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Nilpotent blocks and their source algebras

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§1. Introduction

1.1. Let p be a prime number, k an algebraically closed field of characteristic p and $\mathcal{O}$ a complete discrete valuation ring with residue field k. In [3] Broué and ourself commenced the study of a block b of a finite group G from a new point of view, the hypothesis being no longer on the structure of a defect group P of b but on the kind of embedding of P in $\mathcal{O}Gb$, namely on the so called "local structure" of b which was implicitely represented in [3] by the equivalent class of the *Brauer category* (see [1] for a formal definition). Precisely, assuming that for any subgroup Q of P and any block f of $C_G(Q)$ associated with b, the quotient $N_G(Q,f)/C_G(Q)$ of the stabilizer of (Q,f) in G by the centralizer of Q is a p-group, we determined (up to signs) the full matrix of generalized decomposition numbers of b, regardless the structure of P.

1.2. In [9] we modified the notion of "local structure" in order to enlarge its area of application to any interior G-algebra (actually, to any G-algebra) and in particular, to the full algebra of $\mathcal{O}$-endomorphisms of any $\mathcal{O}G$-module: the "local structure" of b was implicitely represented in [9] by the equivalent class of the so called *local category* (see [11] for a formal definition), which is in general finer than the Brauer one. As in the $\mathcal{O}G$-module case, the concept of "source" arises naturally in the new context, and the source of the interior G-algebra $\mathcal{O}Gb$ – which is an interior P-algebra – turns out to be a (the most?) powerful invariant of the block b (see for instance the introduction in [11]).

1.3. The main purpose of this paper is to give the structure of the source algebras of the blocks considered in [3], but to be coherent with our approach in [9] we with state our hypothesis on b in terms of the local category: we will assume indeed that the quotient $N_G(Q_\delta)/C_G(Q)$ is a p-group for any local pointed group Q_δ on $\mathcal{O}Gb$. This change of point of view does not change the blocks that we consider since in both cases

(1.3.1) *Brauer and local categories coincide.*

Actually, it would be not surprising that they coincide *only* in these blocks, as we explain in 1.9 below.

1.4. Our main theorem shows in particular that for these blocks (see 1.8 below)

(1.4.1) *the $\mathcal{O}$-algebra $\mathcal{O}Gb$ is isomorphic to a full matrix algebra over $\mathcal{O}P$,*

proving a conjecture stated by Broué in a lecture at Yale University in May 1978 (when P is abelian, a proof of (1.4.1) was already in [3]). But our result is more precise that (1.4.1) since it describes the source algebra as an *interior P-algebra* – not only as an $\mathcal{O}$-algebra – which allows us, for instance, to compute the generalized decomposition numbers of b (see (1.12.3) below). The first time we conjectured such a description was in the Midwest group theory seminar at Chicago University in April 1979. In June 1981, in an Oberwolfach meeting, we announced (and issued a preprint [10] on) a complete proof of the main theorem below (provided that $\mathcal{O}$ was "big enough"), where we made use of a consequence of the main result in [3]. From that time the underlying ideas have been developed (see the introductions to Sects. 4 and 5) and the proof we present here takes advantage of a better understanding on what is going on (although the steps are essentially the same as in [10]). The main difference from [10] is that here we do not need to quote [3], which allows us to supply a *new proof* of the main result in [3]: actually we improve this result since we do not assume that $\mathcal{O}$ contains the group of $|G|_p$-roots of unity (see remark 1.14 below).

1.5. In the sequel we will freely use notation and terminology introduced in Sect. 2. Consider $\mathcal{O}G$ as interior G-algebra and set $\alpha = \{b\}$, so that α is a *point* of G on $\mathcal{O}G$. Let P_γ be a defect pointed group of G_α and denote by $\tilde{e}\colon B \to \mathrm{Res}_P^G(\mathcal{O}Gb)$ an embedded algebra associated with P_γ (as a pointed group on $\mathcal{O}Gb$). In this paper we will prove the following statement.

1.6. Main theorem. *With the notation above, the following two conditions on α are equivalent:*

(1.6.1) *For any local pointed group Q_δ on $\mathcal{O}G$ such that $Q_\delta \subset G_\alpha$, the quotient $N_G(Q_\delta)/C_G(Q)$ is a p-group.*

(1.6.2) *There is an $\mathcal{O}$-simple P-algebra S such that $B \cong SP$ as interior P-algebras.*

In that case, the P-algebra S is unique up to isomorphism and has a P-stable $\mathcal{O}$-basis which contains the unity as the unique P-fixed element.

1.7. In Sect. 3 we will prove by the so called "local methods" – which just involve induction arguments on the partially ordered set of local pointed groups on $\mathcal{O}Gb$ and the analogous statement to Sylow's theorem – that (1.6.1) is equivalent to each of the following conditions on α:

(1.7.1) *For any b-Brauer pair (Q,f), the quotient $N_G(Q,f)/C_G(Q)$ is a p-group.*

(1.7.2) *For any local pointed group Q_δ on $\mathcal{O}G$ such that $Q_\delta \subset P_\gamma$ and any element x of G such that $(Q_\delta)^x \subset P_\gamma$ we have $x = zu$ where $z \in C_G(Q)$ and $u \in P$.*

The equivalence between (1.6.1) and (1.7.1) shows that the blocks considered in the main theorem above are exactly the *nilpotent blocks* in [3]. Condition (1.7.2) should be considered as the genuine definition of a *nilpotent block* (or *block with nilpotent local structure*): it is the condition on α which plays an effective rôle in the proof of (1.6.2). On the other hand, this condition has several "local consequences" which are actually equivalent to (1.7.2), condition (1.6.1) being just one of them (which we have chosen to state the main theorem by evident historical reasons).

1.8. Let us show now that (1.6.2) and the last statement in 1.6 imply (1.4.1). Indeed, they imply that $\mathrm{rank}_{\mathcal{O}}(S) \equiv 1 \bmod (p)$ and therefore that (see 6.2 below for a more detailed argument)

(1.8.1) *there is a unique group homomorphism $\varrho: P \to S^*$ lifting the action of P on S such that $\det(\varrho(u)) = 1$ for any $u \in P$; in particular, there is a unique $\mathcal{O}$-algebra isomorphism $SP \cong S \bigotimes_{\mathcal{O}} \mathcal{O}P$ mapping su on $s\varrho(u) \otimes u$ for any $s \in S$ and any $u \in P$.*

Now (1.6.2) and (1.8.1) imply that B and therefore $\mathrm{Ind}_P^G(B)$ are isomorphic to full matrix algebras over $\mathcal{O}P$; then, since there is a canonical embedding from $\mathcal{O}Gb$ to $\mathrm{Ind}_P^G(B)$ (cf. [9], th. 3.4), $\mathcal{O}Gb$ is a full matrix algebra over $\mathcal{O}P$ too. In [7] Okuyama and Tsushima proved that, if P is abelian, (1.4.1) implies (1.6.1) (notice that, if P is abelian, it suffices to prove that the inertial index of b is one to get (1.6.1)). It is not difficult to prove that (1.4.1) implies (1.6.1) whenever G is p-solvable. So, a question arises: is condition (1.4.1) on α always equivalent to conditions (1.6.1) and (1.6.2)?

1.9. In particular, condition (1.4.1) on α implies that

(1.9.1) *the quotient $\mathcal{O}Gb/J(\mathcal{O}Gb)$ is a simple k-algebra,*

but the converse is definitely not true: example 1.3 in [3] supplies a counterexample. However, it is not difficult to see that condition (1.7.1) on b implies condition (1.7.1) on f for any block f of any centralizer $C_G(Q)$ whenever (Q,f) is a b-Brauer pair (cf. [3], Th. 1.2.(4)); that is, if b is a *nilpotent block*, we have:

(1.9.2) *For any b-Brauer pair (Q,f), the quotient $\mathcal{O}C_G(Q)f/J(\mathcal{O}C_G(Q)f)$ is a simple k-algebra.*

Conversely, it is probably true that condition (1.9.2) on b implies that b is a *nilpotent block*. Moreover, notice that (1.9.2) is equivalent to the following *local* condition on α:

(1.9.3) *For any subgroup Q of P there is a unique $\delta \in \mathcal{LP}_{\mathcal{O}G}(Q)$ such that $Q_\delta \subset P_\gamma$.*

Indeed, if (Q,f) is a b-Brauer pair, there is a unique point β of $Q.C_G(Q)$ on $\mathcal{O}G$ such that $\mathrm{Br}_Q(\beta) = \{\mathrm{Br}_Q(f)\}$ (since $\mathrm{Br}_Q((\mathcal{O}G)^{Q.C_G(Q)}) = Z((\mathcal{O}G)(Q))$ and

$Z((\mathcal{O}G)(Q)) \cong Z k C_G(Q)$ by [12], (2.9.2)); hence, if R_ε is a defect pointed group of $Q.C_G(Q)_\beta$, we have $Q_\delta \subset R_\varepsilon$ for any $\delta \in \mathcal{LP}_{\mathcal{O}G}(Q)$ fulfilling

(1.9.4) $$\mathrm{Br}_Q(f) \cdot \mathrm{Br}_Q(\delta) = \mathrm{Br}_Q(\delta)$$

since (1.9.4) implies $Q_\delta \subset Q.C_G(Q)_\beta$ and therefore, R_ε contains a $Q.C_G(Q)$-conjugate of Q_δ (cf. [9], Th. 1.2); consequently, there is $x \in G$ such that $(Q_\delta)^x \subset P_\gamma$ for any $\delta \in \mathcal{LP}_{\mathcal{O}G}(Q)$ fulfilling (1.9.4) (cf. [9], Th. 1.2), and it suffices to apply [2], Th. 1.8 and [12], (2.9.2) and (2.10.1). Finally, notice that (1.9.3) and (1.3.1) are clearly equivalent.

1.10. Assume now that $\mathcal{O}$ is a ring of characteristic zero and denote by $\mathcal{K}$ its quotient field. Then, assuming that α fulfills conditions (1.7.2) and (1.6.2), with the last statements in 1.6 (which are actually easy consequences of (1.6.2), as we show in 7.5 below), we will show how to compute the full matrix of generalized decomposition numbers of the block b of G over $\mathcal{O}$, and get the formulae giving the irreducible characters of $\mathcal{K}Gb$ in terms of the family of absolutely irreducible Brauer characters in any block f of any centralizer $C_G(u)$ where (u, f) runs over the set of b-Brauer elements.

1.11. First of all, notice that the uniqueness of S implies that $\mathrm{tr}(\varrho(u))$ is a rational integer for any $u \in P$, where $\varrho\colon P \to S^*$ is the group homomorphism described in (1.8.1). Indeed, $\mathcal{O}$ is always an extension of a complete unramified discrete valuation ring $\mathcal{O}'$ having the same residue field k (cf. [15], Ch. I, §5, Th. 4); as $b \in \mathcal{O}'G$, $\gamma' = \gamma \cap \mathcal{O}'G$ is a local point of P on $\mathcal{O}'G$ (cf. (2.13.1)), and $P_{\gamma'}$ is still a defect pointed group of G_α on $\mathcal{O}'G$ (cf. (2.13.1)), we may assume that $B = \mathcal{O} \bigotimes_{\mathcal{O}'} B'$ where $\tilde{e}'\colon B' \to \mathrm{Res}_P^G(\mathcal{O}'Gb)$ is an embedded algebra associated with $P_{\gamma'}$, and therefore that $S = \mathcal{O} \bigotimes_{\mathcal{O}'} S'$ where S' is an $\mathcal{O}'$-simple P-algebra obtained from (1.6.2); in that case, we have $\varrho = \mathrm{id} \otimes \varrho'$ where $\varrho'\colon P \to (S')^*$ is the group homomorphism described in (1.8.1); but it is now clear that $\mathrm{tr}(\varrho'(u))$ is a rational integer for any $u \in P$[1]. Then, we have:

(1.11.1) *For any $u \in P$, $\mathrm{tr}(\varrho(u)) = \omega(u)\, m_{\delta(u)}^{\gamma}$ where $\omega(u) \in \{1, -1\}$ and $\delta(u)$ is the unique local point of $\langle u \rangle$ on $\mathcal{O}G$ such that $\langle u \rangle_{\delta(u)} \subset P_\gamma$.*

Indeed, the uniqueness of $\delta(u)$ follows already from (1.9.3); moreover, since P stabilizes an $\mathcal{O}$-basis of S containing the unity, we have $\mathrm{tr}(\varrho(u))^2 = \dim_k(S(\langle u \rangle)) \neq 0$ for any $u \in P$ (cf. [12], (2.8.4)); but $S(\langle u \rangle)$ is a simple k-algebra (cf. Cor. 5.8 below) and as $(SP)(\langle u \rangle) \cong S(\langle u \rangle) \bigotimes_k k C_P(u)$ (cf. (1.8.1), Prop. 5.6 below and [12], (2.9.2)), we have $S(\langle u \rangle) \cong B(\langle u \rangle_{\delta(u)})$ (cf. (1.6.2)); consequently, we get $\mathrm{tr}(\varrho(u))^2 = (m_{\delta(u)}^{\gamma})^2$ (cf. 2.6 below).

1.12. On the other hand, if M is a simple $\mathcal{K}Gb$-module, we know that $\mathrm{Res}_{\mathrm{id} \otimes e}(M)$ is a simple $\mathcal{K} \bigotimes_{\mathcal{O}} B$-module (cf. [9], Cor. 3.5) and therefore, by (1.6.2) there is a simple $\mathcal{K}P$-module N such that

(1.12.1) $$\mathrm{Res}_{\mathrm{id} \otimes e}(M) \cong V \bigotimes_{\mathcal{O}} N$$

[1] Another proof of the inclusion $\mathrm{tr}(\varrho(P)) \subset \mathbf{Z}$ can be obtained from the fact that P and therefore any cyclic subgroup of P stabilizes an $\mathcal{O}$-basis of S (a more detailed discussion on the function $\mathrm{tr} \circ \varrho$ will appear in [14])

where V is a projective indecomposable S-module and we identify B with $S \bigotimes_{\mathcal{O}} \mathcal{O}P$ (cf. (1.8.1) and [9], Prop. 2.1). Consequently, denoting by χ the character of M, by λ the character of N and by i the element $e(1)$ of γ, for any $u \in P$ we have (cf. (1.11.1), (1.12.1) and [9], Th. 4.3)

(1.12.2) $$\chi^{\delta(u)}(u)\, m^{\gamma}_{\delta(u)} = \chi(ui) = \mathrm{tr}_{V \underset{\mathcal{O}}{\otimes} N}(u) = \omega(u)\, m^{\gamma}_{\delta(u)}\, \lambda(u)\,.$$

Hence, denoting by U a set of representatives for the conjugacy classes of P, we get:

(1.12.3) *The full matrix of generalized decomposition numbers of the block b of G over $\mathcal{O}$ is* $(\omega(u)\,\lambda(u))_{u \in U, \lambda \in \mathrm{Irr}_{\mathscr{K}}(P)}$.

Indeed, it follows from (1.7.2) and (1.9.3) that $\{u_{\delta(u)}\}_{u \in U}$ is a set of representatives for the G-conjugacy classes of local pointed elements in G_α, and we know that the correspondence induced by (1.12.1) mapping χ on λ is a bijection from $\mathrm{Irr}_{\mathscr{K}}(G, b)$ onto $\mathrm{Irr}_{\mathscr{K}}(P)$ (cf. [9], Prop. 2.1 and Cor. 3.5); then (1.12.3) follows from (1.12.2) and [9], Cor. 4.4.

1.13. Finally, by (1.7.2) and (1.9.3), for any p-element u of G there is a bijection from $E_G(\langle u\rangle, P)$ onto $\mathscr{L}\mathscr{P}_{\mathcal{O}Gb}(\langle u\rangle)$ mapping $\tilde{\sigma} \in E_G(\langle u\rangle, P)$ on the unique local point δ of $\langle u\rangle$ on $\mathcal{O}Gb$ fulfilling $(u_\delta)^x \in P_\gamma$ and $\sigma(u) = u^x$ for some $x \in G$ (i.e. $\delta = \delta(\sigma(u))^{x^{-1}}$), and we denote by $\varphi_{\tilde{\sigma}}$ the irreducible Brauer character of $C_G(u)$ determined by δ (cf. [12], (2.9.2) and (2.10.1)). Then, with the notation above, it follows from [9], Cor. 4.4 again and (1.12.3) above that:

(1.13.1) *For any p-element u of G and any p'-element s of $C_G(u)$,*

$$\chi(us) = \sum_{\tilde{\sigma} \in E_G(\langle u\rangle, P)} \omega(\sigma(u))\, \lambda(\sigma(u))\, \varphi_{\tilde{\sigma}}(s)\,.$$

1.14. *Remark.* Although we assume that k is algebraically closed (whereas in [3] we just assumed that the field k was generated by the group of $|G|_{p'}$ roots of unity), the interested reader will convince himself that our arguments extend easily to the case where k is just perfect and all the algebras we consider are "split" ("déployées") in the sense developed in [8]. Moreover, the classical results on splitting fields show that all the algebras we consider in the proof of the main theorem are split whenever k contains the group of $|G|_{p'}$-roots of unity: indeed, in that case for any subgroup H of G the $\mathcal{O}$-algebra $\mathcal{O}H$ is split and it suffices to apply systematically the following fact:

(1.14.1) *Let k be a perfect field of characteristic p, $\mathcal{O}$ a complete discrete valuation ring with residue field k and A a G-algebra over $\mathcal{O}$. Assume that, for any local pointed group P_γ on A, $A(P_\gamma)$ is a matrix algebra over k and $k_* \hat{N}_G(P_\gamma)$ is a split algebra. Then the algebra A^G is split too.*

(We denote by $k_* \hat{N}_G(P_\gamma)$ the twisted algebra associated with the central k^*-extension $\hat{N}_G(P_\gamma)$ of $N_G(P_\gamma)$ defined by the action of $N_G(P_\gamma)$ on the simple k-algebra $A(P_\gamma)$; see [12], 5.12 and 6.2 for a more detailed definition).

This paper is divided in seven sections mostly devoted to prove the main theorem. Sects. 4 and 5 are significant exceptions: their contents have been developed in more general frames than needed here to provide handy references in forthcoming papers, avoiding tedious rewritings.

§2. Notation and terminology

2.1. The notation and terminology we need here are mostly contained in [9], in [11] and specially in [12] where we made a particular effort to be complete. There is no sense in repeating such an effort but we recall (rewrite) briefly all the necessary definitions and just comment with more detail some extra specific notation. Throughout the paper p is a prime number, k a field of characteristic p that we assume algebraically closed (except in Remark 1.14!) and $\mathcal{O}$ a complete discrete valuation ring with residue field k (we allow the case $\mathcal{O}=k$).

2.2. All the $\mathcal{O}$-algebras we consider are associative with unity, and $\mathcal{O}$-free of finite rank as $\mathcal{O}$-modules. An $\mathcal{O}$-algebra isomorphic to a finite direct product of full matrix algebras over $\mathcal{O}$ is shortly called *$\mathcal{O}$-semisimple*, and *$\mathcal{O}$-simple* if there is just one factor. If A is an $\mathcal{O}$-algebra, all the A-modules we consider here are $\mathcal{O}$-free of finite rank as $\mathcal{O}$-modules. We denote by A^* the group of invertible elements of A, by A^0 the opposite $\mathcal{O}$-algebra, by ZA the center, by $\mathrm{Aut}(A)$ the group of automorphisms, by $J(A)$ the Jacobson radical and by $\mathcal{P}(A)$ the set of A^*-conjugacy classes of primitive idempotents of A. For any $\alpha\in\mathcal{P}(A)$, we denote by $A(\alpha)$ the simple factor of A associated with α, by $s_\alpha\colon A\to A(\alpha)$ the canonical homomorphism and by $A\cdot\alpha\cdot A$ the two sided ideal generated by α (cf. [9], p. 266), and we set $\dim_k(A(\alpha))=(m_\alpha)^2$ and $J(A\cdot\alpha\cdot A)=J(A)\cap A\cdot\alpha\cdot A$. A *decomposition of unity* in A is a set I of pairwise orthogonal primitive idempotents of A such that $\sum_{i\in I} i=1$; notice that

(2.2.1) *for any $\alpha\in\mathcal{P}(A)$ we have $m_\alpha=|I\cap\alpha|$.*

2.3. A homomorphism $f\colon A\to B$ between $\mathcal{O}$-algebras is not required to be unitary, and we denote by $f^*\colon A^*\to B^*$ the group homomorphism mapping $a^*\in A^*$ on $f(a^*-1)+1$. If N is a B-module, $\mathrm{Res}_f(N)$ denotes $f(1)\cdot N$ endowed with the evident A-module structure. An *exomorphism* $\tilde{f}$ from A to B is the set of homomorphisms obtained by composing a homomorphism $f\colon A\to B$ with all the inner automorphisms of A and B (cf. [9], Def. 3.1 or [11], p. 360); we denote by $\widetilde{\mathrm{Hom}}(A,B)$ the set of exomorphisms from A to B. We say that $\tilde{f}\in\widetilde{\mathrm{Hom}}(A,B)$ is an *embedding* if $\mathrm{Ker}(f)=\{0\}$ and $\mathrm{Im}(f)=f(1)\,Bf(1)$. If $\tilde{f}\colon A\to B$ is an $\mathcal{O}$-algebra exomorphism, for any $\alpha\in\mathcal{P}(A)$ and any $\beta\in\mathcal{P}(B)$ we set $m(\tilde{f})_\beta^\alpha=|J\cap\beta|$ where, choosing $i\in\alpha$, J is a decomposition of unity in $f(i)\,Bf(i)$ (cf. [9], Def. 2.2); notice that, if $\tilde{g}\colon B\to C$ is another $\mathcal{O}$-algebra exomorphism,

(2.3.1) *for any $\alpha\in\mathcal{P}(A)$ and any $\gamma\in\mathcal{P}(C)$ we have*

$$m(\tilde{g}\circ\tilde{f})_\gamma^\alpha=\sum_{\beta\in\mathcal{P}(B)} m(\tilde{f})_\beta^\alpha\, m(\tilde{g})_\gamma^\beta .$$

2.4. Let G be a finite group. As usual we denote by $|G|$ the order of G, by ZG the center, by $\Phi(G)$ the Frattini subgroup, by $N_G(H)$ and $C_G(H)$ the normalizer and the centralizer of a subgroup H of G, and by x^y and $[x,y]$ the elements $y^{-1}xy$ and $x^{-1}y^{-1}xy$ where $x, y\in G$. As above, an *exomorphism* $\tilde{\varphi}\colon G\to H$ is the set of homomorphisms from G to H obtained by composing a group homomorphism $\varphi\colon G\to H$ with all the inner automorphisms of G and H; we denote by $\widetilde{\mathrm{Hom}}(G,H)$ the set of *exomorphisms* from G to H.

2.5. A *G-algebra* A (over $\mathcal{O}$) is an $\mathcal{O}$-algebra endowed with a group homomorphism φ: $G \to \mathrm{Aut}(A)$; we usually write a^x instead of $\varphi(x^{-1})(a)$. If H is a subgroup of G, A^H denotes the unitary subalgebra of H-fixed elements of A and, for any subgroup K of H, Tr_K^H: $A^K \to A^H$ denotes the relative trace map and A_K^H its image (cf. [9], p. 266). For any p-subgroup P of G we set $A(P) = A^P \Big/ \Big(\sum_Q A_Q^P + J(\mathcal{O}) \cdot A^P \Big)$ where Q runs over the set of proper subgroups of P, and we denote by Br_P: $A^P \to A(P)$ (or Br_P^A to avoid confusion) the canonical homomorphism.

2.6. Let A be a G-algebra. A *pointed group H_β on A* is a pair formed by a subgroup H of G and an element β of $\mathscr{P}(A^H)$ (cf. [9], Def. 1.1); we say that β is a *point of H on A*, set $A(H_\beta) = A^H(\beta)$ and denote by $N_G(H_\beta)$ the stabilizer of β in $N_G(H)$; if $H = \langle x \rangle$ we say that x_β is a *pointed element on A*. Moreover, we set $\mathscr{P}_A(H) = \mathscr{P}(A^H)$. If K_γ is a pointed group on A such that $K \subset H$ we write m_γ^β insteed of $m(\tilde{f})_\gamma^\beta$ where f is the inclusion map $A^H \subset A^K$ (cf. 2.3); then, we say that *K_γ is contained in H_β*, and write $K_\gamma \subset H_\beta$ (or $y_\gamma \in H_\beta$ if $K = \langle y \rangle$) whenever $m_\gamma^\beta \neq 0$. A pointed group P_γ (or a point γ of P) on A is *local* if $\mathrm{Br}_P(\gamma) \neq \{0\}$; we denote by $\mathscr{LP}_A(P)$ the set of local points of P on A. A *defect pointed group P_γ* of H_β is a maximal local pointed group on A such that $P_\gamma \subset H_\beta$ (cf. [9], Th. 1.2).

2.7. An *interior G-algebra* A (over $\mathcal{O}$) is an $\mathcal{O}$-algebra endowed with a group homomorphism φ: $G \to A^*$ (cf. [11], p. 359); we usually write $x \cdot a \cdot y$ instead of $\varphi(x)\, a \varphi(y)$; in particular, A becomes a G-algebra setting $a^x = x^{-1} \cdot a \cdot x$. If ψ: $H \to G$ is a group homomorphism, we denote by $\mathrm{Res}_\psi(A)$ the interior H-algebra defined by $\varphi \circ \psi$: $H \to A^*$, and we set $\mathrm{Res}_H^G(A) = \mathrm{Res}_\psi(A)$ when H is a subgroup of G and ψ the inclusion map. If B is a G-algebra, we denote by BG the interior G-algebra formed by the free B-module over G endowed with the distributive product fulfilling

$$(bx)(cy) = bc^{x^{-1}}xy$$

for any $x, y \in G$ and any $b, c \in B$, and with the canonical map from G to BG. Notice that the tensor product of interior G-algebras has a structure of interior G-algebra fulfilling $x \cdot (a \otimes b) \cdot y = x \cdot a \cdot y \otimes x \cdot b \cdot y$, and that if A is an interior G-algebra then, denoting by e the unit element of G,

(2.7.1) *there is a unique interior G-algebra isomorphism* $AG \cong A \bigotimes_{\mathcal{O}} \mathcal{O}G$ *mapping* $ae \in AG$ *on* $a \otimes e$.

