorname{Ker}\delta_g^* + J(\operatorname{Ker}\delta_g^*) = H^0(M, \Theta)$. Therefore the normality of the family J_t implies the normality of the family g_t. Let $v \in T_t P$ and assume that $[v, g_t] \in \operatorname{Im}\delta_{g_t}^*$. I.e., there is a vector field X on M such that $[v, g_t] = [X, g_t]$. Then the assumption and Proposition 9.9 imply that $[v, J_t]$ is decomposed into $I_S + [Y, J_t]$, where $I_S \in \mathrm{CEID}(J_t, g_t)$ is Kähler related to $0 \in \mathrm{EEID}(g_t)$. But then Lemma 9.3 says that $I_S = 0$, which contradicts the stability of the family J_t. Q.E.D.

12.3. **Theorem.** *Let (J_t, g_t) be a normal and stable family of Kähler-Einstein structures on M. Then the canonical riemannian metric g^P on the parameter space P is a Kähler metric compatible with the complex structure J^P on P.*

Proof. For $v \in T_t P$, let v^H be the vector field defined by Eq. (4.1.3). Since $[v^H, J_t] \in \mathrm{CID}(J_t)$ is Kähler related with $[v^H, g_t] \in \mathrm{EEID}_A(J_t, g_t)$, we see, by Proposition 9.9, that $[v^H, J_t] \in \mathrm{CEID}_S(J_t, g_t)$. Therefore, the notation v^H does not contradict that induced by Eq. 11.1.3. Moreover, by Lemma 9.3,

$$(12.3.1)\qquad [v^H, \omega_t] = [v^H, g_t J_t] = [v^H, g_t]\, J_t + g_t [v^H, J_t] = 0.$$

and so

$$(12.3.2)\qquad g^P(v,w)=\int_M ([v^H,g_t],[w^H,g_t])\,v_{g_t}$$
$$=\int_M (g_t[v^H,J_t]\,J_t,\,g_t[w^H,J_t]\,J_t)\,v_{g_t}$$
$$=\int_M ([v^H,J_t],[w^H,J_t])\,v_{g_t}.$$

Therefore,

$$(12.3.3)\qquad g^P(J^Pv,J^Pw)=\int_M ([J^Pv^H,J_t],[J^Pw^H,J_t])\,v_{g_t}$$
$$=\int_M (J_t[v^H,J_t],J_t[w^H,J_t])\,v_{g_t}$$
$$=g^P(v,w),$$

i.e., g^P is a hermitian metric. Now we calculate the exterior derivative $d\omega^P$ of the Kähler form ω^P of g^P.

$$(12.3.4)\qquad \omega^P(v,w)=g^P(v,J^Pw)$$
$$=\int_M ([v^H,J_t],[J^Pw^H,J_t])\,v_{g_t}$$
$$=\int_M ([v^H,J_t],J_t[w^H,J_t])\,v_{g_t}.$$

Then assuming that $[v,w]=[w,z]=[z,v]=0$, we see

$$(12.3.5)\qquad (d\omega^P)(v,w,z)$$
$$=v\int_M ([w^H,J_t],J_t[z^H,J_t])\,v_{g_t}$$
$$+\text{alternating terms.}$$

Here,

$$(12.3.6)\quad v\{([w^H,J_t],J_t[z^H,J_t])\,v_{g_t}\}$$
$$=-[v^H,g_t]^{ij}(g_t)_{km}[w^H,J_t]^k{}_i(J_t[z^H,J_t])^m{}_jv_{g_t}$$
$$+(g_t)^{ij}[v^H,g_t]_{km}[w^H,J_t]^k{}_i(J_t[z^H,J_t])^m{}_jv_{g_t}$$
$$+([v^H,[w^H,J_t]],J_t[z^H,J_t])\,v_{g_t}$$
$$+([w^H,J_t],[v^H,J_t]\,[z^H,J_t])\,v_{g_t}$$
$$+([w^H,J_t],J_t[v^H,[z^H,J_t]])\,v_{g_t}$$
$$+([w^H,J_t],\ J_t[z^H,J_t])\,(1/2)\,[v^H,g_t]^m{}_mv_{g_t},$$

