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The Birkhoff-Lewis Fixed Point Theorem and a Conjecture of V.I. Arnold

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1. Introduction, Results

The aim of this note is to prove two fixed point theorems for symplectic maps which are generated by timedependent globally Hamiltonian vectorfields, which however are not assumed to be close to the identity map. In the case that ψ is a symplectic diffeomorphism on the torus $T^{2n}=\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ this requires that $\psi=\varphi^1$, where φ^t is the flow satisfying

$$\frac{d}{dt}\varphi^t(x)=J\nabla h(t,\varphi^t(x)), \qquad \varphi^0=\mathrm{id},$$

the function $h\in C^2(\mathbb{R}\times\mathbb{R}^{2n})$ being periodic in all the variables of period 1. The skew-symmetric matrix $J\in\mathfrak{L}(\mathbb{R}^{2n})$ defined by $J=\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$, 1 being the unit matrix on $\mathbb{R}^n$, is the standard symplectic structure on $\mathbb{R}^{2n}$. Clearly, a periodic solution on T^{2n} having period 1 of the Hamiltonian equation

$$\dot{x}=J\nabla h(t,x), \qquad x\in\mathbb{R}^{2n}, \tag{1}$$

gives rise to a fixed point of ψ, and the problem is to find 1-periodic solutions of the Hamiltonian system, which is periodic in time of period 1. The first result is as follows.

Theorem 1. *The Hamiltonian vectorfield* (1) *on* T^{2n}, *with the function* $h(t,x)\in C^2(\mathbb{R}\times\mathbb{R}^{2n})$ *being periodic of period* 1 *possesses at least* $2n+1$ *periodic solutions of period* 1.

The periodic solutions found by the theorem are contractible loops on T^{2n}, i.e. are given as periodic functions on $\mathbb{R}^{2n}$. One expects more periodic solutions, if all the periodic solutions are known to be nondegenerate. Here we call a 1-periodic solution nondegenerate, if it has no Floquet-multiplier equal to 1. Recall that $\lambda\in\mathbb{C}$ is a Floquet-multiplier of a periodic solution $x(t)=x(t+1)$, if λ

is an eigenvalue of $d\varphi^1(x(0))$, where φ^t is the flow of the corresponding timedependent vectorfield. Indeed, the following statement holds true:

Theorem 2. *Assume that all the periodic solutions having period* 1 *of the system* (1) *are nondegenerate, then there are at least* 2^{2n} *of them.*

From these existence statements for periodic solutions one deduces immediately the following Corollary for the symplectic map ψ. We call a fixed point $x=\psi(x)$ nondegenerate if 1 is not an eigenvalue of $d\psi(x)$.

Corollary 1. *Every symplectic* C^1*-diffeomorphism* ψ *on the torus* $T^{2n}=\mathbb{R}^{2n}/Z^{2n}$, *which is generated by a globally Hamiltonian vectorfield, possesses at least* $2n+1$ *fixed points. If, moreover, all the fixed points of* ψ *are nondegenerate, then there are at least* 2^{2n} *of them.*

The symplectic diffeomorphism which meet the assumption of the Corollary can be characterized as follows. If (M,ω) is any compact, symplectic and smooth manifold, we denote by $\mathrm{Diff}^\infty(M,\omega)$ the topological group of symplectic C^∞-diffeomorphisms ψ, i.e. $\psi^*\omega=\omega$. Let $\mathrm{Diff}_0^\infty(M,\omega)$ be the identity component in $\mathrm{Diff}^\infty(M,\omega)$, which can be shown to be the identity component by smooth arcs in $\mathrm{Diff}^\infty(M,\omega)$. It has been proved by A. Banyaga [4], that the commutator-subgroup of $\mathrm{Diff}_0^\infty(M,\omega)$ consists precisely of those symplectic diffeomorphisms, which are generated by globally Hamiltonian vectorfields on M, and hence agrees with the subgroup of symplectic diffeomorphisms having vanishing so-called Calabi-invariant.

As a special case we consider a measure preserving diffeomorphism of T^2, which is homologeous to the identity map on T^2 and hence is, on the covering space $\mathbb{R}^2$, of the form

$$\psi\colon\ x\mapsto x+f(x),\qquad x\in\mathbb{R}^2, \tag{2}$$

with f being periodic. As observed by V.I. Arnold, see [2], this map ψ is generated by a globally Hamiltonian vectorfield on T^2 if and only if the meanvalue of f over the torus vanishes, i.e. $[f]=0$. This fact will be proved in the appendix. We therefore conclude from Theorem 1 the following result, which was conjectured by V.I. Arnold in [2] and [3].

Corollary 2. *Every measure preserving* C^1*-diffeomorphism of* T^2 *which is of the form* (2) *with* $[f]=0$ *has at least* 3 *fixed points.*

The condition $[f]=0$ is clearly necessary in order to guarantee a fixed point, as the translation map $x\mapsto x+c$ shows, which has no fixed points on T^2, if $c\notin Z^2$.

We point out that it is not assumed that the symplectic maps considered are C^1-close to the identity map. Indeed under this additional assumption the above fixed points can easily been found as critical points of a so-called generating function, which is defined on the torus. The idea of relating fixed points of symplectic maps to critical points of a related function defined on the corresponding manifold goes back to H. Poincaré [11]. It has been exploited by A. Banyaga [5], J. Moser [8] and A. Weinstein [12] in order to guarantee fixed points for symplectic maps, which are however assumed to be C^1-close to the identity map.

The second result is related to the Birkhoff-Lewis fixed point theorem, for which we refer to J. Moser [9]. The problem can be reduced to an exact symplectic diffeomorphism on $T^n \times D$, $D \subset \mathbb{R}^n$ being a disc, which is assumed to be close to an integrable map. Instead we would like to replace such a smallness condition by a condition at the boundary $T^n \times \partial D$ only. To be precise we consider on the symplectic manifold $T^n \times \mathbb{R}^n$ the time-dependent globally Hamiltonian vectorfield given by

$$h(t, x, y) \in C^2(R \times T^n \times \mathbb{R}^n),$$

and periodic in t and x of period 1. We suppose that

$$h(t, x, y) = \tfrac{1}{2}\langle y, by\rangle + \langle a, y\rangle \tag{3}$$

if $|y| \geqq C > 0$, where $b \in \mathfrak{L}(\mathbb{R}^n)$ is a time-independent, symmetric and non-singular matrix, and where $a \in \mathbb{R}^n$ are constants. Set

$$T^n \times D = \{(x, y) \in T^n \times \mathbb{R}^n \mid |y| < C\}.$$

Under these assumptions, the following statement holds true.

