Invent. math. 70, 319 – 340 (1983)

Inventiones mathematicae

The Number and Linking of Periodic Solutions of Periodic Systems

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1. Introduction

In this paper, we consider equations of the form: $dx/dt = f(t,x)$, where t is a real number, x is a point in $\mathbf{R}^2$, f is of class C^1 and periodic in t of period 1. We assume that there is a closed set of R^2 homeomorphic to a closed disk and invariant under the Poincaré transformation. It is known that this condition is satisfied by many dissipative systems as Duffing's equation. See for example [6].

The purpose of this paper is to discuss the relation between the number and the manner of linking of periodic solutions. Our main theorem, Theorem 1, states that the equation has periodic solutions of every integer period if there are three periodic solutions of period 1 which are attractor or repeller and link together in a certain kind of complex manner.

On the number of periodic solutions of general dissipative systems, the following results are known. The number of periodic solutions of every integer period is divisible by twice the period if all periodic solutions are hyperbolic (Levinson [3], Massera [4]), and is finite if f is real analytic in x and the trace of the Jacobian matrix of f is negative (Nakajima and Seifert [5]).

Our theorem is derived from Theorem 2 in Sect. 3. There we define a polynomial which describes how given periodic solutions and others link together, and in particular gives an information about the number of periodic solutions.

We prove Theorem 2 in Sect. 4, 5 and Theorem 1 in Sect. 6. In Sect. 7, a detailed estimation for the number of periodic solutions of period 1 is given.

2. Theorem 1

Consider the following differential system:

$$\frac{dx}{dt} = f(t,x) \qquad t \in \mathbf{R},\ x \in \mathbf{R}^2. \tag{2.1}$$

0020-9910/83/0070/0319/$04.40

We assume throughout the paper that

(2.2) 1) $f(t,x)$ is an $\mathbf{R}^2$-valued function of class C^1.

2) $f(t,x)$ is periodic in t of period 1, that is, $f(t+1,x)=f(t,x)$.

3) There exists a solution $x=\phi(t;t_0,x_0)$ of the equation defined on $-\infty<t<\infty$ with any initial condition $(t_0,x_0)\in\mathbf{R}\times\mathbf{R}^2$.

Defining a C^1-diffeomorphism $T\colon\mathbf{R}^2\to\mathbf{R}^2$, called the *Poincaré transformation*, by $T(x)=\phi(1;0,x)$, $x\in\mathbf{R}^2$, we also assume that

(2.3) there exists a closed set K of $\mathbf{R}^2$ satisfying that K is homeomorphic to a closed disk and $T(K)\subset K$.

Definition 1. Let p be a natural number. A continuous curve $x\colon\mathbf{R}\to\mathbf{R}^2$ is *p-periodic* if it satisfies for every $t\in\mathbf{R}$ and every natural number $q<p$ that

$$x(t+p)=x(t) \quad\text{and}\quad x(t+q)\neq x(t).$$

A solution $x(t)$ of (2.1) is p-periodic if it is a p-periodic curve, and is *periodic* if it is p-periodic for some natural number p.

Clearly we have by the uniqueness of solution:

Proposition 1. *Let $x(t)$ be a solution of* (2.1). *Then it is p-periodic if and only if $x(0)$ is a periodic point of T of minimal period p.*

Let $c_1,c_2,\dots,c_n$ be periodic solutions of (2.1) and p a natural number. We assume the following conditions in Theorem 1.

(2.4) There exist disjoint closed sets $K_1,\dots,K_n$ of K homeomorphic to a closed disk and satisfying the followings for $i=1,\dots,n$:

$$c_i(0)\in K_i,$$
$$T^p(K_i)\subset K_i \quad\text{or}\quad T^p(K_i)\supset K_i \quad\text{if}\quad c_i(p)=c_i(0),$$
$$T^p(K_i)\ \text{and}\ K_i\ \text{have no intersection if}\ c_i(p)\neq c_i(0).$$

Here, $T^p=T\circ\dots\circ T$ (p-times). Set $K'=K-\bigcup_{i=1}^{n}\operatorname{Int}K_i$, where Int denotes interior. A p-periodic solution is *hyperbolic* if $x(0)$ is a hyperbolic fixed point of T^p.

(2.5) If $x(t)$ is a p-periodic solution of (2.1) with $x(0)\in K'$, then it is hyperbolic and $x(0)$ is not on the boundary of K'.

Remark. The assumption (2.4) is satisfied if the periodic solutions are hyperbolic attractor or repeller.

Now let c_1,c_2 and c_3 be distinct 1-periodic solutions of (2.1). We express topological complexity of these solutions as a series of two letters a,b and their inverses. Let

$$e_1(t)=c_3(t)-c_1(t), \qquad e_2(t)=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}e_1(t).$$

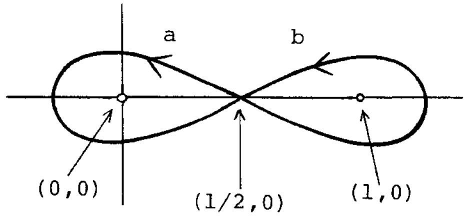

Fig. 1

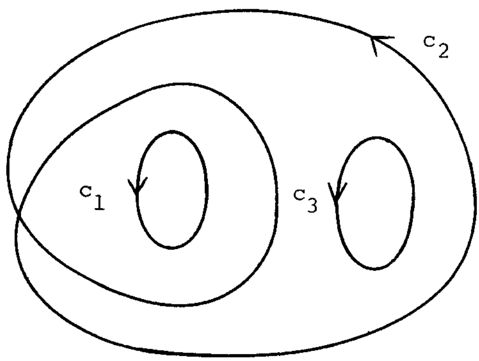

Fig. 2

Then, since $e_1(t)$ and $e_2(t)$ are linearly independent, there is a unique $u(t) = (u_1(t), u_2(t))$ in $\mathbf{R}^2$ such that

$$c_2(t) - c_1(t) = u_1(t)\, e_1(t) + u_2(t)\, e_2(t).$$

It is trivial that $u(t) \neq (0,0), (1,0)$. Therefore $u(t)$ is a 1-periodic curve in $X = \mathbf{R}^2 - \{(0,0), (1,0)\}$. We now define a conjugate class $[c_1, c_2, c_3]$ of the fundamental group $\pi_1(X, (1/2, 0))$ as follows. Choose a curve $v(t)$ $(0 \leqq t \leqq 1)$ from $(1/2, 0)$ to $u(0)$ and consider the closed curve which passes the point $v(3t)$ at $0 \leqq t \leqq 1/3$, $u(3t-1)$ at $1/3 \leqq t \leqq 2/3$ and $v(3-3t)$ at $2/3 \leqq t \leqq 1$, that is, the closed curve which starts $(1/2, 0)$, follows v to $u(0)$, then runs along $u(t)$ $(0 \leqq t \leqq 1)$ and retrace v back to $(1/2, 0)$. Then clearly the conjugate class of the class of this closed curve dose not depend on the choice of v. We denote it by $[c_1, c_2, c_3]$. As is well known, the fundamental group is a free group of rank 2 on generators a and b, which are defined as the classes of closed curves circling $(0,0)$ and $(1,0)$ once in a counterclockwise direction respectively. See Fig. 1. Then clearly

$$[c_1, c_2, c_3] = [a^j], [b^j] \quad \text{or } \sigma(I, J),$$

where [ ] denotes conjugate class, j is an integer, $I = (i_1, \ldots, i_d)$, $J = (j_1, \ldots, j_d)$ are sequences of non-zero integers of length d and

$$\sigma(I, J) = [a^{i_1} b^{j_1} a^{i_2} b^{j_2} \ldots a^{i_d} b^{j_d}].$$

For example, if c_1, c_2 and c_3 are link together as in Fig. 2, then

$$[c_1, c_2, c_3] = \sigma((2), (1)) = [a^2 b].$$

Theorem 1. *Assume the conditions* (2.2), (2.3). *Let* c_1, c_2 *and* c_3 *be* 1-*periodic solutions of* (2.1) *such that* $[c_1, c_2, c_3] = \sigma(I, J)$, *where* I *and* J *are sequences of non-zero integers of the same length with*

$$(I, J) \neq (1, 1, \dots, 1), \quad (-1, -1, \dots, -1).$$

Let p *be a natural number and let* c_1, c_2, c_3 *satisfy* (2.4). *Then the number of* p-*periodic solutions passing a point in* K' *at* $t=0$ *is not smaller than* p^2. *Moreover, it is not smaller than* $p \cdot 2^{p-1}$ *if* (2.5) *is also satisfied.*

3. Theorem 2

In this section, we consider the general case where no assumptions are made on the number and period of the given periodic solutions. In the previous section, where the number is three and the period is one, the type of link of periodic solutions is expressed as an element of the free group of rank two. In the general case, it is expressed as a "braid", as we see in the following.

A continuous map from the unit interval $[0, 1]$ to a topological space X is a *path* in X. A path c is a *loop* if the initial point $c(0)$ and the terminal point $c(1)$ coincide. This point is called the *base point* of c.

Let n be a natural number. Define an open set V_n of $\mathbf{R}^{2n}$ by

$$V_n = \{(x_1, \dots, x_n) | x_i \in \mathbf{R}^2,\ x_i \neq x_j \text{ if } i \neq j\}.$$

Let Σ_n denote the symmetric group of degree n and act on V_n by

$$\tau(x_1, \dots, x_n) = (x_{\tau(1)}, \dots, x_{\tau(n)}),$$

where $\tau \in \Sigma_n$, $(x_1, \dots, x_n) \in V_n$. We denote by V_n/Σ_n the quotient space by the above action of Σ_n and by $\pi\colon V_n \to V_n/\Sigma_n$ the projection. The fundamental group $\pi_1(V_n/\Sigma_n)$ is called the *braid group* and its element a *braid*. In the following, we denote by B_n the braid group $\pi_1(V_n/\Sigma_n, e)$, where $e = \pi((1,0), \dots, (n,0))$.

