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Reduction of Hamiltonian Systems, Affine Lie Algebras and Lax Equations

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Recently M. Adler [1] and B. Kostant pointed out a connection between completely integrable systems of Lax type and the orbit method in representation theory. In a joint article with I. Frenckel [17] the authors suggested an extension of this approach to affine Lie algebras resulting in a large number of finite-dimensional completely integrable systems.[1] In the present paper we give an alternative realization of these systems based on reduction of bi-invariant dynamical systems on Lie groups. Our approach follows the general pattern of [7] (cf. also an earlier paper [18]). The Lie groups we need are the infinite-dimensional loop groups of semisimple Lie groups and their subgroups. The corresponding analytical tools are provided by factorization theory of matrix-valued analytic functions [4].

The reduction formalism applies both to classical and quantum versions of the problem. For systems associated with finite-dimensional Lie algebras the quantization problem can be solved completely. For systems associated with the affine Lie algebras the problems is, however, much more complicated; the spectrum of such systems remains to be determined. In the paper we do not give a full treatment of the quantum case but only suggest a construction of the quantum mechanical integrals of motion.

The contents of our paper is as follows.

In n° 1 we recall some basic facts on reduction of Hamiltonian systems with symmetry. As compared to [7], we have added a reduction scheme for quantum bundles, which is necessary for geometric quantization of Lax equations. In n° 2 we discuss reduction of bi-invariant systems on the cotangent bundle of a Lie group. This provides a natural framework for the Adler-Kostant scheme, which is presented in n° 3. As an example, in n° 4 we consider the open Toda lattice, our results generalizing those of [16].[2] In n° 5 we give a quantum mechanical

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[1] From a remark in the paper [1] (which was known to the authors in the form of a preprint) we learned that M. Adler, P. Moerbeke and T. Ratu are preparing a paper on the same subject

[2] After this paper had been written, the authors received from B. Kostant his preprint "The solution to a generalized Toda Lattice and Representation Theory" which contains a detailed study of open Toda lattices. We also realize that B. Kostant was the first to use group-representation methods for their quantization

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version of n°3 and obtain quantum integrals of motion for geometrically quantized Lax systems; some of the presented results originate from B. Kostant's paper [9]. In n°6 we describe infinite-dimensional Lie groups associated with the affine Lie algebras. The study of reduced Hamiltonian systems on these groups leads to factorization problems for group-valued functions.

Some important examples of dynamical systems associated with algebras of height 2 are given in Appendix 1. Finally, in Appendix 2 we suggest a construction of the quantum integrals of motion for systems associated with the affine Lie algebras.

Notation. Troughout the paper we denote Lie groups by capital Latin letters and their Lie algebras by the corresponding Gothic ones.

1°. We recall some known facts on reduction of the Hamiltonian systems with symmetry.

Let $G\times M\to M$ be a Hamiltonian action of a connected Lie group G on a symplectic manifold (M,ω). This means that for each $x\in\mathfrak{g}$ the corresponding vector field on M is generated by a Hamiltonian H_x, the mapping $x\mapsto H_x$ is linear and $H_{[x,y]}=\{H_x,H_y\}$. Under these conditions the momentum map Φ: $M\to\mathfrak{g}^*$, $\Phi m(x)=H_x(m)$, commutes with the action of G. (Recall that $\mathfrak{g}^*$ is a natural G-module with respect to the coadjoint representation of G).

Let $f\in\mathfrak{g}^*$ be a regular value of Φ, so that $M_f=\Phi^{-1}(f)$ is a smooth submanifold of M. Let ω_f be the restriction of ω to M_f. Let G_f be the stationary subgroup of f. Clearly, G_f leaves M_f invariant and G_f-orbits in M_f are intersections of M_f with G-orbits in M. For simplicity we assume throughout this section that G_f is connected.

Proposition 1. *The null distribution of ω_f coincides with the tangent distribution to G_f-orbits in M_f.*

It follows that all the G_f-orbits in (a connected component of) M_f have the same dimension and form a foliation. Suppose that it defines a smooth fibration over the base $\overline{M}_f$. We shall call $\overline{M}_f$ the quotient manifold.