Theorem 3. *The Hamiltonian vectorfield on $T^n \times \mathbb{R}^n$ admits at least $n+1$ periodic solutions, of period 1, which are contained in $T^n \times D$.*

Again the periodic solutions found are special: their projections onto T^n are contractible loops on T^n. The symplectic diffeomorphism generated by the above time-dependent Hamiltonian vectorfield admits then at least $n+1$ fixed points in $T^n \times D$.

2. Idea of the Proof

The proof of these theorems is based on a variational principle for which the periodic orbits are critical points. This variational problem differs from that customarily used in mechanics, which in the example of the geodesic flow on a manifold M is the length integral or the energy integral. In contrast, the variational principle used here is defined in the loop space over a symplectic space, in the above example, over T^*M. In our problem of the torus $T^{2n} = \mathbb{R}^{2n}/Z^{2n}$ we consider, on the covering space $\mathbb{R}^{2n}$, the action functional, defined on periodic functions $x(0) = x(1)$:

$$f(x) = \int_0^1 \{\tfrac{1}{2}\langle \dot{x}, Jx\rangle - h(t, x(t))\}\, dt,$$

whose Euler-equations are indeed the Hamiltonian Eqs. (1). This functional is neither bounded from above nor from below. That it still can be used effectively for existence proofs was first shown by P. Rabinowitz and subsequently used by many authors. Since h is periodic, following the ideas of [1], it will be shown by a Lyapunov-Schmidt reduction that the required critical points of f are in one-to-one correspondence with the critical points of a function g, which approximates f but is defined on the finite dimensional manifold $M = T^{2n} \times \mathbb{R}^N \times \mathbb{R}^N$ for some large N. The critical points of g are then found as the rest points of the gradient flow ∇g on M. From the fact, that h and its derivatives are uniformly bounded it follows, that the set of bounded solutions of this gradient flow is compact and contained in the compact set $B := T^{2n} \times D \times D$, where D is a disc in $\mathbb{R}^N$. Moreover $B^- := T^{2n} \times \partial D \times D$ is the exit set and $B^+ = T^{2n} \times D \times \partial D$ is the entrance set, so that B is an isolating block in the sense of [6]. The proof now follows from two general statements for flows, which are not necessarily gradient flows. First consider any continuous flow which admits the above very special isolating block B, with exit set B^- and entrance set B^+. Then the invariant set S of the flow contained in B carries cohomology which it obtains from the torus T^{2n}. In fact Theorem 4 states that

$$l(S) \geqq l(B) = l(T^{2n}) = 2n+1,$$

where $l(X)$ denotes the cup long of a compact space X. The second statement concerns Morse-decompositions. If $\{M_1, \ldots, M_k\}$ is an ordered Morse-decomposition of a compact, isolated invariant set S of a continuous flow, then

$$l(S) \leqq \sum_{j=1}^{k} l(M_j),$$

by Theorem 5. If, in addition, the flow on S is gradientlike with finitely many rest points then these rest points consititute a Morse-decomposition of S. In this case $l(M_j) = 1$ and we obtain the estimate

$$l(S) \leqq \sum_{j=1}^{k} 1 = \#\{\text{rest points}\}.$$

We conclude that the gradient flow ∇g possesses at least $2n+1$ rest points and theorem 1 follows. Theorem 3 is proved similarly. Theorem 2 is a simple application of the Morse-theory as developed in [6] and [7] to the compact set S of all bounded solutions of ∇g on M.

3. The Variational Principle and the Reduction

We shall look for special periodic solutions of the Hamiltonian system (1) on $T^{2n} = \mathbb{R}^{2n}/Z^{2n}$, namely for those whose orbits are contractible. In the covering space $\mathbb{R}^{2n}$ of the torus these solutions are described by periodic functions $t \mapsto x(t) \in \mathbb{R}^{2n}$, $x(0) = x(1)$. The required periodic solutions are the critical points of the functional

$$f(x) := \int_0^1 \{\tfrac{1}{2}\langle \dot{x}, Jx\rangle - h(t, x(t))\}\, dt, \tag{4}$$

defined on the space of periodic curves in $\mathbb{R}^{2n}$, i.e. $x(0) = x(1)$. Indeed one verifies immediately that

$$\nabla f(x) = -J\dot{x} - \nabla h(t, x). \tag{5}$$

To be precise we introduce the Hilbert space $H = L_2((0,1); \mathbb{R}^{2n})$. Define in H the linear operator A: $\mathrm{dom}(A) \subset H \to H$ by setting

$$\mathrm{dom}(A) = \{u \in H^1([0,1]; \mathbb{R}^{2n}) \mid u(0) = u(1)\}$$

and $\mathrm{Au} = -J\dot{u}$ if $u \in \mathrm{dom}(A)$. The space H^1 is the Sobolev space of absolutely continuous functions whose first derivative is in L^2. The continuous operator $F: H \to H$ is defined by $F(u)(t) := \nabla h(t, u(t))$, $u \in H$. Its potential $\Phi(u)$ is given by $\Phi(u) := \int_0^1 h(t, u(t))\, dt$, so that $F(u) = \nabla \Phi(u)$.

Since $J^2 = -1$ we can write the equation (1) in the form $-J\dot{x} = \nabla h(t, x)$ and one sees that every solution $u \in \mathrm{dom}(A)$ of the equation

$$\mathrm{Au} = F(u) \tag{6}$$

defines (by periodic continuation) a classical 1-periodic solution of (1). Conversely, every 1-periodic solution on T^{2n} of (1), which is contractible on T^{2n} defines (by restriction) a solution u of the Eq. (6). With these notations the functional f defined by (4) becomes

$$f(u) = \tfrac{1}{2}\langle \mathrm{Au}, u\rangle - \Phi(u), \tag{7}$$

for $u \in \mathrm{dom}(A)$. We look for critical points of f.

Since h is periodic, there is a constant $\alpha > 0$ such that

$$|h''(t, x)| \leqq \alpha \tag{8}$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^{2n}$, where $'$ stands for the derivative in the x-variable. We shall use this estimate in order to reduce the problem of finding critical points of the functional f on $\mathrm{dom}(A)$ to the problem of finding critical points of a related functional, which is defined on a finite dimensional subspace of the Hilbert space H.