Let $c_1, c_2, \dots, c_n$ be periodic solutions satisfying that

(3.1) for any $i = 1, \dots, n$ and any integer m, there is a natural number j with $1 \leqq j \leqq n$ such that $c_i(t+m) = c_j(t)$, $t \in \mathbf{R}$.

This means that any periodic solution, which is obtained from c_i by simply translating time by m, also belongs to the set of periodic solutions. Therefore, the loop $\pi(c_1(t), \dots, c_n(t))$ in V_n/Σ_n determines an element of the braid group $\pi_1(V_n/\Sigma_n)$. Hence the type of link of periodic solutions is expressed as a braid.

The group structure of the braid group is known as follows. Let B_n'' be the finitely generated group with generators $\sigma_1', \dots, \sigma_{n-1}'$ and defining relations

(3.2)
$$\sigma_i'\sigma_j' = \sigma_j'\sigma_i' \quad \text{if } |i-j| \geqq 2, \quad 1 \leqq i,\ j \leqq n-1,$$
$$\sigma_i'\sigma_{i+1}'\sigma_i' = \sigma_{i+1}'\sigma_i'\sigma_{i+1}' \quad \text{if } 1 \leqq i \leqq n-2.$$

Define a path l_i in V_n by

$$l_i(t)=((1,0),\ldots,(i-1,0),l_i^1(t),l_i^2(t),(i+2,0),\ldots,(n,0)) \tag{3.3}$$

where

$$l_i^1(t)=(i+t,\ -(t-t^2)^{1/2}),\ l_i^2(t)=(i+1-t,(t-t^2)^{1/2}).$$

Then $\pi\circ l_i$ is a loop in V_n/Σ_n. Denote the class of this loop in the braid group B_n by σ_i. Then it is known [1, Theorem 1.8] that the homomorphism sending σ_i' to σ_i is an isomorphism from B_n'' to B_n.

Now assume that $n\geqq 3$. Let Λ denote the ring $\mathbf{Z}[a_1,a_1^{-1},\ldots,a_n,a_n^{-1}]$ of integer polynomials in the a_i's and their inverses. Let $v\colon B_n\to\Sigma_n$ be the homomorphism which carries σ_i to the transposition of i and $i+1$. We say that two elements σ and σ' in B_n are equivalent if there is an α in the kernel of v such that $\sigma'=\alpha^{-1}\sigma\alpha$. Denote the set of all equivalence classes of B_n by B_n'. Then a map $A\colon B_n'\to\Lambda$ is defined in the following way. Let $GL(n-1,\Lambda)$ be the group of all invertible matrices of size $n-1$ whose entries are elements of Λ. Let Σ_n act on Λ by

$$\tau\cdot a_i=a_{\tau(i)}, \tag{3.4}$$

and on $GL(n-1,\Lambda)$ by $\tau(\lambda_{ij})=(\tau\,\lambda_{ij})$. Set

$$S_1=\left(\begin{array}{cc|c} -a_1 & 1 & \\ 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{n-3} \end{array}\right),\qquad S_i=\left(\begin{array}{c|ccc|c} I_{i-2} & & 0 & & 0 \\ \hline & 1 & 0 & 0 & \\ 0 & a_i & -a_i & 1 & 0 \\ & 0 & 0 & 1 & \\ \hline 0 & & 0 & & I_{n-i-2} \end{array}\right),$$

$$S_{n-1}=\left(\begin{array}{c|cc} I_{n-3} & \multicolumn{2}{c}{0} \\ \hline & 1 & 0 \\ 0 & a_{n-1} & -a_{n-1} \end{array}\right)$$

where $i=2,\ldots,n-2$, I_k is the identity matrix of size k. Then it is easy to see that the formula

$$\begin{aligned} B(\sigma_i)&=S_i \qquad i=1,\ldots,n-1,\\ B(\sigma\,\sigma')&=B(\sigma')\,(v(\sigma')\,B(\sigma)) \qquad \sigma,\sigma'\in B_n \end{aligned} \tag{3.5}$$

implies that $B(e)=I$, $B(\sigma^{-1})=v(\sigma)\,B(\sigma)^{-1}$, where e is the unit element, and hence defines a map $B\colon B_n\to GL(n-1,\Lambda)$ uniquely. Define a homomorphism inv_τ from Λ to the subring Λ_τ consisting of all τ-invariant elements under the action (3.4), where $\tau\in\Sigma_n$, by

$$\mathrm{inv}_\tau(a_i)=a_i\,a_{\tau(i)}\ldots a_{\tau^{s-1}(i)},$$

where s is the cardinal number of the orbit of τ at i, $\{\tau^u(i)\mid u \text{ is an integer}\}$. We now define $A\colon B_n\to\Lambda$ by

$$A(\sigma)=-\mathrm{inv}_{v(\sigma)}(\mathrm{trace}\,B(\sigma)).$$

It is easily shown that $B(\alpha^{-1}\sigma\alpha)=B(\sigma)$ for every $\sigma\in B_n$, $\alpha\in\operatorname{Ker}\nu$. Hence A induces a map from B'_n to Λ denoted by the same letter A.

Remark. The representation of B_n obtained from B by replacing every a_i by one symbol t coincides with the reduced Burau representation given in [1, Lemma 3.11.1].

For distinct periodic solutions c and c', define an integer $d(c,c')$ as the degree of the loop in $\mathbf{R}^2-\{0\}$: $c(qt)-c'(qt)$, where q is the least common multiple of the periods of c and c'.

Now assume that $C=\{c_1,\ldots,c_n\}$ is a set of periodic solutions of (2.1) satisfying (3.1) in the remainder of the section.

Definition 2. For a sequence of integers $I=(i_1,\ldots,i_n)\in\mathbf{Z}^n$, a periodic solution c of (2.1) is of *degree I*, if $d(c,c_k)=i_k$ for $k=1,\ldots,n$.

Let Σ_n act on $\mathbf{Z}^n$ by

$$\tau(i_1,\ldots,i_n)=(i_{\tau(1)},\ldots,i_{\tau(n)})$$

and for $\tau\in\Sigma_n$ denote by $\mathbf{Z}^n(\tau)$ the subgroup of all τ-invariant elements of $\mathbf{Z}^n$. Define $\tau_C\in\Sigma_n$ by $T(c_i(0))=c_{\tau_C(i)}(0)$. Let p be a natural number. We define a subset $\mathbf{Z}^n(p)$ of $\mathbf{Z}^n$ as follows. For a divisor q of p, let $\eta_{q,p}\colon\mathbf{Z}^n\to\mathbf{Z}^n$ be the homomorphism defined by

$$\eta_{q,p}(e_i)=((p_i,p)/(p_i,q))\,e_i,$$

where $e_i=(0,\ldots,0,1,0,\ldots,0)$ (i-th component$=1$), p_i is the period of c_i and $(\ ,\ )$ denotes least common multiple. Then clearly $\eta_{q,p}(\mathbf{Z}^n(\tau_C^q))\subset\mathbf{Z}^n(\tau_C^q)\subset\mathbf{Z}^n(\tau_C^p)$. We define $\mathbf{Z}^n(p)$ as the set of all elements I of $\mathbf{Z}^n(\tau_C^p)$ which do not belong to the image $\eta_{q,p}(\mathbf{Z}^n(\tau_C^q))$ for every divisor q of p with $q<p$. Note that $\mathbf{Z}^n(1)=\mathbf{Z}^n(\tau_C)$.

Now define an element $A(C,p)$ of Λ as follows. Define a loop C' in V_n/Σ_n by $C'(t)=\pi(c_1(t),\ldots,c_n(t))$ and choose a path v in V_n from $(c_1(0),\ldots,c_n(0))$ to $((1,0),\ldots,(n,0))$. Then $(\pi\circ v)^{-1}C'(\pi\circ v)$ is a loop in V_n/Σ_n based at $e=\pi((1,0),\ldots,(n,0))$. Denote by σ_C the class in B_n of this loop and let $A(C,p)=A(C_p)$, where C_p denotes the element of B'_n represented by σ_C^p. It follows from the equality $\operatorname{Im}[\pi_*:\pi_1(V_n)\to B_n]=\operatorname{Ker}\nu$ that C_p and hence $A(C,p)$ do not depend on the choice of the path v. Let $r_I(C,p)$ denote the coefficient of the term $a_1^{i_1}\ldots a_n^{i_n}$ of $A(C,p)$, that is $A(C,p)=\sum\limits_{I\in\mathbf{Z}^n} r_I(C,p)\,a_1^{i_1}\ldots a_n^{i_n}$.

Theorem 2. *Assume* (2.2), (2.3). *Let p be a natural number and $C=\{c_1,\ldots,c_n\}$ a set of periodic solutions of* (2.1) *with $n\geqq 3$ satisfying* (2.4), (3.1). *Then, for every $I\in\mathbf{Z}^n(p)$ with $r_I(C,p)\neq 0$, there exists a p-periodic solution of degree I which passes K' at $t=0$. The number of such p-periodic solutions is not smaller than the absolute value of $r_I(C,p)$, if the assumption* (2.5) *is added.*

As an immediate consequence, we have

Corollary. *Under the same assumptions as in Theorem 2, the number of p-periodic solutions, passing K' at $t=0$, is not smaller than p times the number of elements I of $\mathbf{Z}^n(p)$ with $r_I(C,p)\neq 0$. It is not smaller than $\sum\limits_{I\in\mathbf{Z}^n(p)}|r_I(C,p)|$ if* (2.5) *is also assumed.*

Remark. When $n=1,2$, the type of link of given periodic solutions does not affect the number of other periodic solutions. When $n=1$, it is clear since B_1 is the trivial group. For any $\sigma\in B_2$, a C^1-map f is easily constructed so that the Eq. (2.1) has only three periodic solutions and that the braid of two of those periodic solutions is equal to σ.

The proof of Theorem 2 depends heavily on Proposition 2, which is concerned with fixed points of continuous maps, in the next section.