Proposition 2. *The 2-form ω_f projects into a closed nondegenerate form $\bar{\omega}_f$ on $\overline{M}_f$.*

Let F be a smooth G-invariant function on M. The corresponding Hamiltonian vector field X_F is G-invariant and tangent to M_f. The projections of F and X_F onto $\overline{M}_f$ are defined in a natural way; we denote them by $\bar{F}$ and $\bar{X}_F$, respectively. Let $X_{\bar{F}}$ be the Hamiltonian vector field on $\overline{M}_f$ generated by $\bar{F}$.

Proposition 3. $\bar{X}_F=X_{\bar{F}}$, *i.e. the flow of the reduced Hamiltonian coincides with the quotient flow.*

Corollary 4. *For G-invariant functions F_1, F_2 on M one has $\overline{\{F_1,F_2\}}=\{\bar{F}_1,\bar{F}_2\}$.*

Now we extend the reduction procedure to the quantum problem.

Let p: $E\to M$ be a quantum bundle over M i.e. a principal $U(1)$-bundle with a connection form α whose curvature is $(2\pi)^{-1}\omega$. Let E_f be the restriction of E to M_f. From Proposition 1 it follows that E_f is flat over each G_f-orbit in M_f. We say, that the prequantization condition holds, if the monodromy of these flat bundles is trivial. Recall that we assume G_f to be connected.

Proposition 5. *If the prequantization condition holds, the quantum bundle $E_f \to M_f$ may be reduced to a quantum bundle over the quotient manifold $\bar{M}_f$.*

Proof. For a function F on M let $\lambda(F)$ be a vector field on E such that $p_* \lambda(F) = X_F$ and $\alpha(\lambda(F)) = F \circ p$. It is easy to verify that the Lie derivative of α with respect to $\lambda(F)$ is zero. Now we take into account the prequantization condition and use parallel transport to lift the action of G_f from M_f to E_f. This action leaves the connection in E_f invariant because the generating vector field on E_f corresponding to $x \in \mathfrak{g}_f$ differs from $\lambda(H_x)$ by a constant vertical field. The orbits of G_f in E_f being horizontal, the connection form may be projected into the quotient bundle $\bar{E}_f \to \bar{M}_f$.

We note that the mapping $F \mapsto \lambda(F)$ satisfies $\lambda(\{F_1, F_2\}) = [\lambda(F_1), \lambda(F_2)]$.

Lemma 6. *Suppose that* i) *The action of G_f on M_f is free;* ii) *The mapping $x \mapsto \lambda(H_x)$ of $\mathfrak{g}_f$ into the Lie algebra of vector fields on E_f extends to an action of G_f on E_f.*

Then the prequantization condition holds if and only if the function $\chi_f(\exp x) = \exp if(x)$ extends to a character of G_f.

2°. As an example, consider the action of a Lie group G on its cotangent bundle $M = T^*G$. We identify T^*G with $G \times \mathfrak{g}^*$ via right translations. G acts on M via left and right translations: $h(g, \xi) = (hg, \mathrm{Ad}^* h(\xi))$, $(g, \xi)h = (gh^{-1}, \xi)$. This action gives rise to the left and right momenta maps Φ_l, Φ_r:

$$\Phi_l(g, \xi) = \xi; \qquad \Phi_r(g, \xi) = \mathrm{Ad}^* g^{-1}(\xi).$$

For brevity, we shall speak of the left and right action of G, respectively.

Let θ be canonical 1-form on T^*G. The quantum bundle over T^*G is the trivial (and trivialized) bundle $E = T^*G \times U(1)$ with the connection form $\alpha = (2\pi)^{-1}\theta + (2\pi iz)^{-1} dz$. The action of G on T^*G naturally lifts to E.

Fix a point $f \in \mathfrak{g}^*$ and put $M_f = \Phi_l^{-1}(f)$.

Lemma 7. (i) *The Poisson bracket in the space of right-invariant functions on T^*G (i.e. such that $F(g, \xi) = F(\xi)$) coincides with the Kirillov bracket on $\mathfrak{g}^*$* ([2]).

