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Varieties with small dual varieties, I

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Introduction

Let X be a complex projective nonlinear n-fold in $\mathbb{P}^n$. Let $X^* \subseteq \mathbb{P}^{N^*}$ be the dual variety of X. Landman defines the defect of X to be $\operatorname{def}(X) = N-1-\dim X^*$. For most examples, $\operatorname{def}(X)=0$ (i.e. X^* is a hypersurface). The main purpose of this paper is to investigate those varieties with positive defect.

Assume that $\operatorname{def}(X)=k>0$. Let H be a general tangent hyperplane of X. The contact locus of H with X is a k dimensional linear space L in X [15]. We show that $N_{L/X}$, the normal sheaf of L, is isomorphic to $N^*_{L/X} \otimes \mathcal{O}_L(1)$. Furthermore, $N_{L/X}$ is a uniform vector bundle on $\mathbb{P}^k$ and $K_X|_L = \mathcal{O}_L\left(\frac{-n-k-2}{2}\right)$. In particular, $n \equiv k \bmod 2$. The pairity theorem was first proved by A. Landman, using the Picard-Lefschetz theory (unpublished). Zak and Landman had observed that $\operatorname{def}(X) \leqq n-2$. We show that if $\operatorname{def}(X)=n-2$, then X is a scroll ($n \geqq 3$). This theorem was first proved by Griffiths and Harris in the case $n=3$. In [4], we shall show that if $\operatorname{def}(X)=k \geqq \frac{n}{2}$, X is a $\mathbb{P}^{n+k/2}$-bundle over a $\frac{n-k}{2}$-fold.

As a consequence of his theorem on tangencies, Zak proved that $\dim X^* \geqq \dim X$. In particular, if X^* is smooth, then $\dim X = \dim X^*$. He also classified those varieties with the properties $\dim X = \frac{2(N-2)}{3}$ and $\dim \operatorname{Sec}(X) = N-1$ [19, 28].

In §4, using the isomorphism between $N_{L/X}$ and $N^*_{L/X} \otimes \mathcal{O}_L(1)$ and the Belinson spectral sequence, we show that if $\dim X = \dim X^* \leqq \frac{2}{3}N$, then X is one of the following varieties:

(a) X is a hypersurface.
(b) X is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ in $\mathbb{P}^{2n-1}$.

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(c) X is the Plücker embedding of $G(2, 5)$ in $\mathbb{P}^9$ [22].
(d) X is the 10-dimensional spinor variety in $\mathbb{P}^{15}$ [19, 25].

Hartshorne conjectures that if $\dim X > \frac{2}{3}N$, then X is a complete intersection. The conjecture will imply that the above list is the complete list of nonsingular projective varieties satisfying the property $\dim X = \dim X^*$. We are able to show that if $\operatorname{codim}(X) = 2$, then $\operatorname{def}(X) = 0$, unless X is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$. Throughout the paper, we shall assume the base field is the complex numbers.

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§ 1.

The following proposition is well known.

Proposition 1.1. *Let X be an irreducible reduced subvariety of $\mathbb{P}^N$.*

(a) *Assume that X is contained in a hyperplane H. If $X^{*\prime}$ is the dual variety of X, when we consider X as a subvariety of $\mathbb{P}^{N-1}$, then X^* is the cone over $X^{*\prime}$ with vertex p corresponding to H.*

(b) *Conversely, if X^* is a cone with vertex p, then X is contained in the corresponding hyperplane H. In particular,* $\operatorname{def}(X)$ *is the same whether we consider X as a subvariety of $\mathbb{P}^N$ or $\mathbb{P}^{N-1}$.*

Proof. (a) *If* $H_1 \neq H$ is a tangent hyperplane of X, then $H_1 \cap H$ is a tangent hyperplane of X in $\mathbb{P}^{N-1}$. Conversely, if T is a tangent hyperplane of X in H, then each hyperplane H_1 in $\mathbb{P}^N$ containing T is tangent to X. Thus X^* is a cone over $X^{*\prime}$.

(b) Each hyperplane which is tangent to X^* at a smooth point will contain the point p. Hence $X = (X^*)^*$ is contained in the hyperplane corresponding to p.

Proposition 1.2. *(Adjunction mapping theorem.) Let Y be a projective n-fold. Suppose that $\mathcal{O}_Y(1)$ is a very ample line bundle on Y and K_Y is the canonical line bundle on Y.*

(a) *If $|K_Y \otimes \mathcal{O}_Y(n-1)| = \emptyset$, then $(Y, \mathcal{O}_Y(1))$ is isomorphic to one of the following:*

1. $(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$.
2. $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.
3. $(Q_n, \mathcal{O}_{Q_n}(1))$, *where Q_n is a quadric hypersurface.*
4. $(\mathbb{P}_C(F), \mathcal{O}(1))$, *where F is a vector bundle of rank n on a curve C and $\mathcal{O}(1)$ is the tautological line bundle.*

(b) *If $|K_Y \otimes \mathcal{O}_Y(n-1)| \neq \emptyset$ then it has no base points.*

Proof. The proposition is a fairly straightforward generalization of the theorem in [26, 27]. One can find a proof in [14].

Theorem 1.3. (*Zak's theorem on tangencies* [6], § 7.)