and

$$(12.3.7)\qquad [v^H,g_t]^{ij}(g_t)_{km}[w^H,J_t]^k{}_i(J_t[z^H,J_t])^m{}_j$$
$$=2\,\mathrm{Re}\{[v^H,g_t]^{\alpha\beta}(g_t)_{\bar\gamma\delta}[w^H,J_t]^{\bar\gamma}{}_\alpha(J_t)^\delta{}_\varepsilon[z^H,J_t]^\varepsilon{}_\beta\}=0,$$

$$(12.3.8)\qquad (g_t)^{ij}[v^H,g_t]_{km}[w^H,J_t]^k{}_i(J_t[z^H,J_t])^m{}_j=0,$$

$$(12.3.9)\quad ([w^H,J_t],[v^H,J_t]\,[z^H,J_t])=2\,\mathrm{Re}\{[w^H,J_t]_\alpha{}^{\bar\beta}[v^H,J_t]^\alpha{}_{\bar\gamma}[z^H,J_t]^{\bar\gamma}{}_{\bar\beta}\}=0,$$

$$(12.3.10)\qquad [v^H,g_t]^m{}_m=0.$$

Therefore,

$$(d\omega^P)(v,w,z)=\int_M ([v^H,[w^H,J_t]],J_t[z^H,J_t])\,v_{g_t} - \int_M ([v^H,[z^H,J_t]],J_t[w^H,J_t])\,v_{g_t} + \text{alternating terms} = \int_M ([A(v,w),J_t],[J^P z^H,J_t])\,v_{g_t} + \text{alternating terms}. \tag{12.3.11}$$

But here $[J^P z^H, J_t] \in \mathrm{CEID}(J_t, g_t)$ and $A(v,w)$ is a vector field on M. Thus

$$d\omega^P = 0. \quad \text{Q.E.D.} \tag{12.3.12}$$

12.4. **Corollary.** *In the situation of* Corollary 10.7, *the space* ELPM(g) *regarded as a family* (J_t, g_t) *of Kähler-Einstein structures on* M *becomes a Kähler manifold. Moreover, the complex structure is real analytic with respect to the real analytic structure of* ELPM(g).

Proof. By Theorem 4.13, the canonical riemannian metric, which is a Kähler metric, is real analytic. Q.E.D.

Assume that $H^0(M, \Theta(J_t)) = 0$. In Proposition 11.4 we defined a complex structure J^T on $T = M \times P$. Here we define a riemannian metric g^T on T by

$$g^T(v^H, w^H) = g^P(v,w) \quad \text{for } v, w \in TP, \tag{12.4.1}$$

$$g^T(X,Y) = g_t(X,Y) \quad \text{for } X, Y \in TM, \tag{12.4.2}$$

$$g^T(v^H, X) = 0 \quad \text{for } v \in TP,\ X \in TM. \tag{12.4.3}$$

Obviously, the metric g^T is a hermitian metric.

12.5. **Proposition.** *Let* (J_t, g_t) *be a normal and stable family of Kähler-Einstein structures on* M. *Assume that* $H^0(M, \Theta(J_t)) = 0$. *Then the hermitian metric* g^T *is a Kähler metric if and only if* $A = 0$, *where* A *is defined by* Eq. (11.2.1).

Proof. Denote by ω^T the Kähler form of g^T. First we see that

$$(d\omega^T)(X,Y,Z) = 0 \quad \text{for } X, Y, Z \in TM \tag{12.5.1}$$

by definition. Next, for $v \in TP$, we extend X and $Y \in TM$ so that $[X,Y]=0$, $[v^H, X] = 0$ and $[v^H, Y] = 0$. Then

$$(d\omega^T)(v^H,X,Y) = v^H(\omega^T(X,Y)) + X(\omega^T(Y,v^H)) + Y(\omega^T(v^H,X)) = v^H(\omega_t(X,Y)) = [v^H,\omega_t](X,Y) = 0. \quad (12.3.1) \tag{12.5.2}$$