2.8. A *homomorphism* of interior G-algebras f: $A \to A'$ is an $\mathcal{O}$-algebra homomorphism fulfilling $f(x \cdot a \cdot y) = x \cdot f(a) \cdot y$ for any $x, y \in G$ and any $a \in A$. As above, an *exomorphism* $\tilde{f}$: $A \to A'$ is the set of homomorphisms obtained composing a homomorphism f: $A \to A'$ with all the inner automorphisms of A and A' (that is, induced by $(A^G)^*$ and $(A'^G)^*$); if ψ: $H \to G$ is a group homomorphism, we denote by $\mathrm{Res}_\psi(\tilde{f})$ the exomorphism of interior H-algebras from $\mathrm{Res}_\psi(A)$ to $\mathrm{Res}_\psi(A')$ containing $\tilde{f}$ (cf. [11], p. 360), and we set $\mathrm{Res}_H^G(\tilde{f}) = \mathrm{Res}_\psi(\tilde{f})$ whenever H is a subgroup of G and ψ the inclusion map. We say that $\tilde{f}$ is an *embedding* of interior G-algebras if $\mathrm{Res}_1^G(\tilde{f})$ is an embedding of $\mathcal{O}$-algebras; in this case, for any pointed group H_β on A, there is a unique pointed group $H_{\beta'}$ on A' fulfilling $f(\beta) \subset \beta'$, and we usually denote β and β' by the same letter.

2.9. Let H be a subgroup of G and B an interior H-algebra; the *induced* interior G-algebra $\mathrm{Ind}_H^G(B)$ is formed by the tensor product $\mathcal{O}G \bigotimes_{\mathcal{O}H} B \bigotimes_{\mathcal{O}H} \mathcal{O}G$ endowed with the distributive product fulfilling

$$(x\otimes b\otimes y)(x'\otimes b'\otimes y') = \begin{cases} x\otimes b\cdot yx'\cdot b'\otimes y' & \text{if} \quad yx'\in H \\ 0 & \text{otherwise} \end{cases}$$

for any x, y, x', $y' \in G$ and any b, $b' \in B$, and with the group homomorphism mapping $x\in G$ on $\sum_y xy\otimes 1\otimes y^{-1}$ where y runs over a set of representatives for G/H in G. Moreover, we denote by

$$(2.9.1) \qquad \tilde{d}_H^G(B)\colon\ B\to \mathrm{Res}_H^G\,\mathrm{Ind}_H^G(B)$$

the *canonical embedding* determined by the interior H-algebra homomorphism mapping $b\in B$ on $1\otimes b\otimes 1$ (cf. [9], Def. 3.3 or [11], p. 360). Finally, if $\tilde{g}\colon B\to B'$ is an interior H-algebra exomorphism, we denote by

$$(2.9.2) \qquad \mathrm{Ind}_H^G(\tilde{g})\colon\ \mathrm{Ind}_H^G(B)\to \mathrm{Ind}_H^G(B')$$

the interior G-algebra exomorphism determined by the correspondence mapping $x\otimes b\otimes y$ on $x\otimes g(b)\otimes y$ for any $b\in B$ and any x, $y\in G$; clearly,

$$(2.9.3) \qquad \mathrm{Res}_H^G\,\mathrm{Ind}_H^G(\tilde{g})\circ \tilde{d}_H^G(B) = \tilde{d}_H^G(B')\circ\tilde{g}$$

which proves by the way that 2.9.2 does not depend on the choice of g in $\tilde{g}$ (cf. [12], (2.3.4) and (2.12.2)).

2.10. Let A be an interior G-algebra. If H_β is a pointed group on A, an *embedded algebra* $(B,\tilde{g})$ *associated with* H_β is a pair formed by an interior H-algebra B and an embedding $\tilde{g}\colon B\to \mathrm{Res}_H^G(A)$ such that $g(1)\in\beta$ (cf. [11], 1.6); then, we have:

(2.10.1) *If* $\tilde{h}\colon C\to \mathrm{Res}_H^G(A)$ *is an interior* H*-algebra exomorphism such that* $h(1)j=h(1)=jh(1)$ *for some* $j\in\beta$, *there is a unique interior* H*-algebra exomorphism* $\tilde{f}\colon C\to B$ *such that* $\tilde{h}=\tilde{g}\circ\tilde{f}$.

Indeed assuming that $g(1)=j$, it is clear that G induces an isomorphism $B\cong jAj$ whereas $\mathrm{Im}(h)\subset jAj$, which proves the existence of $\tilde{f}$; the uniqueness follows from [12], (2.3.3) and (2.12.2). Usually we denote by $(A_\beta,\tilde{f}_\beta)$ an embedded algebra associated with H_β chosen once for ever, and still denote by β the unique point of H on A_β (cf. 2.8).

2.11. Let H_β and K_γ be pointed groups on A; an *A-fusion* $\tilde{\varphi}$ *from* K_γ *to* H_β is a group exomorphism $\tilde{\varphi}\colon K\to H$ such that φ is injective and there is an exomorphism $\tilde{f}_\varphi\colon A_\gamma\to \mathrm{Res}_\varphi(A_\beta)$ fulfilling

$$(2.11.1) \qquad \mathrm{Res}_1^K(\tilde{f}_\gamma) = \mathrm{Res}_1^H(\tilde{f}_\beta)\circ \mathrm{Res}_1^K(\tilde{f}_\varphi)$$

(cf. [11], Def. 2.5); we denote by $F_A(K_\gamma, H_\beta)$ the set of A-fusions from K_γ to H_β, and set $F_A(H_\beta)=F_A(H_\beta,H_\beta)$. On the other hand, we denote by $E_G(K_\gamma,H_\beta)$ the set of $\tilde{\varphi}\in\tilde{\mathrm{H}}\mathrm{om}(K,H)$ such that there is $x\in G$ fulfilling $(K_\gamma)^x\subset H_\beta$ and $\varphi(y)=y^x$ for any $y\in K$ (cf. [11], Def. 2.1), and set $E_G(H_\beta)=E_G(H_\beta,H_\beta)$; moreover, if $A=\mathcal{O}$, the trivial interior G-algebra, we set $E_G(K,H)=E_G(K_\gamma,H_\beta)$ where $\gamma=\{1\}=\beta$.

2.12. This paragraph is only needed in sections 4 and 5 for statements which do no concern the proof of the main theorem (and which are discussed there for the sake of completeness). A k^*-*group* is a group $\hat{G}$ endowed with an injective group homomorphism $\theta: k^* \to Z\hat{G}$ (cf. [12], 5.2). If A is an interior G-algebra, P_γ a pointed p-group on A, $N_{A_\gamma}(P)$ the subgroup of $b \in A_\gamma^*$ such that $P \cdot b = b \cdot P$, $\bar{N}_{A_\gamma}(P)$ the quotient $N_{A_\gamma}(P)/P \cdot (1 + J(A_\gamma^P))$, and E a subgroup of $F_A(P_\gamma)$ then we denote by $\hat{E}^\gamma$ the k^*-group formed by the subgroup of $(\bar{b}, \tilde{\varphi}) \in \bar{N}_{A_\gamma}(P) \times E$ where $\bar{b}$ is the image in $\bar{N}_{A_\gamma}(P)$ of $b \in N_{A_\gamma}(P)$ fulfilling $b \cdot u = \varphi(u) \cdot b$ for any $u \in P$, endowed with the injective group homomorphism mapping $\lambda \in k^*$ on $(\lambda, \tilde{\mathrm{id}})$ where we identify k^* with the image of $(A_\gamma^P)^*$ in $\bar{N}_{A_\gamma}(P)$; moreover, if $E = F_A(P_\gamma)$ we set $\hat{F}_A(P_\gamma) = \hat{E}^\gamma$ (cf. [12], 7.1).

2.13. According 2.7 we denote by $\mathcal{O}G$ the group algebra of G over $\mathcal{O}$, considered as an interior G-algebra. Notice that, as the canonical homomorphism $\mathcal{O}G \to kG$ maps $(\mathcal{O}G)^H$ onto $(kG)^H$ for any subgroup H of G,

(2.13.1) *the canonical homomorphism $\mathcal{O}G \to kG$ induces a bijection between the sets of pointed groups on $\mathcal{O}G$ and kG which preserves inclusions and localness.*

A *block* of G is for us a primitive idempotent b of $Z\mathcal{O}G$; a b-*Brauer pair* is a pair (P, e) where P is a p-subgroup of G fulfilling $\mathrm{Br}_P(b) \neq 0$ (cf. 2.5) and e a block of $C_G(P)$ such that $\mathrm{Br}_P(be) \neq 0$, the normalizer $N_G(P, e)$ of (P, e) being the stabilizer of e in $N_G(P)$ (cf. [2], Def. 1.6); if $P = \langle u \rangle$ we say that (u, e) is a b-*Brauer element* (cf. [2], Def. 2.1).

2.14. Assume that $\mathcal{O}$ is of characteristic zero and denote by $\mathcal{K}$ its quotient field. We denote respectively by $\mathbf{L}_\mathcal{K}(G)$ and $\mathbf{L}_k(G)$ the Grothendieck rings of the categories of $\mathcal{K}G$- and kG-modules (cf. [12], 2.4); recall that (cf. [16], §16.1, Th. 33)

(2.14.1) *Brauer's decomposition homomorphism $\mathbf{L}_\mathcal{K}(G) \to \mathbf{L}_k(G)$ is surjective.*

If b is an idempotent of $Z\mathcal{O}G$, we denote respectively by $\mathrm{Irr}_\mathcal{K}(G, b)$ and $\mathrm{Irr}_k(G, b)$ the sets of Frobenius and Brauer characters of the simple $\mathcal{K}Gb$- and $kG\bar{b}$-modules, where $\bar{b}$ is the image of b in kG, and we simply write $\mathrm{Irr}_\mathcal{K}(G)$ and $\mathrm{Irr}_k(G)$ when $b = 1$.

2.15. Let b be a block of G, so that $\alpha = \{b\}$ is a point of G on $\mathcal{O}G$, and choose a set U of representatives for the G-conjugacy classes of local pointed elements u_δ on $\mathcal{O}G$ such that $u_\delta \in G_\alpha$ (cf. 2.6). The full matrix of *Brauer's generalized decomposition numbers* of b is the matrix $(\chi^\delta(u))_{\chi \in \mathrm{Irr}_\mathcal{K}(G,b),\, u_\delta \in U}$ where for any $\chi \in \mathrm{Irr}_\mathcal{K}(G, b)$ and any $u_\delta \in U$, choosing $j \in \delta$, we have $\chi^\delta(u) = \chi(uj)$ (cf. [9], Cor. 4.4). Now the *generalized Cartan integers* of b may be defined by the equalities (cf. [16], §15.4)

(2.15.1) $$c(u_\delta, v_\varepsilon) = \sum_{\chi \in \mathrm{Irr}_\mathcal{K}(G,b)} \chi^\delta(u)\, \chi^\varepsilon(v)$$

where u_δ and v_ε run over U; notice that (cf. [16], §15.1 and §18.3)

(2.15.2) *if u and v are not G-conjugate then $c(u_\delta, v_\varepsilon) = 0$, whereas if $u = v$ we have $c(u_\delta, u_\varepsilon) = \mathrm{rank}_\mathcal{O}(j\mathcal{O}C_G(u)\, l)$ where $j \in \delta$ and $l \in \varepsilon$.*

Finally, if λ and μ are Frobenius characters of $\mathcal{K}Gb$-modules, we denote by $(\lambda, \mu)_G$ the usual scalar product; it is not difficult to check that, denoting by $(c^\circ(u_\delta, v_\varepsilon))_{u_\delta, v_\varepsilon \in U}$ the inverse (over $\mathbf{Q}$) of the generalized Cartan matrix, we have

(2.15.3) $$(\lambda, \mu)_G = \sum_{u_\delta, v_\varepsilon \in U} c^\circ(u_\delta, v_\varepsilon)\, \lambda^\delta(u)\, \mu^\varepsilon(v).$$

§ 3. Local control

3.1. In this section we prove the equivalence between conditions (1.6.1), (1.7.1) and (1.7.2), and show a relationship between condition (1.7.2) and induction from P to G which plays a crucial rôle in proving that (1.6.1) implies (1.6.2) (see Corollary 4.23 and the proof of Proposition 7.2 below). Actually, this relationship was already stated (in a slightly different form) in [9], Prop. 3.9, but for the convenience of the reader we do not quote this result. From the point of view of our previous preprint [10], this section develops the contents of Sect. 2 and proposition 7.1 in [10], although we will introduce here *two* slightly different notions of *local control*, in order to clarify the arguments.

3.2. Let G be a finite group, H a subgroup of G and B an interior H-algebra (over $\mathcal{O}$). We say that *G is locally controlled by H on B* if for any pair of local pointed groups P_γ and Q_δ on B we have

(3.2.1) $$F_B(Q_\delta, P_\gamma) \cap E_G(Q, P) = E_H(Q_\delta, P_\gamma).$$

Notice that (3.2.1) is equivalent to the following equality over the interior G-algebra $\mathrm{Ind}_H^G(B)$

(3.2.2) $$E_G(Q_\delta, P_\gamma) = E_H(Q_\delta, P_\gamma)$$

where, as usual, we identify P_γ and Q_δ with their images over $\mathrm{Ind}_H^G(B)$ through the canonical embedding $d_H^G(B) \to \mathrm{Res}_H^G \mathrm{Ind}_H^G(B)$ (cf. 2.8 and (2.9.1)); indeed, by [11], 2.10 and Prop. 2.14, setting $A = \mathrm{Ind}_H^G(B)$ we have

$$F_B(Q_\delta, P_\gamma) = F_A(Q_\delta, P_\gamma) \quad \text{and} \quad F_A(Q_\delta, P_\gamma) \cap E_G(Q, P) = E_G(Q_\delta, P_\gamma).$$

3.3. The point is that equalities (3.2.2) are equivalent to easy formulae to compute the *multiplicity algebras* (cf. [12], 2.10) of local pointed groups on $\mathrm{Ind}_H^G(B)$ from the *multiplicity algebras* of local pointed groups on B. Precisely, let L_ε be a pointed group on B; since

$$(\mathrm{Res}_H^G \mathrm{Ind}_H^G(B))(L_\varepsilon) = \mathrm{Res}_{C_H(L)}^{C_G(L)} (\mathrm{Ind}_H^G(B)(L_\varepsilon)),$$

the canonical embedding $d_H^G(B)$: $B \to \mathrm{Res}_H^G \mathrm{Ind}_H^G(B)$ (cf. 2.9.1) induces a unique embedding of interior $C_G(L)$-algebras

(3.3.1) $$\tilde{e}_H^G(L_\varepsilon)\colon \mathrm{Ind}_{C_H(L)}^{C_G(L)}(B(L_\varepsilon)) \to \mathrm{Ind}_H^G(B)(L_\varepsilon)$$

such that the following diagram commutes (cf. [12], 2.12.3)

$$\begin{array}{ccc} \mathrm{Res}^{C_G(L)}_{C_H(L)}\,\mathrm{Ind}^{C_G(L)}_{C_H(L)}(B(L_\varepsilon)) & \xrightarrow{\mathrm{Res}^{C_G(L)}_{C_H(L)}(\tilde{e}^G_H(L_\varepsilon))} & \mathrm{Res}^{C_G(L)}_{C_H(L)}(\mathrm{Ind}^G_H(B)(L_\varepsilon)) \\ {\scriptstyle d^{C_G(L)}_{C_H(L)}(B(L_\varepsilon))}\nwarrow & & \nearrow{\scriptstyle d^G_H(B)(L_\varepsilon)} \\ & B(L_\varepsilon); & \end{array} \tag{3.3.2}$$

indeed, the existence follows from [12], (2.12.3) and (2.14.2), and the uniqueness from [12], (2.3.4) and (2.12.2). Then, we have:

3.4. Proposition. *With the notation above, local pointed groups on* $\mathrm{Ind}^G_H(B)$ *are the G-conjugate of local pointed groups on B. Moreover, the following conditions are equivalent:*

(3.4.1) *The group G is locally controlled by H on B.*

(3.4.2) *For any local pointed group P_γ on B we have*

$$\tilde{e}^G_H(P_\gamma)\colon \mathrm{Ind}^{C_G(P)}_{C_H(P)}(B(P_\gamma)) \cong \mathrm{Ind}^G_H(B)(P_\gamma).$$

Proof. Let Q_δ be a local pointed group on $\mathrm{Ind}^G_H(B)$; as $\mathrm{Tr}^G_H(1\otimes 1\otimes 1)$ is the unity of $\mathrm{Ind}^G_H(B)$, $s_\delta(\mathrm{Tr}^G_H(1\otimes 1\otimes 1))$ is the unity of $\mathrm{Ind}^G_H(B)(Q_\delta)$; but δ being local, we have

$$s_\delta(\mathrm{Tr}^G_H(1\otimes 1\otimes 1)) = \sum_{x\in X} s_\delta(x\otimes 1\otimes x^{-1}) \tag{3.4.3}$$

where X is a set of representatives in G for the double cosets QyH such that $Q^y\subset H$ and $s_\delta(y\otimes 1\otimes y^{-1})\neq 0$ or equivalently $s_{\delta^y}(1\otimes 1\otimes 1)\neq 0$; in particular, $(Q_\delta)^y$ is a local pointed group on B for any $y\in X$, and the first statement follows from $X\neq\emptyset$.

Assume now that Q_δ is already a local pointed group on B. If G is locally controlled by H in B, it follows from (3.2.2) that for any $x\in X$ there is $h\in H$ such that $(Q_\delta)^x=(Q_\delta)^h$ and $u^x=u^h$ for any $u\in Q$ (since $(Q_\delta)^x$ is a local pointed group on B), and therefore we may assume that $X\subset C_G(Q)$ which implies clearly

$$\sum_{x\in X} s_\delta(x\otimes 1\otimes x^{-1}) = \mathrm{Tr}^{C_G(Q)}_{C_H(Q)}(s_\delta(1\otimes 1\otimes 1)); \tag{3.4.4}$$

so, in this case $\tilde{e}^G_H(Q_\delta)$ is an isomorphism (since it is a unitary embedding). Conversely, if $\tilde{e}^G_H(Q_\delta)$ is an isomorphism, the equality (3.4.4) holds and therefore for any $y\in X$ we have $QyH\cap C_G(Q)\neq\emptyset$; so, if P_γ is a local pointed group on B, $\tilde{\varphi}$ an element of $E_G(Q_\delta, P_\gamma)$ (over $\mathrm{Ind}^G_H(B)$) and x an element of G such that $(Q_\delta)^x\subset P_\gamma$ and $\varphi(u)=u^x$ for any $u\in Q$, we have $x=zh$ where $z\in C_G(Q)$ and $h\in H$, and therefore $\tilde{\varphi}\in E_H(Q_\delta, P_\gamma)$; consequently, (3.2.2) holds.

3.5. Let A be an interior G-algebra, α a point of G on A and β a point of H on A such that $H_\beta\subset G_\alpha$. We say that G_α *is locally controlled by* H_β (or that H_β *is a control pointed subgroup of* G_α) if $\alpha\subset\mathrm{Tr}^G_H(A^H\cdot\beta\cdot A^H)$ and, for any pair of local pointed groups P_γ and Q_δ on A contained in H_β, we have

$$E_G(Q_\delta, P_\gamma) = E_H(Q_\delta, P_\gamma). \tag{3.5.1}$$

In this case, there is a unique embedding (cf. [9], Prop. 3.6)

$$\tilde{g}_\alpha^\beta \colon A_\alpha \to \mathrm{Ind}_H^G(A_\beta) \tag{3.5.2}$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Res}_H^G(A_\alpha) & \longrightarrow & \mathrm{Res}_H^G \mathrm{Ind}_H^G(A_\beta) \\ {\scriptstyle \tilde{f}_\beta^\alpha} \nwarrow & & \nearrow {\scriptstyle \tilde{d}_H^G(A_\beta)} \\ & A_\beta & \end{array} \tag{3.5.3}$$

and in particular, G is locally controlled by H on A_β (since (3.5.1) remains true over $\mathrm{Ind}_H^G(A_\beta)$, after the usual identifications). Conversely, it is quite clear that if such an embedding exists and G is locally controlled by H on A_β then G_α is locally controlled by H_β (cf. (3.2.2) and [9], Prop. 3.6). Moreover, the inclusion $\alpha \subset \mathrm{Tr}_H^G(A^H \cdot \beta \cdot A^H)$ implies that H_β contains a defect pointed group of G_α (cf. [9], Th. 1.2) and this statement has the following partial converse, which provides a criterion on the existence of control pointed subgroups of G_α.

3.6. Lemma. *Let A be an interior G-algebra, α a point of G on A and P_γ a defect pointed group of G_α. If $P \subset H$ there is $\beta \in \mathscr{P}_A(H)$ such that*

$$P_\gamma \subset H_\beta \subset G_\alpha \quad \textit{and} \quad \alpha \subset \mathrm{Tr}_H^G(A^H \cdot \beta \cdot A^H). \tag{3.6.1}$$

Proof. We may assume that $\alpha = \{1\}$. Set $\bar{N} = N_G(P_\gamma)/P$ and $\bar{M} = N_H(P_\gamma)/P$; as $A(P_\gamma)_1^{\bar{N}}$ contains the unity (cf. [9], Prop. 1.3), we have $A(P_\gamma)^{\bar{N}} = A(P_\gamma)_{\bar{M}}^{\bar{N}}$ and $A(P_\gamma)^{\bar{M}} = A(P_\gamma)_1^{\bar{M}} = s_\gamma(A_P^H)$, and therefore there is $\beta \in \mathscr{P}_A(H)$ such that

$$\mathrm{Tr}_{\bar{M}}^{\bar{N}}(s_\gamma(A^H \cdot \beta \cdot A^H)) = A(P_\gamma)^{\bar{N}}; \tag{3.6.2}$$

in particular, we have $P_\gamma \subset H_\beta$ and therefore, $\beta \subset \mathrm{Tr}_P^H(A^P \cdot \gamma A^P)$ (since $H_\beta \subset G_\alpha$ and P_γ is a maximal local pointed subgroup of G_α); but, for any $a \in A^P \cdot \gamma \cdot A^P$, we have (cf. [9], Prop. 1.3)

$$\mathrm{Tr}_{\bar{M}}^{\bar{N}}(s_\gamma(\mathrm{Tr}_P^H(a))) = \mathrm{Tr}_1^{\bar{N}}(s_\gamma(a)) = s_\gamma(\mathrm{Tr}_H^G(\mathrm{Tr}_P^H(a))); \tag{3.6.3}$$

so, by (3.6.2) and (3.6.3), we get $s_\gamma(\mathrm{Tr}_H^G(A^H \cdot \beta \cdot A^H)) = A(P_\gamma)^{\bar{N}}$ and therefore, $\mathrm{Tr}_H^G(A^H \cdot \beta \cdot A^H) = A^G$; consequently, β fulfills condition (3.6.1).

3.7. Proposition. *Let A be an interior G-algebra, α a point of G on A and P_γ a defect pointed group of G_α. Assume that $P \subset H$ and that, for any local pointed group Q_δ on A such that $Q_\delta \subset P_\gamma$ and any element x of G such that $(Q_\delta)^x \subset P_\gamma$ we have $x = zh$ where $z \in C_G(Q)$ and $h \in H$. Then there is $\beta \in \mathscr{P}_A(H)$ such that $P_\gamma \subset H_\beta \subset G_\alpha$ and G_α is locally controlled by H_β.*

Proof. Let β be a point of H on A fulfilling condition (3.6.1). As P_γ is still a defect pointed group of H_β (cf. [9], Th. 1.2), if Q_δ and R_ε are local pointed groups on A contained in H_β, there are elements h and k of H such that $(Q_\delta)^h \subset P_\gamma$ and $(R_\varepsilon)^k \subset P_\gamma$; so, if $\tilde{\varphi}$ is an element of $E_G(R_\varepsilon, Q_\delta)$ and x an element of G such that $(R_\varepsilon)^x \subset Q_\delta$ and $\varphi(u) = u^x$ for any $u \in R$, we have $(R_\varepsilon)^{xh} \subset P_\gamma$ and by hypothesis, we get $k^{-1}xh = z^k l$ where $z \in C_G(R)$ and $l \in H$; hence, $x = zklh^{-1}$ and therefore, $\tilde{\varphi} \in E_H(R_\varepsilon, Q_\delta)$. Consequently, G_α is locally controlled by H_β.

Our last result states the announced equivalence between conditions (1.6.1), (1.7.1) and (1.7.2). Notice that, by Proposition 3.7, conditions (1.7.2) and (3.8.2) below are equivalent.