(a) *Suppose that X is a nondegenerate projective n-fold in $\mathbb{P}^N$. If H is a k-plane in $\mathbb{P}^N$ $(k \geqq n)$, then* $\dim \operatorname{Sing}(H \cap X) \leqq k - n$.

(b) *If X is a nonlinear n-fold in $\mathbb{P}^N$, then* $\dim X^* \geqq \dim X$.

Corollary 1.4. *Suppose that X is a nonlinear projective n-fold in $\mathbb{P}^N$. If X^* is smooth, then* $\dim X = \dim X^*$.

Proof. Since $(X^*)^* = X$, $\dim X \geqq \dim X^*$, by 1.3(b). So $\dim X = \dim X^*$, by 1.3(b).

§ 2.

In the rest of the paper, we shall assume X is a nonlinear projective n-fold in $\mathbb{P}^N$. We shall also assume that $\operatorname{def}(X) = k$. If q is a general point of X and H is a general tangent hyperplane of X at q, then the contact locus of H with X is a k-dimensional linear space L. The main purpose of the section is to show that $N_{L/X}$, the normal sheaf of L in X, is isomorphic to $N_{L/X}^* \otimes \mathcal{O}_L(1)$. If $k > 0$, then we will show that $N_{L/X}$ is a uniform vector bundle on $\mathbb{P}^k$ and $K_X|_L = \mathcal{O}_L\left(\frac{-n-k-2}{2}\right)$. In particular, if $k > 0$, then $n \equiv k \bmod 2$. The pairity result was first observed by A. Landman (unpublished).

Theorem 2.1. *Let X, H, and L be as defined above. Assume* $\operatorname{def}(X) = k$.

(a) *If p is a point in L, then the tangent cone of the hyperplane section $H \cap X$ at p is a quadric hypersurface of rank $n-k$ in $\mathbb{P}(\Omega_X^1(p))$.*

(b) *Let s_h: $\mathcal{O}_X(-1) \to \mathcal{O}_X$ be the section defining $H \cap X$. Then s_h factors through I_L^2, where I_L is the ideal sheaf of L in X.*

(c) *Let t_h be the section of $I_L^2/I_L^3 \cong S^2(N_{L/X}^*)$ induced by s_h. Then t_h defines a nonsingular quadric hypersurface in $\mathbb{P}(N_{L/X}^*(p))$.*

Proof. (a) Let C_X be the conormal variety of X. Then $C_X = C_{X^*}$ [15]. *Let* p_2 be the projection map from C_X to X^* and let h be the point in X^* corresponding to H. We may assume that p_2 is smooth along $p_2^{-1}(h)$. In [13], Kleiman showed that rank of the Hessian of s_h at p is equal to $n - \operatorname{rank}(\Omega^1_{C_X/X^*}(p,h)) = n - k$.

(b) We choose a local coordinate system $\{x_1, x_2, \ldots, x_n\}$ for X at p. We shall assume I_L is generated by $x_1, x_2, \ldots, x_{n-k}$. Using the fact that $L \subseteq H \cap X$, we can write the power series of s_h in the following form,

$$s_h = x_1 f_1 + x_2 f_2 + \ldots + x_{n-k} f_{n-k} + \sum_{i=1}^{n-k} \sum_{j=1}^{n-k} x_i x_j (g_{i,j}),$$

where $f_1, f_2, \ldots, f_{n-k}$ are power series with the variables $x_{n-k+1}, \ldots, x_n$ only. But $\operatorname{Sing}(H \cap X) = L$. Thus $\left.\frac{\partial s_h}{\partial x_i}\right|_L = 0$ for $i = 1, 2, \ldots, n-k$. Hence $f_1 = f_2 = \ldots = f_{n-k} = 0$. Now $s_h = \sum_{i=1}^{n-k} \sum_{j=1}^{n-k} x_i x_j (g_{i,j})$. Thus s_h factors through I_L^2.

(c) We can write $g_{i,j} = a_{i,j} + h_{i,j}$, where $(a_{i,j})$'s are constants and $(h_{i,j})$'s are power series without the constant term. Now

$$s_h = \sum_{i=1}^{n-k} \sum_{j=1}^{n-k} x_i x_j (a_{i,j} + h_{i,j}).$$

By 2.1a, $\sum_{i=1}^{n-k}\sum_{j=1}^{n-k} a_{i,j}x_i x_j$ is a quadratic form of rank $n-k$. But this is also the equation for the quadric hypersurface in $\mathbb{P}(I_L/I_L^2(p))$ induced by s_h.

Theorem 2.2 $N_{L/X}\cong N_{L/X}^*\otimes\mathcal{O}_L(1)$.

Proof. We shall continue to use the notations in 2.1. By 2.1(b) and (c), s_h gives a section of

$$I_L^2/I_L^3\otimes\mathcal{O}_L(1)=S^2(N_{L/X}^*)\otimes\mathcal{O}_L(1).$$

Since we assume the base field is not of characteristic two, $S^2 N_{L/X}^*\otimes\mathcal{O}_L(1)$ is a direct summand of

$$N_{L/X}^*\otimes N_{L/X}^*\otimes\mathcal{O}_L(1)\cong\mathrm{Hom}(N_{L/X}, N_{L/X}^*(1)).$$

Let g_h be the map from $N_{L/X}$ to $N_{L/X}^*(1)$ induced by s_h. Then g_h is an isomorphism by 2.1(c).