4. Fixed points

We assume that $n\geqq 3$. Let L be a topological space homeomorphic to a n-times punctured disk. Let x_0 be a point in L and $\alpha_1,\dots,\alpha_n$ be a free basis of the fundamental group $\pi_1(L,x_0)$. Denote by b_i the homology class in $H_1(L;\mathbf{Z})$ determined by α_i. Let ρ be the homomorphism from B_n to the group of all automorphisms of $\pi_1(L,x_0)$ defined by

(4.1)
$$\begin{aligned}\rho(\sigma_i)(\alpha_j)&=\alpha_i\,\alpha_{i+1}\,\alpha_i^{-1} && j=i\\ &=\alpha_i && j=i+1\\ &=\alpha_j && j\neq i,\ i+1.\end{aligned}$$

For $\tau\in\Sigma_n$, let $\mathrm{inv}_\tau\colon \mathbf{Z}^n\to\mathbf{Z}^n$ and $\mathrm{inv}_\tau\colon H_1(L;\mathbf{Z})\to H_1(L;\mathbf{Z})$ be the homomorphisms defined by

(4.2)
$$\mathrm{inv}_\tau(e_i)=\sum_{k=0}^{s-1} e_{\tau^k(i)},\quad \mathrm{inv}_\tau(b_i)=\sum_{k=0}^{s-1} b_{\tau^k(i)},$$

where s is the cardinal number of the set $\{\tau^u(i)|u\in\mathbf{Z}\}$, $e_i=(0,\dots,1,\dots,0)$ (i-th component$=1$).

A path v in a topological space X defines a homomorphism v_* from $\pi_1(X,v(0))$ to $\pi_1(X,v(1))$ by

(4.3)
$$v_*([w])=[v^{-1}\,w\,v],$$

where w is a loop in X based at $v(0)$, -1 denotes inverse path, $v^{-1}wv$ is the product of the paths v^{-1}, w and v.

Let $S\colon L\to L$ be a continuous map with the image $S(L)$ compact, v a path in L from x_0 to $S(x_0)$ and σ an element of B_n. For $I\in\mathbf{Z}^n$, define an integer r_I by

$$A(\sigma)=\sum r_I\,a_1^{i_1}\dots a_n^{i_n}.$$

Proposition 2. *Assume that*

(4.4)
$$S_*=v_*\circ\rho(\sigma)\colon \pi_1(L,x_0)\to\pi_1(L,S(x_0)).$$

If I is an element of $\mathbf{Z}^n(v(\sigma))$ with $r_I\neq 0$, then the map S has a fixed point in L such that

(4.5) *for any path h in L from x_0 to the fixed point, the equality*

$$\mathrm{inv}_{v(\sigma)}[h^{-1}\, v(S\circ h)] = \sum_{k=1}^{n} i_k b_k$$

holds.

Moreover, there exist at least $|r_I|$ *fixed points in L satisfying* (4.5), *provided that*

(4.6) *S has no fixed points on the boundary of L and all fixed points are hyperbolic.*

Proof. Let $\tau = v(\sigma)$. We fix $I_0 \in \mathbf{Z}^n(\tau)$. Let $\theta\colon \pi_1(L, x_0) \to H_1(L; \mathbf{Z})$ be the Hurewicz homomorphism. Then, for every natural number r, there is a finite regular covering space $p_r\colon L_r \to L$ such that

$$(4.7) \qquad p_{r*}(\pi_1(L_r, y)) = \theta^{-1}\langle r\, b_i, b_i - b_{\tau(i)} | i = 1, \ldots, n\rangle,$$

for every point y in the inverse image of x_0 under p_r, where $\langle\ \rangle$ denotes generated subgroup. Let n' be the rank of the free abelian group $\mathbf{Z}^n(\tau)$. Then the covering is $r^{n'}$-sheeted. Fix a point y_0 with $p_r(y_0) = x_0$. Let J be an element of $\mathbf{Z}^n$ satisfying $-\mathrm{inv}_\tau(J) = I_0$. By (4.4), $S_*(b_i) = b_{\tau(i)}$, so the subgroup $H_r = \theta^{-1}\langle r\, b_i, b_i - b_{\tau(i)} | i = 1, \ldots, n\rangle$ of $\pi_1(L, x_0)$ is invariant under S_*. Therefore, there is a unique lift $S_r\colon L_r \to L_r$ of S such that $S_r(y_0) = (\alpha^J v)'(1)$, where $\alpha^J = \alpha_1^{j_1} \ldots \alpha_n^{j_n}$ and $(\alpha^J v)'$ is the lift of the path $\alpha^J v$ with the initial point y_0. S_r sends $y \in L_r$ to the terminal point of the lift from y_0 of $\alpha^J v(S \circ p_r \circ l)$, where l is a path from y_0 to y.

Lemma 1. *Let r be a sufficiently large number. If the Lefschetz number* $\Lambda(S_r)$ *is not zero, then there is a fixed point of S satisfying* (4.5) *for* I_0. *The number of such fixed points is not smaller than* $|\Lambda(S_r)|/R$ *if* (4.6) *is satisfied, where* $R = r^{n'}$.

Proof. Since $S(L)$ is compact, $S_r(L_r)$ is contained in a compact deformation retract of L_r. Therefore if $\Lambda(S_r) \neq 0$, then the Lefschetz fixed point theorem [2, Chap. VII, Prop. 6.6] implies the existence of a fixed point y_1 of S_r in L_r. Let $x_1 = p_r(y_1)$. Then every point in the inverse image of x_1 is a fixed point of S_r. This is because the covering transformation group of p_r acts transitively on each fiber of p_r and S_r commutes with any covering transformation. Let h be a path as in (4.5) and h' a lift of h with $h'(0) = y_0$. Then since $p_r(h'(1)) = x_1$, the definition of S_r implies that $h'^{-1}(\alpha^J v)'(S_r \circ h')$ is a loop in L_r. Hence the homology class $[h^{-1}\alpha^J v(S\circ h)] = [h^{-1} v(S\circ h)] + [\alpha^J]$ belongs to $\langle r\, b_i, b_i - b_{\tau(i)}\rangle$. From this

$$\mathrm{inv}_\tau[h^{-1} v(S\circ h)] = -\mathrm{inv}_\tau[\alpha^J] = \sum_{k=1}^{n} i_k^0 b_k \,\mathrm{modulo}\, \langle r\, b_i\rangle,$$

where $(i_1^0, \ldots, i_n^0) = I_0$. Therefore Lemma 2 below completes the proof of the first half of the lemma.

We prove the second assertion. By the Lefschetz fixed point theorem, $\Lambda(S_r) = \sum_y \mathrm{ind}_y S_r$, where summation takes over all fixed points of S_r and $\mathrm{ind}_y S_r$ is the fixed point index of S_r at y. By [6, Prop. 5], if S satisfies (4.6), then $\mathrm{ind}_y S_r = 1$ or -1. Therefore, the second assertion holds from the fact shown before that, for a fixed point y of S_r, every element in the fiber over $p_r(y)$ is also a fixed point. q.e.d.

Lemma 2. *For $I \in \mathbf{Z}^n(\tau)$, let F_I be the set of all fixed points of S satisfying* (4.5). *Then F_I is empty, if the norm $\|I\|$ of I is sufficiently large.*

Proof. Suppose the conclusion does not hold. Then there are sequences $x(n)$, $I(n)$, $n=1,2,\ldots$, of elements of L and $\mathbf{Z}^n(\tau)$ respectively with $x(n) \in F_{I(n)}$ and $\|I(n)\| \to \infty$ if $n \to \infty$. Since $\{x(n)\}$ are contained in the compact set $S(L)$, we can assume that the sequence $\{x(n)\}$ has the limit point $x(\infty)$ in L. Choose a path h from x_0 to the fixed point $x(\infty)$ and let $I(\infty)=(i'_1,\ldots,i'_n)$ satisfy $\mathrm{inv}_\tau[h^{-1} v(S \circ h)] = \sum_{k=1}^{n} i'_k b_k$. Then $x(\infty) \in F_{I(\infty)}$, because $\mathrm{inv}_\tau = \mathrm{inv}_\tau \circ S_*$ on $H_1(L;\mathbf{Z})$. It is clear that if $I' \in \mathbf{Z}^n(\tau)$ and there is a point in $F_{I'}$ sufficiently near to $x(\infty)$, then $I' = I(\infty)$. This implies that $I(n)=I(\infty)$ for n sufficiently large and leads to the contradiction. q.e.d.

Since $\Lambda(S_r)=1-\mathrm{trace}[S_{r*}\colon H_1(L_r;\mathbf{Q}) \to H_1(L_r;\mathbf{Q})]$, to prove the proposition, we must compute the trace of S_{r*}. Let $V_r = H_r/[H_r, H_r]$, where $[\ ,\ ]$ denotes commutator subgroup, and $\Lambda_r = \Lambda/(a_i^r - 1, a_i - a_{\tau(i)}\, i=1,\ldots,n)$, where $(\ )$ is generated ideal. Then V_r is a Λ_r-module by $a_i[\alpha]=[\alpha_i \alpha \alpha_i^{-1}]$, $\alpha \in H_r$. Since $\rho(\sigma)(H_r) \subset H_r$, $\rho(\sigma)$ induces a Λ_r-homomorphism $\rho_r\colon V_r \to V_r$. For the sake of simplicity, we use the same symbol a_i for the image of a_i under the projection $\Lambda \to \Lambda_r$ and let $a^I = a_1^{i_1} \ldots a_n^{i_n} \in \Lambda$ or Λ_r for $I \in \mathbf{Z}^n$. The following diagram clearly commutes.

$$
\begin{array}{ccccc}
H_1(L_r;\mathbf{Z}) & \xrightarrow{S_{r*}} & H_1(L_r;\mathbf{Z}) & \xleftarrow{\mathrm{id}} & H_1(L_r;\mathbf{Z}) \\
\uparrow{\scriptstyle\theta} & & \uparrow{\scriptstyle\theta} & & \uparrow{\scriptstyle\theta} \\
\pi_1(L_r, y_0) & \xrightarrow{S_{r*}} & \pi_1(L_r, S_r(y_0)) & \xleftarrow{(\alpha^J v)'_*} & \pi_1(L_r, y_0) \\
\downarrow{\scriptstyle p_{r*}} & & \downarrow{\scriptstyle p_{r*}} & & \downarrow{\scriptstyle p_{r*}} \\
H_r & \xrightarrow{S_*} & p_{r*}\pi_1(L_r, S_r(y_0)) & \xleftarrow[(\alpha^J v)_*]{} & H_r.
\end{array}
$$

This and (4.4) implies the following commutative diagram.

$$
\begin{array}{ccc}
H_1(L_r;\mathbf{Z}) & \xrightarrow{S_{r*}} & H_1(L_r;\mathbf{Z}) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
V_r & \xrightarrow{a^J \rho_r} & V_r
\end{array}
$$

where a^J denotes the scalar multiplication. Thus it suffices to consider the Λ_r-homomorphism $a^J \rho_r$ for the computation of trace S_{r*}.