First observe that the operator A is selfadjoint, $A^* = A$. It has closed range and a compact resolvent. The spectrum of A, $\sigma(A)$, is a pure point spectrum and $\sigma(A) = 2\pi\mathbb{Z}$. Every eigenvalue $\lambda \in \sigma(A)$ has multiplicity $2n$ and the eigenspace $E(\lambda) := \ker(\lambda - A)$ is spanned by the orthogonal basis given by the loops:

$$t \mapsto e^{t\lambda J} e_k = (\cos \lambda t)\, e_k + (\sin \lambda t)\, J e_k,$$

$k = 1, 2, \ldots, 2n$, where $\{e_k \mid 1 \leqq k \leqq 2n\}$ is the standard basis in $\mathbb{R}^{2n}$. In particular $\ker(A) = \mathbb{R}^{2n}$, the kernel of A consist precisely of the constant loops in $\mathbb{R}^{2n}$. Denoting by $\{E_\lambda \mid \lambda \in \mathbb{R}\}$ the spectral resolution of A we define the orthogonal projection $P \in \mathfrak{L}(H)$ by

$$P = \int_{-\beta}^{\beta} dE_\lambda, \quad \text{with} \quad \beta \geqq 2\alpha, \quad (\alpha \text{ as in } (8)),$$

where $\beta \notin 2\pi\mathbb{Z}$. Let $P^\perp = 1 - P$ and set $Z = P(H)$ and $Y = P^\perp(H)$. Then $H = Z \oplus Y$ and $\dim Z < \infty$. With these notations the equation $\mathrm{Au} - F(u) = 0$, for $u \in \mathrm{dom}(A)$

is equivalent to the pair of equations

$$\begin{aligned} APu - PF(u) &= 0 \\ AP^{\perp}u - P^{\perp}F(u) &= 0. \end{aligned} \tag{9}$$

Now writing $u = Pu + P^{\perp}u = z + y \in Z \oplus Y$ we shall solve, for fixed $z \in Z$, the second equation of (9) which becomes $Ay - P^{\perp}F(z+y) = 0$. With $A_0 := A|Y$ this equation is equivalent to

$$y = A_0^{-1} P^{\perp} F(z+y). \tag{10}$$

Observe that $|A_0^{-1}| \leqq \beta^{-1}$ and $|P^{\perp}| = 1$. Also, from (8) we conclude that $|F(u) - F(v)| \leqq \alpha |u - v|$ for all $u, v \in H$. Consequently, in view of $\beta \geqq 2\alpha$, the right hand side of (10) is a contraction operator in H having contraction constant 1/2. We conclude, for fixed $z \in Z$, that the equation (10) has a unique solution $y = v(z) \in Y$. Since $(A_0^{-1} y)(t) = \int_0^t J y(s)\, ds$, we have $A_0^{-1}(Y) \subset H^1$ and therefore $v(z) \in \mathrm{dom}(A)$. Moreover, the map $z \mapsto v(z)$ from Z into Y is Lipschitz-continuous. In fact we have

$$|v(z_1) - v(z_2)| \leqq \tfrac{1}{2}\{|z_1 - z_2| + |v(z_1) - v(z_2)|\}.$$

Setting

$$u(z) = z + v(z)$$

we now have to solve the first equation of (9), namely $Az - PF(u(z)) = 0$, which in view of (10) is equivalent to the equation $Au(z) - F(u(z)) = 0$. One verifies readily that

$$\nabla g(z) = Az - PF(u(z)) \quad \text{with} \quad g(z) := f(u(z)). \tag{11}$$

It remains to find critical points of the function g, which is defined on the finite dimensional space Z.

The following observation is crucial. Since h is periodic we conclude by uniqueness that $v(z+j) = v(z)$ for every $j \in \mathbb{Z}^{2n}$ and for every $z \in Z$. Therefore $u(z+j) = u(z) + j$ and consequently

$$\nabla g(z+j) = \nabla g(z), \qquad j \in \mathbb{Z}^{2n}, \tag{12}$$

for all $z \in Z$. If $z \in Z$, we set $z = x + \xi$, with $x = [z]$ being the mean value of z. Hence $x \in \mathrm{Ker}(A)$ and $\xi \in \mathrm{Ker}(A)^{\perp} \cap Z$. Writing $z = (x, \xi)$ we conclude from (12) that $\nabla g(z) = \nabla g(x, \xi)$ is a vectorfield on $(x, \xi) \in T^{2n} \times \mathbb{R}^{2M}$, where $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$. Summarizing we have proved

Lemma 1. *The rest points of the Lipschitz-continuous vectorfield $\nabla g(z) = \nabla g(x, \xi)$ on $T^{2n} \times \mathbb{R}^{2M}$ are in one-to-one correspondence with those periodic solutions having period* 1 *of the Hamiltonian Eq.* (1) *on T^{2n} which are contractible.*

In order to find the rest points of ∇g we study the gradient flow $\frac{d}{ds} z = \nabla g(z)$ on $(x, \xi) \in T^{2n} \times \mathbb{R}^{2M}$, which, explicitely, is given by

$$\frac{d}{ds}x = -Q_0 F(u(z)) \tag{13}$$
$$\frac{d}{ds}\xi = A\xi - QF(u(z)),$$

where Q_0 is the orthogonal projection onto the constants, i.e. onto the kernel of the operator A, and where Q is the orthogonal projection onto the complement of $\ker(A)$ in Z. We denote the splitting $\xi=(\xi_+, \xi_-)\in\mathbb{R}^M\times\mathbb{R}^M$ of QAQ into the positive and negative part by:

$$A\xi = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} \tag{14}$$
$$\langle A_+\xi_+, \xi_+\rangle \geqq 2\pi|\xi_+|^2$$
$$\langle A_-\xi_-, \xi_-\rangle \leqq -2\pi|\xi_-|^2.$$

Since h is uniformly bounded, there is a constant $K>0$ such that $|F(u(z))|\leqq K$ for every $z\in Z$, and we conclude from (13) and (14), that with $\varepsilon=(2\pi-1)K^2>0$ one has

$$\frac{d}{ds}|\xi_+|^2 \geqq \varepsilon \qquad \text{if } |\xi_+|\geqq K \tag{15}$$
$$\frac{d}{ds}|\xi_-|^2 \leqq -\varepsilon \qquad \text{if } |\xi_-|\geqq K.$$

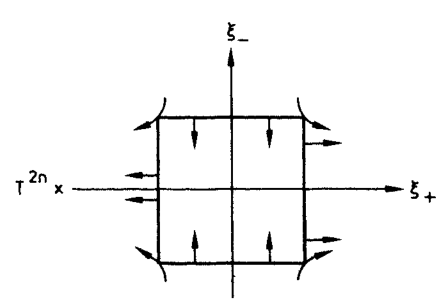
In fact,

$$\frac{d}{ds}\frac{1}{2}|\xi_+|^2 = \langle \xi_+, A_+\xi_+ - Q_+F(u(z))\rangle = \langle \xi_+, A_+\xi_+\rangle - \langle \xi_+, Q_+F(u(z))\rangle$$
$$\geqq 2\pi|\xi_+|^2 - |\xi_+|\cdot K \geqq \varepsilon.$$

Similarly,

$$\frac{d}{ds}\frac{1}{2}|\xi_-|^2 \leqq -2\pi|\xi_-|^2 + |\xi_-|\cdot K \leqq -\varepsilon,$$

and (15) follows. Clearly all the rest points of ∇g are contained in the compact set $B=T^{2n}\times D_1\times D_2$, where D_1 and D_2 are the discs of radius K, $D_1=\{\xi_+\in\mathbb{R}^M \mid |\xi_+|\leqq K\}$ and $D_2=\{\xi_-\in\mathbb{R}^M \mid |\xi_-|\leqq K\}$. Moreover $B^-:=T^{2n}\times\partial D_1\times D_2$ is the exit set of B and $B^+=T^{2n}\times D_1\times\partial D_2$ is the entrance set, so that B is an isolating block in the sense of [6].