Remark. Let E be a vector bundle on $\mathbb{P}^m$. E is said to be a uniform bundle if $E|_T$ is isomorphic to a fixed bundle $\mathcal{O}_{\mathbb{P}^1}(a_1)\oplus\ldots\oplus\mathcal{O}_{\mathbb{P}^1}(a_r)$ for all lines T in $\mathbb{P}^m$.

Theorem 2.3. *Assume* $\mathrm{def}(x)=k>0$.

(a) *If T is a line in L, then*

$$N_{L/X}|_T=\frac{n-k}{2}\mathcal{O}_{\mathbb{P}^1}\oplus\frac{n-k}{2}\mathcal{O}_{\mathbb{P}^1}(1)$$

(i.e. $N_{L/X}$ is a uniform vector bundle).

$$N_{T/X}=\frac{n-k}{2}\mathcal{O}_{\mathbb{P}^1}\oplus\frac{n+k-2}{2}\mathcal{O}_{\mathbb{P}^1}(1).$$

(b) *There is an irreducible $\frac{3n+k-4}{2}$ dimensional family of lines in X. If p is a general point in X, then there is an $\frac{n+k-2}{2}$ dimensional family of lines in X through p.*

Proof. (a) Let N_L and N_X be the normal sheaves of L and X in $\mathbb{P}^N$ respectively. Suppose T is a line in L. Then there is the following exact sequence,

$$0\to N_{L/X}|_T\to N_L|_T\to N_X|_T\to 0,$$

where $N_L|_T=N-k\,\mathcal{O}_{\mathbb{P}^1}(1)$. If $N_{L/X}|_T\cong\bigoplus_{i=1}^{n-k}\mathcal{O}_{\mathbb{P}^1}(a_i)$, then $a_i\leqq 1$. Using the isomorphism between $N_{L/X}$ and $N_{L/X}^*\otimes\mathcal{O}_L(1)$, we observe that $a_i\geqq 0$. Hence

$$N_{L/X}|_T=\frac{n-k}{2}\mathcal{O}_T\oplus\frac{n-k}{2}\mathcal{O}_T(1).$$

This implies that

$$N_{T/X}\cong\frac{n-k}{2}\mathcal{O}_{\mathbb{P}^1}\oplus\frac{n+k-2}{2}\mathcal{O}_{\mathbb{P}^1}(1).$$

(b) Let $p \in L$ and let T_0 be a line in L through p. Since $h^1(N_{T_0/X})=0$, the Hilbert scheme of lines in X is smooth at the point t_0 corresponding to T_0. Hence there is a unique irreducible component $\mathcal{F}$ of the Hilbert scheme containing the point t_0. Also

$$\dim \mathcal{F} = h^0(N_{T_0/X}) = \frac{3n+k-4}{2}.$$

Consider the following closed subscheme of the $\mathcal{F}$:

$$\mathcal{H} = \{T \mid T \text{ is a line in the family } \mathcal{F} \text{ and } p \in T\}.$$

Since $h^1(N_{T_0/X} \otimes I_{p/T_0})=0$, $\mathcal{H}$ is smooth at the point corresponding to t_0. Hence there is a unique irreducible component $\mathcal{H}_0$ of $\mathcal{H}$ containing the point t_0. $\dim \mathcal{H}_0 = h^0(N_{T/X} \otimes I_{p/T}) = \dfrac{n+k-2}{2}$.

Theorem 2.4. *Assume that* $\operatorname{def}(X)=k>0$.

(a) $n \equiv k \bmod 2$.

(b) $K_X|_L = \mathcal{O}_L\left(\dfrac{-n-k-2}{2}\right)$.

(c) *The Kodaira dimension of* X *is negative.*

(d) *If* $K_X = \mathcal{O}_X(a)$, *then* $a = \dfrac{-n-k-2}{2}$.

(e) *If* $\dim X > \dfrac{N}{2}+1$, *then* $K_X = \mathcal{O}_X\left(\dfrac{-n-k-2}{2}\right)$.

Proof. (a) By 2.3. $n \equiv k \bmod 2$.

(b) $\Lambda^{n-k} N_{L/X} = \mathcal{O}_L\left(\dfrac{n-k}{2}\right)$. Thus $K_X|_L = \mathcal{O}_L\left(\dfrac{-n-k-2}{2}\right)$ by the adjunction formula.

(c) Since there is such a k-plane L through a general point p, $|K_X^m| = \emptyset$ for $m \geqq 0$.

(d) and (e) If $\dim X \geqq \dfrac{N}{2}+1$, then Barth's theorem [2] asserted that $\operatorname{Pic} X$ is generated by $\mathcal{O}_X(1)$. Thus $K_X \cong \mathcal{O}_X\left(\dfrac{-n-k-2}{2}\right)$ by (b). Also (d) follows from (b).

Remark. 2.4(c) was first observed by Griffiths and Harris [10].

§ 3.

In this section, we shall apply the result in § 2 to obtain information about varieties with small dual varieties. Again we shall assume X is a nonlinear projective n-fold in $\mathbb{P}^N$.