Let $W_r = \bigoplus \Lambda_r e_{i,j}$ be the free Λ_r-module generated by symbols $e_{i,j}$, $1 \leqq i < j \leqq n$ and $\beta_r\colon W_r \to V_r$ be a Λ_r-homomorphism defined by

(4.8) $$\beta_r(e_{i,j})=[\alpha_i\alpha_j\alpha_i^{-1}\alpha_j^{-1}].$$

Let $e'_{i,j}=(\prod_{k\neq i,j}(a_k-1))e_{i,j}$ and W'_r, W''_r be the submodules of W_r generated by $e'_{i,i+1}(i=1,\dots,n-1)$, $e'_{i,j}(1\leqq i<j\leqq n)$ respectively.

Let $V_r^1=V_r/\beta_r(W_r)$, $V_r^2=\beta_r(W_r)/\beta_r(W'_r)$, $V_r^3=\beta_r(W'_r)$ and, for $i=1,2$ and 3, ρ_r^i: $V_r^i\to V_r^i$ be the Λ_r-homomorphism induced from ρ_r: $V_r\to V_r$. Then we have

(4.9) $$V_r=V_r^1\oplus V_r^2\oplus V_r^3,\quad \rho_r=\rho_r^1\oplus\rho_r^2\oplus\rho_r^3.$$

Lemma 3. $|\mathrm{trace}[\rho_r^1\otimes 1_{\mathbf{Q}}: V_r^1\otimes\mathbf{Q}\to V_r^1\otimes\mathbf{Q}]|\leqq n.$

Proof. Clearly the Λ_r-module V_r is generated by $\beta_r(e_{i,j})$, $[\alpha_i^r]$, $[\alpha_i\alpha_{\tau(i)}^{-1}], i,j=1,\dots,n$. Therefore it follows from the equality $[\alpha_i^r]-[\alpha_{\tau(i)}^r]=r[\alpha_i\alpha_{\tau(i)}^{-1}]$ modulo $\beta_r(W_r)$ that the $(\Lambda_r\otimes\mathbf{Q})$-module $V_r\otimes\mathbf{Q}$ is generated by $\beta_r(e_{i,j})$ and $[\alpha_i^r]$. Hence the $\mathbf{Q}$-vector space $V_r^1\otimes\mathbf{Q}$ has a basis $[\alpha_i^r], i=1,\dots,n$. Since $\rho_r^1[\alpha_i^r]=[\alpha_{\tau(i)}^r]$, $\rho_r^1\otimes 1_{\mathbf{Q}}$ is identified with τ: $\mathbf{Q}^n\to\mathbf{Q}^n$. Therefore the lemma holds. q.e.d.

Lemma 4. *There is a positive number M independent of r such that* $|\mathrm{trace}(\rho_r^2\otimes 1_{\mathbf{Q}})|<MR/r$, *where* $R=r^{n'}$.

Proof. Define a homomorphism C_r from B_n to the group of all Λ_r-automorphisms of W_r by

$$\begin{aligned}
C_r(\sigma_k)(e_{i,j}) &= -a_k e_{k,k+1} && i=k,\ j=k+1\\
&= e_{k,j} && i=k+1,\ j\neq k\\
&= e_{i,k} && i\neq k,\ j=k+1\\
&= (1-a_j)e_{k,k+1}+e_{k+1,j} && i=k,\ j\neq k+1\\
&= (a_i-1)e_{k,k+1}+e_{i,k+1} && i\neq k+1,\ j=k\\
&= e_{i,j} && \text{otherwise.}
\end{aligned}$$

A straightforward calculation shows that $\beta_r\circ C_r(\sigma)=\rho_r\circ\beta_r$.

For a linear map A: $V\to V$, where V is a $\mathbf{Q}$-vector space with a basis $v_1,\dots,v_m$ and a norm

$$\left\|\sum_{i=1}^m s_i v_i\right\|_1=\sum_{i=1}^m |s_i|,\quad s_i\in\mathbf{Q},$$

let $\|A\|_1$ be the operator norm of A. Then it is easy to see that the absolute value of any eigenvalue of A is not greater than $\|A\|_1$ and that

$$\|A\|_1=\max\left\{\sum_{i=1}^m |a_{ij}| : j=1,\dots,m\right\},$$

where a_{ij} is the matrix representation of A. The former fact implies that, for a subspace W of V invariant under A,

(4.10) $$|\mathrm{trace}\,A'|\leqq\|A\|_1\dim V/W$$

where A': $V/W\to V/W$ is induced from A.

It is easy to verify that $\|C_r(\sigma)\otimes 1_{\mathbf{Q}}\|_1$ is smaller than a constant independent of r for suitable bases of $W_r\otimes\mathbf{Q}$. On the other hand, it is verified that there is a number N independent of r such that $\dim(W_r/Z\otimes\mathbf{Q})<NR/r$, where Z is the kernel of $W_r \xrightarrow{\beta_r} \beta_r(W_r)\to V_r^2$, in the following way. Let

$$\alpha_{ijk}=(\alpha_i\alpha_j\alpha_k\alpha_j^{-1}\alpha_k^{-1}\alpha_i^{-1})(\alpha_i\alpha_k\alpha_i^{-1}\alpha_k^{-1})\in H_r.$$

Then clearly $\alpha_{ijk}\alpha_{kij}\alpha_{jki}$ is the unit element of H_r. This implies that

$$(a_i-1)\beta_r(e_{j,k})-(a_j-1)\beta_r(e_{i,k})+(a_k-1)\beta_r(e_{i,j})=0 \quad \text{in } V_r.$$

Hence $\beta_r(e'_{i,j})=\sum_{k=i}^{j-1}\beta_r(e'_{k,k+1})$ holds and so $\beta_r(W'_r)=\beta_r(W''_r)$. Therefore $W''_r\subset Z$ and $\dim(W_r/Z\otimes\mathbf{Q})\leqq\dim(W_r/W''_r\otimes\mathbf{Q})$. The latter does not exceed $(n-2)\,n(n-1)\,R/2r$, since

$$\dim((\Lambda_r/\Lambda_r(\prod_{k\neq i,j}(a_k-1)))\otimes\mathbf{Q})\leqq(n-2)\,R/r.$$

Hence the required estimation holds.

Since $\rho_r^2\otimes 1_{\mathbf{Q}}$ is identified with the linear map on $W_r/Z\otimes\mathbf{Q}$ induced from $C_r(\sigma)\otimes 1_{\mathbf{Q}}$, by replacing V, W, A with $W_r, Z, C_r(\sigma)\otimes 1_{\mathbf{Q}}$, (4.10) and the above inequalities prove the lemma. q.e.d.

Let $\oplus\Lambda_r e_i$ be the free Λ_r-module with a free basis e_i, $i=1,\dots,n-1$ and let $B_r\colon\oplus\Lambda_r e_i\to\oplus\Lambda_r e_i$ be a Λ_r-homomorphism defined by

(4.11) $$B_r(e_i)=\sum_{j=1}^{n-1}b_{ji}e_j,$$

where b_{ij} is the image of the (i,j)-component of the matrix $B(\sigma)$ under the projection $\Lambda\to\Lambda_r$.

Lemma 5. *There is a number M' independent of r such that*

$$|\mathrm{trace}(a^J\rho_r^3\otimes 1_{\mathbf{Q}})-\mathrm{trace}(a^J B_r\otimes 1_{\mathbf{Q}})|<M'R/r.$$

Proof. Define a surjective Λ_r-homomorphism

$$\gamma_r\colon\oplus\Lambda_r e_i\to V_r^3$$

by $\gamma_r(e_i)=\beta_r(e'_{i,i+1})$. Then calculation shows $\rho_r^3\circ\gamma_r=\gamma_r\circ B_r$. Since $\chi(L_r)=R\chi(L)=R(1-n)$, where χ is the Euler number, we get $\dim(\beta_r(W_r)\otimes\mathbf{Q})=\dim(V_r\otimes\mathbf{Q})-n=(n-1)R+1-n$. Using this and the facts shown in the proof of Lemma 4, we have

$$\begin{aligned}\dim(\beta_r(W'_r)\otimes\mathbf{Q})&=\dim(\beta_r(W''_r)\otimes\mathbf{Q})\\&\geqq\dim(\beta_r(W_r)\otimes\mathbf{Q})-\dim(W_r/W''_r\otimes\mathbf{Q})\\&\geqq(n-1)(R-1)-(n-2)\,n(n-1)\,R/2r.\end{aligned}$$

Therefore

$$\dim(\operatorname{Ker}\gamma_r \otimes \mathbf{Q}) = \dim(\bigoplus \Lambda_r\, e_i \otimes \mathbf{Q}) - \dim(\beta_r(W_r') \otimes \mathbf{Q})$$
$$\leqq (n-1)(1+n(n-2)R/2r).$$

Using this and the fact that $\|a^J B_r \otimes 1_{\mathbf{Q}}\|_1$ are bounded to the above, we can complete the proof by the inequality

$$|\operatorname{trace}(a^J B_r|_{\operatorname{Ker}\gamma_r} \otimes 1_{\mathbf{Q}})| \leqq \|a^J B_r \otimes 1_{\mathbf{Q}}\|_1 \dim(\operatorname{Ker}\gamma_r \otimes \mathbf{Q}). \quad \text{q.e.d.}$$