3.8. Theorem. *Let $\alpha = \{b\}$ be a point of G on $\mathcal{O}G$ and P_γ a defect pointed group of G_α. The following three conditions on α are equivalent:*

(3.8.1) *For any local pointed subgroup Q_δ of G_α, $E_G(Q_\delta)$ is a p-group.*

(3.8.2) *The pointed group G_α is locally controlled by P_γ.*

(3.8.3) *For any b-Brauer pair (Q, f), the quotient $N_G(Q, f)/C_G(Q)$ is a p-group.*

In order to prove theorem 3.8 we need the following two lemmas. Recall that, for any p-subgroup Q of G, we have $(\mathcal{O}G)(Q) \cong k C_G(Q)$ (cf. [12], (2.9.2)) and therefore any simple $k C_G(Q)$-module is associated with a local point of Q on $\mathcal{O}G$ (cf. [12], (2.10.1)).

3.9. Lemma. *Let Q_δ be a local pointed group on $\mathcal{O}G$ such that $Q.C_G(Q) \subset H$ and V a simple $k(C_G(Q)/ZQ)$-module associated with δ. There is a unique point β of H on $\mathcal{O}G$ such that $Q_\delta \subset H_\beta$, and then Q_δ is a defect pointed group of H_β if and only if V is a projective module and $E_H(Q_\delta)$ is a p'-group.*

Proof. Set $A = \mathcal{O}G$, $C = Q.C_G(Q)$, $\bar{C} = C/Q$ and $\bar{N} = N_H(Q_\delta)/Q$; as $A(Q_\delta) \cong \mathrm{End}_k(V)$, we have $A(Q_\delta)^{\bar{C}} \cong k$; so, on one hand we get $s_\delta(A^H) \cong k$ which proves the uniqueness of β (the existence being trivially true), and on the other hand we have $A(Q_\delta)_1^{\bar{N}} = |E_H(Q_\delta)| \, A(Q_\delta)_1^{\bar{C}}$; but Q_δ is a defect pointed group of H_β if and only if $A(Q_\delta)_1^{\bar{N}} \cong k$ (cf. [9], Prop. 1.3), and by Higman's criterion, V is projective if and only if $A(Q_\delta)_1^{\bar{C}} \cong k$.

3.10. Lemma. *Let Q_δ be a local pointed group on $\mathcal{O}G$ and $\tilde{R}$ a p-subgroup of $E_G(Q_\delta)$. If H is the inverse image of $\tilde{R}$ in $N_G(Q_\delta)$, β the point of H on $\mathcal{O}G$ such that $Q_\delta \subset H_\beta$ and R_ε a defect pointed group of H_β, then $H = R.C_G(Q)$.*

Proof. First of all notice that $Q_\delta \subset R_\varepsilon$ (since R_ε contains an H-conjugate of Q_δ). Now set $L = R.C_G(Q)$ and $N = N_H(L)$, and denote respectively by λ and ν the points of L and N on $\mathcal{O}G$ such that $R_\varepsilon \subset L_\lambda \subset N_\nu \subset H_\beta$ (cf. Lemma 3.9), or equivalently $Q_\delta \subset L_\lambda \subset N_\nu$; on one hand, as N normalizes Q_δ and L, N normalizes L_λ and by Frattini's argument, we get $N = L.N_N(R_\varepsilon)$; on the other hand, R_ε is still a defect pointed group of N_ν and therefore $E_N(R_\varepsilon)$ is a p'-group (cf. Lemma 3.9). Consequently, N/L is both a p-group and a p'-group and therefore, $N = L$; but, as $H/Q.C_G(Q) \cong \tilde{R}$, L is subnormal in H; hence, $H = L$.

Proof of Theorem 3.8. Assume that (3.8.1) holds; by Proposition 3.7, to prove statement (3.8.2) it suffices to prove that, if Q_δ is a local pointed group on $\mathcal{O}G$ such that $Q_\delta \subset P_\gamma$ and x is an element of G such that $Q_\delta \subset (P_\gamma)^x$, we have $x = uz$ where $u \in P$ and $z \in C_G(Q)$. We argue by induction on $|P:Q|$; as $N_G(P_\gamma) = P.C_G(P)$ (cf. (3.8.1) and Lemma 3.9), we may assume that $Q \neq P$. Assume that $H = N_G(Q_\delta)$ and denote by β the point of H on $\mathcal{O}G$ such that $Q_\delta \subset H_\beta$ (cf. Lemma 3.9); by

Lemma 3.10, if R_ε is a defect pointed group of H_β, we have $H = R.C_G(Q)$; but there are local pointed groups $R'_{\varepsilon'}$ and $R''_{\varepsilon''}$ on $\mathcal{O}G$ such that (cf. [9], Cor. 1.5)

(3.8.4) $$Q_\delta \lneqq R'_{\varepsilon'} \subset P_\gamma \quad \text{and} \quad Q_\delta \lneqq R''_{\varepsilon''} \subset (P_\gamma)^x;$$

so, there are elements n' and n'' of $C_G(Q)$ such that

(3.8.5) $$R'_{\varepsilon'} \subset (R_\varepsilon)^{n'} \quad \text{and} \quad R''_{\varepsilon''} \subset (R_\varepsilon)^{n''}$$

(since H_β contains both $R'_{\varepsilon'}$ and $R''_{\varepsilon''}$). Consequently, if y is an element of G such that $R_\varepsilon \subset (P_\gamma)^y$, we have (cf. (3.8.5))

(3.8.6) $$R'_{\varepsilon'} \subset (P_\gamma)^{yn'} \quad \text{and} \quad R''_{\varepsilon''} \subset (P_\gamma)^{yn''}$$

and by (3.8.4) and the induction hypothesis, we get $yn' = u'z'$ and $yn''x^{-1} = u''(z'')^{x^{-1}}$ where u', $u'' \in P$ and z', $z'' \in C_G(Q)$ (since $C_G(Q)$ contains $C_G(R')$ and $C_G(R'')$); hence, $x = uz$ where $u = (u'')^{-1}u' \in P$ and $z = z'(n')^{-1}n''(z'')^{-1} \in C_G(Q)$.

Assume that (3.8.2) holds and set $A = \mathcal{O}G$; as $A(Q) \cong k C_G(Q)$, $\mathrm{Br}_Q(f)$ is a primitive idempotent of $ZA(Q)$ and there is $\delta \in \mathscr{LP}_A(Q)$ such that $s_\delta(f) \neq 0$ (and then, $s_\delta(f) = 1$). Assume that $H = N_G(Q_\delta)$ and denote by β the point of H on $\mathcal{O}G$ such that $Q_\delta \subset H_\beta$ (cf. Lemma 3.9); as $s_\delta(\beta) \neq \{0\}$ and $\mathrm{Br}_Q(A^H) \subset ZA(Q)$, we get $\mathrm{Br}_Q(\beta) = \{\mathrm{Br}_Q(f)\}$ and therefore, if $x \in N_G(Q, f)$ we have $Q_{\delta^x} \subset H_\beta$ too. Let R_ε be a defect pointed group of H_β such that $Q_{\delta^x} \subset R_\varepsilon$; on one hand we have $Q_\delta \subset R_\varepsilon$ too (since R_ε contains an H-conjugate of Q_δ), and on the other hand there is $y \in G$ such that $(R_\varepsilon)^y \subset P_\gamma$. Now P_γ contains Q_δ, $(Q_\delta)^y$ and $(Q_\delta)^{xy}$, and by (3.8.2) and Proposition 3.7 we have $y = z'u'$ and $xy = z''u''$ where u', $u'' \in P$ and z', $z'' \in C_G(Q)$; hence, $x = (z'^{-1})^{x^{-1}} z''u''(u')^{-1} \in C_G(Q).P$. Consequently, we get $N_G(Q, f) = C_G(Q).N_P(Q, f)$ and therefore $N_G(Q, f)/C_G(Q)$ is a p-group.

Finally, assume that (3.8.3) holds and let Q_δ be a local pointed group on $\mathcal{O}G$ such that $Q_\delta \subset G_\alpha$, or equivalently $s_\delta(b) \neq 0$; it is clear that there is a unique block f of $C_G(Q)$ such that $s_\delta(f) \neq 0$; as $s_\delta(b) = 1 = s_\delta(f)$, (Q, f) is a b-Brauer pair and the uniqueness of f forces $N_G(Q_\delta) \subset N_G(Q, f)$, so that $E_G(Q_\delta)$ is a p-group.

§ 4. $(\hat{\mathfrak{L}}, \mathfrak{L})$-covering exomorphisms

4.1. Let G be a finite group. As we said in [10] the most important tool to prove the main theorem is a class of exomorphisms of interior G-algebras that we name *covering exomorphisms*. This class allows us to lift pointed groups preserving multiplicities (see Proposition 4.18 below), and is stable by induction from a subgroup H of G whenever G is locally controlled by H on the arrowhead interior H-algebra (see Corollary 4.23 below). By (3.8.2) the last statement applies when inducing a source interior P-algebra of a nilpotent block b of G from a defect group P of b (see Proposition 7.2 below).

4.2. The surprising fact which was not yet clear in [10] is that a slightly more general class of exomorphisms – named $(\hat{\mathfrak{L}}, \mathfrak{L})$-*covering exomorphisms*, where $\mathfrak{L}$ and $\hat{\mathfrak{L}}$ are suitable sets of local pointed groups which are just empty in *covering exomorphisms* – plays a significant rôle in situations which have nothing to do with our proof of the main theorem (see example 4.24 below). Since the arguments to study $(\hat{\mathfrak{L}}, \mathfrak{L})$-*covering exomorphisms* are essentially the same that we employed in [10] to study *coverings*, we think reasonable to develop already here the general notion, thus providing a useful reference.

4.3. First of all, it is handy to consider the $\mathcal{O}$-algebra case (i.e. the case where $G=\{1\}$). Let B and $\hat{B}$ be $\mathcal{O}$-algebras and $\tilde{g}: \hat{B} \to B$ an $\mathcal{O}$-algebra exomorphism. We say that $\tilde{g}$ is a *covering exomorphism* if $B = \mathrm{Im}(g) + J(B)$. Then, we say that $\tilde{g}$ is *strict* if moreover $\mathrm{Ker}(g) \subset J(B)$.

4.4. Proposition. *With the notation above, the following conditions on* $\tilde{g}$ *are equivalent:*

(4.4.1) *The exomorphism* $\tilde{g}$: $\hat{B} \to B$ *is a covering.*

(4.4.2) *We have* $g(1)=1$ *and for any* $\beta \in \mathcal{P}(B)$ *there is a unique* $\hat{\beta} \in \mathcal{P}(\hat{B})$ *such that* $g(\hat{\beta}) \subset \beta$ *and* $m_{\hat{\beta}} = m_\beta$.

(4.4.3) *There is an injective map* $\tilde{g}^*$: $\mathcal{P}(B) \to \mathcal{P}(\hat{B})$ *such that, for any* $\beta \in \mathcal{P}(B)$, *setting* $\hat{\beta} = \tilde{g}^*(\beta)$ *we have* $g(\hat{\beta}) \subset \beta$ *and* $m_{\hat{\beta}} = m_\beta$.

In that case if $\hat{\beta} \in \mathcal{P}(\hat{B})$, $\hat{\beta} \not\subset \mathrm{Ker}(g)$ *is equivalent to* $\hat{\beta} \in \mathrm{Im}(\tilde{g}^*)$; *in particular,* $\tilde{g}$ *is strict if and only if* $\tilde{g}^*$ *is bijective.*

4.5. *Remark.* Notice that if $\beta \in \mathcal{P}(B)$ and $\hat{\beta} \in \mathcal{P}(\hat{B})$ fulfill $m(\tilde{g})_\beta^{\hat{\beta}} \neq 0$ and $m_{\hat{\beta}} = m_\beta$ then g induces an isomorphism $\hat{B}(\hat{\beta}) \cong B(\beta)$ and for any $\hat{\beta}' \in \mathcal{P}(\hat{B})$, we have $m(\tilde{g})_\beta^{\hat{\beta}'} = \delta_{\hat{\beta}, \hat{\beta}'}$.

Proof. If $B = \mathrm{Im}(g) + J(B)$, g induces a surjective homomorphism $\hat{B} \to B/J(B) \cong \prod_{\beta \in \mathcal{P}(B)} B(\beta)$, and therefore for any $\beta \in \mathcal{P}(B)$, $B(\beta)$ is a simple factor of $\hat{B}$ which corresponds to some $\hat{\beta} \in \mathcal{P}(\hat{B})$ such that $m(\tilde{g})_{\beta'}^{\hat{\beta}} = \delta_{\beta, \beta'}$ for any $\beta' \in \mathcal{P}(B)$ and $m_{\hat{\beta}} = m_\beta$; moreover, $g(1)$ and 1 lift the unity of $B/J(B)$ in B and therefore, $g(1)=1$.

Assume now that (4.4.2) holds. For any $\beta \in \mathcal{P}(B)$ set $\tilde{g}^*(\beta) = \hat{\beta}$ where $\hat{\beta} \in \mathcal{P}(\hat{B})$ fulfills $m(\tilde{g})_{\beta'}^{\hat{\beta}} = \delta_{\beta, \beta'}$ for any $\beta' \in \mathcal{P}(B)$ and $m_{\hat{\beta}} = m_\beta$; so $\tilde{g}^*$ is a map from $\mathcal{P}(B)$ to $\mathcal{P}(\hat{B})$ fulfilling $g(\hat{\beta}) \subset \beta$ and $m_\beta = m_{\hat{\beta}}$ which shows that $\tilde{g}^*$ is injective and that $\hat{\beta} \in \mathrm{Im}(\tilde{g}^*)$ is equivalent to $\hat{\beta} \not\subset \ker(g)$ for any $\hat{\beta} \in \mathcal{P}(\hat{B})$ (since $\hat{\beta} \notin \mathrm{Im}(\tilde{g}^*)$ implies $m(\tilde{g})_{\beta'}^{\hat{\beta}} = 0$ for any $\beta' \in \mathcal{P}(B)$).

Finally, assume that (4.4.3) holds. By Remark 4.5, g induces an isomorphism $B(\beta) \cong \hat{B}(\hat{\beta})$ where $\hat{\beta} = \tilde{g}^*(\beta)$ for any $\beta \in \mathcal{P}(B)$, and therefore an isomorphism $B/J(B) \cong \prod_{\hat{\beta} \in \mathrm{Im}(\tilde{g}^*)} \hat{B}(\hat{\beta})$; hence, $B = \mathrm{Im}(g) + J(B)$.

4.6. Proposition. *With the notation above, let* $\hat{\hat{B}}$ *be an* $\mathcal{O}$*-algebra and* $\tilde{h}$: $\hat{\hat{B}} \to \hat{B}$ *an* $\mathcal{O}$*-algebra exomorphism.*

(4.6.1) *If* $\tilde{g}$ *and* $\tilde{h}$ *are covering exomorphisms then* $\tilde{g} \circ \tilde{h}$ *is also a covering exomorphism and* $(\tilde{g} \circ \tilde{h})^* = \tilde{h}^* \circ \tilde{g}^*$. *In particular, if two of them are strict, the third is strict too.*

(4.6.2) *If* $\tilde{g} \circ \tilde{h}$ *is a covering exomorphism then* $\tilde{g}$ *is a covering exomorphism too. If moreover* $\tilde{g}$ *is strict then* $\tilde{h}$ *is also a covering exomorphism.*

Proof. If $\tilde{g}$ and $\tilde{h}$ are covering exomorphisms, they induce surjective homomorphisms $\hat{\hat{B}}/J(\hat{\hat{B}}) \to \hat{B}/J(\hat{B}) \to B/J(B)$ and therefore $B = \mathrm{Im}(g \circ h) + J(B)$; moreover, we have $g(h(\hat{\hat{\beta}})) \subset g(\hat{\beta}) \subset \beta$ where $\beta \in \mathscr{P}(B)$, $\hat{\beta} = \tilde{g}^*(\beta)$ and $\hat{\hat{\beta}} = \tilde{h}^*(\hat{\beta})$, which proves the equality $(\tilde{g} \circ \tilde{h})^* = \tilde{h}^* \circ \tilde{g}^*$.

Clearly $B = \mathrm{Im}(g \circ h) + J(B)$ implies $B = \mathrm{Im}(g) + J(B)$; if moreover $\tilde{g}$ is strict, g induces an isomorphism $\hat{B}/J(\hat{B}) \cong B/J(B)$ which maps the image of $\mathrm{Im}(h)$ onto the image of $\mathrm{Im}(g \circ h)$, forcing $\hat{B} = \mathrm{Im}(h) + J(\hat{B})$.

4.7. Proposition. *With the notation above, let C and $\hat{C}$ be $\mathcal{O}$-algebras, $\tilde{e}$: $C \to B$ and $\tilde{\hat{e}}$: $\hat{C} \to \hat{B}$ embeddings, and $\tilde{h}$: $\hat{C} \to C$ a unitary $\mathcal{O}$-algebra exomorphism such that $\tilde{e} \circ \tilde{h} = \tilde{g} \circ \tilde{\hat{e}}$. If $\tilde{g}$ is a covering exomorphism then $\tilde{h}$ is also a covering exomorphism, which is strict if $\tilde{g}$ is so.*

4.8. *Remark.* If $\tilde{g}$ and $\tilde{h}$ are covering exomorphism, the injective maps $\tilde{g}^*$ and $\tilde{h}^*$ are clearly compatible with the injective maps $\mathscr{P}(C) \to \mathscr{P}(B)$ and $\mathscr{P}(\hat{C}) \to \mathscr{P}(\hat{B})$ induced respectively by $\tilde{e}$ and $\tilde{\hat{e}}$ (cf. 2.8). So, we may identify as usual $\mathscr{P}(C)$ and $\mathscr{P}(\hat{C})$ with their respective images in $\mathscr{P}(B)$ and $\mathscr{P}(\hat{B})$.

Proof. Assume that $e \circ h = g \circ \hat{e}$. As h is unitary, we have $e(1) = g(\hat{e}(1))$ and therefore, $e(1)\,\mathrm{Im}(g)\,e(1) = e(\mathrm{Im}(h))$; but clearly $e(1)\,J(B)\,e(1) = e(J(C))$; consequently, $B = \mathrm{Im}(g) + J(B)$ implies $C = \mathrm{Im}(h) + J(C)$. Similarly, $\mathrm{Ker}(g) \subset J(\hat{B})$ implies $\mathrm{Ker}(h) \subset J(\hat{C})$.

4.9. We are ready to discuss the interior G-algebra case. Let A be an interior G-algebra and $\mathfrak{L}$ a set of local pointed groups on A fulfilling the following condition

(4.9.1) *If $P_\gamma \in \mathfrak{L}$ and Q_δ is a local pointed group on A such that $(Q_\delta)^x \subset P_\gamma$ for some $x \in G$ then $Q_\delta \in \mathfrak{L}$.*

For any subgroup H of G, we set

(4.9.2) $$A_{\mathfrak{L}}^H = \sum_{P_\gamma \in \mathfrak{L}_H} \mathrm{Tr}_P^H(A^P \cdot \gamma \cdot A^P)$$

where $\mathfrak{L}_H$ is the set of $P_\gamma \in \mathfrak{L}$ such that $P \subset H$, and we denote by $\mathscr{P}_A^{\mathfrak{L}}(H)$ the set of points $\beta \in \mathscr{P}_A(H)$ such that $\beta \not\subset A_{\mathfrak{L}}^H$: that is, if Q_δ is a defect pointed group of H_β, $\beta \in \mathscr{P}_A^{\mathfrak{L}}(H)$ is equivalent to $Q_\delta \notin \mathfrak{L}$ (cf. [9], Th. 1.2). Notice that $\mathfrak{L}_H$ is a set of local pointed groups on $\mathrm{Res}_H^G(A)$ fulfilling condition (4.9.1). For any p-subgroup P of G, we set

(4.9.3) $$A(P)_{\mathfrak{L}} = \mathrm{Br}_P(A_{\mathfrak{L}}^P) = \sum_{\gamma \in \mathscr{LP}_A(P) - \mathscr{P}_A^{\mathfrak{L}}(P)} A(P) \cdot \mathrm{Br}_P(\gamma) \cdot A(P)$$

and we denote by $\mathscr{LP}_A^{\mathfrak{L}}(P)$ the intersection of $\mathscr{LP}_A(P)$ and $\mathscr{P}_A^{\mathfrak{L}}(P)$.

4.10. If B is an interior G-algebra and $\tilde{e}$: $B \to A$ an interior G-algebra embedding, it is clear that the set $\tilde{e}^{-1}(\mathfrak{L})$ of local pointed groups P_γ on B such that $P_\gamma \in \mathfrak{L}$ fulfills Condition (4.9.1). Moreover, for any subgroup H of G, setting $\mathfrak{M} = \tilde{e}^{-1}(\mathfrak{L})$ we have

(4.10.1) $$\mathscr{P}_B^{\mathfrak{M}}(H) = \mathscr{P}_B(H) \cap \mathscr{P}_A^{\mathfrak{L}}(H) \quad \textit{and} \quad B_{\mathfrak{M}}^H \subset e^{-1}(A_{\mathfrak{L}}^H) \subset B_{\mathfrak{M}}^H + J(B^H).$$

Indeed, if $\beta \in \mathscr{P}_B(H)$ and Q_δ is a defect pointed group of H_β, we have $\beta \in \mathscr{P}_A^{\mathfrak{L}}(H)$ if and only if $Q_\delta \notin \mathfrak{L}$ (cf. 4.9) or equivalently $Q_\delta \notin \mathfrak{M}$, whereas we have $\beta \subset A_{\mathfrak{L}}^H$ if and only if $Q_\delta \in \mathfrak{L}$ (cf. [9], Th. 1.2) or equivalently $Q_\delta \in \mathfrak{M}$ which is again equivalent to $\beta \subset B_{\mathfrak{M}}^H$ (cf. [9], Th. 1.2).

4.11. Similarly, if H is a subgroup of G, B an interior H-algebra and $\mathfrak{M}$ a set of local pointed groups on B fulfilling condition (4.9.1), the set $\mathfrak{M}^G$ of local pointed groups $(P_\gamma)^x$ on $\mathrm{Ind}_H^G(B)$ where P_γ runs over $\mathfrak{M}$ and x over G, fulfills condition (4.9.1) too. Notice that if G is locally controlled by H on B (cf. 3.2) then $\mathfrak{M} = \tilde{d}_H^G(B)^{-1}((\mathfrak{M}^G)_H)$ (cf. (2.9.1)).

4.12. Let $\hat{A}$ be an interior G-algebra, $\hat{\mathfrak{L}}$ a set of local pointed groups on $\hat{A}$ fulfilling condition (4.9.1), and $\tilde{f}\colon \hat{A} \to A$ an interior G-algebra exomorphism. We say that $\tilde{f}$ is $(\hat{\mathfrak{L}}, \mathfrak{L})$-*compatible* if and only if, for any subgroup H of G, we have $f(\hat{A}_{\hat{\mathfrak{L}}}^H) \subset A_{\mathfrak{L}}^H$ or equivalently, $\tilde{f}$ induces an $\mathcal{O}$-algebra exomorphism

$$\tilde{f}_{\hat{\mathfrak{L}},\mathfrak{L}}^H\colon \hat{A}^H/\hat{A}_{\hat{\mathfrak{L}}}^H \to A^H/A_{\mathfrak{L}}^H; \tag{4.12.1}$$

in that case, for any p-subgroup P of G, $\tilde{f}_{\hat{\mathfrak{L}},\mathfrak{L}}^P$ induces a k-algebra exomorphism

$$\tilde{f}(P)_{\hat{\mathfrak{L}},\mathfrak{L}}\colon \hat{A}(P)/\hat{A}(P)_{\hat{\mathfrak{L}}} \to A(P)/A(P)_{\mathfrak{L}}. \tag{4.12.2}$$

4.13. Proposition. *With the notation above, the following conditions on $\tilde{f}$ are equivalent:*

(4.13.1) *The exomorphism $\tilde{f}\colon \hat{A} \to A$ is $(\hat{\mathfrak{L}}, \mathfrak{L})$-compatible.*

(4.13.2) *For any $P_{\hat{\gamma}} \in \hat{\mathfrak{L}}$ and any subgroup Q of P we have* $\mathrm{Br}_Q(f(\hat{\gamma})) \subset A(Q)_{\mathfrak{L}}$.

(4.13.3) *If $P_{\hat{\gamma}} \in \hat{\mathfrak{L}}$, any local pointed group Q_δ on A such that $Q \subset P$ and $s_\delta(f(\hat{\gamma})) \neq \{0\}$ belongs to $\mathfrak{L}$.*

Proof. If $\tilde{f}\colon \hat{A} \to A$ is $(\hat{\mathfrak{L}}, \mathfrak{L})$-compatible, by (4.12.1) we have $f(\hat{\gamma}) \subset A_{\mathfrak{L}}^P \subset A_{\mathfrak{L}}^Q$ and therefore, $\mathrm{Br}_Q(f(\hat{\gamma})) \subset A(Q)_{\mathfrak{L}}$ for any $P_{\hat{\gamma}} \in \hat{\mathfrak{L}}$ and any subgroup Q of P. Moreover, with the same notation, if $\mathrm{Br}_Q(f(\hat{\gamma})) \subset A(Q)_{\mathfrak{L}}$ and $s_\delta(f(\hat{\gamma})) \neq \{0\}$ for $\delta \in \mathscr{LP}_A(Q)$, we get $s_\delta(A_{\mathfrak{L}}^Q) \neq \{0\}$ which implies $Q_\delta \in \mathfrak{L}$ since δ is local.