Summarising we have proved:

Lemma 2. *The compact set* $B = T^{2n} \times D_1 \times D_2$ *is an isolating block for the flow of* ∇g *with exit set* $B^- = T^{2n} \times \partial D_1 \times D_2$ *and with entrance set* $B^+ = T^{2n} \times D_1 \times \partial D_2$.

We shall prove that every continuous flow which admits the above very special isolating block (B, B^-, B^+) contains at least $2n+1$ rest points in B, provided the flow on the invariant set in B is gradient like.

4. Two Statements for Flows

We consider, more generally, on $M = T^m \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ with coordinates (x, y_1, y_2) a continuous flow, which is defined in an open neighborhood of the compact set $B = T^m \times D_1 \times D_2$, where D_1 and D_2 are discs, $D_1 = \{y_1 \in \mathbb{R}^{N_1} \,|\, |y_1| \leqq K_1\}$ and $D_2 = \{y_2 \in \mathbb{R}^{N_2} \,|\, |y_2| \leqq K_2\}$. If $\gamma \in M$ and if $\varphi^t(\gamma)$ is the orbit of the flow through $\gamma = \varphi^0(\gamma)$ we shall write $\varphi^t(\gamma) = \gamma \cdot t$. For an interval $J \subset \mathbb{R}$ we set $\gamma \cdot J = \{\gamma \cdot t \,|\, t \in J\}$. The invariant set contained in B is defined to be

$$S = \{\gamma \in B \,|\, \gamma \cdot \mathbb{R} \subset B\}.$$

Recall the

Definition. Let $\check{H}^*(X)$ be the Alexander cohomology of a compact topological space X with real coefficients. Then the cup long of X is defined as

$$l(X) = 1 + \sup\{k \in \mathbb{N} \,|\, \exists \alpha_1, \ldots, \alpha_k \in H^*(X) \setminus 1 \quad \text{with} \quad \alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_k \neq 0\},$$

and $l(X) = 1$ if no such class exists.

Theorem 4. *Assume a continuous flow on* $T^m \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ *admits the isolating block* $B = T^m \times D_1 \times D_2$ *with exit set* $B^- := T^m \times \partial D_1 \times D_2$ *and with entrance set* $B^+ := T^m \times D_1 \times \partial D_2$. *Let* S *be the invariant set in* B, *then:*

$$l(S) \geqq l(B) = l(T^m) = m + 1.$$

We need a Lemma and define by means of the flow the compact sets

$$\begin{aligned} A^+ &:= \{\gamma \in B \,|\, \gamma \cdot \mathbb{R}^+ \subset B\} \\ A^- &:= \{\gamma \in B \,|\, \gamma \cdot \mathbb{R}^- \subset B\}. \end{aligned} \tag{16}$$

It then follows, that

Lemma 3 (Wazewski's Principle). *Assume* B *is an isolating block with exit set* B^- *and entrance set* B^+. *Then* B^+ *is a strong deformation retract of* $B \setminus A^-$ *and* B^- *is a strong deformation retract of* $B \setminus A^+$.

Proof. The proof follows immediately from the definition of an isolating block. In fact, in order to prove that B^- is a strong deformation retract of $B \setminus A^+$ define the continuous function $\tau^+ : B \setminus A^+ \to \mathbb{R}$ by setting $\tau^+(x) = \sup\{t \,|\, x \cdot [0, t]$

$\subset B\}$. Then $\tau^+(x)=0$ if and only if $x\in B^-$. The deformation retraction F: $(B\backslash A^+)\times[0,1]\to B\backslash A^+$ is simply given by $F(x,s)=x\cdot\{s\tau^+(x)\}$. The other part of the lemma is proved similarly. □

Proof of Theorem 4. Note that $\check{H}^*(B)=\check{H}^*(T^m)$, since $B=T^m\times D_1\times D_2$. Let $\check{x}_1,\dots,\check{x}_m$ be the classes in $\check{H}^*(B)$ which correspond to the de Rham classes dx_i on T^m. Then $\check{\vartheta}:=\check{x}_1\cup\check{x}_2\cup\dots\cup\check{x}_m\neq 0$ in $\check{H}^*(B)$ since $dx_1\wedge\dots\wedge dx_m$ is a volume form on T^m. We shall show that $\check{\vartheta}$ maps nontrivially to $\check{H}^*(S)$ under the inclusion induced map i^*: $\check{H}^*(B)\to\check{H}^*(S)$. The statement $l(S)\geqq m+1$ then follows.

To see this consider the following two diagrams, in which H^* and H_* denote the singular cohomology and homology respectively with real coefficients. Observe that $\partial B=B^+\cup B^-$, and

$$(B\backslash A^-)\cup(B\backslash A^+)=B\backslash(A^-\cap A^+)=B\backslash S.$$

Observe also, that the compact set S is contained in the interior of B.

$$\begin{tikzcd}
H^*(B,B^+) \otimes H^*(B,B^-) \arrow[r, "\cup"] & H^*(B,\partial B) \arrow[r, phantom, "\cong"] & H_*(B,\partial B) \arrow[r, "\check{D}"] \arrow[d, "j_*"] & \check{H}^*(B) \arrow[d, "i^*"] \\
H^*(B,B\backslash A^-) \otimes H^*(B,B\backslash A^+) \arrow[u, "i_1^*\otimes i_2^*"] \arrow[r, "\cup"] & H^*(B,B\backslash S) \arrow[u, "j^*"] \arrow[r, phantom, "\cong"] & H_*(B,B\backslash S) \arrow[r, "\check{D}"] & \check{H}^*(S).
\end{tikzcd}$$

The vertical maps are inclusion induced. The maps i_1^* and i_2^* are isomorphisms, by Lemma 3. The isomorphism $\check{D}$ is the Alexander duality map: $\check{D}$: $H_j(B,\partial B)\to\check{H}^{d-j}(B)$, with $d=m+N_1+N_2$.