(ii) *The quotient manifold $\bar{M}_f$ is isomorphic to the orbit $\mathcal{O}_f \subset \mathfrak{g}^*$. The projection $M_f \to \mathcal{O}_f$ is given by the right momentum map* ([2]).

(iii) *The quantum bundle E_f reduces to a quantum bundle over $\mathcal{O}_f$ if and only if f defines a character of G_f.*

(iv) *f defines a character of G_f if and only if the restriction $(2\pi)^{-1}\theta_f$ of the 1-form $(2\pi)^{-1}\theta$ to G_f-orbits in M_f is integral (from Proposition 1 it follows that the restricted form is closed). The 2-form $\bar{\omega}_f$ on $\bar{M}_f \simeq \mathcal{O}_f$ is obtained from θ_f by transgression. If $H^1(G, \mathbf{Z}) = 0$, f defines a character of G_f if and only if $(2\pi)^{-1}\bar{\omega}_f$ is integral* [10].

Assume now that the orbit $\mathcal{O}_f$ is equipped with a real G-invariant polarization [8]. Let L_f be the corresponding Lagrangian subgroup, $\mathfrak{l}_f$ its Lie algebra, π: $\mathfrak{g}^* \to \mathfrak{l}_f^*$ the natural projection. Note that T^*G admits a standard real polarization associated with the projection $T^*G \to G$. Consider the reduction of T^*G with respect to the right action of L_f over $f = \pi(f)$.

Lemma 8. (i) *The standard polarization of T^*G projects onto a real polarization of the quotient symplectic manifold $\overline{M}_{\bar{f}}$.*

(ii) *Suppose that the polarization of $\mathcal{O}_f$ satisfies the Pukanszky condition; $\pi^{-1}(\bar{f}) \subset \mathcal{O}_f$. Then $\overline{M}_{\bar{f}}$ and $\mathcal{O}_f$ are isomorphic as polarized symplectic manifolds.*

Proof. The inverse image of $\bar{f}$ under the right momentum map $\pi \circ \Phi_r$: $T^*G \to \mathfrak{l}_f^*$ is

$$M_{\bar{f}} = \{(g, \mathrm{Ad}^* g(\xi)) \colon g \in G, \pi(\xi) = \bar{f}\}.$$

The linear spaces $\mathrm{Ad}^* g(\pi^{-1}(\bar{f}))$ are intersections of $M_{\bar{f}}$ with the leafs of the standard polarization of T^*G. This makes (i) evident. To prove (ii) consider the left momentum map Φ_l: $T^*G \to \mathfrak{g}^*$. If the Pukanszky condition holds, Φ_l maps $M_{\bar{f}}$ onto $\mathcal{O}_f$ and $\mathrm{Ad}^* g(\pi^{-1}(\bar{f}))$ onto the leafs of the L_f-polarization of $\mathcal{O}_f$. The fibers of Φ_l coincide with the right L_f-orbits. In other words. $\overline{M}_{\bar{f}} \simeq \mathcal{O}_f$ and Φ_l is the reduction map. ■

Let E be a quantum bundle over a polarized symplectic manifold (M, L). Denote by $S(M)$ the space of smooth sections of the associated line bundle over M covariantly constant on the leafs of L. The space $S(T^*G)$ is naturally isomorphic to $C^\infty(G)$. From the above lemma we get

Proposition 9. *Suppose that the character of $\mathfrak{l}_f$ defined by $\bar{f} = \pi(f)$ extends to a 1-dimensional representation χ_f of L_f. Then $S(\mathcal{O}_f)$ may be identified with the subspace $W_f \subset C^\infty(G)$ consisting of the functions φ such that*

$$\varphi(gh) = \chi_f(h^{-1})\varphi(g), \qquad h \in L_f.$$

3°. Now we apply the reduction technics to get an algebra of commuting Hamiltonians described in [17].

We recall briefly some notions from [17].