Finally assume that (4.13.3) holds; to prove (4.13.1) it suffices to prove that if $P_{\hat{\gamma}} \in \hat{\mathfrak{L}}$ then $f(\hat{\gamma}) \subset A_{\mathfrak{L}}^P$. Choose $\hat{\imath} \in \hat{\gamma}$ and let T be a maximal P-stable abelian $\mathcal{O}$-semisimple subalgebra of $f(\hat{\imath})Af(\hat{\imath})$. If j is a primitive idempotent of T and Q the stabilizer of j in P, we know that j belongs to a local point δ of Q on A (cf. [12], (2.9.3)); as $s_\delta(f(\hat{\imath})) \neq 0$, we have $Q_\delta \in \mathfrak{L}$ and so, $\mathrm{Tr}_Q^P(j) \in A_{\mathfrak{L}}^P$. Consequently, we have $T^P \subset A_{\mathfrak{L}}^P$ and in particular, $f(\hat{\imath}) \in A_{\mathfrak{L}}^P$.

4.14. With the notation above, we say that $\tilde{f}\colon \hat{A} \to A$ is an $(\hat{\mathfrak{L}}, \mathfrak{L})$-*covering exomorphism* if $\tilde{f}$ is $(\hat{\mathfrak{L}}, \mathfrak{L})$-compatible and $\tilde{f}_{\hat{L},\mathfrak{L}}^H\colon \hat{A}^H/\hat{A}_{\hat{\mathfrak{L}}}^H \to A^H/A_{\mathfrak{L}}^H$ is a covering exomorphism of $\mathcal{O}$-algebras for any subgroup H of G; in that case, we denote by

$$(\tilde{f}_{\hat{\mathfrak{L}},\mathfrak{L}}^H)^*\colon \mathscr{P}_A^{\mathfrak{L}}(H) \to \mathscr{P}_{\hat{A}}^{\hat{\mathfrak{L}}}(H) \tag{4.14.1}$$

the injective map mapping $\beta \in \mathscr{P}_A^{\mathfrak{L}}(H)$ on $\hat{\beta} \in \mathscr{P}_{\hat{A}}^{\hat{\mathfrak{L}}}(H)$ such that $f(\hat{\beta}) \subset \beta + A_{\mathfrak{L}}^H$ (cf. Prop. 4.4), and by

$$\tilde{f}(H_\beta)\colon \hat{A}(H_{\hat{\beta}}) \cong A(H_\beta) \tag{4.14.2}$$

the exoisomorphism induced by $\tilde{f}^H_{\hat{\mathfrak{L}},\mathfrak{L}}$ (cf. Remark 4.5). Moreover, we say that $\tilde{f}$ is *strict* if $\tilde{f}^H_{\hat{\mathfrak{L}},\mathfrak{L}}$ is strict for any subgroup H of G or equivalently, if the maps (4.14.1) are bijective (cf. Prop. 4.4). Notice that $\tilde{f}$ is always $(\emptyset, \mathfrak{L})$-compatible and is a $(\hat{\mathfrak{L}}, \mathfrak{L})$-covering exomorphism if and only if is both $(\hat{\mathfrak{L}}, \mathfrak{L})$-compatible and a $(\emptyset, \mathfrak{L})$-covering exomorphism (cf. (4.6.2)); in particular, if $\hat{\mathfrak{L}}$ is the set of local pointed groups $P_{\hat{\gamma}}$ on $\hat{A}$ such that $f(\hat{\gamma}) \subset A^P_{\mathfrak{L}}$ (which fulfills clearly condition (4.9.1)) then $\tilde{f}$ is a strict $(\hat{\mathfrak{L}}, \mathfrak{L})$-covering exomorphism if and only if is an $(\emptyset, \mathfrak{L})$-covering exomorphism; so, the only interest in choosing $\hat{\mathfrak{L}}$ no empty is to be able to consider *strict* $(\hat{\mathfrak{L}}, \mathfrak{L})$-covering exomorphisms (see Remark 4.16 below). When $\mathfrak{L}$ and $\hat{\mathfrak{L}}$ are empty – the only case we need to prove the main theorem – we say *covering* instead of $(\emptyset, \emptyset)$-*covering* and write $\tilde{f}^H$ instead of $\tilde{f}^H_{\emptyset,\emptyset}$. First of all we generalize Propositions 4.6 and 4.7 to the new context.

4.15. Proposition. *With the notation above, let $\hat{\hat{A}}$ be an interior G-algebra, $\hat{\hat{\mathfrak{L}}}$ a set of local pointed groups on $\hat{\hat{A}}$ fulfilling condition* (4.9.1), *and $\tilde{g}$: $\hat{\hat{A}} \to \hat{A}$ an interior G-algebra exomorphism.*

(4.15.1) *If $\tilde{f}$ and $\tilde{g}$ are respectively $(\hat{\mathfrak{L}}, \mathfrak{L})$- and $(\hat{\hat{\mathfrak{L}}}, \hat{\mathfrak{L}})$-covering exomorphisms then $\tilde{f} \circ \tilde{g}$ is an $(\hat{\hat{\mathfrak{L}}}, \mathfrak{L})$-covering exomorphism and we have*

$$((\tilde{f} \circ \tilde{g})^H_{\hat{\hat{\mathfrak{L}}},\mathfrak{L}})^* = (\tilde{g}^H_{\hat{\hat{\mathfrak{L}}},\hat{\mathfrak{L}}})^* \circ (\tilde{f}^H_{\hat{\mathfrak{L}},\mathfrak{L}})^*$$

for any subgroup H of G. In particular, if two of them are strict, the third is strict too.

(4.15.2) *If $\tilde{f} \circ \tilde{g}$ is an $(\hat{\hat{\mathfrak{L}}}, \mathfrak{L})$-covering exomorphism then $\tilde{f}$ is an $(\emptyset, \mathfrak{L})$-covering exomorphism. If moreover $\tilde{f}$ is a strict $(\hat{\mathfrak{L}}, \mathfrak{L})$-covering exomorphism then $\tilde{g}$ is an* $(\hat{\hat{\mathfrak{L}}}, \hat{\mathfrak{L}})$-covering exomorphism.

4.16. *Remark.* Statement (4.15.2) is the main reason to consider *strict* $(\hat{\mathfrak{L}}, \mathfrak{L})$-*covering exomorphisms*, and we will use this result as follows: if $\tilde{f}$: $\hat{A} \to A$ is a strict $(\hat{\mathfrak{L}}, \mathfrak{L})$-covering exomorphism, $A = \mathcal{O}Gb$ where $\{b\} = \alpha$ is a point of G on $\mathcal{O}G$ and $\hat{\alpha} \in \mathcal{P}(\hat{A})$ fulfills $f(\hat{\alpha}) \subset \alpha$ (cf. (4.4.3)) then there is clearly a unique interior G-algebra exomorphism $\tilde{g}$: $A \to \hat{A}$ such that $g(b) \in \hat{\alpha}$ which fulfills $\tilde{f} \circ \tilde{g} = \widetilde{\mathrm{id}}_A$ and therefore is a strict $(\mathfrak{L}, \hat{\mathfrak{L}})$-covering exomorphism (see the proof of Proposition 7.2 below).

Proof. If all the exomorphisms $\tilde{f}$, $\tilde{g}$ and $\tilde{h} = \tilde{f} \circ \tilde{g}$ are compatible, for any subgroup H of G we have the following commutative diagram

$$\begin{array}{ccc} \hat{\hat{A}}^H/\hat{\hat{A}}^H_{\hat{\hat{\mathfrak{L}}}} & \xrightarrow{\tilde{h}^H_{\hat{\hat{\mathfrak{L}}},\mathfrak{L}}} & A^H/A^H_{\mathfrak{L}} \\ & {\scriptstyle \tilde{g}^H_{\hat{\hat{\mathfrak{L}}},\hat{\mathfrak{L}}}} \searrow \quad \nearrow {\scriptstyle \tilde{f}^H_{\hat{\mathfrak{L}},\mathfrak{L}}} & \\ & \hat{A}^H/\hat{A}^H_{\hat{\mathfrak{L}}} & \end{array}$$

and it suffices to apply Proposition 4.6. Moreover, if $\tilde{f}$ and $\tilde{g}$ are respectively $(\hat{\mathfrak{L}}, \mathfrak{L})$- and $(\hat{\hat{\mathfrak{L}}}, \hat{\mathfrak{L}})$-compatible, $\tilde{h}$ is clearly $(\hat{\hat{\mathfrak{L}}}, \mathfrak{L})$-compatible.

So, assume that $\tilde{h}$ and $\tilde{f}$ are respectively $(\hat{\hat{\mathfrak{L}}}, \mathfrak{L})$- and strict $(\hat{\mathfrak{L}}, \mathfrak{L})$-covering exomorphisms; we will prove that $\tilde{g}$ fulfills condition (4.13.3). Let $P_{\hat{\gamma}}$ be an element of $\hat{\hat{\mathfrak{L}}}$ and $Q_{\hat{\delta}}$ a local pointed group on $\hat{A}$ such that $Q \subset P$ and $Q_{\hat{\delta}} \notin \hat{\mathfrak{L}}$; as $\tilde{f}$ is a strict $(\hat{\mathfrak{L}}, \mathfrak{L})$-covering exomorphism, there is $\delta \in \mathcal{P}^{\mathfrak{L}}_A(Q)$ such that

$f(\hat{\delta}) \subset \delta$ and $m_{\hat{\hat{\delta}}} = m_\delta$ (cf. 4.14 and Prop. 4.4); moreover, as $\tilde{h}$ is an $(\hat{\hat{\mathfrak{L}}}, \mathfrak{L})$-covering exomorphism, there is $\hat{\hat{\delta}} \in \mathscr{P}_{\hat{\hat{A}}}^{\hat{\hat{\mathfrak{L}}}}(Q)$ such that $f(g(\hat{\hat{\delta}})) \subset \delta$ and $m_{\hat{\hat{\delta}}} = m_\delta$. Consequently, we get $m_{\hat{\hat{\delta}}} = m_{\hat{\delta}}$ and $m(\tilde{g})_{\hat{\delta}}^{\hat{\hat{\delta}}} \neq 0$ (cf. (2.3.1)), and therefore $\tilde{g}$ induces an exoisomorphism $\hat{\hat{A}}(Q_{\hat{\hat{\delta}}}) \cong \hat{A}(Q_{\hat{\delta}})$ mapping $s_{\hat{\hat{\delta}}}(\hat{\hat{\gamma}})$ on $s_{\hat{\delta}}(g(\hat{\hat{\gamma}}))$; but, as $P_{\hat{\hat{\gamma}}} \in \hat{\hat{\mathfrak{L}}}$ and $Q_{\hat{\hat{\delta}}} \notin \hat{\hat{\mathfrak{L}}}$, we have $s_{\hat{\hat{\delta}}}(\hat{\hat{\gamma}}) = \{0\}$ (cf. (4.9.1)); consequently, $s_{\hat{\delta}}(g(\hat{\hat{\gamma}})) = \{0\}$.

4.17. Proposition. *With the notation above, let B and $\hat{B}$ be interior G-algebras, $\tilde{e}$: $B \to A$ and $\tilde{\hat{e}}$: $\hat{B} \to \hat{A}$ interior G-algebra embeddings, and $\tilde{g}$: $\hat{B} \to B$ an interior G-algebra exomorphism such that $\tilde{e} \circ \tilde{g} = \tilde{f} \circ \tilde{\hat{e}}$; set $\mathfrak{M} = \tilde{e}^{-1}(\mathfrak{L})$ and $\hat{\mathfrak{M}} = \tilde{\hat{e}}^{-1}(\hat{\mathfrak{L}})$, and assume that $1 - g(1) \in B_{\hat{\mathfrak{M}}}^G$. If $\tilde{f}$ is an $(\hat{\mathfrak{L}}, \mathfrak{L})$-covering exomorphism then $\tilde{g}$ is an $(\hat{\mathfrak{M}}, \mathfrak{M})$-covering exomorphism which is strict if $\tilde{f}$ is so.*

Proof. If the exomorphisms $\tilde{f}$ and $\tilde{g}$ are compatible, for any subgroup H of G we have the following commutative diagram

$$\begin{array}{ccc}
\hat{A}^H/\hat{A}^H_{\hat{\mathfrak{L}}} & \xrightarrow{\tilde{f}^H_{\hat{\mathfrak{L}},\mathfrak{L}}} & A^H/A^H_{\mathfrak{L}} \\
\tilde{\hat{e}}^H_{\mathfrak{L}} \uparrow & & \uparrow \tilde{e}^H_{\mathfrak{L}} \\
\hat{B}^H/\hat{e}^{-1}(\hat{A}^H_{\hat{\mathfrak{L}}}) & \xrightarrow{\tilde{g}^H_{\hat{\mathfrak{L}},\mathfrak{L}}} & B^H/e^{-1}(A^H_{\mathfrak{L}}) \\
\uparrow & & \uparrow \\
\hat{B}^H/\hat{B}^H_{\hat{\mathfrak{M}}} & \xrightarrow{\tilde{g}^H_{\hat{\mathfrak{M}},\mathfrak{M}}} & B^H/B^H_{\mathfrak{M}}
\end{array}$$

where $\tilde{e}^H_{\mathfrak{L}}$ and $\tilde{\hat{e}}^H_{\hat{\mathfrak{L}}}$ are the $\mathcal{O}$-algebra embeddings induced respectively by $\tilde{e}$ and $\tilde{\hat{e}}$, and $\tilde{g}^H_{\hat{\mathfrak{L}},\mathfrak{L}}$ is the $\mathcal{O}$-algebra exomorphism induced by $\tilde{g}^H_{\hat{\mathfrak{M}},\mathfrak{M}}$, which is a (strict) covering exomorphism if and only if $\tilde{g}^H_{\hat{\mathfrak{M}},\mathfrak{M}}$ is so (cf. (4.10.1)). Hence, in this case it suffices to apply Proposition 4.7.

So, assume that $\tilde{f}$ is $(\hat{\mathfrak{L}}, \mathfrak{L})$-compatible; we will prove that $\tilde{g}$ fulfills condition (4.13.3). Let $P_{\hat{\gamma}}$ be an element of $\hat{\mathfrak{M}}$ and Q_δ a local pointed group on B such that $Q \subset P$ and $s_\delta(g(\hat{\gamma})) \neq \{0\}$; with the usual identifications, $P_{\hat{\gamma}}$ belongs to $\hat{\mathfrak{L}}$ (cf. 4.10) and therefore Q_δ belongs to $\mathfrak{L}$ by (4.13.3) applied to $\tilde{f}$; so, $Q_\delta \in \mathfrak{M}$.

The following proposition summarizes the lifting features of $(\hat{\mathfrak{L}}, \mathfrak{L})$-covering exomorphisms. Statement (4.18.3) below is only considered here for the sake of completeness. Recall that if P_γ is a local pointed group on A then $N_{A_\gamma}(P)$ is the subgroup of $b \in A_\gamma^*$ fulfilling $b \cdot P = P \cdot b$, $\bar{N}_{A_\gamma}(P)$ is the quotient $N_{A_\gamma}(P)/P \cdot (1 + J(A_\gamma^P))$, and $\hat{F}_A(P_\gamma)$ is the k^*-group formed by the subgroup of $(\bar{b}, \varphi) \in \bar{N}_{A_\gamma}(P) \times F_A(P_\gamma)$ where $\bar{b}$ is the image in $\bar{N}_{A_\gamma}(P)$ of an element b of $N_{A_\gamma}(P)$ such that $b \cdot u = \varphi(u) \cdot b$ for any $u \in P$, endowed with the injective group homomorphism mapping $\lambda \in k^*$ on $(\lambda, \widetilde{\mathrm{id}})$ where we identify k^* with the image of $(A_\gamma^P)^*$ in $\bar{N}_{A_\gamma}(P)$ (cf. 2.12).

4.18. Proposition. *With the notation above, assume that $\tilde{f}$ is an $(\hat{\mathfrak{L}}, \mathfrak{L})$-covering exomorphism.* Let *$H_{\hat{\beta}}$ and P_γ be respectively pointed groups on $\hat{A}$ and A such that $\hat{\beta} \in \mathscr{P}_{\hat{A}}^{\hat{\mathfrak{L}}}(H)$ and $\gamma \in \mathscr{P}_A^{\mathfrak{L}}(P)$, and denote by $\hat{\gamma}$ the point of P on $\hat{A}$ such that $f(\hat{\gamma}) \subset \gamma$.*

(4.18.1) *If $P_{\hat{\gamma}} \subset H_{\hat{\beta}}$ there is $\beta \in \mathscr{P}_A^{\mathfrak{L}}(H)$ such that $f(\hat{\beta}) \subset \beta$.*

(4.18.2) *If $f(\hat{\beta}) \subset \beta$ for some $\beta \in \mathscr{P}_A^{\mathfrak{L}}(H)$ we have*

$$E_G(P_{\hat{\gamma}}, H_{\hat{\beta}}) = E_G(P_\gamma, H_\beta) \quad \text{and} \quad F_{\hat{A}}(P_{\hat{\gamma}}, H_{\hat{\beta}}) \subset F_A(P_\gamma, H_\beta).$$

In particular, $P_{\hat\gamma} \subset H_{\hat\beta}$ *if and only if* $P_\gamma \subset H_\beta$, *and then* $m_{\hat\gamma}^{\hat\beta} = m_\gamma^\beta$. *Moreover,* $P_{\hat\gamma}$ *is a defect pointed group of* $H_{\hat\beta}$ *if and only if* P_γ *is a defect pointed group of* H_β.

(4.18.3) *If* P *is a p-group, there is a unique* k^**-group homomorphism* $\hat F_{\hat A}(P_{\hat\gamma}) \to \hat F_A(P_\gamma)$ *mapping* $(\bar{\hat b}, \tilde\varphi)$ *on* $(\bar b, \tilde\varphi)$ *where* $\tilde\varphi \in F_{\hat A}(P_{\hat\gamma})$, $\bar{\hat b}$ *is the image in* $\bar N_{\hat A_{\hat\gamma}}(P)$ *of* $\hat b \in \hat A^*_{\hat\gamma}$ *such that* $\hat b \cdot u = \varphi(u) \cdot \hat b$ *for any* $u \in P$, *and* $\bar b$ *is the image in* $\bar N_{A_\gamma}(P)$ *of the unique* $b \in A_\gamma$ *such that* $f_\gamma(b) = f_\gamma(1) f(f_{\hat\gamma}(\hat b)) f_\gamma(1)$, *the representatives* f_γ, f *and* $f_{\hat\gamma}$ *fulfilling* $s_\gamma(f_\gamma(1)) = s_\gamma(f(f_{\hat\gamma}(1)))$.

4.19. *Remark.* Notice that, when $P = H$, the last equivalence in (4.18.2) states that $P_{\hat\gamma}$ is local if and only if P_γ is local. Moreover, in (4.18.3) our hypothesis insure clearly that $\hat b \in N_{\hat A_{\hat\gamma}}(P)$ and $(\bar{\hat b}, \tilde\varphi) \in \hat F_{\hat A}(P_{\hat\gamma})$, whereas we prove below that $b \in N_{A_\gamma}(P)$ and $(\bar b, \tilde\varphi) \in \hat F_A(P_\gamma)$.

Proof. First of all assume that $P \subset H$; then we have clearly (cf. (2.3.1))

(4.18.4)
$$\sum_{\hat\gamma' \in \mathcal{P}_{\hat A}(P)} m_{\hat\gamma'}^{\hat\beta}\, m(\hat f)_\gamma^{\hat\gamma'} = m(\hat f)_\gamma^{\hat\beta} = \sum_{\beta' \in \mathcal{P}_A(H)} m(\hat f)_{\beta'}^{\hat\beta}\, m_\gamma^{\beta'}.$$

But if $\hat\gamma' \in \mathcal{P}_{\hat A}(P) - \mathcal{P}_{\hat A}^{\hat{\mathfrak{L}}}(P)$ we have $f(\hat\gamma') \subset A_{\mathfrak{L}}^P$ (cf. 4.12) and therefore, $m(\hat f)_\gamma^{\hat\gamma'} = 0$; moreover, if $\beta' \in \mathcal{P}_A(H) - \mathcal{P}_A^{\mathfrak{L}}(H)$, any local pointed group Q_δ on A such that $Q_\delta \subset H_{\beta'}$ belongs to $\mathfrak{L}$ (cf. 4.9) and therefore, $m_\gamma^{\beta'} = 0$ (since a defect pointed group of P_γ does not belong to $\mathfrak{L}$). On the other hand, it follows from proposition 4.4 that $m(\hat f)_\gamma^{\hat\gamma'} = \delta_{\hat\gamma, \hat\gamma'}$ for any $\hat\gamma' \in \mathcal{P}_{\hat A}^{\hat{\mathfrak{L}}}(P)$, and if $f(\hat\beta) \subset \beta$ for some $\beta \in \mathcal{P}_A^{\mathfrak{L}}(H)$ we get similarly $m(\hat f)_{\beta'}^{\hat\beta} = \delta_{\beta, \beta'}$, whereas $\hat\beta \in \mathcal{P}_{\hat A}^{\hat{\mathfrak{L}}}(H) - \mathrm{Im}((\hat f_{\mathfrak{L}, \mathfrak{L}}^H)^*)$ implies $m(\hat f)_{\beta'}^{\hat\beta} = 0$, for any $\beta' \in \mathcal{P}_A^{\mathfrak{L}}(H)$. Hence, by (4.18.4) we have $m_{\hat\gamma}^{\hat\beta} = m(\hat f)_\gamma^{\hat\beta}$ and therefore, if $P_{\hat\gamma} \subset H_{\hat\beta}$ there is $\beta \in \mathcal{P}_A^{\mathfrak{L}}(H)$ such that $f(\hat\beta) \subset \beta$.

Henceforth assume that $f(\hat\beta) \subset \beta$ for some $\beta \in \mathcal{P}_A^{\mathfrak{L}}(H)$. If $P \subset H$, we get again from (4.18.4) $m_{\hat\gamma}^{\hat\beta} = m(\hat f)_\gamma^{\hat\beta} = m_\gamma^\beta$ and in particular, $P_{\hat\gamma} \subset H_{\hat\beta}$ is equivalent to $P_\gamma \subset H_\beta$. So, for any $x \in G$ we have $(P_{\hat\gamma})^x \subset H_{\hat\beta}$ if and only if $(P_\gamma)^x \subset H_\beta$ (since $\gamma^x \in \mathcal{P}_A^{\mathfrak{L}}(P^x)$ by (4.9.1)) and therefore, $E_G(P_{\hat\gamma}, H_{\hat\beta}) = E_G(P_\gamma, H_\beta)$.

Now, to prove the inclusion $F_{\hat A}(P_{\hat\gamma}, H_{\hat\beta}) \subset F_A(P_\gamma, H_\beta)$ we may assume that $|P| = |H|$ (cf. [11], 2.11); then, if $\tilde\varphi \in F_{\hat A}(P_{\hat\gamma}, H_{\hat\beta})$ there is $\hat a \in \hat A^*$ such that $(\varphi(u) \cdot \hat\imath)^{\hat a} = u \cdot \hat\jmath$ for any $u \in P$, where $\hat\imath \in \hat\beta$ and $\hat\jmath \in \hat\gamma$ (cf. [11], Prop. 2.12); moreover, by (4.14.1) we have $f(\hat\imath) = i + l$ where $i \in \beta$ and l is an idempotent of $A_{\mathfrak{L}}^H$ such that $il = 0 = li$, and therefore $f(\hat\jmath) = j + f(l)^a$ where $a = f^*(\hat a) \in A^*$ (cf. 2.3) and $j = i^a \in \gamma$; it follows easily that $(\varphi(u) \cdot i)^a = u \cdot j$ for any $u \in P$ and therefore, $\tilde\varphi \in F_A(P_\gamma, H_\beta)$ (cf. [11], Prop. 2.12).

To prove (4.18.3) keep the notation above, assume that P is a p-group, $H = P$ and $\hat\imath = \hat\jmath$, and choose $c \in 1 + J(A^P)$ such that $iac = iai = aci$ (cf. [12], Lemma 6.3). If $f_{\hat\gamma}(1) = \hat\imath$ we may assume that $f_{\hat\gamma}(\hat b) = \hat\imath\hat a = \hat a\hat\imath$, and if moreover $f_\gamma(1) = i$ we get $f_\gamma(b) = iai = iac$, which implies that $b \in A_\gamma^*$ and $b \cdot u = \varphi(u) \cdot b$ for any $u \in P$; so, in this case b belongs to $N_{A_\gamma}(P)$ and has the same action as φ on the image of P in A_γ^*, which proves that $(\bar b, \tilde\varphi) \in \hat F_A(P_\gamma)$ (cf. [12], 7.1). But, if we modify our choices of f_γ, f and $f_{\hat\gamma}$, we have just to consider the unique element b' of A_γ such that $f_\gamma(b') = ida(i+l)d^{-1}i$ for some $d \in (A^P)^*$ fulfilling $s_\gamma(i^d) = s_\gamma(i)$, and since $ida(i+l)d^{-1}i = iac\, id^{ac}c^{-1}(i+l)d^{-1}i$, we get $b' = bb''$ where b'' is the unique element of A_γ such that $f_\gamma(b'') = id^{ac}c^{-1}(i+l)d^{-1}i$; finally, as

$$s_\gamma(id^{ac}c^{-1}(i+l)d^{-1}i) = s_\gamma(id^{ac})\, s_\gamma(i)\, s_\gamma(d^{-1}i) = s_\gamma((id)^{ac})\, s_\gamma(d^{-1}i) = s_\gamma(i)$$

we obtain $\bar b'' = 1$ and so, $\bar b' = \bar b$.