Proposition 3.1. $\operatorname{def}(X)=0$, *if* X *is one of the following varieties:*

(a) X *is a complete intersection.*

(b) X *is a curve.*

(c) X *is a surface.*

Proof. (a) We may assume X is nondegenerate by 1.1. Then $N_{X/\mathbb{P}^N}(-1)$ is an ample bundle. Let $C_X = \mathbb{P}(N_{N/\mathbb{P}^N}(-1))$ be the conormal variety of X and let p_2: $C_X \to X^*$ be the projective map. $p_2^* \mathcal{O}_{X^*}(1)$ is the tautological line bundle of $\mathbb{P}(N_{X/\mathbb{P}^N}(-1))$. Hence p_2 is finite.

(b) A general tangent hyperplane can only be tangent to X at a point. Thus $\operatorname{def}(X)=0$.

(c) A general tangent hyperplane can only be tangent to X along a subvariety. Thus $\operatorname{def}(X) \leqq 1$. Then $\operatorname{def}(X)=0$ by 2.4.a.

Remark. 3.1.c. is a theorem of Griffiths, Harris, Landman, and Marchionna [10].

Theorem 3.2. *Assume* $n \geqq 2$. *Then* $\operatorname{def}(X) \leqq n-2$. *Furthermore,* $\operatorname{def}(X)=n-2$, *if and only if* X *is a scroll (i.e.* $X=\mathbb{P}_C(F)$ *where* F *is a rank* n *vector bundle on a curve* C *and the fibers are embedded linearly).*

Proof. It is clear that $\operatorname{def}(X) \leqq n-1$. Then $\operatorname{def}(X) \leqq n-2$ by 2.4.a.

If $\operatorname{def}(X)=n-2>0$, then there is a $n-2$-plane L through a general point p such that

$$K_X \otimes \mathcal{O}_X(n-1)|_L \cong \mathcal{O}_L(-1) \quad \text{by 2.3.}$$

Thus X is a scroll by 1.2. The converse is well known.

Remark. In [4], we shall show that if $\operatorname{def}(X)=k \geqq \frac{n}{2}$, then X is a $\mathbb{P}^{n+k/2}$ bundle over a $\frac{n-k}{2}$-fold.

Theorem 3.3. (a) *If* X *is a* 3-*fold and* $\operatorname{def}(X)>0$, *then* X *is a scroll.*

(b) *If* X *is a* 4-*fold and* $\operatorname{def}(X)>0$, *then* X *is a scroll.*

(c) *Assume that* $n \geqq 3$ *and* $N=2n-1$. *If* $\dim X = \dim X^*$, *then* X *is the Segre embedding of* $\mathbb{P}^1 \times \mathbb{P}^{n-1}$.

Proof. (a) If $n=3$ then $\operatorname{def}(X)=1$ and X is a scroll by 3.2.

(b) If $n=4$ then $\operatorname{def}(X)=2$ by 2.4. Hence X is a scroll.

(c) $\operatorname{def}(X)=n-2$. Thus X is a scroll. Then X is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ by a theorem of S. Kleiman ([16], 4.3).

Remark. 3.3.a was first proved by Griffiths and Harris. The fact $\operatorname{def}(X) \leqq n-2$ was first observed by Zak and Landman.

Theorem 3.4. *If the codimension of* X *is two, then* $\operatorname{def}(X)=0$, *unless* X *is the Segre embedding of* $\mathbb{P}^1 \times \mathbb{P}^2$ *in* $\mathbb{P}^5$.

Proof. Assume that $\dim X \geqq 4$ and $\operatorname{def}(X)=k>0$. Then $K_X \cong \mathcal{O}_X\left(\frac{-n-k-2}{2}\right)$ by 2.4. But Ballico and Chiantini [1] have proved that if $K_X = \mathcal{O}_X(-a)$ with $a>0$, then X is a complete intersection. This contradicts 3.1. If $\dim X = 1$ or 2, then $\operatorname{def}(X)=0$ by 3.1. If X is a 3-fold in $\mathbb{P}^5$ and $\operatorname{def}(X)>0$, then X is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ by 3.3.

Remark. Holme and Schneider have independently observed that if $\operatorname{codim}(X)=2$ and $\dim X \geqq 4$, then $\operatorname{def}(X)=0$.

Theorem 3.5. *If* $k=\operatorname{def}(X)\geqq\frac{N}{2}$, *then* $N_{L/X}=\frac{n+k}{2}\mathcal{O}_L\oplus\frac{n-k}{2}\mathcal{O}_L(1)$.

Proof. $N_{L/X}$ is a uniform bundle by 2.3.(a). The classification of uniform bundles ([3] and [5]) implies that $N_{L/X}$ is isomorphic to either $\frac{n-k}{2}\mathcal{O}_L\oplus\frac{n-k}{2}\mathcal{O}_L(1)$ or $\Omega^1_{\mathbb{P}^2}(2)$. Assume for contradiction that $N_{L/X}\cong\Omega^1_{\mathbb{P}^2}(2)$. Then $n=4$ and X is a scroll by 3.1 and 3.3. Say $X=\mathbb{P}_C(F)$ where F is a rank 4 locally free sheaf on a curve C. Then L is embedded as a 2-plane in a fibre f of $\mathbb{P}_C(F)$. Consider the exact sequence,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_{L/f} & \longrightarrow & N_{L/X} & \longrightarrow & N_{f/X}|_L & \longrightarrow & 0\\
 & & \wr\| & & \| & & \wr\| & & \\
0 & \longrightarrow & \mathcal{O}_L(1) & \longrightarrow & N_{L/X} & \longrightarrow & \mathcal{O}_L & \longrightarrow & 0.
\end{array}$$

We observe that $N_{L/X}=\mathcal{O}_L\oplus\mathcal{O}_L(1)$.