By Lemma 3, 4, 5 and (4.9), we see

$$\begin{aligned} &|\operatorname{trace}(a^J B_r \otimes 1_{\mathbf{Q}}) + \Lambda(S_r)| \\ &= |\operatorname{trace}(a^J B_r \otimes 1_{\mathbf{Q}}) - \operatorname{trace}(a^J \rho_r \otimes 1_{\mathbf{Q}}) + 1| \\ &< M'' R/r \end{aligned} \tag{4.12}$$

for a number M'' independent of r. Now let r be sufficiently large. Since B_r is the Λ_r-homomorphism on the free Λ_r-module, $\operatorname{trace}(a^J B_r \otimes 1_{\mathbf{Q}})$ is divisible by R. Also, $\Lambda(S_r)$ is divisible by R, because S can be deformed to satisfy (4.6). Hence (4.12) implies

$$\Lambda(S_r) = -\operatorname{trace}(a^J B_r \otimes 1_{\mathbf{Q}}). \tag{4.13}$$

Let $\operatorname{trace} B(\sigma) = \sum_{I \in \mathbf{Z}^n} r_I' a^I$. Then

$$\begin{aligned} &\operatorname{trace}(a^J B_r \otimes 1_{\mathbf{Q}}) \\ &= \sum_{I \in \mathbf{Z}^n} r_I' \operatorname{trace}[a^{I+J} \otimes 1_{\mathbf{Q}} \colon \Lambda_r \otimes \mathbf{Q} \to \Lambda_r \otimes \mathbf{Q}]. \end{aligned} \tag{4.14}$$

Since $a^{I+J} = 1$ in Λ_r if and only if all components of $\operatorname{inv}_\tau(I+J) = \operatorname{inv}_\tau I - I_0$ are divisible by r, noticing that r is sufficiently large, we get

$$\begin{aligned} r_I' \operatorname{trace}(a^{I+J} \otimes 1_{\mathbf{Q}}) &= 0 \qquad \text{if } \operatorname{inv}_\tau I \neq I_0, \\ &= r_I' R \qquad \text{if } \operatorname{inv}_\tau I = I_0. \end{aligned}$$

Therefore (4.13), (4.14) imply that

$$\Lambda(S_r) = -\Big(\sum_{\operatorname{inv}_\tau I = I_0} r_I'\Big) R = r_{I_0} R.$$

This and Lemma 1 complete the proof of the proposition.

5. Proof of Theorem 2

We first prove the theorem in the case of $p=1$. Assume (2.2), (2.3). Let $C = \{c_1, \ldots, c_n\}$ be a set of periodic solutions of (2.1) satisfying (2.4) and (3.1). Assume $n \geqq 3$. Let

$$X = \mathbf{R}^2 - \{c_1(0), \ldots, c_n(0)\}.$$

Choose a point x_0' in $X \cap (\{0\} \times \mathbf{R})$ such that $\phi(t; 0, x_0') \notin K$ for $0 \leqq t \leqq 1$. This is possible because K is compact. Fix an orientation preserving homeomorphism $\psi: \mathbf{R}^2 \to \mathbf{R}^2$ isotopic to the identity such that $\psi(c_i(0)) = (i, 0)$ for $i = 1, \ldots, n$ and $\psi(x_0') = x_0'$. Such homeomorphism clearly exists. For $i = 1, \ldots, n$, let d_i be a loop in X based at x_0' which follows the straight line from x_0' to very near $(i, 0)$, then circle it once in a counterclockwise direction and retrace the same line back to x_0'. Denote by β_i the class of the loop $\psi^{-1} \circ d_i$ in $\pi_1(X, x_0')$. Then $\beta_1, \ldots, \beta_n$ gives a basis of the free group $\pi_1(X, x_0')$. Let ρ' be the homomorphism from B_n to the group of all automorphisms of $\pi_1(X, x_0')$ defined by

$$\begin{aligned} \rho'(\sigma_i)(\beta_j) &= \beta_i \beta_{i+1} \beta_i^{-1} && j = i, \\ &= \beta_i && j = i+1, \\ &= \beta_j && j \neq i,\ i+1. \end{aligned} \tag{5.1}$$

Let $u(t) = \phi(t; 0, x_0')$. Since ψ is isotopic to the identity, the braid σ_C equals, after a suitable choice of v in the definition of it, to the class of $\pi(\psi(c_1(t), \ldots, \psi(c_n(t)))$. Noticing this fact, we have

Lemma 1. $T_* = u_* \circ \rho'(\sigma_C)$: $\pi_1(X, x_0') \to \pi_1(X, T(x_0'))$.

Proof. For a real number t, define a C^1-diffeomorphism $\Phi_t: \mathbf{R}^3 \to \mathbf{R}^3$ by $\Phi_t(s, x) = (t+s, \phi(t+s; s, x))$. Then this diffeomorphism preserves the open set

$$Y = \mathbf{R}^3 - \bigcup_{i=n}^{n} \{(t, c_i(t)) | t \in \mathbf{R}\}.$$

For $k = 0, 1$, defines a homotopy equivalence $i_k: X \to Y$ by $i_k(x) = (k, x)$. Since $i_1 \circ T = \Phi_1 \circ i_0$ we have

$$T_* = i_{1*}^{-1} \circ \Phi_{1*} \circ i_{0*}: \pi_1(X, x_0') \to \pi_1(X, T(x_0')). \tag{5.2}$$

Define $u_0(t) = \Phi_t(0, x_0')$, $u_1(t) = (t, x_0')$. These are paths in Y. Then the following is clear:

$$\Phi_{1*} = u_{0*}: \pi_1(Y, (0, x_0')) \to \pi_1(Y, (1, T(x_0'))). \tag{5.3}$$

From (5.2), (5.3) and the fact the the paths $u_1(i_1 \circ u)$ and u_0 are homotopic with end points held fixed, we have

$$\begin{aligned} T_* &= i_{1*}^{-1} \circ u_{0*} \circ i_{0*} = i_{1*}^{-1} \circ (i_1 \circ u)_* \circ u_{1*} \circ i_{0*} \\ &= i_{1*}^{-1} \circ (i_1 \circ u)_* \circ i_{1*} \circ i_{1*}^{-1} \circ u_{1*} \circ i_{0*} \\ &= u_* \circ i_{1*}^{-1} \circ u_{1*} \circ i_{0*}. \end{aligned} \tag{5.4}$$

In the following, we show that $i_{1*}^{-1} \circ u_{1*} \circ i_{0*} = \rho'(\sigma_C)$.

For a path $w = (w_1, \ldots, w_n)$ in V_n such that the image $w_i([0, 1])$ dose not contain x_0' for any $i = 1, \ldots, n$, let

$$Y(w) = [0, 1] \times \mathbf{R}^2 - \bigcup_{i=n}^{n} \{(t, w_i(t)) | 0 \leqq t \leqq 1\},$$

define a path $u(w)$ in $Y(w)$ by $u(w)(t)=(t,x'_0)$ and define $i(w,k)$: $\mathbf{R}^2-\{w_1(0),\ldots,w_n(0)\}\to Y(w)$ by $i(w,k)(x)=(k,x)$ for $k=0,1$. Then it is easy to see by (3.3) that for $k=1,\ldots,n-1$.

$$\begin{aligned} i(l_k,1)_*^{-1}\circ u(l_k)_*\circ i(l_k,0)_*([d_j]) &= [d_k d_{k+1} d_k^{-1}] && j=k,\\ &=[d_k] && j=k+1,\\ &=[d_j] && j\neq k,\ k+1. \end{aligned}$$

Since $\psi_*\beta_j=[d_j]$, this implies that for $k=1,\ldots,n-1$,

$$\rho'(\sigma_k)=(i(l_k,1)\circ\psi)_*^{-1}\circ u(l_k)_*\circ(i(l_k,0)\circ\psi)_*. \tag{5.5}$$

Express σ_C as $\sigma_C=\sigma_{i_1}^{\varepsilon_1}\ldots\sigma_{i_d}^{\varepsilon_d}$, where $\varepsilon_1,\ldots,\varepsilon_d=1,-1$, and let $\sigma(k)=\sigma_{i_k}^{\varepsilon_k}$, $l(k)=l_{i_k}^{\varepsilon_k}$, $l=l(1)\ldots l(d)$. Define a path v_k in $Y(l)$ by $v_k(t)=((k-1+t)/d,x'_0)$ and define $j_k\colon X\to Y(l)$ by $j_k(x)=((k-1)/d,\psi(x))$. Then (5.5) implies that

$$\rho'(\sigma(k))=(j_{k+1})_*^{-1}\circ v_{k*}\circ j_{k*}.$$

Therefore we get

$$\begin{aligned}\rho'(\sigma_C)&=\rho'(\sigma(d))\ldots\rho'(\sigma(1))\\ &=(j_{d+1})_*^{-1}\circ(v_d\circ\ldots\circ v_1)_*\circ j_{1*}\\ &=(j_{d+1})_*^{-1}\circ u(l)_*\circ j_{1*}.\end{aligned} \tag{5.6}$$

If we can show that there is a homeomorphism Ψ from $Y\cap([0,1]\times\mathbf{R}^2)$ to $Y(l)$ such that j_1 and j_{d+1} are homotopic to $\Psi\circ i_0$ and $\Psi\circ i_1$ respectively and $\Psi\circ u_1=u(l)$, then (5.6) implies that

$$\begin{aligned}\rho'(\sigma_C)&=i_{1*}^{-1}\circ\Psi_*^{-1}\circ u(l)_*\circ\Psi_*\circ i_{0*}\\ &=i_{1*}^{-1}\circ\Psi_*^{-1}\circ(\Psi\circ u_1)_*\circ\Psi_*\circ i_{0*}\\ &=i_{1*}^{-1}\circ u_{1*}\circ i_{0*}.\end{aligned}$$

This and (5.4) prove the lemma. The homeomorphism Ψ is constructed as follows. Let $H_t=(H_t^1,\ldots k,H_t^n)$ be a homotopy of class C^1 in V_n between l and $(\psi\circ c_1,\ldots,\psi\circ c_n)|_{[0,1]}$ with $H_t(k)=((1,0),\ldots,(n,0))$, $k=0,1$. By means of a partition of unity, we can construct a C^1-map $F\colon\mathbf{R}\times[0,1]\times\mathbf{R}^2\to\mathbf{R}^2$ such that for every $(t,s)\in\mathbf{R}\times[0,1]$

$$F(t,s,H_t^i(s))=\frac{d}{dt}H_t^i(s),$$

$$F(t,s,x'_0)=0.$$

Let T' be the Poincaré transformation of the time dependent equation on $[0,1]\times\mathbf{R}^2$:

$$dx/dt=F(t,s,x),$$

$$ds/dt=0.$$

Then the restriction of $T'\circ(\mathrm{id}\times\psi)$ to $Y\cap([0,1]\times\mathbf{R}^2)$ gives the desired homeomorphism Ψ. q.e.d.