On the other hand, if $\hat{a} \in 1 + J(\hat{A}^P)$ then $a \in 1 + J(A^P)$ (cf. 2.3 and (4.14.2)) and therefore $\bar{b} = 1$. Finally, if $\tilde{\varphi}'$, $\hat{b}'$, $\hat{a}'$, a', c' and b' is another family fulfilling the conditions above (i.e. such that $\tilde{\varphi}' \in F_{\hat{A}}(P_{\hat{\gamma}})$, $\hat{b}' \in \hat{A}^*_{\hat{\gamma}}$, $\hat{a}' \in \hat{A}^*$, $f_{\hat{\gamma}}(\hat{b}') = \hat{i}\hat{a}' = \hat{a}'\hat{i}$, $(\varphi'(u) \cdot \hat{i})^{\hat{a}'} = u \cdot \hat{i}$ for any $u \in P$, $a' = f^*(\hat{a}')$, $c' \in 1 + J(A^P)$, $ia'c' = ia'i = a'c'i$, $b' \in A_\gamma$ and $f_\gamma(b') = ia'i$) it is clear that $f_{\hat{\gamma}}(\hat{b}\hat{b}') = \hat{a}\hat{a}'\hat{i}$ and $(\varphi(\varphi'(u)) \cdot \hat{i})^{\hat{a}\hat{a}'} = u \cdot \hat{i}$, whence $\hat{b}\hat{b}' \cdot u = \varphi(\varphi'(u)) \cdot \hat{b}\hat{b}'$ for any $u \in P$; then, if b'' is the unique element of A_γ such that $f_\gamma(b'') = iaa'i$, we have

$$f_\gamma((bb')^{-1}b'') = i(a'c')^{-1}\, i(ac)^{-1} iaa'i = i(c')^{-1}(c^{-1})^{a'} i$$

and therefore, $(bb')^{-1}b'' \in 1 + J(A^P_\gamma)$ whence $\bar{b}'' = \overline{bb'}$. The prove of (4.13.3) is complete.

Finally, it suffices to prove that $\hat{\gamma}$ is local if and only if γ is local. Indeed, in that case, if $P_{\hat{\gamma}}$ is a defect pointed group of $H_{\hat{\beta}}$ and Q_δ is a local pointed group on A such that $P_\gamma \subset Q_\delta \subset H_\beta$, Q_δ does not belong to $\mathfrak{L}$ and therefore setting $\hat{\delta} = (\tilde{f}^Q_{\mathfrak{L},\mathfrak{L}})^*(\delta)$ we have $P_{\hat{\gamma}} \subset Q_{\hat{\delta}} \subset H_{\hat{\beta}}$ which implies $P = Q$; conversely, if P_γ is a defect pointed group of H_β and $Q_{\hat{\delta}}$ is a local pointed group on $\hat{A}$ such that $P_{\hat{\gamma}} \subset Q_{\hat{\delta}} \subset H_{\hat{\beta}}$, there is $\delta \in \mathscr{P}^{\mathfrak{L}}_A(Q)$ such that $f(\hat{\delta}) \subset \delta$, δ is local and we have $P_\gamma \subset Q_\delta \subset H_\beta$ which implies again $P = Q$.

If $\hat{\gamma}$ is not local there is a proper subgroup Q of P such that $\hat{\gamma} \in \hat{A}^P_Q$ and therefore, $f(\hat{\gamma}) \subset A^P_Q$ which implies $\gamma \subset A^P_Q$ (since $m(\tilde{f})^{\hat{\gamma}}_\gamma \neq 0$). Conversely, if Q is a proper subgroup of P such that $\gamma \subset A^P_Q$, it follows from (4.4.2), (4.12.1) and (4.20.1) below that

$$\gamma \subset \mathrm{Tr}^P_Q(f(\hat{A}^Q) + A^Q_{\mathfrak{L}} + J(A^Q)) \subset f(\hat{A}^P_Q) + A^P_{\mathfrak{L}} + J(A^P) \qquad (4.18.3)$$

and therefore, we get $\tilde{f}(P_\gamma)(s_{\hat{\gamma}}(\hat{A}^P_Q)) = s_\gamma(f(\hat{A}^P_Q)) \neq \{0\}$ (cf. (4.14.2) and (4.18.3)), which implies $s_{\hat{\gamma}}(\hat{A}^P_Q) = \hat{A}(P_{\hat{\gamma}})$; so, $\hat{\gamma}$ is not local.

4.20. Lemma. *With the notation above, if P is a subgroup of G and Q a subnormal subgroup of P, we have:*

$$\mathrm{Tr}^P_Q(J(A^Q)) \subset J(A^P) \qquad (4.20.1)$$

Proof. Arguing by induction on $|P:Q|$ we may assume that Q is normal in P; in that case the $\mathcal{O}$-algebra A^Q is P-stable and therefore,

$$\mathrm{Tr}^P_Q(J(A^Q)) \subset A^P \cap J(A^Q) \subset J(A^P).$$

It is clearly hopeless to get inclusion (4.20.1) without suitable hypothesis. However we have the following general result which will be useful in the proof of Theorem (4.22) below. Recall that if P_γ is a pointed group on A, $A^P \cdot \gamma \cdot A^P$ denotes the two-sided ideal of A^P generated by γ, and $J(A^P \cdot \gamma \cdot A^P)$ the intersection $J(A^P) \cap A^P \cdot \gamma \cdot A^P$ (cf. 2.2).

4.21. Lemma. *With the notation above, if P_γ is a local pointed group on A and H a subgroup of G containing P, we have:*

$$\mathrm{Tr}^H_P(J(A^P \cdot \gamma \cdot A^P)) \subset J(A^H) + \sum_{Q_\delta} \mathrm{Tr}^H_Q(A^Q \cdot \delta \cdot A^Q) \qquad (4.21.1)$$

where Q_δ runs over the set of local pointed groups on A such that $Q_\delta \subsetneqq P_\gamma$.

Proof. Let a be an element of $J(A^P \cdot \gamma \cdot A^P)$ and since $A^H = \sum_{\beta \in \mathscr{P}_A(H)} A^H \cdot \beta \cdot A^H$, set $\mathrm{Tr}_P^H(a) = \sum_{\beta \in \mathscr{P}_A(H)} b_\beta$ where $b_\beta \in A^H \cdot \beta \cdot A^H$ for any $\beta \in \mathscr{P}_A(H)$. As $s_\beta(\mathrm{Tr}_P^H(a)) = s_\beta(b_\beta)$ for any $\beta \in \mathscr{P}_A(H)$, it suffices to prove that if $\beta \in \mathscr{P}_A(H)$ and $s_\beta(\mathrm{Tr}_P^H(a)) \neq 0$ then $b_\beta \in \mathrm{Tr}_Q^H(A^Q \cdot \delta \cdot A^Q)$ for some local pointed group Q_δ on A such that $Q_\delta \subsetneqq P_\gamma$. Let β be a point of H on A such that $s_\beta(\mathrm{Tr}_P^H(a)) \neq 0$; as $\mathrm{Tr}_P^H(J(A^P \cdot \gamma \cdot A^P))$ is an ideal of A^H, we have $s_\beta(\mathrm{Tr}_P^H(J(A^P \cdot \gamma \cdot A^P)) = A(H_\beta)$ and therefore, $\beta \subset \mathrm{Tr}_P^H(J(A^P \cdot \gamma \cdot A^P))$; hence, there is a defect pointed group Q_δ of H_β such that $Q_\delta \subset P_\gamma$. Then, on one hand we have $b_\beta \in A^H \cdot \beta \cdot A^H \subset \mathrm{Tr}_Q^H(A^Q \cdot \delta \cdot A^Q)$; on the other hand, setting $\bar{N} = N_H(P_\gamma)/P$ we have (cf. [9], Prop. 1.3)

$$s_\gamma(\beta) \subset s_\gamma(\mathrm{Tr}_P^H(J(A^P \cdot \gamma \cdot A^P))) = \mathrm{Tr}_1^{\bar{N}}(s_\gamma(J(A^P \cdot \gamma \cdot A^P))) = \{0\}$$

and therefore $P_\gamma \not\subset H_\beta$; so, $Q_\delta \neq P_\gamma$.

The following theorem may be considered as the main result of this section. It shows that the notion of $(\hat{\mathfrak{L}}, \mathfrak{L})$-*covering exomorphism* is essentially "local".

4.22. Theorem. *With the notation above, the following conditions on $\tilde{f}$ are equivalent*

(4.22.1) *The exomorphism $\tilde{f}$: $\hat{A} \to A$ is an $(\hat{\mathfrak{L}}, \mathfrak{L})$-covering.*

(4.22.2) *For any p-subgroup P of G we have $\tilde{f}(P)(\hat{A}(P)_{\hat{\mathfrak{L}}}) \subset A(P)_{\mathfrak{L}}$ and the induced map $\tilde{f}(P)_{\hat{\mathfrak{L}},\mathfrak{L}}$: $\hat{A}(P)/\hat{A}(P)_{\hat{\mathfrak{L}}} \to A(P)/A(P)_{\mathfrak{L}}$ is a covering exomorphism of $\mathscr{k}$-algebras.*

(4.22.3) *For any local pointed group P_γ on A such that $P_\gamma \notin \mathfrak{L}$ there is $\hat{\gamma} \in \mathscr{P}_{\hat{A}}^{\hat{\mathfrak{L}}}(P)$ such that $m_{\hat{\gamma}} = m_\gamma$ and* $\mathrm{Br}_P(f(\hat{\gamma})) \subset \mathrm{Br}_P(\gamma) + A(P)_{\mathfrak{L}}$.

(4.22.4) *The exomorphism $\tilde{f}$: $\hat{A} \to A$ is $(\hat{\mathfrak{L}}, \mathfrak{L})$-compatible and, for any subgroup H of G and any subgroup K of H, we have*

$$A_K^H \subset f(\hat{A}_K^H) + A_{\mathfrak{L}}^H + J(A^H).$$

In that case, $\tilde{f}$ is strict if and only if $\tilde{f}(P)_{\hat{\mathfrak{L}},\mathfrak{L}}$ is strict for any p-subgroup P of G.

Proof. If $\tilde{f}$: $\hat{A} \to A$ is an $(\hat{\mathfrak{L}}, \mathfrak{L})$-covering exomorphism, for any p-subgroup P of G, $\tilde{f}$ induces an exomorphism $\mathscr{k} \otimes \tilde{f}_{\hat{\mathfrak{L}},\mathfrak{L}}^P$: $\mathscr{k} \otimes (\hat{A}^P/\hat{A}_{\hat{\mathfrak{L}}}^P) \to \mathscr{k} \otimes (A^P/A_{\mathfrak{L}}^P)$ which is a covering exomorphism of k-algebras (cf. 4.14); hence, the exomorphism $\mathscr{k} \otimes (\hat{A}^P/\hat{A}_{\hat{\mathfrak{L}}}^P) \to A(P)/A(P)_{\mathfrak{L}}$ induced by $\tilde{f}$ is a covering too and by (4.6.2), the exomorphism $\tilde{f}(P)_{\hat{\mathfrak{L}},\mathfrak{L}}$: $\hat{A}(P)/\hat{A}(P)_{\hat{\mathfrak{L}}} \to A(P)/A(P)_{\mathfrak{L}}$ is also a covering of k-algebras. In that case, if $\tilde{f}$ is strict then $\mathscr{k} \otimes \tilde{f}_{\hat{\mathfrak{L}},\mathfrak{L}}^P$ is strict too, and by Propositions 4.4 and 4.18, $(\mathscr{k} \otimes \tilde{f}_{\hat{\mathfrak{L}},\mathfrak{L}}^P)^*$ induces a bijection from $\mathscr{L}\mathscr{P}_{\hat{A}}^{\hat{\mathfrak{L}}}(P)$ onto $\mathscr{L}\mathscr{P}_A^{\mathfrak{L}}(P)$; consequently, $(\tilde{f}(P)_{\hat{\mathfrak{L}},\mathfrak{L}})^*$ is a bijection too for any p-subgroup P of G. Conversely, assume that $\tilde{f}(P)_{\hat{\mathfrak{L}},\mathfrak{L}}$ is strict for any p-subgroup P of G; if $H_{\hat{\beta}}$ is a pointed group on $\hat{A}$ such that $\hat{\beta} \in \mathscr{P}_{\hat{A}}^{\hat{\mathfrak{L}}}(H)$ and $Q_{\hat{\delta}}$ is a defect pointed group of $H_{\hat{\beta}}$, we have $Q_{\hat{\delta}} \notin \hat{\mathfrak{L}}$ and therefore, $\mathrm{Br}_Q(\hat{\delta}) \not\subset \hat{A}(Q)_{\hat{\mathfrak{L}}}$; as $\tilde{f}(Q)_{\hat{\mathfrak{L}},\mathfrak{L}}$ is strict, there is $\delta \in \mathscr{L}\mathscr{P}_A^{\mathfrak{L}}(Q)$ such that $f(\hat{\delta}) \subset \delta$ and $m_{\hat{\delta}} = m_\delta$ (cf. Prop. 4.4) and in particular, f induces an isomorphism $\hat{A}(Q_{\hat{\delta}}) \cong A(Q_\delta)$ (cf. Remark 4.5) mapping $s_{\hat{\delta}}(\hat{\beta})$ on $s_\delta(f(\hat{\beta}))$; so, $s_\delta(f(\hat{\beta})) \neq \{0\}$ and therefore, $\hat{\beta} \not\subset \hat{A}_{\hat{\mathfrak{L}}}^H + \mathrm{Ker}(f)$. Consequently, $\tilde{f}_{\hat{\mathfrak{L}},\mathfrak{L}}^H$ is strict for any subgroup H of G (cf. Prop. 4.4).

Assume that (4.22.2) holds. If P_γ is a local pointed group on A such that $P_\gamma \notin \mathfrak{L}$, $\mathrm{Br}_P(\gamma)$ is a point of $A(P)$ of multiplicity m_γ such that $\mathrm{Br}_P(\gamma) \not\subset A(P)_{\mathfrak{L}}$; so, (4.22.3) follows from Proposition 4.4 applied to the k-algebra exomorphism $\tilde{f}(P)_{\hat{\mathfrak{L}},\mathfrak{L}}$.

Assume that (4.22.3) holds; first of all we will prove that $\tilde{f}$ fulfills condition (4.13.3). Let $P_{\hat{\gamma}}$ be an element of $\hat{\mathfrak{L}}$ and Q_δ a local pointed group on A such that $Q \subset P$ and $Q_\delta \notin \mathfrak{L}$; by (4.22.3) there is $\hat{\delta} \in \mathscr{P}_{\hat{A}}^{\hat{\mathfrak{L}}}(Q)$ such that $m_{\hat{\delta}} = m_\delta$ and $m(\tilde{f})_{\hat{\delta}}^{\delta} \neq 0$; so, f induces an isomorphism $\hat{A}(Q_{\hat{\delta}}) \cong A(Q_\delta)$ mapping $s_{\hat{\delta}}(\hat{\gamma})$ on $s_\delta(f(\hat{\gamma}))$ (cf. Remark 4.5) and therefore $s_\delta(f(\hat{\gamma})) = \{0\}$ since $Q_{\hat{\delta}} \not\subset P_{\hat{\gamma}}$ (cf. (4.9.1)).

To prove the inclusion $A_K^H \subset f(\hat{A}_K^H) + A_{\mathfrak{L}}^H + J(A^H)$, we argue by induction on $|K|$; as $A^K = \sum_{\varepsilon \in \mathscr{P}_A(K)} A^K \cdot \varepsilon \cdot A^K$, it suffices to prove that if $\varepsilon \in \mathscr{P}_A^{\mathfrak{L}}(K)$ we have

$$\mathrm{Tr}_K^H(A^K \cdot \varepsilon \cdot A^K) \subset f(\hat{A}_K^H) + A_{\mathfrak{L}}^H + J(A^H). \tag{4.22.5}$$

Let P_γ be a defect pointed group of K_ε; as $P_\gamma \notin \mathfrak{L}$ (cf. 4.9), there is $\hat{\gamma} \in \mathscr{P}_{\hat{A}}^{\hat{\mathfrak{L}}}(P)$ such that $m_{\hat{\gamma}} = m_\gamma$ and $\mathrm{Br}_P(f(\hat{\gamma})) \subset \mathrm{Br}_P(\gamma) + A(P)_{\mathfrak{L}}$, which implies that $s_\gamma(f(\hat{A}^P \cdot \hat{\gamma} \cdot \hat{A}^P)) = A(P_\gamma)$ (cf. Remark 4.5) and $s_{\gamma'}(f(\hat{A}^P \cdot \hat{\gamma} \cdot \hat{A}^P)) = \{0\}$ for any $\gamma' \in \mathscr{L}\mathscr{P}_A^{\mathfrak{L}}(P) - \{\gamma\}$; it follows easily that

$$A^P \cdot \gamma \cdot A^P \subset f(\hat{A}^P \cdot \hat{\gamma} \cdot \hat{A}^P) + \sum_{\gamma' \in \mathscr{L}\mathscr{P}_A^{\mathfrak{L}}(P)} J(A^P \cdot \gamma' A^P) + A_{\mathfrak{L}}^P + \mathrm{Ker}(\mathrm{Br}_P^A); \tag{4.22.6}$$

then, as $A^K \cdot \varepsilon \cdot A^K \subset \mathrm{Tr}_P^K(A^P \cdot \gamma \cdot A^P)$, applying Tr_P^H to (4.22.6) we get (cf. Lemma 4.21)

$$\mathrm{Tr}_K^H(A^K \cdot \varepsilon \cdot A^K) \subset f(\hat{A}_P^H) + A_{\mathfrak{L}}^H + J(A^H) + \sum_Q A_Q^H \tag{4.22.7}$$

where Q runs over the set of proper subgroups of P; as $\hat{A}_Q^H \subset \hat{A}_K^H$ for any subgroup Q of K, (4.22.5) follows now from (4.22.7) and the induction hypothesis.

Finally, (4.22.1) follows from (4.22.4) taking $K = H$.

4.23. Corollary. *Let H be a subgroup of G, B and $\hat{B}$ interior H-algebras, $\mathfrak{M}$ and $\hat{\mathfrak{M}}$ sets of local pointed groups on B and $\hat{B}$ respectively, fulfilling condition* (4.9.1), *and $\tilde{g}: \hat{B} \to B$ an interior H-algebra exomorphism. Assume that G is locally controlled by H on B. Then $\mathrm{Ind}_H^G(\tilde{g})$: $\mathrm{Ind}_H^G(\hat{B}) \to \mathrm{Ind}_H^G(B)$ is an $(\hat{\mathfrak{M}}^G, \mathfrak{M}^G)$-covering exomorphism if and only if $\tilde{g}: \hat{B} \to B$ is an $(\hat{\mathfrak{M}}, \mathfrak{M})$-covering exomorphism. In that case if $\tilde{g}$ is strict then $\mathrm{Ind}_H^G(\tilde{g})$ is strict too.*

Proof. With the notation above, we set $A = \mathrm{Ind}_H^G(B)$, $\hat{A} = \mathrm{Ind}_H^G(\hat{B})$, $\mathfrak{L} = \mathfrak{M}^G$, $\hat{\mathfrak{L}} = \hat{\mathfrak{M}}^G$ and $f = \mathrm{Ind}_H^G(g)$; moreover, we have $\hat{\mathfrak{M}} \subset \tilde{d}_H^G(\hat{B})^{-1}(\hat{\mathfrak{L}}_H)$ and as G is locally controlled by H on B, $\mathfrak{M} = \tilde{d}_H^G(B)^{-1}(\mathfrak{L}_H)$. If $\tilde{f}$ is an $(\hat{\mathfrak{L}}, \mathfrak{L})$-covering exomorphism, it is clear that $\mathrm{Res}_H^G(\tilde{f})$ is an $(\hat{\mathfrak{L}}_H, \mathfrak{L}_H)$-covering exomorphism and that $1 - f(1)$ belongs to $A_{\mathfrak{L}}^G$ (cf. (4.4.2) and (4.14); in particular, as $1 - f(1) = \mathrm{Tr}_H^G(1 \otimes (1 - g(1)) \otimes 1)$ and $A_{\mathfrak{L}}^G \subset A_{\mathfrak{L}}^H$, we get $1 - g(1) \in B_{\mathfrak{M}}^H$ (cf. (4.10.1)); hence, it follows from Proposition 4.17 that $\tilde{g}$ is an $(\hat{\mathfrak{M}}, \mathfrak{M})$-covering exomorphism.

Assume now that $\tilde{g}$ is an $(\hat{\mathfrak{M}}, \mathfrak{M})$-covering exomorphism; first of all we will prove that $\tilde{f}$ fulfills condition (4.13.3). Let P_γ be an element of $\hat{\mathfrak{L}}$ and Q_δ a local pointed group on A such that $Q \subset P$ and $s_\delta(f(\hat{\gamma})) \neq \{0\}$; by 4.11 we may assume that $P_{\hat{\gamma}} \in \hat{\mathfrak{M}}$ (since we may replace $P_{\hat{\gamma}}$ and Q_δ by $(P_{\hat{\gamma}})^x$ and $(Q_\delta)^x$ for some $x \in G$); so, with the usual identifications, $P_{\hat{\gamma}}$ becomes a local pointed group on $\hat{B}$ such that $s_\delta(1 \otimes g(\hat{\gamma}) \otimes 1) \neq \{0\}$ and in particular, we get $s_\delta(1 \otimes 1 \otimes 1) \neq 0$; hence, with the usual identifications again, Q_δ becomes a local pointed group on B such that $s_\delta(g(\hat{\gamma})) \neq \{0\}$ and therefore, $Q_\delta \in \mathfrak{M}$ (since $\tilde{g}$ fulfills condition (4.13.3)).

Secondly, we will prove that $\tilde{f}$ fulfills condition (4.22.3). Let P_γ be a local pointed group on A such that $P_\gamma \notin \mathfrak{L}$; by Proposition 3.4 we may assume that P_γ is a local pointed group on B such that $P_\gamma \notin \mathfrak{M}$; then, as $\tilde{g}$ fulfills condition (4.22.3), there is $\hat{\gamma} \in \mathscr{P}_{\hat{B}}^{\hat{\mathfrak{L}}}(P)$ such that

$$m_{\hat{\gamma}}(\hat{B}) = m_\gamma(B) \qquad \text{and} \qquad \mathrm{Br}_P(g(\hat{\gamma})) \subset \mathrm{Br}_P(\gamma) + B(P)_{\mathfrak{M}};$$

in particular, g induces an isomorphism $\hat{B}(P_{\hat{\gamma}}) \cong B(P_\gamma)$ (cf. Remark 4.5). Hence, with the usual identifications, it follows from (3.4.2) that f induces also an isomorphism $\hat{A}(P_{\hat{\gamma}}) \cong A(P_\gamma)$ (since a surjective homomorphism $\hat{A}(P_{\hat{\gamma}}) \to A(P_\gamma)$ is bijective) and in particular, we get $m_{\hat{\gamma}}(\hat{A}) = m_\gamma(A)$ Moreover, as $1 \otimes B(P)_{\mathfrak{M}} \otimes 1 \subset A(P)_{\mathfrak{L}}$, with the usual identifications again, we get also $\mathrm{Br}_P(f(\hat{\gamma})) \subset \mathrm{Br}_P(\gamma) + A(P)_{\mathfrak{L}}$ and in particular, $\hat{\gamma} \in \mathscr{P}_{\hat{A}}^{\hat{\mathfrak{L}}}(P)$ (since $\tilde{f}$ is $(\hat{\mathfrak{L}}, \mathfrak{L})$-compatible).

Finally, assume furthermore that $\tilde{g}$ is strict. By Theorem 4.22 and Proposition 4.4, it suffices to prove that if $P_{\hat{\gamma}}$ is a local pointed group on $\hat{A}$ such that $P_{\hat{\gamma}} \notin \hat{\mathfrak{L}}$ then there is $\gamma \in \mathscr{L}\mathscr{P}_A^{\mathfrak{L}}(P)$ such that $f(\hat{\gamma}) \subset \gamma$; but, as above we may assume that $P_{\hat{\gamma}}$ is a local pointed group on $\hat{B}$ such that $P_{\hat{\gamma}} \notin \hat{\mathfrak{M}}$; in that case, by Theorem 4.22 applied to $\tilde{g}$, there is $\gamma \in \mathscr{L}\mathscr{P}_B^{\mathfrak{M}}(P)$ such that $g(\hat{\gamma}) \subset \gamma$; then, with the usual identifications, we get $f(\hat{\gamma}) \subset \gamma$ and $\gamma \in \mathscr{L}\mathscr{P}_A^{\mathfrak{L}}(P)$ since $\mathfrak{M} = \bar{d}_H^G(B)^{-1}(\mathfrak{L}_H)$.

4.24. *Example.* Let M, P and Q be $\mathcal{O}G$-modules and

$$Q \xrightarrow{d} P \xrightarrow{e} M \longrightarrow 0$$

an exact sequence of $\mathcal{O}G$-module homomorphisms, and assume that P and Q are projective. Assume that A is the interior G-algebra $\mathrm{End}_{\mathcal{O}}(M)$, $\hat{A}$ the interior G-subalgebra of $\mathrm{End}_{\mathcal{O}}(P) \times \mathrm{End}_{\mathcal{O}}(Q)$ formed by the pairs (a, b) such that $a \circ d = d \circ b$, $\mathfrak{L}$ and $\hat{\mathfrak{L}}$ the respective sets $\{1_\delta\}_{\delta \in \mathscr{P}(A)}$ and $\{1_{\hat{\delta}}\}_{\hat{\delta} \in \mathscr{P}(\hat{A})}$, and $f\colon \hat{A} \to A$ the homomorphism mapping $(a, b) \in \hat{A}$ on the unique $c \in A$ such that $c \circ e = e \circ a$. Then it is quite clear that $\tilde{f}$ is a *strict* $(\hat{\mathfrak{L}}, \mathfrak{L})$*-covering exomorphism.* Actually, for any subgroup H of G, the exomorphism $\tilde{f}_{\hat{\mathfrak{L}}, \mathfrak{L}}^H$ is surjective; but we are not able to prove that *surjectivity* would be preserved by induction under control, as in Corollary 4.13 above.