§ 4.

First we will construct a 10-dimensional variety S_4 in $\mathbb{P}^{15}$. Later on in the section we will prove that if X is a 10-fold in $\mathbb{P}^{15}$ such that $\dim X=\dim X^*$. Then we shall show that $X\cong S_4$.

Let W be a five dimensional vector space. Set $T\underset{\text{def}}{=}\mathbb{P}(W)\cong\mathbb{P}^4$ and $D\underset{\text{def}}{=}\mathbb{P}(\Lambda^3 W)=\mathbb{P}^9$. Denote by G the Plücker embedding of the Grassman variety of 2-planes in T. If I is the incidence correspondence between T and G, then $I=\mathbb{P}_G(Q)$ where Q is the universal rank 3 quotient bundle on G. Consider the following diagram:

$$\begin{array}{ccccc}
E\underset{\text{def}}{=}\mathbb{P}(\Omega^2_{\mathbb{P}^4}(2)) & & \xrightarrow{\;f\;} & & D=\mathbb{P}^9\\
 & \cup & & & \cup\\
\Big\downarrow h & I & \longrightarrow & & G\\
 & \Big\downarrow & & & \\
T=\mathbb{P}^4 & =\mathbb{P}^4 & & &
\end{array} \tag{4.0.1}$$

Observe that E is just the blowing up of $\mathbb{P}^9$ along G. $I\subseteq E$ is just the exceptional divisor. Let $\mathcal{O}_E(0,1)$ be the tautological line bundle of $\mathbb{P}(\Omega^2_T(2))$. Observe that

$$f^*\mathcal{O}_D(1)=\mathcal{O}_E(0,1)\otimes h^*\mathcal{O}_T(1). \tag{4.0.2}$$

Let $t\in T$ and $k(t)$ be its residue field. Then t corresponds to a 1-dimensional quotient space of W. Consider the standard exact sequence

$$0\to\Omega^1_T(1)\to W\otimes\mathcal{O}_T\to\mathcal{O}_T(1)\to 0.$$

Then the fibre of I over t, $I_t=\{$2-planes in T through $t\}\cong\{$2-dimensional quotient spaces of $\Omega^1_T(1)\otimes k(t)\}=\operatorname{Gr}(2,\Omega^1_T(1)\otimes k(t))$. In fact $I=\operatorname{Gr}(2,\Omega^1_T(1))$

$\subseteq \Omega_T^2(2)) = E$. The inclusion map $I \subseteq \mathbb{P}(\Omega_T^2(2))$ is just given by the Plücker embedding. Observe that

$$
\begin{aligned}
H^0(\mathcal{O}_E(0,2)\otimes h^*\mathcal{O}_T(1)) &\cong H^0(h_*\mathcal{O}_E(0,2)\otimes\mathcal{O}_T(1)) \\
&\cong H^0(S^2(\Omega_T^2)\otimes\mathcal{O}_T(5)) \\
&\cong \mathrm{Hom}(\Omega_T^4, S^2\Omega_T^2) \subseteq \mathrm{Hom}(\Omega_T^4, \mathrm{Hom}((\Omega_T^2)^*, \Omega_T^2)).
\end{aligned}
$$

It follows from the Plücker relations, $I \subseteq E$ is defined by the sections $H^0(\mathcal{O}_E(0,2)\otimes h^*\mathcal{O}_T(1))$ corresponding to the map from Ω_T^4 to $\mathrm{Hom}((\Omega_T^2)^*, \Omega_T^2)$ given by the exterior product. Now

$$\mathcal{O}_E(I) = \mathcal{O}_E(0,2)\otimes h^*\mathcal{O}_T(1), \tag{4.0.3}$$

and

$$h^*\mathcal{O}_T(1) = \mathcal{O}_E(0,2)\otimes\mathcal{O}_E(-I). \tag{4.0.4}$$

Since I is exceptional divisor for the map f, $h^0(\mathcal{O}_E(I)) = 1$ and

$$I \in |\mathcal{O}_E(0,2)\otimes h^*\mathcal{O}_T(1)| \tag{4.0.5}$$

is the unique divisor. Now we embed D as a hyperplane in $\mathbb{P}^{10}$. Let $\tilde{\mathbb{P}}^{10}$ be the blowing up of $\mathbb{P}^{10}$ along G. Denote by F the exceptional divisor and denote by E the proper transform of D in $\tilde{\mathbb{P}}^{10}$. Consider the following diagram:

$$
\begin{array}{ccccc}
F & \subseteq & \tilde{\mathbb{P}}^{10} & \supseteq & E \\
\downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow \\
G & \subseteq & \mathbb{P}^{10} & \supseteq & D.
\end{array}
$$