Moreover, for v, w, $z \in TP$,

$$(12.5.3)\qquad \begin{aligned}(d\omega^T)(v^H, w^H, z^H) &= v^H(\omega^T(w^H, z^H)) - \omega^T([v^H, w^H], z^H)\\ &\quad + \text{alternating terms}\\ &= v(\omega^P(w, z)) - \omega^P([v, w], z)\\ &\quad + \text{alternating terms}\\ &= (d\omega^P)(v, w, z) = 0. \quad (12.3.12)\end{aligned}$$

Finally,

$$(12.5.4)\quad \begin{aligned}(d\omega^T)(v^H, w^H, X) &= v^H(\omega^T(w^H, X)) + w^H(\omega^T(X, v^H)) + X(\omega^T(v^H, w^H))\\ &\quad - \omega^T([v^H, w^H], X) - \omega^T([w^H, X], v^H) - \omega^T([X, v^H], w^H)\\ &= -\omega_t(A(v, w), X). \quad \text{Q.E.D.}\end{aligned}$$

It seems to the author that the condition $A=0$ is rather strong and probably does not occur except on the one dimensional complex torus.

12.6. *Example.* In the situation of Example 10.8, if $d>m+2$, then the space $MH_{m,d}$ is identified with the space $H_{m,d}$. They canonically become Kähler manifolds.

13. **Appendix** - Proof of Theorem 3.1

To prove Theorem 3.1, we recall some basic definitions and facts in the theory of real analytic objects in Banach spaces, for which we refer to [14, Chap. IV]. This category is effectively used for the Plateau problem (c.f. [30]).

13.1. *Definition.* Let V and W be Banach spaces and U an open set of V. A mapping f: $U \to W$ is said to be *real analytic* if for each point $x \in U$ f can be represented by a convergent power series around x.

13.2. *Definition.* Let V and W be complex Banach spaces and U an open set of V. A mapping f: $U \to W$ is said to be *holomorphic* if f is of class C^1 and the derivative f'_x at each point $x \in U$ commutes with the almost complex structures.

13.3 **Lemma** ([14, p. 134, Theorem 3.7]). *Let V and W be complex Banach spaces and U an open set of V. A holomorphic mapping f: $U \to W$ is real analytic.*

13.4 **Lemma** ([1, Theorem 5.7], c.f. [14, p. 144, Theorem 3.11]). *Let V and W be Banach spaces and $V^{\mathbf{C}}$, $W^{\mathbf{C}}$ their complexifications. Let U be an open set of V and f: $U \to W$ a real analytic map. Then there exists an open set $U^{\mathbf{C}}$ of $V^{\mathbf{C}}$ which contains U such that f can be extended to a holomorphic map $f^{\mathbf{C}}$: $U^{\mathbf{C}} \to W^{\mathbf{C}}$.*

The most important fact is the following

13.5. **Lemma** ([14, p. 145, Theorem 3.12]). *In the real analytic category in Banach spaces, the implicit function theorem holds.*

So we see as a corollary the following

13.6. **Lemma.** *Let V and W be Hilbert spaces and f a real analytic mapping from V to W defined on an open neighbourhood of the origin $0 \in V$. Assume that $f(0)=0$ and that the image of the differential f'_0 at 0 is closed in W. Then there is an open neighbourhood U of $0 \in V$ such that the set $f^{-1}(0) \cap U$ is a real analytic set in a real analytic submanifold Z of U whose tangent space $T_0 Z$ coincides with* $\mathrm{Ker} f'_0$.

Proof. Let $p\colon W \to \operatorname{Im} f_0'$ be a projection map and set $q = \mathrm{id}_W - p$. Applying Lemma 13.5 to the map $p \circ f$, we see that there is an open neighbourhood U of $0 \in V$ such that the set $(p \circ f)^{-1}(0) \cap U$ forms a real analytic submanifold of U. If we set $Z = (p \circ f)^{-1}(0) \cap U$, then $T_0 Z = \operatorname{Ker} f_0'$ and $f^{-1}(0) \cap U = (q \circ f | Z)^{-1}(0)$. Q.E.D.

To work in this category, the following Lemma is basic.