We shall find cohomology in $H^*(S)$ by means of the following argument. Let $\alpha\in H^*(B,B^+)$ and $\beta\in H^*(B,B^-)$ such that $\alpha\cup\beta\neq 0$ in $H^*(B,\partial B)$. Then $\alpha=i_1^*(\alpha^*)$ for some $\alpha^*\in H^*(B,B\backslash A^-)$ and $\beta=i_2^*(\beta^*)$ for some β^* in $H^*(B,B\backslash A^+)$, since i_1^* and i_2^* are isomorphisms. Hence $\alpha\cup\beta=j^*(\alpha^*\cup\beta^*)$. If now $\alpha\cup\beta\in H^j(B,\partial B)$ then, going to the dual spaces (in the sense of vector spaces), there is a $\mu\in H_j(B,\partial B)$ such that $j_*(\mu)=:\mu^*\neq 0$ in $H_j(B,B\backslash S)$. Consequently, the Alexander dual $\check{D}(\mu)\in\check{H}^{d-j}(B)$ is maped onto $i^*(\check{D}(\mu))=\check{D}(\mu^*)\neq 0$ in $\check{H}^{d-j}(S)$. If, in particular, $j=N_1+N_2$, then $d-j=m$ and $\check{D}(\mu)\in\check{H}^m(B)\cong\check{H}^m(T^m)$ which is generated by $\check{\vartheta}=\check{x}_1\cup\dots\cup\check{x}_m$. Consequently $i^*(\check{\vartheta})=i^*(\check{x}_1\cup\dots\cup\check{x}_m)\neq 0$ in $\check{H}^m(S)$ and so $l(S)\geqq l(T^m)=m+1$ as claimed in the theorem.

In order to carry out this argument concretely we first recall that

$$H^*(B,B^+)\cong H^*(T^m)\otimes H^*(D_2,\partial D_2)\otimes H^*(D_1) \quad\text{and}$$
$$H^*(B,B^-)\cong H^*(T^m)\otimes H^*(D_2)\otimes H^*(D_1,\partial D_1),$$

also

$$\begin{aligned}H^*(B,\partial B)&\cong H^*(T^m)\otimes H^*(D_2,\partial D_2)\otimes H^*(D_1,\partial D_1)\\&\cong H^*(T^m)\otimes H^*(D_1\times D_2,(D_1\times\partial D_2)\cup(\partial D_1\times D_2)).\end{aligned}$$

We choose now α to be the image $\hat{\xi}$ in $H^*(B,B^+)$ of the generator ξ of $H^{N_1}(D_2,\partial D_2)\cong H^{N_1}(S^{N_1},*)$, and we choose β to be the image $\hat{\eta}$ in $H^*(B,B^-)$ of the generator η of $H^{N_2}(D_1,\partial D_1)\cong H^{N_2}(S^{N_2},*)$. Then $\alpha\cup\beta$ is equal to the image

$\hat{\zeta}$ in $H^*(B,\partial B)$ of the generator ζ of $H^{N_1+N_2}((D_1,\partial D_1)\times(D_2,\partial D_2))$, and so $\alpha\cup\beta \neq 0$ in $H^{N_1+N_2}(B,\partial B)$, so the theorem follows, by the above argument. □

Definition. Let S be a compact invariant set of a continuous flow. A Morse decomposition of S is a finite collection $\{M_p\}_{p\in P}$ of disjoint, compact and invariant subsets of S, which can be ordered, say $(M_1, M_2, \ldots, M_k)$, $k=|P|$, so that the following property holds true. If

$$\gamma\in S\setminus\bigcup_{p\in P} M_p,$$

then there is a pair of indices $i<j$ such that the positive $(t\to+\infty)$ and negative $(t\to-\infty)$ limit sets $\omega(\gamma)$ and $\omega^*(\gamma)$ of γ satisfy:

$$\omega(\gamma)\subset M_i \quad\text{and}\quad \omega^*(\gamma)\subset M_j.$$

Theorem 5. *Let S be any compact invariant set of a continuous flow, with Morse-decomposition $\{M_p\}_{p\in P}$. Then*

$$l(S)\leqq\sum_{p\in P} l(M_p).$$

In particular, if $|P|<l(S)$, then some M_p has non-trivial Alexander cohomology (so contains a continuum of points).

Postponing the proof of this theorem we first derive the corollary:

Corollary. *Let S be a compact invariant set of a continuous flow. Assume, in addition, that S is gradient like (i.e. there exists a continuous real valued function on S which is strictly decreasing on non-constant orbits). Then S contains at least $l(S)$ rest points.*

In fact, assume there are only finitely many rest points in S. Then they form a Morse-decomposition of S, since S is assumed to be gradient like. As none of the sets of this decomposition has nontrivial cohomology, hence $l(M_p)=1$, there must be at least $l(S)$ sets in the decomposition, hence $l(S)$ rest points, proving the corollary. It remains to prove Theorem 5.

Proof of Theorem 5. First observe that any decomposition of S can be obtained by first decomposing it into two sets, then decomposing one of these and continuing until the decomposition is reached. Therefore one needs only prove the theorem for decompositions into two sets. Thus let (M_1, M_2) be an ordered Morse-decomposition of S. From the definition we conclude that there is a compact neighborhood S_1 of M_1 in S and a compact neighborhood S_2 of M_2 in S with $S_1\cup S_2=S$ and such that

$$M_1=\bigcap_{t>0} S_1\cdot t \quad\text{and}\quad M_2=\bigcap_{t>0} S_2\cdot(-t).$$

Consequently, by the continuity property of the Alexander cohomology, $\check{H}^*(S_1)=\check{H}^*(M_1)$ and $\check{H}^*(S_2)=\check{H}^*(M_2)$, and it is therefore sufficient to prove that $l(S_1)+l(S_2)\geqq l(S)$ for $S_1\cup S_2=S$. This will follow from the following general observation:

Lemma 4. *Let $S_1 \cup S_2 \subset S$ be three compact sets. Denote by $i_1: S_1 \to S$, $i_2: S_2 \to S$ and $i: S_1 \cup S_2 \to S$ the inclusion maps. Let $\alpha, \beta \in \check{H}^*(S)$. Then $i_1^* \alpha = 0$ and $i_2^* \beta = 0$ imply $i^*(\alpha \cup \beta) = 0$.*

Proof. Consider the following diagram

$$\begin{array}{ccccc}
\check{H}^*(S, S_1) & \otimes & \check{H}^*(S, S_2) & \xrightarrow{\cup} & \check{H}^*(S, S_1 \cup S_2) \\
\downarrow{\scriptstyle j_1^*} & & \downarrow{\scriptstyle j_2^*} & & \downarrow{\scriptstyle j^*} \\
\check{H}^*(S) & \otimes & \check{H}^*(S) & \xrightarrow{\cup} & \check{H}^*(S) \\
\downarrow{\scriptstyle i_1^*} & & \downarrow{\scriptstyle i_2^*} & & \downarrow{\scriptstyle i^*} \\
\check{H}^*(S_1) & \otimes & \check{H}^*(S_2) & \xrightarrow{\cup} & \check{H}^*(S_1 \cup S_2).
\end{array}$$