E is the blowing up of D along G. So $E = \mathbb{P}(\Omega_{\mathbb{P}^4}^2(2))$ and $F \cap E = I$ is the incidence correspondence between $\mathbb{P}^4$ and G. The ideal sheaf $I_{G/\mathbb{P}^{10}}(2)$ is generated by its sections and $h^0(I_{G/\mathbb{P}^{10}}(2)) = 16$. Thus the complete linear system $|\pi^*\mathcal{O}_{\mathbb{P}^{10}}(2)\otimes\mathcal{O}(-F)|$ gives a morphism $\phi\colon \tilde{\mathbb{P}}^{10} \to \mathbb{P}^{15}$. Let $S_4 \underset{\text{def}}{=} \phi(\tilde{\mathbb{P}}^{10})$. Let $\mathcal{L} = \phi^*\mathcal{O}_{S_4}(1)$. Then

$$\mathcal{L} = \pi^*\mathcal{O}_{\mathbb{P}^{10}}(2)\otimes\mathcal{O}(-F) = \pi^*\mathcal{O}_{\mathbb{P}^{10}}(1)\otimes\mathcal{O}(-E).$$

By (4.0.2) $\mathcal{L}|_E = h^*\mathcal{O}_{\mathbb{P}^4}(1)$. Thus $\phi(E) \underset{\text{def}}{=} L$ is a 4-plane in S_4. Also that $\mathcal{O}_E(-E)$ is just the tautological line bundle of $\mathbb{P}(\Lambda^2\Omega_{\mathbb{P}^4}^1\otimes\mathcal{O}_{\mathbb{P}^4}(2))$. As in the classical cases [8, 19], one can show that ϕ is just the blowing down of $\tilde{\mathbb{P}}^{10}$ along E. So in fact S_4 is a smooth 10-fold in $\mathbb{P}^{15}$. (See [29] for an elegant proof that S_4 is isomorphic to the 10-dimensional spinor variety.)

Let X be a nonlinear n-fold in $\mathbb{P}^N$ such that $\mathrm{def}(X) = k > 0$. Let H_1 be a general tangent hyperplane of X. Then the contact locus of H_1 with X is a k-plane L. Let $\tilde{X}$ be the blowing up of X along L. Denote by E the exceptional divisor and denote by F the proper transform of $H_1 \cap X$. Consider the following diagram:

$$
\begin{array}{ccccccc}
\mathbb{P}(N_{L/X}^*) = & E & \subseteq & \tilde{X} & \supseteq & F \\
& \downarrow & & \downarrow{\scriptstyle p} & & \downarrow \\
& L & \subseteq & X & \supseteq & H_1 \cap X.
\end{array}
\tag{4.0.6}
$$

We shall denote by $\mathcal{O}_{\tilde{X}}(a,b)$ the line bundle $p^*\mathcal{O}_X(a)\otimes\mathcal{O}_{\tilde{X}}(-bE)$. Then $\mathcal{O}_{\tilde{X}}(F)=\mathcal{O}_{\tilde{X}}(1,2)$.

Let $f\colon \tilde{X}\to\mathbb{P}^{N-1-k}$ be the projection with center L. *Let* $Y=f(\tilde{X})$. Then $f^*\mathcal{O}_Y(1)=\mathcal{O}_{\tilde{X}}(1,1)$. The hyperplane section $H_1\cap X$ will correspond to a hyperplane section D of Y. Observe that $f^{-1}(D)=E+F$.

Lemma 4.1. (a) *If Z is a positive dimensional fibre of f, then $Z\subseteq E\cup F$.*

(b) $\dim Y=\dim X$.

Proof. Let $y\in Y-D$. Assure that $Z=f^{-1}(y)$ and $\dim Z\geqq 1$. Since $Z\cap(E\cup F)=\emptyset$, p maps Z isomorphically to a variety in X. So $\mathcal{O}_{\tilde{X}}(1,0)|_Z$ is nontrivial. But $\mathcal{O}_{\tilde{X}}(1,1)|_Z=f^*\mathcal{O}_Y(1)|_Z$ is trivial. So $\mathcal{O}_{\tilde{X}}(0,1)|_Z$ is nontrivial. Hence $Z\cap E\neq\emptyset$. This is a contradiction.

Lemma 4.2. *Assume that $K_X=\mathcal{O}_X(b)$ for some b. Then*

(a) $K_X=\mathcal{O}_X\left(\frac{-n-k-2}{2}\right)$.

(b) $K_{\tilde{X}}=\mathcal{O}_{\tilde{X}}\left(\frac{-n-k-2}{2},\ -n+k+1\right)$.

(c) $H^i(\mathcal{O}_{\tilde{X}}(a,1))=0$, *if $i>0$ and* $a\geqq\frac{n-3k-2}{2}$.

(d) $H^i(\mathcal{O}_{\tilde{X}}(a,2))=0$, *if $i>0$ and* $a\geqq\frac{n-3k}{2}$.

Proof. (a) Since $K_X|_L=\mathcal{O}_L\left(\frac{-n-k-2}{2}\right)$ by 2.4b, $K_X=\mathcal{O}_X\left(\frac{-n-k-2}{2}\right)$.

(b) This follows from (a) and the fact that X is the blowing up of X along L.

(c) $\mathcal{O}_{\tilde{X}}(a,1)=K_{\tilde{X}}\otimes f^*\mathcal{O}_Y(n-k)\otimes\mathcal{O}_{\tilde{X}}\left(a-\frac{n-3k-2}{2},0\right)$. It follows from the vanishing theorem of Grauert-Rimenschneider ([24], Theorem 3), that $H^i(\mathcal{O}_{\tilde{X}}(a,1))=0$, if $i>0$ and $a\geqq\frac{n-3k-2}{2}$.

(d) The proof is similar to (c). We shall leave it to the readers.