Let a Lie algebra $\mathfrak{g}$ be split into a linear sum of two subalgebras, $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$. Let $f \in \mathfrak{a}^* \simeq \mathfrak{b}^\perp$ be a character of $\mathfrak{a}$. Given a function F on $\mathfrak{g}^*$ let us denote by F_f the function on $\mathfrak{b}^* \simeq \mathfrak{a}^\perp$ defined by $F_f(\xi) = F(\xi + f)$. Let $I(\mathfrak{g}^*)$ be the algebra of smooth invariant functions on $\mathfrak{g}^*$.

Theorem 10. *The functions F_f, $F \in I(\mathfrak{g}^*)$ Poisson commute on $\mathfrak{b}^*$. If there exists a nondegenerate invariant bilinear form on $\mathfrak{g}$, the corresponding equations of motion may be written in the Lax form.*

Let A and B be the subgroups of G corresponding to the subalgebras $\mathfrak{a}$ and $\mathfrak{b}$. To prove the theorem we reduce T^*G with respect to the left action of A. Assume first that the mapping $(a, b) \mapsto ab$ of the product $A \times B$ onto an open subset $AB \subset G$ is one-to-one. Consider the reduction of $T^*(AB) = AB \times \mathfrak{g}^*$ with respect to the left action of A over the character $f \in \mathfrak{a}^*$; let T_f be the quotient manifold. Obviously, T_f is a B-space.

Proposition 11. *T_f is isomorphic to T^*B as a symplectic B-space.*

Proof. The inverse image of f under the left momentum map is $B \times \mathfrak{b}_f^*$, where $\mathfrak{b}_f^* = f + \mathfrak{a}^\perp$. The set $B \times \mathfrak{b}_f^*$ is a cross-section for the action of A and thus provides a model for the quotient manifold T_f. Since $f|_{\mathfrak{b}} = 0$ and $\mathfrak{a}^\perp \simeq \mathfrak{b}^*$, $B \times \mathfrak{b}_f^*$ is isomorphic to T^*B as a B-space. Moreover, the restriction of the canonical 1-form on T^*G to $B\times\mathfrak{b}_f^*$ coincides under this isomorphism with the canonical 1-form on T^*B. Hence also the symplectic structures of the two manifolds coincide. ■

The functions from $I(\mathfrak{g}^*)$ extend to bi-invariant functions on T^*G. Their restrictions to $B\times\mathfrak{b}_f^*$ commute with respect to the Poisson bracket on T^*B (Proposition 4) and are right B-invariant. Hence from Lemma 7(i) we get the first part of the theorem.

Proposition 12. *Let F be a bi-invariant function on T^*G.*

(i) *The trajectory of the Hamiltonian F starting at (g,ξ) has the form*

$$(g(t),\xi(t))=(\exp t\,dF(\xi+f)\cdot g,\xi).$$

(ii) *Let $\xi\in\mathfrak{b}^*$, $\exp t\,dF(\xi+f)=a(t)\,b(t)$, $a(t)$ and $b(t)$ being smooth curves in the subgroups A and B, respectively. Then the trajectory of the reduced Hamiltonian $\bar{F}$ in $\mathfrak{b}^*$ starting at ξ has the form*

$$\xi(t)=\mathrm{Ad}_B^*\,b(t)\,\xi \quad \textit{or, equivalently,}$$

$$\xi(t)+f=\mathrm{Ad}_G^*\,a(t)^{-1}(\xi+f).$$

(iii) *The reduced equations of motion are*

$$\dot{\xi}=\mathrm{ad}^*M(\xi+f);$$

here M is the projection of $dF(\xi+f)$ to $\mathfrak{a}$ along $\mathfrak{b}$.

If there is a nondegenerate invariant form on $\mathfrak{g}$, then $\mathfrak{g}^*\simeq\mathfrak{g}$, $\mathrm{ad}^*\simeq\mathrm{ad}$ and the equations of motion have the Lax form.

Proof. (i) is evident. By Lemma 3 the trajectories of the reduced system are projections of those of the initial one. This proves (ii); (iii) follows from (ii) by taking time derivative.