The vertical sequences are exact. If $\alpha \in \check{H}^*(S)$ satisfies $i_1^*(\alpha) = 0$ then there is an $\hat{\alpha} \in \check{H}^*(S, S_1)$ with $j_1^*(\hat{\alpha}) = \alpha$. Similarly, if $i_2^*(\beta) = 0$, then $j_2^*(\hat{\beta}) = \hat{\alpha}$ for some $\hat{\beta} \in \check{H}^*(S, S_2)$. Since $j^*(\hat{\alpha} \cup \hat{\beta}) = j_1^*(\hat{\alpha}) \cup j_2^*(\hat{\beta}) = \alpha \cup \beta$ we conclude, by exactness, $i^*(\alpha \cup \beta) = i^* \circ j^*(\hat{\alpha} \cup \hat{\beta}) = 0$, as advertized. □

Going back to the proof of Theorem 5 we let $\alpha_1, \alpha_2, \ldots, \alpha_l$ be in $H^*(S)$ such that $\alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_l \neq 0$. Let the α's be ordered so that $\alpha_1 \cup \ldots \cup \alpha_r$ is the longest product not in the kernel of i_1^*. Therefore $l(S_1) \geqq r+1$ and $i_1^*(\alpha_1 \cup \ldots \cup \alpha_r \cup \alpha_{r+1}) = 0$. Since $S = S_1 \cup S_2$ it follows from Lemma 4, that $i_2^*(a_{r+2} \cup \ldots \cup \alpha_l) \neq 0$ and therefore $l(S_2) \geqq l - (r+1) + 1 = l - r$. Hence $l(S_1) + l(S_2) \geqq l + 1$. We have shown that if S admits a nontrivial product with l factors, i.e. if $l(S) \geqq l+1$, then $l(S_1) + l(S_2) \geqq l+1$, so the statement of the theorem follows. □

5. Proof of Theorem 1

By Lemma 1 the 1-periodic solutions, which are contractible on T^{2n}, are in one-to-one correspondence with the rest points of the gradient flow of ∇g on $T^{2n} \times \mathbb{R}^{2M}$. If S is the invariant set of this flow in the block B of Lemma 2 we conclude, by the Corollary of Theorem 5, together with Theorem 4, that S contains at least $2n+1$ rest points. □

6. Proof of Theorem 3

The proof proceeds along the same lines as that of Theorem 1 and we sketch the essential points. On the manifold $M = T^n \times \mathbb{R}^n$ the Hamiltonian vectorfield is given by the Hamiltonian function $h(t, x, y)$, on the covering space $\mathbb{R}^n \times \mathbb{R}^n$. The function h is periodic of period 1 in t and in the x-variables. We look again for special periodic solutions which are, on $\mathbb{R}^n \times \mathbb{R}^n$, given by $(x(t), y(t))$, where both $x(t)$ and $y(t)$ are periodic of period 1. Hence, on T^n, the loop

$t \mapsto x(t)$ is contractible. These periodic solutions are the critical points of f as defined in (4), with x replaced this time by (x, y). We write

$$\begin{aligned} h(t,x,y) &= \tfrac{1}{2}\langle y, by\rangle + \{h(t,x,y) - \tfrac{1}{2}\langle y, by\rangle\} \\ &= h_1(y) + h_0(t,x,y). \end{aligned}$$

It follows from the assumption of Theorem 3 that $|\nabla h_0| \leqq K$. We proceed now as in the proof of Theorem 1. The Eq. (6) looks as follows:

$$Au = F_1(u) + F_0(u) = F(u),$$

where $F_1(u)(t) = \nabla h_1(u(t))$ and $F_0(u)(t) = \nabla h_0(t, u(t))$. Moreover $|F_0(u)| \leqq K$ for all $u \in H$. The operator $F_1(u)$ is a bounded linear operator of the Hilbertspace H:

$$F_1(u) := Bu, \qquad Bu(t) = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} u(t).$$

Since $|h''(t,x,y)| \leqq \alpha$ for some $\alpha > 0$, the sought periodic solutions are again found as the critical points of a function g defined on a finite dimensional space Z. We find

$$\nabla g(z) = Az - Bz - PF_0(u(z)).$$

Here we have used, that B commutes with the projection P of the Hilbert space H onto Z. The vectorfield $\nabla g(z)$ is this time a vectorfield on $T^n \times \mathbb{R}^n \times \mathbb{R}^{2M}$. More specifically we can use as coordinates in Z the Fourier coefficients up to order $N = [\beta]$, with β as in the proof of Theorem 1. If $z(t) = (x(t), y(t))$ we thus have

$$\begin{aligned} x(t) &= x_0 + \sum_{n=1}^{N} (\alpha_n \varphi_n(t) + \beta_n \psi_n(t)) \\ y(t) &= y_0 + \sum_{n=1}^{N} (a_n \varphi_n(t) + b_n \psi_n(t)), \end{aligned}$$

where $\varphi_n(t) = \sin(2\pi n t)$ and $\psi_n(t) = \cos(2\pi n t)$. The meanvalue $x_0 = [x(t)]$ is the variable on the torus T^n. In these coordinates, the gradient equation $\frac{d}{ds} z = \nabla g(z)$ becomes, if we omit the nonlinear term $PF_0(u(z))$, which is uniformly bounded:

$$\frac{d}{ds} x_0 = 0$$

$$\frac{d}{ds} y_0 = -b y_0$$

$$\frac{d}{ds}\begin{pmatrix} \alpha_n \\ \beta_n \\ a_n \\ b_n \end{pmatrix} = \left(\begin{array}{c|c} 0 & 2\pi n J \\ \hline 2\pi n \quad J^T & \begin{matrix} -b & 0 \\ 0 & -b \end{matrix} \end{array}\right) \begin{pmatrix} \alpha_n \\ \beta_n \\ a_n \\ b_n \end{pmatrix};$$