Lemma 4.3. *Assume that $K_X=\mathcal{O}_X(b)$. Also assume that* $\frac{n-3k-2}{2}\leqq 0$. *Then*

(a) $H^0(N^*_{L/X}(a))=0$ *for* $a\leqq 0$.

(b) $H^k(N^*_{L/X}(a))=0$ *for* $a\geqq -k$.

(c) $H^i(N^*_{L/X}(a))=0$ *if* $0<i<k$ *and* $a\geqq\frac{n-3k}{2}$.

(d) $H^i(N^*_{L/X}(a))=0$ *if* $0<i<k$ *and* $a\leqq\frac{k-n}{2}$.

Proof. (a) Consider the exact sequence,

$$0=H^0(\mathcal{O}_{\tilde{X}}(0,1))\to H^0(\mathcal{O}_E(0,1))\to H^1(\mathcal{O}_{\tilde{X}}(0,2)).$$

Now $H^1(\mathcal{O}_{\tilde{X}}(0,2))=0$ by 4.2(d). So $H^0(\mathcal{O}_E(0,1))\cong H^0(N^*_{L/X})=0$. Hence $H^0(N^*_{L/X}(a))=0$ for $a\leqq 0$.

(b) Recall that $N_{L/X}=N^*_{L/X}\otimes\mathcal{O}_L(1)$. So (b) following from (a) and Serre's duality.

(c) Consider the exact sequence

$$H^i(\mathcal{O}_{\tilde{X}}(a,1))\to H^i(\mathcal{O}_E(a,1))\to H^{i+1}(\mathcal{O}_{\tilde{X}}(a,2)).$$

By 4.2(e) and (d), we conclude that

$$H^i(\mathcal{O}_E(a,1))\cong H^i(N^*_{L/X}\otimes\mathcal{O}(a))=0 \quad \text{for } a\geqq\frac{n-3k}{2}.$$

(d) This follows from (c) and Serre's duality.

Theorem 4.4. *Assume that* $K_X=\mathcal{O}_X(b)$ *for some* $b\in\mathbb{Z}$. *Then*

(a) $\operatorname{def}(X)\leqq\frac{n-2}{2}$ $(n\geqq 3)$.

(b) *If* $\dim X=4m+2$ *and* $\operatorname{def}(X)=2m(>0)$, *then*

$$N^*_{L/X}=H^m(N^*_{L/X}(m))\otimes\Omega^m_L(m), \quad \textit{and } m\leqq 2.$$

Proof. Consider the Belinson spectral sequence ([23], 3.1.3.)

$$E_1^{pq}=H^q(N^*_{L/X}(p))\otimes\Omega_L^{-p}(-p)$$

which converges to

$$E^i=\begin{cases}N^*_{L/X} & \text{if } i=0\\ 0 & \text{otherwise} \quad (\text{i.e. } E_\infty^{pq}=0, \text{ if } p+q\neq 0).\end{cases}$$

(a) If $\operatorname{def}(X)\geqq\frac{n-1}{2}$, then $\frac{n-3k}{2}-1\leqq\frac{k-n}{2}$. It follows from 4.3 $H^q(N^*_{L/X}(p))=0$ for $-k\leqq p\leqq 0$. It follogs that $N^*_{L/X}=0$. This is a contradiction.

(b) In this case $H^q(N^*_{L/X}(p))=0$ for $-2m\leqq p\leqq 0$ unless $p=m$. So $E_1^{pq}=E_\infty^{pq}$. This implies that $N^*_{L/X}=H^m(N^*_{L/X})\otimes\Omega^m_L(m)$. So $\operatorname{rank}(N^*_{L/X})=2m+2\geqq\binom{2m}{m}$. We conclude that $m\leqq 2$.

Theorem 4.5. *Let X be a nonlinear n-fold in* $\mathbb{P}^n$. *We assume that* $n\leqq\frac{2}{3}N$. *Suppose that* $\dim X=\dim X^*$. *Then X is one of the following varieties:*

(a) *X is a hypersurface in* $\mathbb{P}^2$ *or* $\mathbb{P}^3$.

(b) *X is the Segre embedding of* $\mathbb{P}^1\times\mathbb{P}^{n-1}$ *in* $\mathbb{P}^{2n-1}$.

(c) *X is the Plücker embedding of* $G(2,5)$ *in* $\mathbb{P}^9$.

(d) *X is the* 10-*dimensional spinor variety* S_4 *in* $\mathbb{P}^{15}$ [19, 25].

Proof. We may assume that $n\geqq 3$. (3.1). Now $\operatorname{def}(X)=N-1-n$. Since $\operatorname{def}(X)\leqq n-2$, we conclude that $n\geqq\frac{N+1}{2}$. If $n=\frac{N+1}{2}$, then $\operatorname{def}(X)=n-2$ and X is a Segre variety by 3.3(c). In the following we shall assume that $n\geqq\frac{N}{2}+1$.