Let $L=K-\{c_1(0),\ldots,c_n(0)\}$.

Lemma 2. *There is a continuous map $S: L \to L$ which coincides with T throughout K' and has no fixed points on $\bigcup_{i=1}^{n} \operatorname{Int} L_i$, where $L_i = K_i - \{c_i(0)\}$, and whose image $S(L)$ is compact.*

Proof. For $i=1,\ldots,n$, choose a homeomorphism $\phi_i: K_i \to D^2$, where $D^2 = \{x \in \mathbf{R}^2 \,|\, |x| \leqq 1\}$, with $\phi_i(c_i(0)) = (0,0)$. Define τ_C by $T(c_i(0)) = c_{\tau_C(i)}(0)$.

For i with $\tau_C(i) \neq i$ or $T(K_i) \supset K_i$, define $T_i: L_i \to L$ by

$$T_i(x) = T(\phi_i^{-1}(\phi_i(x)/\|\phi_i(x)\|)).$$

Next assume that i satisfies $\tau_C(i) = i$ and $T(K_i) \not\supset K_i$. Then (2.4) implies that $T(K_i) \subset K_i$. Then it is clear that there is a continuous map $\xi_i: S^1 \times (0,1] \to \mathbf{R}^2 - \{0\}$, where S^1 is the boundary of D^2, such that for any $x \in S^1$

$$\begin{aligned} \xi_i(x,1) &= \phi_i \circ T \circ \phi_i^{-1}(x) & & \\ \xi_i(x,t) &\neq x & & 0 < t < 1, \\ |\xi_i(x,t)| &\leqq 1/t & & 1/2 \leqq t \leqq 1, \\ &= 1/t & & 0 < t \leqq 1/2. \end{aligned}$$

Define $\xi_i': D^2 - \{0\} \to D^2 - \{0\}$ by $\xi_i'(x) = \|x\|\, \xi_i(x/\|x\|, \|x\|)$ and $T_i: L_i \to L$ by $T_i = \phi_i^{-1} \circ \xi_i' \circ \phi_i$.

T_i has no fixed points on $\operatorname{Int} L_i$ by (2.4) in the former case, by the property of ξ_i in the latter case. Since $\phi_i(\operatorname{Int} K_i) = \operatorname{Int} D^2$ by the invariance theorem of domain and $T_i = T$ on $K' \cap K_i$, the map $S: L \to L$ defined by $S = T$ on K' and $S = T_i$ on L_i is well defined and continuous. Also, clearly the image of S is compact, so we obtain the required map. q.e.d.

Choose a point $x_0 \in K'$ and a path w in $\mathbf{R}^2 - \bigcup_{i=1}^{n} \operatorname{Int} K_i$ from x_0 to x_0'. Since L and X are homotopy equivalent, there is a path v in L from x_0 to $S(x_0) = T(x_0)$ such that

(5.7) The paths v and $wu(T \circ w)^{-1}$ are fixed end-point homotopic.

Define $\alpha_j = (w_* \circ i_*)^{-1} \beta_j$, where $i: L \to X$ is the inclusion. Then, the homomorphism ρ from B_n to the group of all automorphisms of $\pi_1(L, x_0)$ defined by (4.1) satisfies

(5.8) $\rho(\sigma) = (w_* \circ i_*)^{-1} \circ \rho'(\sigma) \circ w_* \circ i_*$.

Lemma 3. $S_* = v_* \circ \rho(\sigma_C): \pi_1(L, x_0) \to \pi_1(L, S(x_0))$.

Proof. From the following commutative diagram:

$$\begin{array}{ccccc} \pi_1(L,x_0) & \xrightarrow{S_*} & \pi_1(L,S(x_0)) & \xleftarrow{v_*} & \pi_1(L,x_0) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ \pi_1(X,x_0) & \xrightarrow{T_*} & \pi_1(X,T(x_0)) & \xleftarrow{v_*} & \pi_1(X,x_0) \\ \downarrow{\scriptstyle w_*} & & \downarrow{\scriptstyle (T\circ w)_*} & & \downarrow{\scriptstyle w_*} \\ \pi_1(X,x_0') & \xrightarrow{T_*} & \pi_1(X,T(x_0')) & \xleftarrow{u_*} & \pi_1(X,x_0'). \end{array}$$

we see that $S_*=v_*\circ(w_*\circ i_*)^{-1}\circ u_*^{-1}\circ T_*\circ w_*\circ i_*$. Therefore by Lemma 1 and (5.8), we complete the proof. q.e.d.

By the above lemma and Proposition 2, in order to prove the theorem in the case of $p=1$, it suffices to show the following.

Lemma 4. *Let $I=(i_1,\dots,i_n)\in\mathbf{Z}^n(\tau_C)$. If $x_1\in L$ is a fixed point of S satisfying* (4.5), *where $\sigma=\sigma_C$, then the 1-periodic solution $\phi(t;0,x_1)$ is of degree I.*

Proof. Let Z be the quotient space $Y/(s,x)\sim(s+1,x)$, $\pi_Z\colon Y\to Z$ the projection and $i_Z=\pi_Z\circ i_0=\pi_Z\circ i_1\colon X\to Z$. Let $c(t)=\pi_Z(\Phi_t(0,x_1))$, h a path as in (4.5) and $d=i_Z\circ(w^{-1}h)$. Since two paths in Y $(i_0\circ(h^{-1}w))u_0(i_1\circ T\circ(w^{-1}h))$ and $\Phi_t(0,x_1)$ are fixed end point homotopic, (5.7) implies that

$$\begin{aligned}(5.9)\qquad c&\sim\pi_Z\circ((i_0\circ(h^{-1}w))u_0(i_1\circ T\circ(w^{-1}h)))=d^{-1}(\pi_Z\circ u_0)i_Z\circ T\circ(w^{-1}h)\\&\sim(d^{-1}(\pi_Z\circ u_0)(i_Z\circ u^{-1})d)(i_Z\circ(h^{-1}wu(T\circ w)^{-1}(T\circ h)))\\&\sim(d^{-1}(\pi_Z\circ u_0)(i_Z\circ u^{-1})d)(i_Z\circ(h^{-1}v(T\circ h))),\end{aligned}$$

where $\sim$ means fixed end point homotopic. Therefore, since S and $T|_L$ are homotopic with x_0 and x_1 fixed, the homology class of c satisfies

$$[c]=[(\pi_Z\circ u_0)(i_Z\circ u^{-1})]+i_{Z*}[h^{-1}v(S\circ h)].$$

If we denote by $b_0\in H_1(Z:\mathbf{Z})$ the homology class of the loop $\pi_Z(t,x_0')$, then clearly $b_0=[(\pi_Z\circ u_0)(i_Z\circ u^{-1})]$. Hence the following lemma completes the proof.

Lemma 5. *Let c be a loop in Z. If $[c]=b_0+i_{Z*}\left(\sum_{k=1}^{n}j_kb_k\right)$, then $\mathrm{inv}_{\tau_C}(j_1,\dots,j_n)=(d(c',c_1),\dots,d(c',c_n))$, where $b_k=i_*[\alpha_k]$ and the 1-periodic curve c' satisfies $c(t)=\pi_Z(t,c'(t))$ for $0\leqq t\leqq 1$.*

Proof. Fix $1\leqq k\leqq n$. Let $P_k=\{\tau_C^s(k)|s\in\mathbf{Z}\}$, then the cardinal number of this set is the period p_k of c_k. Let

$$Y'=\mathbf{R}^3-\bigcup_{j\in P_k}\{(t,c_j(t)|t\in\mathbf{R}\},\ Z_1=Y'/(s,x)\sim(s+1,x)$$

and $Z_2=Y'/(s,x)\sim(s+p_k,x)$. Let $\pi_{Z_\varepsilon}\colon Y'\to Z_\varepsilon$ be the projection, $i_{Z_\varepsilon}=\pi_{Z^\varepsilon}|_{Y\circ i_\theta}\colon X\to Z_\varepsilon$ for $\varepsilon=1,2$, and $\pi'\colon Z_2\to Z_1$ the projection. Then π' is a p_k-fold regular covering map. By the Alexander duality and the exact sequence of homology group for the pair $(S^1\times\mathbf{R}^2, Z_1)$, we have that

$H_1(Z_1;\mathbf{Z})$ is a free abelian group of rank 2 with basis b_0', b_k',

and by the similar way we have that

$H_1(Z_2;\mathbf{Z})$ is a free abelian group of rank p_k+1 with basis b_0'', b_j'' $j\in P_k$,

where b_0', b_0'' are the classes of $\pi_{Z_1}(t,x_0')$, $\pi_{Z_2}(p_kt,x_0')$, $b_k'=i_{Z_1*}b_k$, $b_j''=i_{Z_2*}b_j$. Let $d_1(t)=\pi_{Z_1}(t,c'(t))$, $d_2(t)=\pi_{Z_2}(p_kt,c'(p_kt))$. If the assumption of the lemma is satisfied, then clearly $[d_1]=b_0'+\left(\sum_{s\in P_k}j_s\right)b_k'$. Since $\pi'\circ d_2=d_1\dots d_1$ (p_k times), the loop d_2 is invariant under the covering transformation $\pi_{Z_2}(s,x)\to\pi_{Z_2}(s+1,x)$ and $\pi'_* b'_0=p_k b''_0$, $\pi'_* b''_j=b'_k$, we obtain

$$[d_2]=b''_0+(\sum_{s\in P_k} j_s)(\sum_{t\in P_k} b''_t). \tag{5.10}$$

Let $Z_3=(\mathbf{R}^3-\{(t,c_k(t))|t\in\mathbf{R}\})/(s,x)\sim(s+p_k,x)$ and $\Psi: Z_3\to S^1\times(\mathbf{R}^2-\{0\})$ be a homeomorphism induced from $(s,x)\to(s/p_k,\ x-c_k(s))$. Let $i': Z_2\to Z_3$ be the inclusion. Then the homology class $(\Psi\circ i')_*[d_2]$ is equal to the class of the loop $(t,c'(p_k t)-c_k(p_k t))$ in $S^1\times(\mathbf{R}^2-\{0\})$. Therefore if we identify $H_1(S^1\times(\mathbf{R}^2-\{0\});\mathbf{Z})$ with $\mathbf{Z}^2$ canonically, then $(\Psi\circ i')_*[d_2]=(1,d(c',c_k))$. On the other hand, (5.10) implies that $(\Psi\circ i')_*[d_2]=(1,\sum_{s\in P_k} j_s)$. Thus the lemma is proved. q.e.d.