The assumption that the mapping $A\times B\to AB$ is globally one-to-one may be eliminated by considering an open subset of the level surface M_f which projects onto a sufficiently small neighborhood of the unit element in G.

Remark 1. Assume again that the mapping $A\times B\to AB$ is one-to-one and consider a further reduction of the manifold $\overline{M}_f=A\backslash M_f$ $(M=T^*G)$ with respect to the right action of B over $c\in\mathfrak{b}^*$. Since $T^*B\subset\overline{M}_f$ the quotient space $\overline{M}_{f,c}=A\backslash M_{f,c}/B_c$ contains an open subset isomorphic to the orbit $\mathcal{O}_c$ of B in $\mathfrak{b}^*$. In general, the restrictions of quotient Hamiltonian flows to $\mathcal{O}_c$ are incomplete. If $\overline{M}_{f,c}$ is a smooth manifold, we may consider it as a completion of $\mathcal{O}_c$ with respect to the flows, generated by Hamiltonians $F\in I(\mathfrak{g}^*)$.

Remark 2. If the orbit $\mathcal{O}_c$ admits a real polarization, the space $\overline{M}_f$ may also be reduced with respect to the right action of the corresponding Lagrangian subgroup. This is of special advantage in the quantum version of the problem (cf. n° 5).

4°. Example. Let $\mathfrak{g}$ be a real split semisimple Lie algebra, let $\mathfrak{a}\subset\mathfrak{g}$ be its split Cartan subalgebra, let Δ be the root system of $(\mathfrak{g},\mathfrak{a})$. For $\alpha\in\Delta$ let e_α denote the corresponding root vector. Fix an order in Δ and let P be the set of simple roots. Put $\mathfrak{n}_\pm = \bigoplus_{\alpha\in\Delta_\pm} \mathbf{R}e_\alpha$, $\mathfrak{b}_\pm = \mathfrak{a}+\mathfrak{n}_\pm$. Let θ be the Cartan involution in $\mathfrak{g}$ which fixes $\mathfrak{a}$ and interchanges $\mathfrak{b}_+$ and $\mathfrak{b}_-$.

Let G be the corresponding real semisimple Lie group. We denote the Cartan involution in G which corresponds to θ by the same letter. Let K be the maximal compact subgroup of G which is fixed under θ, let M be the centralizer of $\mathfrak{a}$ in K. Put $A=\exp\mathfrak{a}$, $N_\pm=\exp\mathfrak{n}_\pm$, $B=MAN_-$. Put $f=\sum_{\alpha\in P} e_{-\alpha}$.

Identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the Killing form Q. This induces isomorphisms $\mathfrak{b}_-^*\simeq\mathfrak{b}_+$, $\mathfrak{n}_+^*\simeq\mathfrak{n}_-$. Thus f may be regarded as a character of $\mathfrak{n}_+$.

Let $\Phi\colon T^*G\to\mathfrak{n}_+^*\oplus\mathfrak{b}_-^*$ be the momentum map corresponding to the left action of N_+ and the right action of B. Fix a point $c\in\mathfrak{b}_-^*$ and let $B_c\subset B$ be its stationary subgroup.

Proposition 13. *The space $M_{f,c}=\Phi^{-1}((f,c))$ is a smooth submanifold of T^*G. The quotient space $\overline{M}_{f,c}=N_+\backslash M_{f,c}/B_c$ is a smooth symplectic manifold.*

Proof. It suffices to prove that

(i) (f,c) is a regular value of Φ.

(ii) The group $N_+\times B_c$ acts properly on $M_{f,c}$.