The following example shows the covering situation considered in this paper (see proofs of Propositions 6.10 and 7.2).

4.25. *Example.* Assume that $\hat{A}$ is the interior G-algebra AG (cf. 2.7), $\mathfrak{L}$ and $\hat{\mathfrak{L}}$ the empty sets, and $f\colon \hat{A} \to A$ the homomorphism mapping ax on $a \cdot x$ for any $a \in A$ and any $x \in G$. Then we claim that $\tilde{f}$ is a *covering exomorphism* which is *strict* if G is a p-group. Indeed, the $\mathcal{O}$-algebra homomorphism $A \to \hat{A}$ mapping $a \in A$ on ae where e is the unit element of G is both a G-algebra homomorphism and a section of f, and therefore we have $f(\hat{A}^H) = A^H$ for any subgroup H of G; moreover, it is clear that

$$\mathrm{Ker}(f) = \sum_{a \in A,\, x \in G} \mathcal{O}(ax - (a \cdot x)e)$$

and therefore if G is a p-group then $\operatorname{Ker}(\bar{f}) \subset J(\hat{A})$ (cf. (2.7.1)) which implies that $\bar{f}$ is strict (cf. 4.14 and 4.3).

§5. Fusions in $S \otimes A$

5.1. The main result of this section (see Theorem 5.3 below) provides a tool to prove that condition 1.6.2 implies condition 1.6.1 (or precisely, condition 1.7.2). Actually, the particular case needed here (see Proposition 6.10 below) could be easily handled in Sect. 6 by an argument *ad hoc*, as we did in [10]. But we prefer to set this particular case in its general context, specially to provide a handy reference to study the extensions of nilpotent blocks in [6].

5.2. Let P be a finite p-group, A an interior P-algebra and S an $\mathcal{O}$-simple interior P-algebra having a P-stable $\mathcal{O}$-basis which contains the unity. If $A = \mathcal{O}$, the trivial interior P-algebra, we denote by 1 the unique (local) point of any subgroup Q of P. The main purpose of this section is to prove the following result (where statement (5.3.3) is discussed here for the sake of completeness). We write $S \otimes A$ instead of $S \bigotimes_{\mathcal{O}} A$.

5.3. Theorem. *With the notation above:*

(5.3.1) *For any subgroup Q of P there is a bijection $\mathcal{LP}_A(Q) \to \mathcal{LP}_{S\otimes A}(Q)$ mapping $\delta \in \mathcal{LP}_A(Q)$ on the unique local point $S \times \delta$ of Q on $S \otimes A$ such that, for any $j \in \delta$, there is $j' \in S \times \delta$ fulfilling $(1 \otimes j) j' = j' = j'(1 \otimes j)$.*

(5.3.2) *If Q_δ and R_ε are local pointed groups on A, setting $F = F_S(R_{S\times 1}, Q_{S\times 1})$ we have*

$$F_A(R_\varepsilon, Q_\delta) \cap F = F_{S\otimes A}(R_{S\times\varepsilon}, Q_{S\times\delta}) \cap F.$$

(5.3.3) *If Q_δ is a local pointed group on A, setting $E = F_A(Q_\delta) \cap F_S(Q_{S\times 1})$ there is a k^*-group isomorphism $\hat{E}^\delta \cong \hat{E}^{S\times\delta}$ lifting the identity.*

5.4. *Remark.* In our applications of (5.3.2) (here and in [6]) we are able to prove independently that F contains both $F_A(R_\varepsilon, Q_\delta)$ and $F_{S\otimes A}(R_{S\times\varepsilon}, Q_{S\times\delta})$.

5.5 First of all, we expose some "almost general" facts on tensor products of interior P-algebras: we restrict ourself to the case when at least one factor has a P-stable $\mathcal{O}$-basis. Let A' be an interior P-algebra having a P-stable $\mathcal{O}$-basis B'; our first result describes the local pointed groups on $A \otimes A'$ in terms of the local pointed groups on A and A'.

5.6. Proposition. *With the notation above, for any subgroup Q of P there is a k-algebra isomorphism*

(5.6.1) $$A(Q) \otimes A'(Q) \cong (A \otimes A')(Q)$$

mapping $\operatorname{Br}_Q(a) \otimes \operatorname{Br}_Q(a')$ *on* $\operatorname{Br}_Q(a \otimes a')$. *In particular, there is a bijection*

(5.6.2) $$\mathcal{LP}_A(Q) \times \mathcal{LP}_{A'}(Q) \to \mathcal{LP}_{A\otimes A'}(Q)$$

mapping (δ,δ') *on the unique local point* $\delta\times\delta'$ *of* Q *on* $A\otimes A'$ *such that* $\mathrm{Br}_Q(\delta)\otimes\mathrm{Br}_Q(\delta')\subset\mathrm{Br}_Q(\delta\times\delta')$ *(up to identification through* (5.6.1)), *and then there are k-algebra isomorphisms*

$$(5.6.3)\qquad A(Q_\delta)\otimes A'(Q_{\delta'})\cong(A\otimes A')(Q_{\delta\times\delta'})$$

mapping $s_\delta(a)\otimes s_{\delta'}(a')$ *on* $s_{\delta\times\delta'}(a\otimes a')$, *and embeddings*

$$(5.6.4)\qquad \tilde{g}_{\delta\times\delta'}\colon(A\otimes A')_{\delta\times\delta'}\to A_\delta\otimes A'_{\delta'}$$

fulfilling $(\tilde{f}_\delta\otimes\tilde{f}_{\delta'})\circ\tilde{g}_{\delta\times\delta'}=\tilde{f}_{\delta\times\delta'}$.

Proof. As $(A\otimes A')^Q_R$ contains $A^Q\otimes(A')^Q_R$ and $A^Q_R\otimes(A')^Q$ for any subgroup R of Q, the homomorphism $A^Q\otimes(A')^Q\to(A\otimes A')(Q)$ mapping $a\otimes a'$ on $\mathrm{Br}_Q(a\otimes a')$ induces indeed a $\mathscr{k}$-algebra homomorphism

$$(5.6.5)\qquad A(Q)\otimes A'(Q)\to(A\otimes A')(Q)$$

mapping $\mathrm{Br}_Q(a)\otimes\mathrm{Br}_Q(a')$ on $\mathrm{Br}_Q(a\otimes a')$, and we will prove that this homomorphism is bijective.

For any subgroup R of Q, denote by $(B')^R$ the set of R-fixed elements of B' and by B'_R a set of representatives for the orbits of R on B' (so $(B')^R\subset B'_R$); as $A\otimes A'=\bigoplus_{b\in B'}A\otimes b'$, it is easily checked that

$$(5.6.6)\qquad (A\otimes A')^R=\bigoplus_{b'\in B'_R}\mathrm{Tr}^R_{R_{b'}}(A^{R_{b'}}\otimes b')$$

and therefore, we get

$$(5.6.7)\qquad (A\otimes A')^Q_R=\bigoplus_{b'\in B'_R}\mathrm{Tr}^Q_{Q_{b'}}(A^{Q_{b'}}_{R_{b'}}\otimes b')$$

where $Q_{b'}$ and $R_{b'}$ are respectively the stabilizer of $b'\in B'$ in Q and R; now, it follows from (5.4.6) applied to Q that

$$(5.6.8)\qquad \mathrm{Br}_Q\Big(\bigoplus_{b'\in(B')^Q}A^Q\otimes b'\Big)=(A\otimes A')(Q)$$

and from (5.6.7) that

$$(5.6.9)\qquad \Big(\bigoplus_{b'\in(B')^Q}A^Q\otimes b'\Big)\cap\mathrm{Ker}(\mathrm{Br}^{A\otimes A'}_Q)=\bigoplus_{b'\in(B')^Q}\mathrm{Ker}(\mathrm{Br}^A_Q)\otimes b'.$$

In particular, (5.6.8) and (5.6.9) applied to $A=\mathcal{O}$ prove that $\{\mathrm{Br}_Q(b')\}_{b'\in(B')^Q}$ is a $\mathscr{k}$-basis of $A'(Q)$; consequently, we get

(5.6.10)

$$\bigoplus_{b'\in(B')^Q}(A^Q\otimes b'/\mathrm{Ker}(\mathrm{Br}^A_Q)\otimes b')\cong\bigoplus_{b'\in(B')^Q}A(Q)\otimes\mathrm{Br}_Q(b')=A(Q)\otimes A'(Q).$$

Finally, as (5.6.5) maps $\mathrm{Br}_Q(a)\otimes\mathrm{Br}_Q(b')$ on $\mathrm{Br}_Q(a\otimes b')$ for any $b'\in(B')^Q$, it follows from (5.6.8), (5.6.9) and (5.6.10) that (5.6.5) is bijective.

Moreover bijection (5.6.2) and isomorphisms (5.6.3) are easy consequences of (5.6.1). On the other hand, by (5.6.2) applied to the interior Q-algebras A_δ and $A'_{\delta'}$,

$\delta \times \delta'$ is still a local point of Q on $A_\delta \otimes A'_{\delta'}$ and then (5.6.4) follows (cf. [12], (2.13.1)).

5.7. It is clear that the multiplication induces an interior P-algebra isomorphism

(5.7.1) $$S \otimes S^0 \cong \mathrm{End}_{\mathcal{O}}(S)$$

and as P stabilizes an $\mathcal{O}$-basis of S containing the unity, there is a (unique) embedding from the trivial interior P-algebra $\mathcal{O}$ to $S \otimes S^0$.

5.8. Corollary. *With the notation above, for any subgroup Q of P, $S(Q)$ is a simple k-algebra and in particular, $|\mathcal{LP}_S(Q)| = 1$.*

Proof. By (5.6.1) and (5.7.1) we have $S(Q) \otimes S^0(Q) \cong (\mathrm{End}_{\mathcal{O}}(S))(Q)$; but it is clear that $S^0(Q) = S(Q)^0$ and as P stabilizes an $\mathcal{O}$-basis of S, it is well-known that $(\mathrm{End}_{\mathcal{O}}(S))(Q) \cong \mathrm{End}_k(S(Q))$ (cf. [12], (2.9.1)); consequently, $S(Q)$ has to be simple (since $S(Q) \neq \{0\}$).

We have only partial results both on the relationship between inclusions and on the relationship between fusions of local pointed groups on A, A' and $A \otimes A'$; but they are complete enough to prove Theorem 5.3.

5.9. Proposition. *With the notation above, let Q_δ, R_ε be local pointed groups on A and $Q_{\delta'}$, $R_{\varepsilon'}$ local pointed groups on A'.*

(5.9.1) *If $R \subset Q$ we have $m_{\varepsilon \times \varepsilon'}^{\delta \times \delta'} \leqq m_\varepsilon^\delta m_{\varepsilon'}^{\delta'}$. In particular, if $R_{\varepsilon \times \varepsilon'} \subset Q_{\delta \times \delta'}$ then $R_\varepsilon \subset Q_\delta$ and $R_{\varepsilon'} \subset Q_{\delta'}$.*

(5.9.2) *If $|R| = |Q|$ we have*

$$F_A(R_\varepsilon, Q_\delta) \cap F_{A'}(R_{\varepsilon'}, Q_{\delta'}) \subset F_{A \otimes A'}(R_{\varepsilon \times \varepsilon'}, Q_{\delta \times \delta'})$$

Proof. If $j \in \delta$ and $j' \in \delta'$, it follows from (5.6.4) that there is $j'' \in \delta \times \delta'$ such that $j''(j \otimes j') = j'' = (j \otimes j')j''$; but if $R \subset Q$, the isomorphism $A(R_\varepsilon) \otimes A'(R_{\varepsilon'}) \cong (A \otimes A')(R_{\varepsilon \times \varepsilon'})$ (cf. (5.6.3)) maps $s_\varepsilon(j) \otimes s_{\varepsilon'}(j')$ on $s_{\varepsilon \times \varepsilon'}(j \otimes j')$; then it is clear that $m_{\varepsilon \times \varepsilon'}^{\delta \times \delta'} \leqq m_\varepsilon^\delta m_{\varepsilon'}^{\delta'}$.

Assume now that $|R| = |Q|$ and let $\tilde{\varphi}$ be an element of $F_A(R_\varepsilon, Q_\delta) \cap F_{A'}(R_{\varepsilon'}, Q_{\delta'})$; by definition (cf. [11], Def. 2.5), there are exomorphisms of interior R-algebras

(5.9.3) $$\tilde{f}_\varphi : A_\varepsilon \cong \mathrm{Res}_\varphi(A_\delta) \quad \text{and} \quad \tilde{f}'_\varphi : A'_{\varepsilon'} \cong \mathrm{Res}_\varphi(A'_{\delta'})$$

fulfilling

(5.9.4)
$$\mathrm{Res}_1^R(\tilde{f}_\varepsilon) = \mathrm{Res}_1^Q(\tilde{f}_\delta) \circ \mathrm{Res}_1^R(\tilde{f}_\varphi) \quad \text{and} \quad \mathrm{Res}_1^R(\tilde{f}_{\varepsilon'}) = \mathrm{Res}_1^Q(\tilde{f}_{\delta'}) \circ \mathrm{Res}_1^R(\tilde{f}'_\varphi).$$

But it follows from (5.6.1) applied to $A_\delta \otimes A'_{\delta'}$ that $\delta \times \delta'$ is the unique local point of Q on $A_\delta \otimes A'_{\delta'}$; similarly, $\varepsilon \times \varepsilon'$ is the unique local point of R on $A_\varepsilon \otimes A'_{\varepsilon'}$. Consequently, setting $A'' = A \otimes A'$, $\delta'' = \delta \times \delta'$ and $\varepsilon'' = \varepsilon \times \varepsilon'$, the tensor product of the exomorphisms (5.9.3) induces an exomorphism

(5.9.5) $$\tilde{f}''_\varphi : A''_{\varepsilon''} \cong \mathrm{Res}_\varphi(A''_{\delta''})$$

such that (cf. (5.6.4) and [12], (2.3.3))

(5.9.6) $$\mathrm{Res}_\varphi(\tilde g_{\delta''}) \circ \tilde f''_\varphi = (\tilde f_\varphi \otimes \tilde f'_\varphi) \circ \tilde g_{\varepsilon''}.$$

Now, it is easily checked from (5.6.4), (5.9.6) and (5.9.3) that the exomorphism (5.9.5) fulfills $\mathrm{Res}^R_1(\tilde f_{\varepsilon''}) = \mathrm{Res}^Q_1(\tilde f_{\delta''}) \circ \mathrm{Res}^R_1(\tilde f''_\varphi)$, whence $\tilde\varphi \in F_{A''}(R_{\varepsilon''}, Q_{\delta''})$.

5.10. Let Q_δ and $Q_{\delta'}$ be respectively local pointed groups on A and A', set $A'' = A \otimes A'$, $\delta'' = \delta \times \delta'$ and $E = F_A(Q_\delta) \cap F_{A'}(Q_{\delta'})$, and denote by $\tilde g_{\delta''}$: $A''_{\delta''} \to A_\delta \otimes A'_{\delta'}$ the embedding (5.6.4). Our last result shows that the central extension $\hat E^{\delta''}$ of E by k^* is the "sum" of the central extensions $\hat E^\delta$ and $\hat E^{\delta'}$ (cf. 2.12 and [12], 7.1 and 5.9).

5.11. Proposition. *With the notation above, there is a unique k^*-group isomorphism*

(5.11.1) $$\hat E^\delta * \hat E^{\delta'} \cong \hat E^{\delta''}$$

mapping $(\bar a, \tilde\varphi) \otimes (\bar a', \tilde\varphi) \in \hat E^\delta * \hat E^{\delta'}$ on $(\bar a'', \tilde\varphi) \in \hat E^{\delta''}$ where $\tilde\varphi \in E$, a and a' are respectively elements of A^*_δ and $(A'_{\delta'})^*$ fulfilling

(5.11.2) *for any $u \in Q$, $a \cdot u = \varphi(u) \cdot a$ and $a' \cdot u = \varphi(u) \cdot a'$,*

a'' is the unique element of $A''_{\delta''}$ such that

(5.11.3) $$g_{\delta''}(a'') = g_{\delta''}(1)(a \otimes a') g_{\delta''}(1)$$

and $\bar a$, $\bar a'$ and $\bar a''$ are respectively the images of a, a' and a'' in $\bar N_{A_\delta}(Q)$, $\bar N_{A'_{\delta''}}(Q)$ and $\bar N_{A''_{\delta''}}(Q)$.

5.12. *Remark.* Notice that (5.11.2) guarantees $a \in N_{A_\delta}(Q)$ and $a' \in N_{A'_{\delta'}}(Q)$ (cf. 2.12) and we prove below that (5.11.3) implies $a'' \in N_{A''_{\delta''}}(Q)$.

Proof. Set $i = g_{\delta''}(1)$; as $(A_\delta \otimes A'_{\delta'})(Q_{\delta''}) \cong k$ (cf. (5.6.3)) and, with the usual identification, δ'' is the unique local point of Q on $A_\delta \otimes A'_{\delta'}$ (cf. (5.6.2)), we have

(5.11.4) $$s_{\delta''}(i^{a \otimes a'}) = s_{\delta''}(i);$$

hence, setting

(5.11.5) $$d'' = (i^{a \otimes a'})\, i + (1 - i^{a \otimes a'})(1 - i)$$

it is clear that $d'' \in 1 + J((A_\delta \otimes A'_{\delta'})^Q)$ and it is easily checked that $i(a \otimes a')d'' = i(a \otimes a')i = (a \otimes a')d''i$; in particular, a'' is invertible and we have

(5.11.6) $$(a'')^{-1} = i(d'')^{-1}(a \otimes a')^{-1}.$$

Moreover, $g_{\delta''}(a'' \cdot u) = i(a \otimes a') \cdot u \cdot i = i \cdot \varphi(u) \cdot (a \otimes a')i = g_{\delta''}(\varphi(u) \cdot a'')$ and therefore, $a'' \cdot u = \varphi(u) \cdot a''$ for any $u \in Q$. Consequently, a'' belongs to $N_{A''_{\delta''}}(Q)$ and has the same action as φ on the image of Q in $(A''_{\delta''})^*$, which proves that $(\bar a'', \tilde\varphi)$ is an element of $\hat E^{\delta''}$ (cf. 2.12).

Notice first that $\bar a''$ does not depend on the choice of $g_{\delta''}$ in $\tilde g_{\delta''}$. Indeed, if $g' \in \tilde g_{\delta''}$ there is $c'' \in ((A_\delta \otimes A'_{\delta'})^Q)^*$ such that $g'(b'') = g_{\delta''}(b'')^{c''}$ for any $b'' \in A''_{\delta''}$; hence, if $b'' \in A''_{\delta''}$ is such that $g'(b'') = g'(1)(a \otimes a')\, g'(1)$ we get (cf. (5.11.6))

$$g_{\delta''}((a'')^{-1} b'') = i(d'')^{-1}(a \otimes a')^{-1} i c'' (a \otimes a')(c'')^{-1} i = i(d'')^{-1}(ic'')^{a \otimes a'}(c'')^{-1} i$$

but, as $(A_\delta \otimes A'_{\delta'})(Q_{\delta''}) \cong k$ (cf. (5.6.3)), we have $s_{\delta''}((ic'')^{a\otimes a'}) = s_{\delta''}(ic'')$; consequently, $(a'')^{-1}b''$ belongs to $1 + J((A''_{\delta''})^Q)$, whence $\bar{a}'' = \bar{b}''$.

Secondly, if b and b' are respectively elements of $N_{A_\delta}(Q)$ and $N_{A'_{\delta'}}(Q)$ such that $\bar{a} = \bar{b}$ and $\bar{a}' = \bar{b}'$, and $b'' \in A''_{\delta''}$ is such that $g_{\delta''}(b'') = i(b \otimes b')i$, we claim that $\bar{a}'' = \bar{b}''$. Indeed, we get (cf. (5.11.5))

$$g_{\delta''}((a'')^{-1}b'') = i(d'')^{-1}\,(i^{a\otimes a'})\,(a^{-1}b \otimes (a')^{-1}b')i$$

and therefore, it follows from (5.11.4) that $(a'')^{-1}b''$ belongs to $1 + J((A''_{\delta''})^Q)$, whence $\bar{a}'' = \bar{b}''$.

Consequently, there is a unique map $\mu\colon \hat{E}^\delta * \hat{E}^{\delta'} \to \hat{E}^{\delta''}$ mapping $(\bar{a}, \tilde{\varphi}) \otimes (\bar{a}', \tilde{\varphi})$ on $(\bar{a}'', \tilde{\varphi})$; moreover, it is clear that μ maps $\lambda.(\bar{a}, \tilde{\varphi}) \otimes (\bar{a}', \tilde{\varphi})$ on $\lambda.(\bar{a}'', \tilde{\varphi})$ for any $\lambda \in k^*$, and induces the identity on E. So, it suffices to prove that μ is a group homomorphism.

Let $\tilde{\psi}$ be an element of E, b and b' elements of A^*_δ and $(A'_{\delta'})^*$ fulfilling 5.11.2 with respect to ψ, e'' the element $(i^{b\otimes b'})i + (1 - i^{b\otimes b'})(1 - i)$ of $1 + J((A_\delta \otimes A'_{\delta'})^Q)$ (cf. (5.11.5)), b'' the unique element of $A''_{\delta''}$ such that $g_{\delta''}(b'') = i(b \otimes b')i$ (cf. (5.11.3)), and $\bar{b}$, $\bar{b}'$ and $\bar{b}''$ the respective images of b, b' and b'' in $\bar{N}_{A_\delta}(Q)$, $\bar{N}_{A'_{\delta'}}(Q)$ and $\bar{N}_{A''_{\delta''}}(Q)$, so that μ maps $(\bar{b}, \tilde{\psi}) \otimes (\bar{b}', \tilde{\psi})$ on $(\bar{b}'', \tilde{\psi})$. Then it is clear that ab and $a'b'$ fulfill (5.11.2) with respect to $\varphi \circ \psi$, and if c'' is the unique element of $A''_{\delta''}$ such that $g_{\delta''}(c'') = i(ab \otimes a'b')i$, it follows from (5.11.6) applied to a'' and b'' that

$$\begin{aligned} g_{\delta''}((a''b'')^{-1}c'') &= i(e'')^{-1}(b \otimes b')^{-1}i(d'')^{-1}\,(a \otimes a')^{-1}i(ab \otimes a'b')i \\ &= i(e'')^{-1}\,(i(d'')^{-1})^{b\otimes b'}\,(i^{ab\otimes a'b'})i; \end{aligned}$$

consequently, μ maps $(\overline{ab}, \tilde{\varphi} \circ \tilde{\psi}) \otimes (\bar{a}'\bar{b}', \tilde{\varphi} \circ \tilde{\psi})$ on $(\bar{c}'', \tilde{\varphi} \circ \tilde{\psi})$ and $(a''b'')^{-1}c''$ belongs to $1 + J((A''_{\delta''})^Q)$ whence $\bar{c}'' = \bar{a}''\bar{b}''$; hence, μ is a group homomorphism.

Proof of Theorem 5.3. By corollary 5.8 we have $\mathcal{LP}_S(Q) = \{\sigma\}$ and therefore, by 5.6.2 there is a bijection

(5.3.4) $$\mathcal{LP}_A(Q) \to \mathcal{LP}_{S\otimes A}(Q)$$

mapping $\delta \in \mathcal{LP}_A(Q)$ on $\sigma \times \delta$, which is the unique local point of Q on $S \otimes A$ such that $\mathrm{Br}_Q(i \otimes j) \in \mathrm{Br}_Q(\sigma \times \delta)$ if $i \in \sigma$ and $j \in \delta$; hence, there is $j' \in \sigma \times \delta$ such that $(i \otimes j)j' = j' = j'(i \otimes j)$ and therefore, such that $(1 \otimes j)j' = j' = j'(1 \otimes j)$. Moreover, if k is a primitive idempotent of $(S \otimes A)^Q$ such that $\mathrm{Br}_Q(k) \neq 0$ and $(1 \otimes j)k = k = k(1 \otimes j)$, it follows from (5.6.1) that $\mathrm{Br}_Q(k)$ is a conjugate of some $\mathrm{Br}_Q(l \otimes j)$ where l is a primitive idempotent of S^Q such that $\mathrm{Br}_Q(l) \neq 0$, which implies $l \in \sigma$ by corollary 5.8; consequently, $\mathrm{Br}_Q(k) \in \mathrm{Br}_Q(\sigma \times \delta)$, whence $k \in \sigma \times \delta$.

To prove (5.2.2) it suffices to prove the inclusion

(5.3.5) $$F_A(R_\varepsilon, Q_\delta) \cap F \subset F_{S\otimes A}(R_{S\times\varepsilon}, Q_{S\times\delta});$$

indeed, it follows from (5.3.5) applied to $S \otimes A$ and S^0 that

(5.3.6) $$F_{S\otimes A}(R_{S\times\varepsilon}, Q_{S\times\delta}) \cap F \subset F_{S^0\otimes S\otimes A}\,(R_{S\times(S\times\varepsilon)}, Q_{S\times(S\times\delta)})$$

(since $F_{S^0}(R_{S^0\times 1}, Q_{S^0\times 1}) = F$); but there is an embedding $\mathcal{O} \to S^0 \otimes S$ (cf. 5.7) and therefore we get an embedding

(5.3.7) $$A \to S^0 \otimes S \otimes A$$

mapping Q_δ on $Q_{S^0\otimes S\times\delta}$ and R_ε on $R_{S^0\otimes S\times\varepsilon}$, which implies (cf. [11], 2.14)

$$(5.3.8)\qquad F_A(R_\varepsilon, Q_\delta) = F_{S^0\otimes S\otimes A}(R_{S^0\otimes S\times\varepsilon}, Q_{S^0\otimes S\times\delta});$$

moreover, it is easy to check that

$$(5.3.9)\qquad S^0\times(S\times\delta) = S^0\otimes S\times\delta \quad\text{and}\quad S^0\times(S\times\varepsilon) = S^0\otimes S\times\varepsilon;$$

consequently, by (5.3.6), (5.3.8) and (5.3.9) we have

$$(5.3.10)\qquad F_{S\otimes A}(R_{S\times\varepsilon}, Q_{S\times\delta})\cap F\subset F_A(R_\varepsilon, Q_\delta);$$

now, (5.3.2) follows from (5.3.5) and (5.3.10).