Now we prove the theorem when $p\geqq 2$. Define a C^1-map $f_p\colon \mathbf{R}^3\to\mathbf{R}^2$ by $f_p(t,x)=pf(pt,x)$.
Then the periodic system

$$dx/dt=f_p(t,x) \tag{5.11}$$

satisfies (2.2), (2.3). Clearly $x(t)$ is a solution of (2.1) if and only if $x(pt)$ is a solution of (5.11). Thus, for $k=1,\dots n$, $c'_k(t)=c_k(pt)$ is a periodic solution of (5.11) of period $(p,p_k)/p$ where $(\ ,\ )$ denotes least common multiple. Since Theorem 2 is proved before when $p=1$, applying it to the system (5.11) and the set of periodic solutions $C_p=\{c'_1,\dots,c'_n\}$, we see that, for every element I of $\mathbf{Z}^n(\tau_C^p)$ with $r_I(C_p,1)$ $(=r_I(C,p))\neq 0$, there is a 1-periodic solution c' of (5.11) passing K' at $t=0$ with

$$\begin{aligned} d(c',c'_k)&=\deg(c((p,p_k)t)-c_k((p,p_k)t)\\ &=i_k \end{aligned}$$

for $k=1,\dots,n$, where $c(t)=c'(t/p)$.

Suppose that c is q-periodic, where $q<p$ is a divisor of p, then

$$\begin{aligned} &\deg(c((p,p_k)t)-c_k((p,p_k)t))\\ &\quad=((p,p_k)/(q,p_k))\deg(c((q,p_k)t)-c_k((q,p_k)t))\\ &\quad=((p,p_k)/(q,p_k))\,d(c,c_k). \end{aligned}$$

This implies that $I=\eta_{q,p}(d(c,c_1),\dots,d(c,c_n))$, so $I\in\eta_{q,p}(\mathbf{Z}^n(\tau_C^q))$. Hence if $I\in\mathbf{Z}^n(p)$, then c is p-periodic. Thus the theorem is proved for $p\geqq 2$.

6. Proof of Theorem 1

By the corollary of Theorem 2, it suffices to show that

$$N_p\geqq p \quad\text{and}\quad \sum_{I\in\mathbf{Z}^n(p)}|r_I(C,p)|=N'_p\geqq 2^{p-1},$$

where N_p is the number of $I\in\mathbf{Z}^n(p)$ with $r_I(C,p)\neq 0$.

We can assume without loss of generality that

$$c_1(t)=(1,0),\quad c_3(t)=(3,0) \qquad \text{for every } t\in\mathbf{R}.$$

For, choose a parametrized diffeomorphism $\phi_t\colon \mathbf{R}^2\to\mathbf{R}^2$ periodic in t of period 1 with $\phi_t(c_1(t))=(1,0)$, $\phi_t(c_3(t))=(3,0)$. Then the verification of Theorem 1 for the Eq. (2.1) and the 1-periodic solutions c_1, c_2, c_3 is equivalent to that for the periodic system on $\mathbf{R}^2$, for which $\phi_t(x(t))$ is a solution so long as $x(t)$ is a solution of (2.1), and the 1-periodic solutions $(1,0), \phi_t(c_2(t)), (3,0)$, because

$$[c_1, c_2, c_3]=[(1,0), \phi_t(c_2(t)), (3,0)].$$

Clearly we have, setting $I=(i_1,\dots,i_d)$, $J=(j_1,\dots,j_d)$,

Lemma 1. *The equivalence class* C_p *in* B_n' *of the braid* σ_C^p *contains the braid* $(\alpha^{i_1}\beta^{j_1}\dots\alpha^{i_d}\beta^{j_d})^p$, *where* $\alpha=\sigma_1^2$, $\beta=\sigma_2^2$.

For integers i and m, let

$$\begin{aligned} P(i,m)&=\sum_{s=0}^{m-1}(a_j a_2)^s \qquad m>0\\ &=-\sum_{s=1}^{-m}(a_j a_2)^{-s} \qquad m<0,\\ Q(i,m)&=(a_j a_2)^m, \qquad R=a_2(1-a_1)(1-a_3), \end{aligned}$$

where $j=1$ for i odd, $j=3$ for i even.

For a natural number k, let $P(k)$ denote the polynomial ring $\mathbf{Z}[X_1,\dots,X_k, Y_1,\dots,Y_k, Z]$. For $s=1,\dots,k$ define a group homomorphism $F_s\colon P(k)\to P(k)$, where $P(k)$ is considered as an abelian group, by

$$\begin{aligned} F_s(X(I)\,Y(J)\,Z^l)&=X(I-e_{s-1}-e_s)\,Y(J+e_s)\,Z^{l-1}\\ &\qquad \text{if } i_{s-1}>0, i_s>0 \quad\text{and}\quad l>0,\\ &=0 \qquad \text{otherwise,} \end{aligned}$$

where $I, J\in\mathbf{Z}^k$, l is an integer, $X(I)=X_1^{i_1}\dots X_k^{i_k}$, $Y(J)=Y_1^{j_1}\dots Y_k^{j_k}$ etc., $e_i=(0,\dots,0,1,0,\dots,0)$ (i-th component$=1$) and i_0, e_0 mean i_k, e_k respectively. The following is clear.

(6.1) $$F_s\circ F_{s'}=F_{s'}\circ F_s \quad s, s'=1,\dots,k,$$

$$F_s(WW')=WF_s(W') \text{ if } W \text{ contains no symbols } X_{s-1}, X_s.$$

Define $\Phi\colon P(k)\times\Lambda^{2k+1}\to\Lambda$ by

$$\Phi(V,\lambda_1,\dots,\lambda_{2k+1})=V(\lambda_1,\dots,\lambda_{2k+1}).$$

For $M=(m_1,\dots,m_k)\in\mathbf{Z}^k$, define $\Phi_M\colon P(k)\to\Lambda$ by

$$\Phi_M(V)=\Phi(V, P(1,m_1),\dots,P(k,m_k), Q(1,m_1),\dots,Q(k,m_k), R).$$

Let $\Psi_M=\Phi_M\circ(1+F_2)\circ(1+F_3)\circ\dots\circ(1+F_k)\colon P(k)\to\Lambda$.

Lemma 2. *If* $\sigma=\alpha^{m_1}\beta^{m_2}\dots\alpha^{m_{k-1}}\beta^{m_k}$, *where* $M=(m_1,\dots,m_k)\in\mathbf{Z}^k$, *then* $B(\sigma)$ *equals to*

$$\begin{pmatrix} Q(1,m_1)\,\Psi_M(X_2\dots X_{k-1}Z^{k'}) & (1-a_1)\,\Psi_M(X_1\dots X_{k-1}Z^{k'}) \\ a_2(1-a_3)\,Q(1,m_1)\,\Psi_M(X_2\dots X_k Z^{k'}) & \Psi_M(X_1\dots X_k Z^{k/2}) \end{pmatrix},$$

where $k'=k/2-1$.

Proof. Since

$$B(\alpha^m\beta^n)=\begin{pmatrix} Q(1,m) & (1-a_1)\,P(1,m) \\ a_2(1-a_3)\,Q(1,m)\,P(2,n) & P(1,m)\,P(2,n)\,R+Q(2,n) \end{pmatrix}$$

for integers n,m and $B(\sigma)=B(\alpha^{m_{k-1}}\beta^{m_k})\dots B(\alpha^{m_1}\beta^{m_2})$, this lemma is proved by induction on k with the aid of the following formulas.

$$\begin{aligned}&\Psi_M(X_sX_{s+1}\dots X_tZ^u)\\ &\quad=\Psi_M(X_s\dots X_{t-2}Z^{u-1})\,Q(t,m_t)+\Psi_M(X_s\dots X_{t-1}Z^{u-1})\,P(t,m_t)R\\ &\qquad\text{if } t-s \text{ is odd},\\ &\quad=\Psi_M(X_s\dots X_{t-2}Z^{u-1})\,Q(t,m_t)+\Psi_M(X_s\dots X_{t-1}Z^{u})\,P(t,m_t)\\ &\qquad\text{if } t-s \text{ is even},\end{aligned}$$

where $1\leqq s<t\leqq k$ and $u=(t-s+1)/2$ if $t-s$ is odd, $u=(t-s)/2$ if $t-s$ is even. q.e.d.

In the following, let $k=2pd$, $k'=pd$ and $M=(K,\dots,K)$ (p times), where $K=(i_1,j_1,i_2,j_2,\dots,i_d,j_d)$. By Lemma 1 and 2, we have

Lemma 3. $A(C,p)=-(\Phi_M\circ(1+F_1)\circ(1+F_2)\circ\dots\circ(1+F_k))(X_1\dots X_kZ^{k'})$.