13.7. **Lemma.** *Let E and F be vector bundles over M and $E^{\mathbf{C}}$, $F^{\mathbf{C}}$ their complexifications. Let f be a C^∞-cross section of E and $\psi\colon E \to F$ a fiber preserving C^∞-map defined on an open set of E which contains the image of f. Assume that ψ has an extension to a fiber preserving map $\psi^{\mathbf{C}}\colon E^{\mathbf{C}} \to F^{\mathbf{C}}$ defined on an open set of $E^{\mathbf{C}}$ such that the restriction $\psi_x^{\mathbf{C}}$ to each fiber $E_x^{\mathbf{C}}$ is holomorphic. Then the map $\Psi\colon H^s(E) \to H^s(F)$ defined by*

$$\Psi(u) = \psi \circ u, \tag{13.7.1}$$

defined on an open neighbourhood of f, is real analytic provided that $s > [n/2] + 1$.

Proof. By Lemma 13.3, it is sufficient to prove that the map $\Psi^{\mathbf{C}}\colon H^s(E^{\mathbf{C}}) \to H^s(F^{\mathbf{C}})$ defined by

$$\Psi^{\mathbf{C}}(u) = \psi^{\mathbf{C}} \circ u, \tag{13.7.2}$$

which is an extension of Ψ, is holomorphic. But in fact $\Psi^{\mathbf{C}}$ is C^∞ ([29, Theorem 11.3]), and for each $x \in M$ we have

$$\lim_{z \to 0} \frac{1}{z} \{ \Psi^{\mathbf{C}}(u + zv)(x) - \Psi^{\mathbf{C}}(u)(x) \} = (\psi_x^{\mathbf{C}})'_{u(x)}(v(x)), \tag{13.7.3}$$

where z denotes complex number. Thus $\Psi^{\mathbf{C}}$ is holomorphic. Q.E.D.

13.8. *Remark.* This Lemma, together with the observation that the differentiation is linear, says that ordinary tensor calculus operations on a compact C^∞-manifold are real analytic with respect to some suitable H^s-topology.

Now we come back to our space $\mathscr{M}^s$.

13.9 **Lemma.** *Ebin's slice $\mathscr{S}_g^s$ is a real analytic submanifold of $\mathscr{M}^s$.*

Proof. The definition of Ebin's slice $\mathscr{S}_g^s$ reduces as follows. Let V be a finite dimensional vector space and $S^2 V$ (resp. $S_+^2 V$) the space of all symmetric (resp. positive definite symmetric) bilinear forms on V. If we fix an inner product $g_0 \in S_+^2 V$, we can define a riemannian metric on $S_+^2 V$ by

$$(\psi, \phi)_g = \operatorname{Tr}(g^{-1} \psi g^{-1} \phi) \det(g_0^{-1} g)^{1/2} \tag{13.9.1}$$

for $g \in S_+^2 V$ and $\psi, \phi \in S^2 V$. This riemannian metric depends on g_0, but only up to constant factor. Therefore the exponential map exp does not depend on g_0. Coming back to the manifold M, we define an exponential map Exp on $\mathscr{M}^s$ by

$$(\operatorname{Exp}_g h)_x = \exp_{g_x} h_x \quad \text{for } x \in M, \tag{13.9.2}$$

where $h \in H^s(S^2 M)$. The slice $\mathscr{S}_g^s$ is defined as $\operatorname{Exp}_g(U)$, where U is an open neighbourhood of the origin in $\operatorname{Ker} \delta_g$. But by Lemma 3.12, the map Exp_g is real analytic, hence $\mathscr{S}_g^s$ is a real analytic submanifold of $\mathscr{M}^s$. Q.E.D.

Proof of Theorem 3.1. By Lemma 2.6, we see that

$$\operatorname{ELPM}(g) = (E | \mathscr{S}_g^s \cap \mathscr{M}_1^s)^{-1}(0), \tag{13.9.3}$$

where E is defined by Eq. (1.2.1) and regarded as a real analytic map from $\mathscr{M}^s$ into $H^{s-2}(S^2 M)$. By [24, Proposition 3.2], the image of the differential $(E | \mathscr{S}_g^s \cap \mathscr{M}_1^s)'_g$ at g is closed in $H^{s-2}(S^2 M)$. Therefore the proof reduces to Lemma 13.6. Q.E.D.

13.10. *Remark.* In the proof of Theorem 3.1, we did not effectively use the fact that any Einstein metric is real analytic (Proposition 0.2).

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Oblatum 25-X-1982