Let $d\Phi_{g,\xi}\colon \mathfrak{g}\oplus\mathfrak{g}\to\mathfrak{n}_+^*\oplus\mathfrak{b}_-^*\simeq\mathfrak{n}_-\oplus\mathfrak{b}_+$ be the differential of Φ at $(g,\xi)\in M_{f,c}$. One easily finds that

$$d\Phi_{g,\xi}(x,y)=y|_{\mathfrak{n}_+}+\operatorname{Ad}g^{-1}(y+[x,\xi])|_{\mathfrak{b}_-}.$$

To prove that $d\Phi_{g,\xi}$ is surjective it suffices to check that the map $(x,y)\mapsto y+[x,\xi]$ which maps $\mathfrak{g}\oplus\mathfrak{b}_+$ into $\mathfrak{g}$ is surjective. If $v\in\mathfrak{g}$ is such that $Q(v,y+[x,\xi])=0$ for all $x\in\mathfrak{g}$, $y\in\mathfrak{b}_+$, then $v\in\mathfrak{b}_+^\perp=\mathfrak{n}_+$ and $[v,\xi]=0$. Note that $\xi\in f+\mathfrak{b}_+$. In [9, Th. 1.2] it is proved that the adjoint action of N_+ on $\mathfrak{g}$ induces a free and proper action on the affine space $f+\mathfrak{b}_+\subset\mathfrak{g}$. Hence $v=0$, which proves (i). It follows also that $N_+\times B$ acts freely and properly on $M_f=G\times(f+b_+)$, whence we get (ii). ■

The orbit $\mathcal{O}_c$ is open and dense in the quotient manifold $\overline{M}_{f,c}$. In $\overline{M}_{f,c}$ the reduced Hamiltonian flows generated by $F\in I(\mathfrak{g})$ are complete; their restrictions to $\mathcal{O}_c$ are usually incomplete. An important special case in which they occur to be complete on $\mathcal{O}_c$ is the generalized Toda lattice.

Put $e=\sum_{\alpha\in P} e_\alpha$. Clearly, $e\in\mathfrak{b}_+\simeq\mathfrak{b}_-^*$. Let $B^0=AN_-$ be the identity component of B. Let $\mathcal{O}_e^0$ be the B^0-orbit of e (with respect to the coadjoint action). The typical point of $\mathcal{O}_e^0$ is

$$x=p+\sum_{\alpha\in P} c_\alpha e_\alpha,\qquad p\in\mathfrak{a},\ c_\alpha\in\mathbf{R},\ c_\alpha>0.$$

Recall that $\mathcal{O}_e^0$ naturally embeds onto an open subset of the quotient space $\overline{M}_{f,e}$. Thus, the Hamiltonians $F\in I(\mathfrak{g})$ induce quotient flows on $\mathcal{O}_e^0$.

Proposition 14. *The quotient Hamiltonian flow on $\mathcal{O}_e^0$ induced by a function $F\in I(\mathfrak{g})$ is complete.*

Remark. It is easy to see that the reduced Hamiltonian $\bar{F}$ equals F restricted to $\mathcal{O}_e^0$. In particular, the dynamical system associated with the function $H(\xi) = Q(\xi, \xi)$ is the generalized Toda lattice.

Proof of Proposition 14. We shall use another realization of $\mathcal{O}_e^0$. Consider reduction of T^*G with respect to the left action of the maximal compact subgroup $K \subset G$ and the right action of B^0 over the point $(0, c) \in \mathfrak{k}^* \oplus \mathfrak{b}^*$. Since $G = KAN_-$, the quotient space $\bar{M}_{0,c}$ is isomorphic to $\mathcal{O}_c^0$, the B^0-orbit of c. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. Since $\mathfrak{b}_-^* \simeq \mathfrak{k}^\perp = \mathfrak{p}$, the quotient space $\bar{M}_{0,c}$ naturally embeds into $\mathfrak{p}$. Clearly, the reduced flows on $\bar{M}_{0,c}$ generated by $F \in I(\mathfrak{g})$ are always complete. The spaces $\mathfrak{b}_+$ and $\mathfrak{p}$ are two different models of $\mathfrak{b}_-^*$. Let i: $\mathfrak{b}_+ \to \mathfrak{p}$ be the natural isomorphism $i(x) = x - \theta(x)$. The reduced Hamiltonian on $\mathcal{O}_c^0$ corresponding to $F \in I(\mathfrak{g})$ is $\bar{F}' = F \circ i$. It happens that for $c = e$ the Hamiltonians $\bar{F}$ and $\bar{F}'$ are related by a simple canonical transformation. Namely, define a map $\mathcal{O}_e^0 \to \mathcal{O}_e^0$ by putting $p' = \frac{1}{2}p$, $c'_\alpha = \frac{1}{4}c_\alpha^2$ (We make use of coordinates on $\mathcal{O}_e^0$ introduced above). This map is symplectic. Let F be a homogeneous polynomial of degree d. It is easy to check that $\bar{F}'(p, c) = 2^d \bar{F}(p', c')$. ■