$1 \leqq n \leqq N$. Since, by assumption, $\det b \neq 0$, we see from this representation, that the vectorfield transversal to the torus is hyperbolic, if we omit the nonlinear terms. Since the omitted terms are uniformly bounded, we can construct therefore, as in Lemma 2, a special isolating block B for the gradient flow $\nabla g(z)$ which is of the form: T^n times hyperbolic i.e. $B = T^n \times D_1 \times D_2$ with exit set $B^- = T^n \times \partial D_1 \times D_2$ and entrance set $B^1 = T^n \times D_1 \times \partial D_2$. Application of the Corollary to Theorem 5 together with Theorem 4 yields $(n+1)$ critical points of g, which give rise to $(n+1)$ periodic solutions. We claim that the periodic solutions found are contained in the region $T^n \times D \subset T^n \times \mathbb{R}^n$, where D is as in the assumption of Theorem 3. In fact, on $T^n \times (\mathbb{R}^n \setminus D)$ the Hamiltonian system is integrable;

$$\dot{x} = \frac{\partial}{\partial y} h(t,x,y) = b y + a$$

$$\dot{y} = -\frac{\partial}{\partial x} h(t,x,y) = 0.$$

Hence the tori $T^n \times \{y\}$ are invariant under the flow, and the restriction of the flow onto a torus $T^n \times \{y\}$ is in fact linear: φ^t: $(x,y) \mapsto (x + t(by+a), y)$. In particular, the periodic solutions are not described by periodic functions on $\mathbb{R}^n \times (\mathbb{R}^n \setminus D)$ and hence do not count. Therefore the periodic solutions found above must lie in $T^n \times D$. The proof of Theorem 3 is finished. $\square$

7. Proof of Theorem 2

We shall make use of the Morse-theory for flows as represented in [7]. In order to briefly outline the result we need, we consider a continuous flow on a locally compact and metric space X. A compact and invariant subset $S \subset X$ is called isolated, if it admits a compact neighborhood N such that S is the maximal invariant subset which is contained in N. With an isolated invariant set S a pair (N_1, N_2) of compact spaces can be associated, where $N_2 \subset N_1$ is roughly the "exit set" of N_1 and where $S \subset \operatorname{int}(N_1 \setminus N_2)$ is the invariant set contained in N_1. The homotopy type of the pointed space $(N_1/N_2, *)$ then does not depend on the particular choice of the "index-pair" (N_1, N_2) for S, and is called the index of S. It is denoted by $h(S) = [(N_1/N_2, *)]$. The algebraic invariants of $h(S)$ are defined to be

$$p(t, h(S)) := \sum_{j \geqq 0} t^j \dim \check{H}^j(N_1, N_2),$$

where (N_1, N_2) is any index-pair for S. Let now $\{M_1, \ldots, M_k\}$ be an ordered Morse-decomposition of S. Then the relation between the algebraic invariants of $h(M_k)$ and those of $h(S)$ is described by the following Morse-inequalities (see [7], Theorem 3.3):

$$\sum_{j=1}^{k} p(t, h(M_j)) = p(t, h(S)) + (1+t)\,Q(t), \tag{17}$$

where Q is a formal power series with non-negative integer coefficients.

We shall apply this equation to the flow of ∇g on $T^{2n}\times\mathbb{R}^N\times\mathbb{R}^N$, with S being the set of bounded solutions. Then S is compact, since by (13), $|\nabla g(z)|\geqq\varepsilon$ for all $z=(x,\xi)\in T^{2n}\times\mathbb{R}^{2N}$ with $|\xi|\geqq K$. In particular S is contained in the interior of the compact set $B:=T^{2n}\times D_1\times D_2$, if we choose the radii of the discs sufficiently big. Moreover, by Lemma 2, the compact pair (B,B^-), with the exit set $B^-:=T^{2n}\times\partial D_1\times D_2$, is an index pair for S in the sense of ([7], Definition 3.4). Therefore $h(S)=[(B/B^-,*)]$ and the algebraic invariants of $h(S)$ are easily computed. Namely

$$\check{H}^*(h(S))\cong\check{H}(B/B^-,*)\cong\check{H}^*(B,B^-)\cong\check{H}^*(T^{2n}\times D_1,T^{2n}\times\partial D_1)$$

which, by the Künneth-formula, is isomorphic to $\check{H}^*(T^{2n})\otimes\check{H}^*(D_1,\partial D_1)$. As $\check{H}^*(D_1,\partial D_1)\cong\check{H}^*(S^N,*)$, S^N being a sphere of dimension N, we conclude $\check{H}^j(h(S))\cong\check{H}^{j-N}(T^{2n})$. Consequently, the algebraic invariants of $h(S)$ are given by

$$p(t,h(S))=\sum_{j=0}^{2n}\binom{2n}{j}t^{N+j}, \tag{18}$$

since $\dim\check{H}^j(T^{2n})=\binom{2n}{j}$. Recall Lemma 1 and assume that all the periodic solutions having period 1 are nondegenerate. In this case it can be shown ([7], Lemma 2.6) that the function g is a Morse-function, hence has only non-degenerate critical points. Their number is finite, since the critical points are contained in the compact set B. Therefore the critical points $\{z_j\}=M_j$ can be labeled so that they form an ordered Morse-decomposition of S. It is easily seen ([7], Sect. 3.6), that $h(\{z_j\})=[(S^{d_j},*)]$, where d_j is the Morse-index of the critical point z_j. Consequently $p(t,h(\{z_j\})=t^{d_j}$. Hence, by (18), the Morse-inequalities look as follows:

$$\sum_{j=1}^{k}t^{d_j}=\sum_{j=0}^{2n}\binom{2n}{j}t^{j+N}+(1+t)\,Q(t). \tag{19}$$

Since the polynomial Q has nonnegative integer coefficients we conclude in particular that indeed $k\geqq\sum_{j=0}^{2n}\binom{2n}{j}=2^{2n}$, as we wanted to prove. □

8. Appendix

We prove the following statement which we found in V.I. Arnold's book [2], Appendix 9, without proof however. The idea of the proof was suggested to us by J. Mather.

Theorem 6. *If ψ is a symplectic C^∞-diffeomorphism of $T^2=\mathbb{R}^2/\mathbb{Z}^2$ then the following statements are equivalent.*

(i) *ψ is, on $\mathbb{R}^2$, of the form $x\mapsto x+f(x)$ with f being periodic and $[f]=0$.*

(ii) *ψ is generated by a globally Hamiltonian vectorfield on T^2.*

(iii) *ψ belongs to the commutator subgroup of $\mathrm{Diff}_0^\infty(T^2,\omega)$, the identity component of the group $\mathrm{Diff}^\infty(T^2,\omega)$ of symplectic diffeomorphisms of T^2.*

The statement (ii)$\Leftrightarrow$(iii) is a special case of a result due to A. Banyaga [4]. In order to prove (i)$\Leftrightarrow$(ii) we begin with a Lemma, which is due to J. Moser.

Let M be a compact symplectic manifold. Fixing a symplectic form ω_0 on M we consider the set $\Omega = \{\text{symplectic form } \omega \text{ on } M \mid \int_c \omega = \int_c \omega_0 \text{ for all 2-cycles } c \text{ on } M\}$.