For $G=(g_1,\dots,g_d)\in\mathbf{Z}^d$ and $s=1,\dots,d$, let $G_s=(g_{s+1},\dots,g_d,g_1,\dots,g_s)$. Then $\sigma(G,H)=\sigma(G_s,H_s)$ for $G,H\in\mathbf{Z}^d$. It is clear that the following conditions are equivalent.

i) $[c_1,c_2,c_3]=\sigma(I,J)$.
ii) $[c_3,c_2,c_1]=\sigma(J,I_1)$.
iii) $[c_1(-t),c_2(-t),c_3(-t)]=\sigma(I',J'_1)$,

where $I'=(-i_d,\dots,-i_1)$, $J'=(-j_d,\dots,-j_1)$. Therefore, the verification of Theorem 1 in the case of $[c_1,c_2,c_3]=\sigma(I,J)$ is equivalent to that in the case of $[c_1,c_2,c_3]=\sigma(I_s,J_s)$, $\sigma(J_s,I_{s+1})$, $\sigma(J'_s,I'_s)$ or $\sigma(I'_s,J'_{s+1})$. Thus for the proof of the theorem, it suffices to consider only the following four cases.

I) $|i_s|>1$ and $|j_t|>1$ for some s and t.
II) $i_s=1, j_s>0$ for $s=1,\dots,d$ and $j_t>1$ for some t.
III) $i_s>0$, $|j_s|=1$ for $s=1,\dots,d$ and $j_t<0$ for some t.
IV) $|j_s|=1$ for $s=1,\dots,d$ and $i_t>0$, $i_u<0$ for some t,u.

For a subset Γ of $\mathbf{Z}^3$ and $\lambda=\sum r_I a^I\in\Lambda$, where $a^I=a_1^{i_1}\dots a_n^{i_n}$, let $\Gamma(\lambda)=\sum_{I\in\Gamma} r_I a^I$.

Case I) Define $\gamma: \Lambda \to \mathbf{Z}$ by

$$\gamma(\sum r_I a^I) = \max\{i_2 - i_1 - i_3 | r_I \neq 0\}, \qquad \gamma(0) = 0.$$

Then $\gamma(A(C,p)) = k'$ and $\gamma(\Phi_M(F_s(V))) < \gamma(\Phi_M(V))$ for $V \in P(k)$, $s = 1, \ldots, k$ with $F_s(V) \neq 0$. Hence if we set

$$\Gamma = \{I \in \mathbf{Z}^3 | i_2 - i_1 - i_3 = k'\}.$$

then Lemma 3 implies that

$$-\Gamma(A(C,p)) = \Gamma(\Phi_M(X_1 \ldots X_k Z^{k'})) = a_2^{k'} \Phi_M(X_1 \ldots X_{2d})^p.$$

Therefore, setting

$$u = \left(\sum_{s=1}^{d} \min\{i_s, 0\}\right) p + 1, \Gamma' = \Gamma \cap \{i_1 = u\},$$

we get $\Gamma'(A(C,p)) = v a_1^u a_2^w \Phi_M(X_2 X_4 \ldots X_{2d})^p$, where v, w are integers with $v < 0$. Since $\Phi_M(X_2 X_4 \ldots X_{2d})^p$ has at least p terms by the assumption and $\Gamma' \subset \mathbf{Z}^3(p)$, we get $N_p \geqq p$. Also $N_p' \geqq p 2^p$ is easily verified. Hence the theorem is proved.

To prove the other cases, we need several preparations. Let $\delta: \Lambda \to \mathbf{Z}$ be defined by

$$\delta(\sum r_I a^I) = \max\{i_1 | r_I \neq 0\}, \ \delta(0) = 0.$$

Then $\delta(A(C,p)) = \left(\sum_{s=1}^{d} \max\{i_s, 0\}\right) p$. Let $\Delta = \{i_1 = \delta(A(C,p))\}$. Define subsets $\Omega_i, i = 1, \ldots, 5$, of $\{1, \ldots, k\}$ by

$$\begin{aligned}
\Omega_1 &= \{s\text{: odd with } m_{s-2}, m_s > 0\},\\
\Omega_2 &= \{s\text{: odd with } m_{s-2} < 0, m_s > 0\},\\
\Omega_3 &= \{s\text{: even with } m_{s-1}, m_{s+1} < 0\},\\
\Omega_4 &= \{s\text{: even with } m_{s-1} < 0,\ m_{s+1} > 0\},\\
\Omega_5 &= \{s\text{: even with } m_{s-1} > 0,\ m_{s+1} < 0\}.
\end{aligned}$$

Let

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4.$$

Then $\delta(\Phi_M(F_s(W))) < \delta(\Phi_M(W))$ for any $s \notin \Omega$, $W \in P(k)$. Therefore if we set $V = -X_1 \ldots X_k Z^{k'}$, then

$$\begin{aligned}
\Delta(A(C,p)) &= \Delta(\Phi_M((1+F_1) \circ \ldots \circ (1+F_k)(V))\\
&= \Delta(\Phi_M((\prod_{s \in \Omega} (1+F_s))(V))),
\end{aligned}$$

where $\prod$ means composition of maps. This, (6.1) and the fact that $F_s \circ F_{s+1}(V) = 0$ implies that

$$(6.2)\quad \Delta(A(C,p)) = \Delta(\Phi_M(\prod_{s\in\Omega_1\cup\Omega_3}(1+F_s)\prod_{u\in\Omega_2}(1+F_u+F_{u-1})(V)))$$
$$= -\Delta(\Phi_M(\prod_{s\in\Omega_1}(1+F_s)(X_{s-1}X_sZ)\prod_{t\in\Omega 3}(X_{t-1}X_tZ)$$
$$\cdot\prod_{u\in\Omega_2}(1+F_u+F_{u-1})(X_{u-2}X_{u-1}X_uZ)\prod_{v\in\Omega_5}X_vZ)).$$

Case II). Since Ω_2, Ω_3 and Ω_5 are empty,

$$(6.3)\qquad -\Delta(A(C,p)) = (a_1 a_2)^{k'}\prod_{s=1}^{d}((a_3-1)P(2s,j_s)+1)^p.$$

$$u = k' + \left(\sum_{s=1}^{d}(j_s-1)\right)p - 1 \quad\text{and}\quad \Gamma = \{i_2 = u\}.$$

Then by (6.3),

$$(\Gamma\cap\Delta)(A(C,p)) = \Gamma(\Delta(A(C,p)))$$
$$= a_1^{k'} a_2^{u} a_3^{u-k'}(a_3-1)^{k'-1}(v a_3 + w(a_3-1)),$$

where v, w are non-negative multiples of p with $v+w<0$. Since $\Gamma\cap\Delta\subset\mathbf{Z}^3(p)$ and $k'\geqq p$ by the assumption, $N_p\geqq p$, $N'_p\geqq p2^{p-1}$ for $p\geqq 1$. Hence the proof is completed.

Case III). Since $\Omega_2, \Omega_3, \Omega_5$ are empty, by (6.2) we get

$$-\Delta(A(C,p)) = a_1^u a_2^v a_3^w (a_2 a_3 - a_3 + 1)^x,$$

where u, v, w and x are multiples of p with $x>0$. Let $\Gamma = \{u\}\times\{v+1\}\times\mathbf{Z}$. Then $\Gamma\subset\mathbf{Z}^3(p)$ and

$$-\Gamma(A(C,p)) = x\, a_1^u a_2^{v+1} a_3^{w+1}(1-a_3)^{x-1}.$$

Therefore, for $p\geqq 1$, $N_p\geqq x\geqq p$ and $N'_p\geqq x2^{x-1}\geqq p2^{p-1}$. Thus we complete the proof.

Case IV). By (6.2), we get

$$\pm\Delta(A(C,p)) = a_1^u a_2^v a_2^w (a_3-1)^x (a_3 - a_2 a_3)^x (a_3 - 1 - a_2 a_3)^y,$$

where u, v, w, x, y are multiples of p with $x>0$, $y\geqq 0$. Let $\Gamma = \{u\}\times\{v+1\}\times\mathbf{Z}$. Then $\Gamma\subset\mathbf{Z}^3(p)$ and

$$\pm\Gamma(A(C,p)) = a_1^u a_2^{v+1} a_3^{w+x}(a_3-1)^{x+y-1}(x(a_3-1)+y a_3).$$

Therefore, for $p\geqq 1$, $N_p\geqq x+y\geqq p$, $N'_p\geqq p2^{p-1}$. Thus the proof is completed.

7. 1-periodic Solutions

In this section, we give a sharper estimation for the number of 1-periodic solutions. The following result is the best estimation which is obtained by our method.

Proposition 3. *Assume* (2.2), (2.3). *Let* c_1, c_2 *and* c_3 *be* 1-*periodic solutions of* (2.1) *satisfying* (2.4) *for* $p=1$. *Then the number of* 1-*periodic solutions, passing* K' *at* $t=0$, *is not smaller than*

$$\begin{array}{ll} 3|m|\,|n|+|m|+|n|-3 & \text{if } [c_1, c_2, c_3]=[a^m b^n], mn>0, \\ 3|m|\,|n|+|m|+|n|+1 & \text{if } [c_1, c_2, c_3]=[a^m b^n], mn<0, \\ 2 & \text{if } [c_1, c_2, c_3]=[a^m] \quad \text{or } [b^m], \end{array}$$

where m, n *are non-zero integers,* [ ] *denotes conjugate class.*

Proof. If $[c_1, c_2, c_3]=[a^m b^n]$, then by Lemma 3 in Sect. 6

$$-A(C,1)=P(1,m)\,P(2,n)\,R+Q(1,m)+Q(2,n).$$

Hence the straightforward calculation proves the proposition. q.e.d.

For example, if $[c_1, c_2, c_3]=[a^2 b]$ (see Fig. 2.), then

$$-A(C,1)=a_1^2 a_2^2 a_3-a_1 a_2^2 a_3+a_1 a_2 a_3+a_1 a_2^2-a_1 a_2+a_2.$$

Hence, there are at least six 1-periodic solutions of degree (2, 2, 1), (1, 2, 1), (1, 2, 0), (1, 1, 1), (1, 1, 0), (0, 1, 0), other than c_1, c_2 and c_3. Also if $[c_1, c_2, c_3]=a^{100} b^{100}$, then there are at least 30,197 1-periodic solutions other than c_1, c_2 and c_3.

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Oblatum 13-V-1982