Then $K_X = \mathcal{O}_X\left(\frac{-N-1}{2}\right)$ by 2.4(e). We conclude that $\operatorname{def}(X) = N-1-n \leqq \frac{n-2}{2}$ by 4.4. Hence $n \geqq \frac{2}{3}N$. By our assumption $n \leqq \frac{2}{3}N$. So $n = \frac{2}{3}N$. Now $\operatorname{def}(X) = N - 1 - n = \frac{1}{2}n - 1$. Thus $n \equiv 0 \bmod 2$. Since $\operatorname{def}(X) \equiv n \bmod 2$, we conclude that $n \equiv 2 \bmod 4$. We write $n = 4m+2$. Then $\operatorname{def}(X) = 2m$. So $m \leqq 2$ by 4.4(b). If X is contained in a hyperplane, then the dual variety of X as a subvariety of $\mathbb{P}^{N-1}$ will have dimension smaller than $\dim X$. This will contradict Zak's theorem. So we conclude that X is nondegenerate. By Zak's linear normality theorem, we conclude that (4.5.1) $h^0(\mathcal{O}_X(1)) = N+1$.

Case 1. Assume that $m=1$.

In this case, X is a 6-fold in $\mathbb{P}^9$ and $K_X = \mathcal{O}_X(-5)$. Let G be the Plücker embedding of $G(2,5)$ in $\mathbb{P}^9$. It follows from the Kodaria vanishing theorem,

$$X(\mathcal{O}_X(a)) = X(\mathcal{O}_G(a)) \quad \text{for } -6 \leqq a \leqq 1.$$

So $\deg X = \deg G = 5$. It follows from Fujita's classification of Del Pezzo manifold that $X \cong G$ [7, 8].

Case 2. Assume that $m=2$.

In this case, X is a 10-fold in $\mathbb{P}^{15}$ and $K_X = \mathcal{O}_X(-8)$. As in Case 1, we can show that $\deg X = \deg S_4 = 12$. Also in this case $N^*_{L/X} = \Omega^2_{\mathbb{P}^4} \otimes \mathcal{O}_{\mathbb{P}^4}(2)$. In the following we shall use the notations in (4.0.6) and (4.1). Let $f\colon \tilde{X} \to \mathbb{P}^{10}$. Suppose that $H \in |\mathcal{O}_{\tilde{X}}(1,0)|$. Using the Chern polynomial of $\Omega^2_{\mathbb{P}^4} \otimes \mathcal{O}_{\mathbb{P}^4}(2)$, we find the following intersection product

$$E \cdot (E^6 - 3H \cdot E^5 + 5H^2 \cdot E^4 - 5H^3 E^3) = 0 \qquad ([12], \text{p. } 429) \tag{4.5.2}$$

in the Chow ring of $\tilde{X}$. Also observe that $H^5 \cdot E = 0$ and $H^4 \cdot E^6 = -1$. Using (4.5.2), we conclude that $H^3 \cdot E^7 = -3$, $H^2 \cdot E^8 = -4$, $H \cdot E^9 = -2$ and $E^{10} = -1$. Also $H^{10} = \deg X = 12$. Let $M \in |\mathcal{O}_{\tilde{X}}(1,1)| = |f^* \mathcal{O}_{\mathbb{P}^{10}}(1)|$. We find $M^{10} = (H-E)^{10} = 1$. We conclude that the map $f\colon \tilde{X} \to \mathbb{P}^{10}$ is a birational morphism $\mathcal{O}_{\tilde{X}}(0,1)|_E$ is the tautological line bundle of $\mathbb{P}(\Omega^2_{\mathbb{P}^4} \otimes \mathcal{O}(2))$. Now $\mathcal{O}_{\tilde{X}}(F) = \mathcal{O}_{\tilde{X}}(1,2)$. So $I = E \cap F$ is the unique divisor in $|\mathcal{O}_E(1,2)|$ and I is the incidence correspondence between $\mathbb{P}^4$ and $G(2,5)$ by (4.0.6). Also observe that $f(I) = G$ is the Grassman variety in $\mathbb{P}^9$ by (4.0.2). Also observe that $f(E+F)$ is a hyperplane D in $\mathbb{P}^{10}$ and $f^{-1}(D) = E+F$. We can compute that $(F \cdot M^7) \cdot H^2 = (F \cdot M^7) \cdot E \cdot H = (F \cdot M^7) \cdot E^2 = 0$. Since $aH - E$ is very ample for sufficiently large a, we conclude that $(F \cdot M^7) = 0$. It follows that $\dim f(F) \leqq 6$. Since $f(I) = G$, we conclude that $f(F) = G$. By the construction given at the beginning of this section, we know $f\colon E - I \to D - G$ is an isomorphism. It follows from Lemma 4.1 and the Zariski's main theorem that $f\colon \tilde{X} - F \to \mathbb{P}^{10} - G$ is an isomorphism. We find $H \cdot M^9 = 2$. Thus each hyperplane section of X corresponds to a quadric hypersurface in $\mathbb{P}^{10}$. The birational morphism f and p induces a birational correspondence $g\colon \mathbb{P}^{10} \to X \subset \mathbb{P}^{15}$. Observe that $g^* \mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^{10}}(2)$, this induces a 15-dimensional linear system in $|\mathcal{O}_{\mathbb{P}^{10}}(2)|$. The base locus of this linear system contains G. But $h^0(I_{G/\mathbb{P}^{10}}(2)) = 16$ and $I_{G/\mathbb{P}^{10}}(2)$ is generated by its sections. Thus the base locus of this linear system is G and there is a morphism $\phi\colon \tilde{\mathbb{P}}^{10} \to X$ where $\tilde{\mathbb{P}}^{10}$ is the blowing up of $\mathbb{P}^{10}$ along G. We observe that X is just the variety S_4 we constructed at the beginning of this section.

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