If $\tilde\varphi\in F_A(R_\varepsilon, Q_\delta)\cap F$ we know (cf. [11], 2.11) that there is $\gamma\in\mathscr{LP}_A(\varphi(R))$ such that $\varphi(R)_\gamma\subset Q_\delta$ and $\tilde\varphi$ is the composition of $\tilde\psi\in F_A(R_\varepsilon, \varphi(R)_\gamma)$ (where $\psi(u)=\varphi(u)$ for any $u\in R$) and the A-fusion defined by the inclusion $\varphi(R)_\gamma\subset Q_\delta$. Then, by (5.3.7) we have $\varphi(R)_{S_0\otimes S\times\gamma}\subset Q_{S^0\otimes S\times\delta}$ too and therefore, by (5.3.9) and (5.9.1) applied to $S\otimes A$ and S^0, we get $\varphi(R)_{S\times\gamma}\subset Q_{S\times\delta}$ (notice that applying this result to the case $A=\mathcal{O}$ we get $\varphi(R)_{S\times 1}\subset Q_{S\times 1}$, which can be proved directly too). On the other hand, as $\tilde\varphi\in F$, it is easily checked that $\tilde\psi\in F_S(R_{S\times 1}, \varphi(R)_{S\times 1})$ and therefore, by (5.3.4) and (5.9.2) applied to A and S, we get $\tilde\psi\in F_{S\otimes A}(R_{S\times\varepsilon}, \varphi(R)_{S\times\gamma})$. Consequently, $\tilde\varphi\in F_{S\otimes A}(R_{S\times\varepsilon}, Q_{S\times\delta})$ which proves (5.3.5).

Finally, (5.3.3) follows from Proposition 5.11 applied to A and S, and from the fact that $\hat E^\sigma\cong k^*\times E$ if $\mathscr{LP}_S(Q)=\{\sigma\}$ (cf. [13]).

§ 6. On the interior *P*-algebra *SP*

6.1. Let P be a finite p-group and S an $\mathcal{O}$-simple P-algebra of $\mathcal{O}$-rank prime to p, and denote by B the interior P-algebra SP associated with S (cf. 2.2). In this section we discuss on the special features of B that we need to prove the main theorem.

6.2. First of all notice that, as $\mathrm{Aut}(S)\cong S^*/\mathcal{O}^*$ (cf. [9], Prop. 2.3 or [12], (2.5.3)) and k is algebraically closed, any automorphism of S is induced by an element of S^* of determinant one. But, denoting by U the finite subgroup of $\mathcal{O}^*$ of order $\mathrm{rank}_\mathcal{O}(S)$, U is a cyclic p'-group and therefore, $\mathbf{H}^2(P, U)=\{0\}=\mathbf{H}^1(P, U)$ (cf. [5], Ch. I, Th. 16.19). Consequently, the structural group homomorphism $P\to\mathrm{Aut}(S)$ can be lifted to a unique group homomorphism $\varrho\colon P\to S^*$ such that $\det(\varrho(u))=1$ for any $u\in \mathrm{P}$. Henceforth we consider S endowed with ϱ as an interior P-algebra, and we identify often B with $S\bigotimes_{\mathcal{O}}\mathcal{O}P$ through the canonical interior P-algebra isomorphism (2.7.1). The following statement is our main result on B.

6.3. Theorem. *Let $\hat B$ be an interior P-algebra and $\tilde g\colon \hat B\to B$ an interior P-algebra exomorphism. If $B=\mathrm{Im}(g)+J(B)$ then $\tilde g$ is surjective.*

6.4. *Remark.* With the terminology introduced in § 4 the equality above says that $\tilde g$ is an $\mathcal{O}$-algebra covering exomorphism (cf. 4.3).

In order to prove the theorem we need the following three lemmas.

6.5. Lemma. *Let A and $\hat{A}$ be $\mathcal{O}$-algebras and $f\colon \hat{A} \to A$ an $\mathcal{O}$-algebra homomorphism. If $A = \mathrm{Im}(f) + J(A)^2$ then f is surjective.*

Proof. Arguing by induction on n we prove first that $A = \mathrm{Im}(f) + J(A)^n$; we may assume that $n \geqq 2$ and $J(A)^{n-1} = (\mathrm{Im}(f) \cap J(A)^{n-1}) + J(A)^n$; it follows that

$$\begin{aligned} J(A)^n &= ((\mathrm{Im}(f) \cap J(A)) + J(A)^2)((\mathrm{Im}(f) \cap J(A)^{n-1}) + J(A)^n) \\ &= (\mathrm{Im}(f) \cap J(A)^n) + J(A)^{n+1} \end{aligned}$$

and therefore, $A = \mathrm{Im}(f) + J(A)^{n+1}$. But there is n such that $J(A)^n \subset J(\mathcal{O}).A$ and then the lemma follows from Nakayama's lemma.

6.6. Lemma. *Let A be an $\mathcal{O}$-algebra such that $A/J(A)$ is a simple k-algebra, and M an A-bimodule. Denote by $[A, M]$ the $\mathcal{O}$-submodule of M generated by the set of elements $a.m - m.a$ where a runs over A and m over M. If N is an A-subbimodule such that $M = N + [A, M]$ then $M = N$.*

Proof. As $[A, M/N]$ is the image of $[A, M]$ in M/N, we may assume that $\mathcal{O} = k$ and M is a simple A-bimodule; then there is an A-bimodule isomorphism $M \cong A/J(A)$ and since $A/J(A) \cong \mathrm{End}_k(V)$ where V is a k-vector space, the image of $[A, A/J(A)]$ in $\mathrm{End}_k(V)$ is contained in $\mathrm{Ker}(\mathrm{tr}_V)$, where $\mathrm{tr}_V\colon \mathrm{End}_k(V) \to k$ is the trace map, which proves that $M \neq [A, M]$, and therefore, $M = N$.

6.7. Lemma. *The map $P \to B$ mapping $u \in P$ on $1 - \varrho(u^{-1})u$ induces an S-bimodule isomorphism $S \bigotimes_{\mathbf{Z}} (P/\Phi(P)) \cong J(B)/(J(B)^2 + J(\mathcal{O}) \cdot B)$.*

Proof. We may assume that $\mathcal{O} = k$ and identify B with $S \bigotimes_k kP$; so, we consider the map $P \to S \bigotimes_k kP$ mapping $u \in P$ on $1 \otimes (1 - u)$ and since $1 - u \in J(kP)$, $1 \otimes (1 - u)$ belongs to $J\left(S \bigotimes_k kP\right)$; moreover, as

$$1 - uu' = (1 - u) + (1 - u') + (1 - u)(1 - u')$$

for any $u, u' \in P$ and $J\left(S \bigotimes_k kP\right)^2 = S \bigotimes_k J(kP)^2$, this map induces a $\mathbf{Z}$-module homomorphism from $P/\Phi(P)$ to $J\left(S \bigotimes_k kP\right) \Big/ J\left(S \bigotimes_k kP\right)^2$ and therefore a S-bimodule homorphism

$$S \bigotimes_{\mathbf{Z}} (P/\Phi(P)) \to J\left(S \bigotimes_k kP\right) \Big/ J\left(S \bigotimes_k kP\right)^2 \tag{6.7.1}$$

which is clearly surjective since $J(kP) = \sum_{u \in P} k(1 - u)$ and we have

$$J\left(S \bigotimes_k kP\right) \Big/ J\left(S \bigotimes_k kP\right)^2 \cong S \bigotimes_k (J(kP)/J(kP)^2).$$

Finally, denoting by U a subset of P such that the image of $\{1 - u\}_{u \in U}$ in $J(kP)/J(kP)^2$ is a k-basis, and by Q the subgroup of P generated by U, we get

$\mathscr{k}P = \mathscr{k}Q + J(\mathscr{k}P)^2$ and it follows from Lemma 6.5 that $\mathscr{k}P = \mathscr{k}Q$, whence $P = Q$; in particular, the image of U in $P/\Phi(P)$ is a generator set; consequently

$$\dim_{\mathscr{k}}\left(\mathscr{k}\bigotimes_{\mathbf{Z}}(P/\Phi(P))\right) \leqq \dim_{\mathscr{k}}(J(\mathscr{k}P)/J(\mathscr{k}P)^2)$$

and therefore (6.7.1) is a bijective homomorphism.

Proof of Theorem 6.3. By Nakayama's lemma we may assume that $\mathcal{O} = \mathscr{k}$, and by Lemma 6.5 it suffices to prove that $J(B) = (\mathrm{Im}(g) \cap J(B)) + J(B)^2$. Set $C = \mathrm{Im}(g)$ and denote by $\bar{X}$ the image of any $X \subset J(B)$ in $J(B)/J(B)^2$; then $\overline{J(B)}$ has an evident structure of C-bimodule and $\overline{C \cap J(B)}$ is a C-subbimodule of $\overline{J(B)}$; as $C/(C \cap J(B)) \cong B/J(B) \cong S$ and $J(B)$ annihilates $\overline{J(B)}$, $\overline{J(B)}$ becomes an S-bimodule and $\overline{C \cap J(B)}$ an S-subbimodule of $\overline{J(B)}$; so, by Lemma 6.6 it suffices to prove that $\overline{J(B)} = \overline{C \cap J(B)} + [S, \overline{J(B)}]$.

By Lemma 6.7 there is an S-bimodule isomorphism $S\bigotimes_{\mathbf{Z}}(P/\Phi(P)) \cong \overline{J(B)}$ mapping $s \otimes \tilde{u}$ on $\overline{s(1-\varrho(u^{-1})u)}$, where $s \in S$, $u \in P$ and $\tilde{u}$ is the image of u in $P/\Phi(P)$. But identifying again B with $S\bigotimes_{\mathscr{k}}\mathscr{k}P$ (cf. 6.2), it is easily checked that $[S, \overline{J(B)}] = [S,S]\bigotimes_{\mathbf{Z}}(P/\Phi(P))$; moreover, $S = \mathscr{k}\cdot 1 + [S,S]$ since p does not divide $\dim_{\mathscr{k}}(S)$. Consequently, it suffices to prove that $\overline{C \cap J(B)} + [S, \overline{J(B)}]$ contains $\{\overline{1-\varrho(u^{-1})u}\}_{u \in P}$.

As $B = C + J(B)$, a maximal $\mathcal{O}$-semisimple subalgebra of C is still maximal in B, and therefore there is $n \in J(B)$ such that $S^{1+n} \subset C$ (cf. [9], Cor. 2.4); in particular, $\varrho(u)^{1+n} \in C$ and therefore $\overline{[\varrho(u), 1+n] - \varrho(u)^{-1}u}$ belongs to $\overline{C \cap J(B)}$ (since $1 - \varrho(u)^{-1}u \in J(B)$) and so, $\varrho(u) + J(B) = u + J(B)$; but it is easily checked that

$$\overline{[\varrho(u), 1+n] - \varrho(u)^{-1}u} = (\bar{n} - \varrho(u)^{-1}\cdot\overline{n\varrho(u)}) + \overline{(1-\varrho(u)^{-1}u)};$$

consequently, as $\bar{n} - \varrho(u)^{-1}\cdot\overline{n\varrho(u)} \in [S, \overline{J(B)}]$, the element $\overline{1-\varrho(u)^{-1}u}$ belongs to $\overline{C \cap J(B)} + [S, \overline{J(B)}]$ for any $u \in P$.

We know that B^* acts transitively on the set of $\mathcal{O}$-simple subalgebras S' of B such that $\mathrm{rank}_{\mathcal{O}}(S') = \mathrm{rank}_{\mathcal{O}}(S)$ (cf. [9], Cor. 2.4) and the next result explicits a transversal in B^* to the stabilizer of S.

6.8. Proposition. *Set* $W = 1 + \sum_{u \in P} \mathrm{Ker}(\mathrm{tr})(1 - \varrho(u)^{-1}u)$ *where* $\mathrm{tr}\colon S \to \mathcal{O}$ *denotes the trace map. Then we have* $W \subset B^*$ *and for any maximal* $\mathcal{O}$*-semisimple subalgebra* S' *of* B *there is a unique* $w \in W$ *such that* $S' = S^w$.

Proof. Let us identify B with $S\bigotimes_{\mathcal{O}}\mathcal{O}P$ (cf. 6.2) and set $J = \sum_{u \in P}\mathcal{O}(1-u)$; as S and S' are both maximal $\mathcal{O}$-semisimple subalgebras of B, there is $b \in B^*$ such that $S' = S^b$ (cf. [9], Cor. 2.4); but, as $S \otimes J$ is an ideal of B contained in $J(B)$ and $B/S \otimes J \cong S$, we have $B^* = S^*.(1 + S \otimes J)$ and therefore, we may assume that $b \in 1 + S \otimes J$; moreover, as p does not divide $\mathrm{rank}_{\mathcal{O}}(S)$, we have $S = \mathcal{O} \oplus \mathrm{Ker}(\mathrm{tr})$ and so

$$b = 1 + n + c = (1+n)\,(1 + (1+n)^{-1}c)$$

where $n \in 1 \otimes J$ and $c \in \mathrm{Ker(tr)} \otimes J$; finally, as $1+n$ centralizes S and $(1+n)^{-1}$ belongs to $1+1 \otimes J$, we get $S' = S^{1+m}$ where $m = (1+n)^{-1} c \in \mathrm{Ker(tr)} \otimes J$.

On the other hand, as $\mathrm{Aut}(S) \cong S^*/\mathcal{O}^*$ (cf. [12], (2.5.3)), the stabilizer of S in B^* is the product of S^* by the centralizer, namely the subgroup $S^* \otimes (\mathcal{O}P)^* = S^*(1+1 \otimes J)$ of B^*; hence, if $S^{1+m} = S^{1+m'}$ where $m, m' \in \mathrm{Ker(tr)} \otimes J$, then $(1+m')(1+m)^{-1} \in 1+1 \otimes J$ since the image of $(1+m')(1+m)^{-1}$ in $B/S \otimes J$ is the unity, and therefore there is $n \in 1 \otimes J$ such that

$$1+m' = (1+n)\,(1+m) = 1+n+m+nm,$$

whence $n = m' - m - nm$; but m, m' and nm belong to $\mathrm{Ker(tr)} \otimes J$, whereas $n \in 1 \otimes J$; consequently, $n = 0$ and $m' = m$.

6.9. Corollary. *The group $(B^P)^*$ acts transitively on the set of P-stable maximal $\mathcal{O}$-simple subalgebras of B. In particular, if S' is a P-algebra such that $B \cong S'P$ as interior P-algebras, we have $S \cong S'$ as P-algebras.*

Proof. With the notation of Proposition 6.8, if S' is a P-stable maximal $\mathcal{O}$-simple subalgebra of B, there is a unique $w \in W$ such that $S' = S^w$; but it is clear that P stabilizes W (since ϱ is unique); consequently, P fixes w.

The last result of this section describes the fusions of local pointed groups on B, when S has a P-stable $\mathcal{O}$-basis, which is the main tool to prove (in Sect. 7) that condition (1.6.2) implies condition (1.6.1). Although we will apply Theorem 5.3, a more direct proof of that result could certainly be obtained from Proposition 6.8 above.

6.10. Proposition. *Assume that P stabilizes an $\mathcal{O}$-basis of S and let Q_δ and R_ε be local pointed groups on B. Then $F_B(R_\varepsilon, Q_\delta) = E_P(R, Q)$.*

Proof. As p does not divide $\mathrm{rank}_{\mathcal{O}}(S)$, P fixes at least one element in a P-stable $\mathcal{O}$-basis of S and therefore, $S(P) \neq \{0\}$ (cf. [12], (2.8.4)); it follows easily that P stabilizes an $\mathcal{O}$-basis of S which contains the unity (see 7.5 below for a more detailed argument). Hence, by 6.2 and Theorem 5.3 we have

$$F_{\mathcal{O}P}(R_{\varepsilon'}, Q_{\delta'}) \cap F = F_B(R_\varepsilon, Q_\delta) \cap F$$

where $F = F_S(R_{S\times 1}, Q_{S\times 1})$ and δ', ε' are respectively the local points of Q, R on $\mathcal{O}P$ corresponding to δ, ε through the canonical bijection (5.3.1). But on one hand, it is clear that $\mathcal{LP}_{\mathcal{O}P}(Q) = \{\delta'\}$ and $\mathcal{LP}_{\mathcal{O}P}(R) = \{\varepsilon'\}$, and therefore (cf. [11], 2.10 and Th. 3.1)

$$F_{\mathcal{O}P}(R_{\varepsilon'}, Q_{\delta'}) = E_P(R_{\varepsilon'}, Q_{\delta'}) = E_P(R, Q) \subset F.$$

On the other hand, it follows from example 4.25 and Proposition 4.18 that $F_B(R_\varepsilon, Q_\delta) \subset F$. Consequently, $E_P(R, Q) = F_B(R_\varepsilon, Q_\delta)$.

§7. Proof of the main theorem

7.1. Let G be a finite group, $\alpha = \{b\}$ a point of G on $\mathcal{O}G$ and P_γ a defect pointed group of G_α; denote by $\tilde{e}\colon B \to \mathrm{Res}^G_P(\mathcal{O}Gb)$ an embedded algebra associated with

P_γ (as a pointed group on $\mathcal{O}Gb$). The key step in our proof of the main theorem is the following result (which is itself a consequence of that theorem).

7.2. Proposition. *Assume that B has an $\mathcal{O}$-simple factor S of $\mathcal{O}$-rank prime to p, and consider S endowed with the interior P-algebra structure induced by B. If G_α is locally controlled by P_γ then there is a unique interior P-algebra exoisomorphism $B \cong S \bigotimes_{\mathcal{O}} \mathcal{O}P$.*

Proof. Denote by $\hat{B}$ the interior P-algebra $B \bigotimes_{\mathcal{O}} \mathcal{O}P \cong BP$ (cf. (2.7.1)) and by $\tilde{g}$: $\hat{B} \to B$ the interior P-algebra exomorphism determined by the homomorphism g mapping $b \otimes u$ on b for any $b \in B$ and any $u \in P$; it follows from example 4.25 that $\tilde{g}$ is a strict covering exomorphism.

Set $A = \mathrm{Ind}_P^G(B)$, $\tilde{d} = \tilde{d}_P^G(B)$, $\hat{A} = \mathrm{Ind}_P^G(\hat{B})$, $\tilde{\hat{d}} = \tilde{d}_P^G(\hat{B})$, $\tilde{f} = \mathrm{Ind}_P^G(\tilde{g})$ and $C = \mathcal{O}Gb$. As G_α is locally controlled by P_γ, G is locally controlled by P on B (cf. 3.5) and therefore, $\tilde{f}$: $\hat{A} \to A$ is a strict covering exomorphism too (cf. Cor. 4.23). But it follows from [9], Th. 3.4 that there is a unique embedding $\tilde{c}$: $C \to A$ such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Res}_P^G(C) & \xrightarrow{\mathrm{Res}_P^G(\tilde{c})} & \mathrm{Res}_P^G(A) \\ \tilde{e}\uparrow & \nearrow \tilde{d} & \\ B & & \end{array} \tag{7.2.1}$$

and, as usual, we identify G_α and P_γ with the corresponding pointed groups on A; then $\tilde{c}$: $C \to A$ and $\tilde{d}$: $B \to \mathrm{Res}_P^G(A)$ are respectively embedded algebras associated with G_α and P_γ (cf. 2.10). Consequently, there are $\hat{\alpha} \in \mathcal{P}_{\hat{A}}(G)$ and $\hat{\gamma} \in \mathcal{P}_{\hat{A}}(P)$ such that $f(\hat{\alpha}) \subset \alpha$ and $f(\hat{\gamma}) \subset \gamma$ (cf. 4.14), and it follows from Proposition 4.18 that $P_{\hat{\gamma}}$ is a defect pointed group of $G_{\hat{\alpha}}$; moreover, the following commutative diagram (cf. (2.9.3))

$$\begin{array}{ccc} \mathrm{Res}_P^G(\hat{A}) & \xrightarrow{\mathrm{Res}_P^G(\tilde{f})} & \mathrm{Res}_P^G(A) \\ \tilde{\hat{d}}\uparrow & & \uparrow\tilde{d} \\ \hat{B} & \xrightarrow{\tilde{g}} & B \end{array} \tag{7.2.2}$$

shows that $\tilde{\hat{d}}$: $\hat{B} \to \mathrm{Res}_P^G(\hat{A})$ is an embedded algebra associated with $P_{\hat{\gamma}}$.

Let $\tilde{\hat{c}}$: $\hat{C} \to \hat{A}$ be an embedded algebra associated with $G_{\hat{\alpha}}$. On one hand, as $P_{\hat{\gamma}} \subset G_{\hat{\alpha}}$, there is a unique embedding $\tilde{\hat{e}}$: $\hat{B} \to \mathrm{Res}_P^G(\hat{C})$ such that the following diagram commutes (cf. [11], 1.8 and 1.9)

$$\begin{array}{ccc} \mathrm{Res}_P^G(\hat{C}) & \xrightarrow{\mathrm{Res}_P^G(\tilde{\hat{c}})} & \mathrm{Res}_P^G(\hat{A}) \\ \tilde{\hat{e}}\uparrow & \nearrow \tilde{\hat{d}} & \\ \hat{B} & & \end{array} \tag{7.2.3}$$

On the other hand, as $f(\hat{c}(1))\in\alpha$, there is a unique exomorphism $\tilde{h}$: $\hat{C}\to C$ such that the following diagram commutes (cf. (2.10.1))

(7.2.4)
$$\begin{array}{ccc} \hat{A} & \xrightarrow{\ \tilde{f}\ } & A \\ {\scriptstyle \tilde{\hat{c}}}\uparrow & & \uparrow{\scriptstyle \tilde{c}} \\ \hat{C} & \xrightarrow{\ \tilde{h}\ } & C \end{array}$$

Now, it follows from (7.2.1), (7.2.2), (7.2.3) and the restriction of (7.2.4) to P that the following diagram commutes (cf. [12], (2.3.3) and (2.12.2))

(7.2.5)
$$\begin{array}{ccc} \mathrm{Res}^G_P(\hat{C}) & \xrightarrow{\ \mathrm{Res}^G_P(\tilde{h})\ } & \mathrm{Res}^G_P(C) \\ {\scriptstyle \tilde{\hat{e}}}\uparrow & & \uparrow{\scriptstyle \tilde{e}} \\ \hat{B} & \xrightarrow{\ \tilde{g}\ } & B \end{array}$$

Arguing as in Remark 4.16, the structural homomorphism $G\to\hat{C}^*$ induces clearly a unique interior G-algebra exomorphism $\tilde{h}'$: $C\to\hat{C}$ such that $\tilde{h}\circ\tilde{h}'=\widetilde{\mathrm{id}}_C$, and by Proposition 4.15, $\tilde{h}'$ is a strict covering exomorphism too, fulfilling $h'(\gamma)\subset\hat{\gamma}$ (cf. (7.2.5) and (4.15.1)); so, $h'(e(1))\in\hat{\gamma}$ and there is again a unique exomorphism $\tilde{g}'$: $B\to\hat{B}$ such that the following diagram commutes (cf. (2.10.1))

(7.2.6)
$$\begin{array}{ccc} \mathrm{Res}^G_P(C) & \xrightarrow{\ \mathrm{Res}^G_P(\tilde{h}')\ } & \mathrm{Res}^G_P(\hat{C}) \\ {\scriptstyle \tilde{e}}\uparrow & & \uparrow{\scriptstyle \tilde{\hat{e}}} \\ B & \xrightarrow{\ \tilde{g}'\ } & \hat{B} \end{array}$$

hence, by Proposition 4.15 again, $\tilde{g}'$ is also a strict covering exomorphism (and we get easily from $\tilde{h}\circ\tilde{h}'=\widetilde{\mathrm{id}}_C$ and [12] (2.3.3) and (2.12.2), that $\tilde{g}\circ\tilde{g}'=\widetilde{\mathrm{id}}_B$).

Let S be an $\mathcal{O}$-simple factor of B of $\mathcal{O}$-rank prime to p, and $\tilde{s}$: $B\to S$ the canonical exomorphism (recall that we consider S as an interior P-algebra in such a way that $\tilde{s}$ is an interior P-algebra exomorphism). Set $\hat{S}=S\bigotimes_{\mathcal{O}}\mathcal{O}P\cong SP$ (cf. (2.7.1)) and $\tilde{\hat{s}}=\tilde{s}\otimes\widetilde{\mathrm{id}}_{\mathcal{O}P}$; that is, $\tilde{\hat{s}}$: $\hat{B}\to\hat{S}$ is a surjective interior P-algebra exomorphism. So, the composition $\mathrm{Res}^P_1(\tilde{\hat{s}}\circ\tilde{g}')$: $\mathrm{Res}^P_1(B)\to\mathrm{Res}^P_1(\hat{S})$ is an $\mathcal{O}$-algebra covering exomorphism and therefore, it follows from Theorem 6.3 that $\mathrm{Res}^P_1(\tilde{\hat{s}}\circ\tilde{g}')$ is surjective. Consequently, to prove that $\tilde{\hat{s}}\circ\tilde{g}'$: $B\to\hat{S}$ is an exoisomorphism of interior P-algebras, it suffices to prove now that $\mathrm{rank}_{\mathcal{O}}(B)=\mathrm{rank}_{\mathcal{O}}(\hat{S})$.