Lemma 5. *Let $s \mapsto \omega_s \in \Omega$, $s \in [0,1]$ be a closed curve which is contractible to ω_0 in Ω, i.e. there exist $\omega_{st} \in \Omega$, $s, t \in [0,1]$ with $\omega_{s0} = \omega_0$, $\omega_{s1} = \omega_s$ for $s \in [0,1]$, and $\omega_{0t} = \omega_{1t} = \omega_0$ for $t \in [0,1]$. Then there exists a closed curve $s \mapsto \varphi_s \in \mathrm{Diff}^\infty(M)$ satisfying*

$$\varphi_s^* \omega_s = \omega_0 \quad \text{and} \quad \varphi_0 = \varphi_1 = \mathrm{id}.$$

Remark. For $\dim M = 2$ every closed curve ω_s meets the assumption of the Lemma. In fact, since Ω is convex we can set $\omega_{st} = t\omega_s + (1-t)\omega_0 \in \Omega$. For $\dim M > 2$ the assumption is met, if the component of Ω containing ω_0 is simply connected. When this is the case is not known to us.

Proof. We follow [10]. By the Hodge decomposition theorem with respect to a given metric on M we have the representation $\frac{d}{dt}\omega_{st} = d\alpha_{st} + h_{st}$. The 2-form h_{st} is harmonic. We shall require that $\alpha_{st} = \delta\beta_{st}$ so that the choice of α_{st} is unique. Since by assumption the periods of ω_{st} are independent of t, those of $\frac{d}{dt}\omega_{st}$ are zero and so $h_{st} = 0$. Thus

$$\frac{d}{dt}\omega_{st} = d\alpha_{st}, \tag{20}$$

where, by the above normalization, α_{st} is unique. As $\omega_{1t} = \omega_{0t} = \omega_0$ we therefore have $\alpha_{0t} = \alpha_{1t} = 0$. Let V_{st} be the unique vectorfield satisfying

$$\omega_{st}(V_{st}, \cdot) = -\alpha_{st}, \tag{21}$$

and let φ_{st} be the flow

$$\frac{d}{dt}\varphi_{st} = V_{st} \circ \varphi_{st}, \qquad \varphi_{s0} = \mathrm{id}. \tag{22}$$

Since $d\omega_{st} = 0$ one finds with (20) and (21)

$$\begin{aligned}\frac{d}{dt}(\varphi_{st}^* \omega_{st}) &= \varphi_{st}^* \left\{\frac{d}{dt}\omega_{st} + d(\omega_{st}(V_{st}, \cdot))\right\} \\ &= \varphi_{st}^* \left\{\frac{d}{dt}\omega_{st} - d\alpha_{st}\right\} = 0.\end{aligned}$$

Hence $\varphi_{st}^* \omega_{st} = \omega_{s0} = \omega_0$. Also, from $V_{0t} = V_{1t} = 0$, we conclude $\varphi_{1t} = \varphi_{0t} = \mathrm{id}$. Therefore $\varphi_s = \varphi_{s1}$ is the desired loop of diffeomorphisms. □

Consider now $M = T^{2n}$ and let ω_0 be the standard symplectic form on T^{2n}. Define the subgroups A, A_1, A_2 of $\mathrm{Diff}^\infty(M)$ as follows

$$\begin{aligned}A &= \{\psi \in \mathrm{Diff}^\infty(M) \mid \psi \text{ homotopic to id}\} \\ A_1 &= \{\psi \in A \mid \psi^* \omega_0 = \omega_0\} \\ A_2 &= \{\psi \in \mathrm{Diff}^\infty(M) \mid \psi^* \omega_0 = \omega_0 \text{ and, on } R^{2n},\ \psi(x) = x + p(x) \\ &\qquad \text{with } [p] = 0\}.\end{aligned}$$

Then clearly $A \supset A_1 \supset A_2$.

Lemma 6. *If $M=T^2$, then A_1 and also A_2 are connected by smooth arcs.*

Proof. It is a nontrivial fact, that, since $\dim T^2=2$, the group of diffeomorphisms of T^2 which are homotopic to the identity is equal to the one component of $\mathrm{Diff}^\infty(T^2)$, which is connected by smooth arcs, see C.J. Earle and J. Eells [13]. If $\psi\in A_1$ we take a smooth arc $\chi_s\in A$ with $\chi_0=\mathrm{id}$ and $\chi_1=\psi$ and set $\omega_s=\chi_s^*\omega_0\in\Omega$ so that $\omega_s=\omega_0$ for $s=0,1$. By Lemma 5 there is φ_s with $\varphi_s^*\omega_s=\omega_0$ and $\varphi_0=\varphi_1=\mathrm{id}$. Therefore $\psi_s=\varphi_s\circ\chi_s\in A_1$ and $\psi_0=\mathrm{id}$, $\psi_1=\psi$.

If, moreover $\psi\in A_2$, then ψ_s: $x\mapsto x+a_s+p_s(x)$, $a_s\in\mathbb{R}^2$ and $a_0=a_1=0$, $[p_s]=0$. We set τ_s: $x\mapsto x-a_s$, then $\tilde\psi_s=\tau_s\circ\psi_s\in A_2$ and $\tilde\psi_0=\mathrm{id}$, $\tilde\psi_1=\psi_1=\psi$. □

In view of this lemma, the statement (i)$\Leftrightarrow$(ii) is an immediate consequence of the following simple

Lemma 7. *A smooth arc $\psi_t\in A_1$ with $\psi_0=\mathrm{id}$ is the flow of a globally Hamiltonian vectorfield on T^{2n} if and only if $\psi_t\in A_2$.*

Proof. Let $\psi_t\in A_2$, then ψ_t: $x\mapsto x+p_t(x)$ with $[p_t]=0$. Since $\psi_t^*\omega_0=\omega_0$ we conclude that the vectorfield $\left(\frac{d}{dt}\psi_t\right)\circ\psi_t^{-1}$ is of the form:

$$\left(\frac{d}{dt}\psi_t\right)\circ\psi_t^{-1}=J\nabla h.$$

$\nabla h(t,x)$ is periodic in x, so that $h(t,x)=\langle x,c(t)\rangle+h_1(t,x)$, with h_1 periodic in x. But $[\dot p_t]=0$, hence $[\dot p_t\circ\psi_t^{-1}]=[\dot p_t]=0$ and consequently $c(t)=0$, so that indeed $h=h_1$ is periodic in x. Conversely, if ψ_t is the flow of the globally Hamiltonian vectorfield $J\nabla h$ on T^{2n}, then $\psi_t^*\omega_0=\omega_0$ and $\psi_t\in A_2$. □

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Note Added in Proof

We point out that M. Chaperon used the method introduced above to estimate the number of intersection points of Lagrangian submanifolds. He proved, for example, the following statement for an n-dimensional torus $M=T^n$. If $j: I\times M\to T^*M$ is an exact Lagrangian isotopy such that $j_0(M)$ is the zero section Σ of $T^*M\to M$, then $\Sigma\cap j_1(M)$ contains at least $n+1$ points.

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