5°. Now we discuss a quantum version of the reduction method, namely a reduction of the centre $Z(\mathfrak{g})$ of the universal envelopping algebra of $\mathfrak{g}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$. Following the pattern of n° 3, consider first the quotient manifold $\bar{M}_f$ obtained via reduction of T^*G with respect to the left action of A over a character $f \in \mathfrak{a}^*$.

Suppose that f gives rise to a unitary character of A, $\chi_f(\exp x) = \exp if(x)$ (the prequantization condition). The space of smooth sections of the quotient quantum line bundle over $\bar{M}_f$ which are covariantly constant on the leaves of the standard polarization of T^*G may be identified with the space W_f of smooth functions on G satisfying the functional equation

$$\varphi(ag) = \chi_f(a^{-1})\,\varphi(g).$$

On $AB \subset G$ such a function is determined by its restriction to B.

The space W_f is a $Z(\mathfrak{g})$-module; we now give its algebraic description.

Extend f to a character of the universal envelopping algebra $U(\mathfrak{a})$ and let U_f be its kernel. Then we have the direct decomposition

$$U(\mathfrak{g}) = U(\mathfrak{b}) \oplus U(\mathfrak{g})\,U_f.$$

Let ρ_f: $U(\mathfrak{g}) \to U(\mathfrak{b})$ be the projection onto the first summand. For $\varphi \in W_f$, $u \in U(\mathfrak{g})\,U_f$ we have $u\varphi = 0$, hence $z\varphi = \rho_f(z)\varphi$ for $z \in Z(\mathfrak{g})$. Restricted to B this equation shows that z may be viewed as an operator in $C^\infty(B)$.

Lemma 15. *Let $\varphi \in W_f$ and let φ_B be its restriction to B. Then*

$$z\varphi(b) = \rho_f(z)\,\varphi_B(b).$$

Corollary 16. *The restriction of ρ_f to $Z(\mathfrak{g})$ is an algebra homomorphism, so that $\rho_f(Z(\mathfrak{g}))$ is a commutative subalgebra of $U(\mathfrak{b})$.*

Now let $*$ be the antiautomorphism of $U(\mathfrak{g})$ which is equal to $-\mathrm{id}$ on $\mathfrak{g}$. Obviously, $Z(\mathfrak{g})$ is $*$-invariant. Assume that there exists a G-invariant measure on $A\backslash G$; clearly it coincides on $B \subset A\backslash G$ with a right-invariant Haar measure db_r. Let Δ be a character of B which distinguishes between right and left Haar measures on B, $db_r = \Delta(b)\,db_l$. Put $2\delta(x) = \left(\frac{d}{dt}\Delta(\exp t x)\right)_{t=0}$. Let α be the automorphism of $U(\mathfrak{b})$ defined by $\alpha(x) = x - \delta(x)$ for $x \in \mathfrak{b}$. Put $\gamma_f = \alpha \circ \rho_f$.

Lemma 17. *The homomorphism γ_f: $Z(\mathfrak{g}) \to U(\mathfrak{b})$ is symmetric, i.e. $\gamma_f(z^*) = \gamma_f(z)^*$.*

Proof. Let us identify $C_0^\infty(B)$ with a subspace W_f^0 of W_f. Let π_r be the right regular representation of $U(\mathfrak{g})$ in W_f^0 and π_l be the left regular representation of $U(\mathfrak{b})$ in $C_0^\infty(B)$. If U, V are linear operators in $W_f^0 \simeq C_0^\infty(B)$, we write $V = U^*$ if for any $\varphi, \psi \in C_0^\infty(B)$

$$\int U\varphi \cdot \bar{\psi}\, db_r = \int \varphi \cdot \overline{V\psi}\, db_r$$

and $V = U^\dagger$ if

$$\int U\varphi \cdot \bar{\psi}\, db_l = \int \varphi \cdot \overline{V\psi}\, db_l.$$