Let V be a projective indecomposable S-module; clearly, $V\bigotimes_{\mathcal{O}}\mathcal{O}P$ is an indecomposable $\hat{S}$-module (cf. [9], Prop. 2.1) and all the simple factors of a Jordan-Hölder sequence of $k\bigotimes_{\mathcal{O}}\left(V\bigotimes_{\mathcal{O}}\mathcal{O}P\right)$ are isomorphic to $k\bigotimes_{\mathcal{O}}V$. As $\hat{s}\circ g'$

is surjective, $\mathrm{Res}_{\hat{s}\circ g'}\left(V\bigotimes_{\mathcal{O}}\mathcal{O}P\right)$ is an indecomposable B-module fulfilling the same condition: all the simple factors of a Jordan-Hölder sequence of $k\bigotimes_{\mathcal{O}}\mathrm{Res}_{\hat{s}\circ g'}\left(V\bigotimes_{\mathcal{O}}\mathcal{O}P\right)$ are isomorphic to $k\bigotimes_{\mathcal{O}}\mathrm{Res}_{\hat{s}\circ g'}(V)$; moreover, it is clear that the restriction of $\mathrm{Res}_{\hat{s}\circ g'}\left(V\bigotimes_{\mathcal{O}}\mathcal{O}P\right)$ to $\mathcal{O}P$ through the structural homomorphism $P\to B^*$ is a projective $\mathcal{O}P$-module. Hence, by Lemma 7.3 below, $\mathrm{Res}_{\hat{s}\circ g'}\left(V\bigotimes_{\mathcal{O}}\mathcal{O}P\right)$ is a projective B-module too and by Lemma 7.4 below, we have

$$\mathscr{P}(B)=\{\delta\} \quad \text{and} \quad \mathrm{rank}_{\mathcal{O}}(B)=m_\delta\,|P|\,\mathrm{rank}_{\mathcal{O}}(V)=\mathrm{rank}_{\mathcal{O}}(\hat{S})$$

(since $B(\delta)\cong k\bigotimes_{\mathcal{O}}S$ and therefore, $m_\delta=\mathrm{rank}_{\mathcal{O}}(V)$).

Finally, by Corollary 6.9, any automorphism of $\hat{S}$ as interior P-algebra is an inner one.

7.3. Lemma. *With the notation above, let N be a B-module and assume that the restriction of N through the structural homomorphism $P\to B^*$ is a projective $\mathcal{O}P$-module. Then N is a projective B-module.*

Proof. Set $M=\mathrm{Ind}_P^G(N)$ and $A=\mathrm{Ind}_P^G(B)$; clearly M is an A-module and the restriction of M through the structural homomorphism $G\to A^*$ is a projective $\mathcal{O}G$-module. But it follows from [9], Th. 3.4 that there is a unique embedding $\tilde{c}$: $\mathcal{O}Gb\to A$ such that $\tilde{d}_P^G(B)=\mathrm{Res}_P^G(\tilde{c})\circ\tilde{e}$ (cf. (7.2.1)). Consequently, $c(b)\cdot M$ is a projective $\mathcal{O}Gb$-module (since it is a direct summand of M as $\mathcal{O}G$-module) and therefore, $\mathrm{Res}_e(c(b)\cdot M)$ is a projective B-module; but, as $c\circ e$ is a representative for $\tilde{d}_P^G(B)$, we have $\mathrm{Res}_e(c(b)\cdot M)=\mathrm{Res}_{c\circ e}(M)\cong N$ as B-modules.

7.4. Lemma. *With the notation above, let N be a projective indecomposable B-module and assume that all the simple factors of a Jordan-Hölder sequence of $k\bigotimes_{\mathcal{O}}N$ are isomorphic. Then $\mathscr{P}(B)=\{\delta\}$ and* $\mathrm{rank}_{\mathcal{O}}(B)=m_\delta\,\mathrm{rank}_{\mathcal{O}}(N)$.

Proof. Let δ be the point of B such that $N\cong Bi$ where $i\in\delta$; if j is a primitive idempotent of B, we have $jBi\neq\{0\}$ if and only if $Bj\cong Bi$ as B-modules (since $jBi\cong\mathrm{Hom}_B(Bj,Bi)$); hence, if f is an idempotent of B with multiplicity m_δ on δ and zero everywhere else (cf. (2.2.1)), we have $(1-f)Bf=\{0\}$; but identifying B with its image through e, we get $(1-f)\mathcal{O}Gf=(1-f)Bf=\{0\}$ and therefore, we have also $\{0\}=f\mathcal{O}G(1-f)=fB(1-f)$ since the Cartan matrix is symmetric (cf. (2.15.1) and (2.15.2)); consequently, f belongs to ZB and as $ZB\subset B^P$, we get $f=1$ (cf. 2.20). So, $\mathscr{P}(B)=\{\delta\}$ and $\mathrm{rank}_{\mathcal{O}}(B)=m_\delta\,\mathrm{rank}_{\mathcal{O}}(N)$ (cf. (2.2.1)).

7.5. Henceforth we prove the main theorem. Assume first that (1.6.2) holds. Then, as $\mathrm{rank}_{\mathcal{O}}(B)/|P|\equiv|E_G(P_\gamma)|\bmod(p)$ (cf. [12], Prop. 14.6), p does not divide $\mathrm{rank}_{\mathcal{O}}(S)$ (cf. [12], 14.5) and the uniqueness of S follows now from Corollary 6.9. As P stabilizes by conjugation an $\mathcal{O}$-basis of $\mathcal{O}G$ and B is a direct summand of $\mathcal{O}G$ as $\mathcal{O}P$-module, P stabilizes by conjugation an $\mathcal{O}$-basis of B (cf. [12], (2.8.5)); but we have $B\cong SP=\bigoplus_{u\in P}Su$ and therefore S is still a direct summand of B as $\mathcal{O}P$-module;

so, P stabilizes an $\mathcal{O}$-basis W of S (cf. [12], (2.8.5)). In particular, (1.6.1) follows now from Proposition 6.10 and [11], Cor. 3.6 applied to P_γ. Moreover, the set $\{wu\}_{w\in W, u\in P}$ is a P-stable $\mathcal{O}$-basis of SP and therefore (cf. [12], (2.8.4)),

$$\dim_k(B(P)) = |W\cap S^P||ZP|$$

which implies that (cf. [12], (2.8.4) and (14.5.1))

$$W\cap S^P = \{w\} \quad \text{and} \quad S(P) = k.\mathrm{Br}_P(w);$$

in particular, if $1 = \sum_{w'\in W} \lambda_{w'} w'$ where $\lambda_{w'}\in\mathcal{O}$ then $\lambda_w\in\mathcal{O}^*$ and therefore we may replace w by the unity, getting a P-stable $\mathcal{O}$-basis of S which contains the unity as the unique P-fixed element.

7.6. From now on we assume that (1.6.1) holds. Then, by Theorem 3.8, G_α is locally controlled by P_γ and in particular, $|E_G(P_\gamma)| = 1$. If I is a decomposition of the unity in B (cf. 2.2) then $B = \bigoplus_{i\in I} Bi$ and for any $i\in I$, Bi becomes a projective $\mathcal{O}P$-module by left multiplication (since it is a direct summand of $\mathcal{O}G$ as $\mathcal{O}P$-modules), whence $|P|$ divides $\mathrm{rank}_\mathcal{O}(Bi)$. Hence, as $\mathrm{rank}_\mathcal{O}(B)/|P| \equiv 1 \bmod (p)$, there is $\delta\in\mathcal{P}(B)$ such that p does not divide m_δ (cf. (2.2.1)). So, assuming that $\mathcal{O} = k$ it follows from Proposition 7.2 that $B\cong B(\delta)\bigotimes_k kP$. In general, it is quite clear that the respective images $\bar\alpha = \{\bar b\}$ and $\bar\gamma$ of α and γ in kG are respectively points of G and P on kG, that $P_{\bar\gamma}$ is a defect pointed group of $G_{\bar\alpha}$, that $1\otimes\bar e$: $k\bigotimes_\mathcal{O} B\to \mathrm{Res}^G_P(kG\bar b)$ is an embedded algebra associated with $P_{\bar\gamma}$ (as a pointed group on $kG\bar b$) and that $G_{\bar\alpha}$ is locally controlled by $P_{\bar\gamma}$ (cf. (2.13.1)). Consequently, arguing as above, it follows from Proposition 7.2 that there is an interior P-algebra isomorphism

$$k\bigotimes_\mathcal{O} B\cong \bar S\bigotimes_k kP \tag{7.6.1}$$

where $\bar S = B(\delta)$. Notice that (7.6.1) implies already that (1.9.1) and therefore (1.9.2) and (1.9.3) hold (cf. [9], Cor. 3.5); in particular (cf. [9], Def. 2.5),

(7.6.2) *we have* $\mathrm{Irr}_k(G,b) = \{\varphi\}$ *and p does not divide* $\varphi(i)$ *where* $i = e(1)$.

7.7. With the notation above, assume that:

(7.7.1) *There is an $\mathcal{O}$-simple interior P-algebra S which has a P-stable $\mathcal{O}$-basis W and fulfills* $k\bigotimes_\mathcal{O} S\cong\bar S$ *as interior P-algebras.*

Then we claim that $B\cong SP$ as interior P-algebras, proving (1.6.2). Indeed, by (2.7.1) and (7.6.1) we have $\bar SP\cong k\bigotimes_\mathcal{O} B$, and the set $\{wu\}_{w\in W, u\in P}$ is an $\mathcal{O}$-basis of SP stable by both left and right P-multiplication; so, the isomorphism $B\cong SP$ follows from the Lemma below applied to the interior P-algebra SP.

7.8. Lemma. *With the notation above, let B' be an interior P-algebra having an $\mathcal{O}$-basis W' stable by both left and right P-multiplication. If* $k\bigotimes_\mathcal{O} B\cong k\bigotimes_\mathcal{O} B'$ *then $B\cong B'$ as interior P-algebras.*

Proof. Set $\bar{B} = \mathscr{k} \bigotimes_{\mathcal{O}} B$ and $\bar{B}' = \mathscr{k} \bigotimes_{\mathcal{O}} B'$. It is clear that the set $\{x \otimes w' \otimes y\}_{x,y\in G, w'\in W'}$ is an $\mathcal{O}$-basis of $\mathrm{Ind}_P^G(B')$ stable by both left and right P-multiplication; consequently, $\mathrm{Ind}_P^G(B')^P$ maps onto $\mathrm{Ind}_P^G(\bar{B}')^P$ and therefore, $\mathrm{Ind}_P^G(B')^G$ maps onto $\mathrm{Ind}_P^G(\bar{B}')^G$ too (since $\mathrm{Ind}_P^G(\bar{B}')^G = \mathrm{Ind}_P^G(\bar{B}')_P^G$). Thus, if $\bar{B} \cong \bar{B}'$ then the embedding

$$\bar{\tilde{f}}: \mathscr{k}G\bar{b} \to \mathrm{Ind}_P^G(\bar{B}) \cong \mathrm{Ind}_P^G(\bar{B}'),$$

obtained from the unique embedding $\tilde{c}$: $\mathcal{O}Gb \to \mathrm{Ind}_P^G(B)$ fulfilling $\bar{d}_P^G(B) = \mathrm{Res}_P^G(\tilde{c}) \circ \tilde{e}$ (cf. 7.2.1 and [9], Th. 3.4), can be lifted to an embedding

$$\tilde{f}: \mathcal{O}Gb \to \mathrm{Ind}_P^G(B')$$

since the primitive idempotent $\bar{i} = \bar{f}(\bar{b})$ of $\mathrm{Ind}_P^G(\bar{B}')^G$ can be lifted to a primitive idempotent i of $\mathrm{Ind}_P^G(B')^G$ (and we set $f(xb) = x, i$ for any $x \in G$); in that case, the idempotents $f(e(1))$ and $1 \otimes 1 \otimes 1$ of $\mathrm{Ind}_P^G(B')^P$ are both primitive and conjugate (since the respective images in $\mathrm{Ind}_P^G(\bar{B}')^P$ are so) and therefore, the embedded algebras

$$\mathrm{Res}_P^G(\tilde{f}) \circ \tilde{e}: B \to \mathrm{Res}_P^G \mathrm{Ind}_P^G(B') \quad \text{and} \quad \bar{d}_P^G(B'): B' \to \mathrm{Res}_P^G \mathrm{Ind}_P^G(B')$$

are associated with the same point of P on $\mathrm{Ind}_P^G(B')$, whence $B \cong B'$ as interior P-algebras (cf. [11], 1.6).

7.9. If we assume that the characteristic of $\mathcal{O}$ is the same as $\mathscr{k}$, there is a ring homomorphism $\mathscr{k} \to \mathcal{O}$ which is a section of the canonical map $\mathcal{O} \to \mathscr{k}$ (cf. [15], Ch. II, §4, Th. 2), and therefore we have $\mathscr{k} \bigotimes_{\mathcal{O}} \left(\mathcal{O} \bigotimes_{\mathscr{k}} \bar{S}\right) \cong \bar{S}$; so, in this case condition (7.7.1) is fulfilled setting $S = \mathcal{O} \bigotimes_{\mathscr{k}} \bar{S}$, and (1.6.2) follows from 7.7. But even when $\mathcal{O}$ is of characteristic zero, since the isomorphism (7.6.1) implies that P stabilizes a $\mathscr{k}$-basis of $\bar{S}$ (cf. 7.5), it is not excluded that condition (7.7.1) could be proved directly: for instance, if P is abelian, condition (7.7.1) follows from Dade's classification of endo-permutation modules over abelian p-groups (cf. [4], p. 318). Anyway it suffices now to prove (1.6.2) in characteristic zero.

7.10. Henceforth we assume that $\mathcal{O}$ is of characteristic zero and denote by $\mathscr{K}$ its quotient field. As p does not divide $\dim_{\mathscr{k}}(B/J(B))$ (cf. (7.6.1)), there is an absolutely irreducible character χ of G associated with b such that p does not divide $\chi(i)$ where $i = e(1) \in \gamma$ (cf. (2.14.1) and (7.6.2)). First of all we claim that (cf. 2.15):

(7.10.1) *For any local pointed element u_δ on $\mathcal{O}Gb$ we have $|\chi^\delta(u)| = 1$.*

Indeed, we may assume that $u_\delta \in P_\gamma$ and as (1.9.3) holds (cf. (7.6.1)), we get $\chi(ui) = m_\delta^\gamma \chi^\delta(u)$ (cf. [9], Th. 4.3); but it is clear that $\chi(ui) \equiv \chi(i) \bmod J(\mathcal{O})$ which implies $\chi(ui) \in \mathcal{O}^*$ (since $\chi(i) \notin J(\mathcal{O})$); consequently, $\chi^\delta(u) \in \mathcal{O}^*$ and in particular, we get

$$1 \leq \prod_{u\in P} |\chi^{\delta(u)}(u)|^2 \tag{7.10.2}$$

where $\mathscr{L}\mathscr{P}_B(\langle u\rangle)=\{\delta(u)\}$ for any $u\in P$, since this product is a strictly positive rational integer (cf. [5], Ch. V, Th. 13.1). On the other hand, as k is algebraically closed and (1.9.2) holds, the generalized Cartan integers $c(u_\delta, v_\varepsilon)$ (cf. (2.15.1)) can be computed from (7.6.1) (cf. (2.15.2)), and assuming that $u_\delta\in P_\gamma$ we get $c(u_\delta, v_\varepsilon)=|C_P(u)|\,\delta_{u_\delta, v_\varepsilon}$ by (5.6.1) and Corollary 5.8; consequently, it is easily checked from (1.9.3), (2.15.3) and (3.8.2) that

$$1=(\chi,\chi)_G=\frac{1}{|P|}\sum_{u\in P}|\chi^{\delta(u)}(u)|^2. \tag{7.10.3}$$

Statement (7.10.1) follows now from the well-known theorem about the arithmetical and geometrical means (this argument was already employed in the proof of Lemma 3.8 in [3]).

7.11. Now it suffices to prove that, for a suitable choice of χ, there is an $\mathcal{O}Gb$-module M such that χ is afforded by $\mathscr{K}\bigotimes_{\mathcal{O}}M$. Indeed, by (7.6.2) and (7.10.1) we have $\chi(s)=\varphi(s)$ for any p'-element s of G (cf. [9], Cor. 4.4) and therefore, $k\bigotimes_{\mathcal{O}}M$ is a simple $kG\bar{b}$-module (cf. [16], §15.2); consequently, $kG\bar{b}$ maps onto $\mathrm{End}_k\left(k\bigotimes_{\mathcal{O}}M\right)$ and therefore, $\mathcal{O}Gb$ maps onto $\mathrm{End}_{\mathcal{O}}(M)$ by Nakayama's Lemma; then $S=\mathrm{End}_{\mathcal{O}}(i\cdot M)$ is an $\mathcal{O}$-simple factor of B of $\mathcal{O}$-rank prime to p (cf. (7.6.2)), and Proposition 7.2 applies.

7.12. Let $\mathscr{K}'$ and $\mathscr{K}''$ be respectively the extensions of $\mathscr{K}$ generated by the groups of p- and $|G|_p$-roots of unity, and denote by $\mathcal{O}'$ and $\mathcal{O}''$ the corresponding valuation rings. As k is still the residue field of $\mathcal{O}'$ and $\mathcal{O}''$, it is quite clear (cf. (2.13.1)) that $\alpha=\{b\}$ is still a point of G on $\mathcal{O}'G$ and $\mathcal{O}''G$, that we have $\gamma\subset\gamma'\subset\gamma''$ where γ' and γ'' are respectively points of P on $\mathcal{O}'G$ and $\mathcal{O}''G$, that $P_{\gamma'}$ and $P_{\gamma''}$ are respectively defect pointed groups of G_α on $\mathcal{O}'G$ and $\mathcal{O}''G$, that $\tilde{e}$: $B\to\mathrm{Res}_P^G(\mathcal{O}Gb)$ induces embedded algebras

$$\mathcal{O}'\bigotimes_{\mathcal{O}}B\to\mathrm{Res}_P^G(\mathcal{O}'Gb)\quad\text{and}\quad\mathcal{O}''\bigotimes_{\mathcal{O}}B\to\mathrm{Res}_P^G(\mathcal{O}''Gb)$$

associated respectively with $P_{\gamma'}$ and $P_{\gamma''}$ (as pointed groups on $\mathcal{O}'Gb$ and $\mathcal{O}''Gb$), and that G_α is locally controlled by $P_{\gamma'}$ and $P_{\gamma''}$.

7.13. As k is algebraically closed, $\mathscr{K}$ contains already the group of roots of unity of order prime to p (cf. [16], Ch. II, §4, Prop. 8), and therefore $\mathscr{K}''$ is a splitting field for G (cf. [5], Ch. V, Th. 9.11); hence, there is an $\mathcal{O}''Gb$-module M'' such that χ is afforded by $\mathscr{K}''\bigotimes_{\mathcal{O}''}M''$. It follows now from 7.11 and Proposition 7.2 applied to $\mathcal{O}''$, G and b that there is an $\mathcal{O}''$-simple interior P-algebra S'' such that $\mathcal{O}''\bigotimes_{\mathcal{O}}B\cong S''\bigotimes_{\mathcal{O}''}\mathcal{O}''P$ as interior P-algebras, and we may assume that, denoting by ϱ'': $P\to(S'')^*$ the structural homomorphism, we have $\det(\varrho''(u))=1$ for any $u\in P$ (cf. (2.7.1) and 6.2). So, paragraphs 1.10 to 1.13 apply to $\mathcal{O}''$, G and b; in particular, we have a bijection from $\mathrm{Irr}_{\mathscr{K}''}(G,b)$ onto $\mathrm{Irr}_{\mathscr{K}''}(P)$ induced by (1.12.1) and it is quite clear (cf. (1.12.3)) we may assume that our choice of χ corresponds to the trivial character of P (i.e. to $\lambda=1$). In this case, by (1.13.1) all the values of χ lie in $\mathcal{K}$ (actually in $\mathcal{O}$) and it suffices to prove that the Schur index $s_{\mathcal{K}}(\chi)$ of χ over $\mathcal{K}$ is one (cf. 7.11 and [5], Ch. V, Th. 14.13).

7.14. First of all we claim that $s_{\mathcal{K}'}(\chi) = 1$. Indeed, it is clear that $\dim_{\mathcal{K}'}(\mathcal{K}'')$ is a power of p and therefore, $s_{\mathcal{K}'}(\chi)$ is a power of p too (cf. [5], Ch. V, Th. 14.11); but Brauer's decomposition homomorphism $\mathbf{L}_{\mathcal{K}'}(G) \to \mathbf{L}_{k}(G)$ being surjective (cf. (2.14.1)), there is $\chi' \in \mathrm{Irr}_{\mathcal{K}''}(G, b)$ mapping on a linear character λ' of P such that $\lambda'(P) \subset \mathcal{K}'$, and fulfilling $s_{\mathcal{K}'}(\chi') = 1$; that is, χ' still belongs to $\mathrm{Irr}_{\mathcal{K}'}(G, b)$ and therefore there is an $\mathcal{O}'Gb$-module M' such that χ' is afforded by $\mathcal{K}' \bigotimes_{\mathcal{O}'} M'$. As $\chi'(s) = \varphi(s)$ for any p'-element s of G (cf. (1.13.1)), it follows again from 7.11 and Proposition 7.2 now applied to $\mathcal{O}'$, G and b that there is an $\mathcal{O}'$-simple interior P-algebra S' such that $\mathcal{O}' \bigotimes_{\mathcal{O}} B \cong S' \bigotimes_{\mathcal{O}'} \mathcal{O}'P$ as interior P-algebras, and we may assume again that, denoting by $\varrho'\colon P \to (S')^*$ the structural homomorphism, we have $\det(\varrho'(u)) = 1$ for any $u \in P$ (cf. (2.7.1) and 6.2). Hence, paragraphs 1.10 to 1.13 still apply to $\mathcal{O}'$, G and b and in particular, χ has to be the inverse image of the trivial character of P by the bijection from $\mathrm{Irr}_{\mathcal{K}'}(G, b)$ onto $\mathrm{Irr}_{\mathcal{K}'}(P)$ induced by (1.12.1) (cf. (1.13.1)), proving $s_{\mathcal{K}'}(\chi) = 1$.

7.15. Finally we claim that $s_{\mathcal{K}}(\chi) = 1$. Indeed, on one hand $s_{\mathcal{K}}(\chi)$ divides now $\dim_{\mathcal{K}}(\mathcal{K}')$ (cf. [5], Ch. V, Th. 14.11). On the other hand, if $\lambda \in \mathrm{Irr}_{\mathcal{K}'}(P) - \{1\}$ then $\varrho(P)$ generates $\mathcal{K}'$ over $\mathcal{K}$ (since, if the image of $u \in P$ by the representation map is an element of order p of the center of the image of P, $\lambda(u)/\lambda(1)$ is a primitive p-root of unity), and therefore any orbit of $\mathrm{Gal}(\mathcal{K}', \mathcal{K})$ over $\mathrm{Irr}_{\mathcal{K}'}(G, b) - \{\chi\}$ is regular (cf. (1.13.1)). Consequently, the surjectivity of Brauer's decomposition homomorphism $\mathbf{L}_{\mathcal{K}}(G) \to \mathbf{L}_{k}(G)$ (cf. (2.14.1)) forces $s_{\mathcal{K}}(\chi)$ to be one (cf. [5], Ch. V, Th. 14.13).

References

1. Broué, M.: Les l-blocs des groupes $GL(n,q)$ et $U(n,q^2)$ et leurs structures locales. Séminaire BOURBAKI exp. no 640. Astérisque, 133–134 (1984/85)
2. Broué, M., Puig, L.: Characters and local structure in G-algebras. J. Algebra **63**, 306–317 (1980)
3. Broué, M., Puig, L.: A Frobenius theorem for blocks. Invent. Math. **56**, 117–128 (1980)
4. Dade, E.C.: Endo-permutation modules over p-groups, II. Ann. Math. **108**, 317–346 (1978)
5. Huppert, B.: Endlichen Gruppen I. Berlin-Heidelberg-New York: Springer 1967
6. Külshammer, B., Puig, L.: Extensions of nilpotent blocks. (Preprint 1988)
7. Okuyama, T., Tsushima, Y.: Local properties of p-block algebras of finite groups. Osaka J. Math. **20**, 33–41 (1983)
8. Puig, L.: Sur un théorème de Green. Math. Z. **166**, 117–129 (1979)
9. Puig, L.: Pointed groups and construction of characters. Math. Z. **176**, 265–292 (1981)
10. Puig, L.: The source algebra of a nilpotent block. Preprint 1981
11. Puig, L.: Local fusion in block source algebras. J. of Algebra **104**, 358–369 (1986)
12. Puig, L.: Pointed groups and construction of modules. J. Algebra (Preprint 1988)
13. Puig, L.: Local extensions in endo-permutation modules split: a proof of Dade's theorem, Séminaire Claude Chevalley III, pp. 199–206, Publ. Math. de l'Université de Paris VII, 1987
14. Puig, L.: Some notes on Dade's endo-permutation modules. (Preprint 1988)
15. Serre, J.-P.: Corps locaux. Paris: Hermann 1968
16. Serre, J.-P.: Représentations linéaires des groupes finis. Paris: Hermann 1971

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