Now, for $u \in U(\mathfrak{g})$ $\pi_r(u^*) = \pi_r(u)^*$. For $z \in Z(\mathfrak{g})$ we have $\pi_r(z) = \pi_l(\rho_f(z))$, so that $\pi_l(\rho_f(z^*)) = \pi_l(\rho_f(z))^*$. Since multiplication by $\Delta^{\frac{1}{2}}$ transforms $*$ into $\dagger$ and ρ_f into γ_f, we get $\pi_l(\gamma_f(z^*)) = \pi_l(\gamma_f(z))^\dagger$. On the other hand for $u, v \in U(\mathfrak{b})$, $v = u^*$ if and only if $\pi_l(v) = \pi_l(u)^\dagger$. This proves the lemma.

Further reduction of the classical phase space with respect to the right action of B over $c \in \mathfrak{b}^*$ gives the orbit $\mathcal{O}_c$. In the quantum case it is more convenient to reduce our phase space with respect to the right action of the Lagrangian subgroup L_c (n° 2, Lemma 8). Assume that $\bar{c} = \pi(c)$ extends to a character χ_c of L_c (the quantization condition). Then the space $S(\overline{M}_{f,\bar{c}})$ of smooth sections of the reduced quantum line bundle over $\overline{M}_{f,\bar{c}}$ which are covariantly constant on the leafs of the polarization may be identified with the space

$$W_{f,c} = \{\varphi \in C^\infty(G)\colon\ \varphi(agl) = \chi_f(a^{-1})\,\chi_c(l)\,\varphi(g);\ a \in A,\ l \in L_c\}.$$

Accordingly, the space $S(\mathcal{O}_c)$ is identified with

$$W_c = \{\varphi \in C^\infty(B)\colon\ \varphi(bl) = \chi_c(l)\,\varphi(b),\ l \in L_c\}.$$

$W_{f,c}$ is a $Z(\mathfrak{g})$-module, W_c is a $U(\mathfrak{b})$-module. Restricting the elements of $W_{f,c}$ to B one obtains a ρ_f-equivariant linear map $W_{f,c} \to W_c$.

The projection ρ_f provides a quantum-mechanical analogue of the classical reduction. In general, "quantum operators" $\rho_f(z)$ are not in one-to-one correspondence with reduced classical Hamiltonians. We return to the example of n° 4 where a more precise information is available. Our exposition is based on the results of D. Kazhdan and B. Kostant. We keep to the notation of n° 4, writing $\mathfrak{b}$ instead of $\mathfrak{b}_-$.

Let $S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}$. There is a canonical isomorphism between $S(\mathfrak{g})$ and the polynomial algebra $P(\mathfrak{g}^*)$ on the dual space. In particular, it makes sense to speak of a value of $u \in S(\mathfrak{g})$ at a point $x \in \mathfrak{g}^* \simeq \mathfrak{g}$.

Define an element $x_0 \in \mathfrak{a}$ by $\alpha(x_0)=1$, $\alpha \in P$. The endomorphism $\mathrm{ad}\, x_0$ extends to a differentiation of $S(\mathfrak{g})$ and $U(\mathfrak{g})$ which is compatible with the standard grading in $S(\mathfrak{g})$ (respectively, with the standard filtration in $U(\mathfrak{g})$). Following D. Kazhdan, we introduce in $S(\mathfrak{b})$ *the* x_0*-grading.*

Let $(S_i(\mathfrak{b}))_j$ be the eigenspace of $\mathrm{ad}\, x_0|_{S_i(\mathfrak{b})}$ corresponding to the eigenvalue j. Clearly, all such eigenvalues are integral. Put

$$S_{(k)}(\mathfrak{b}) = \bigoplus_{i+j=k} (S_i(\mathfrak{b}))_j.$$

Evidently, $S_{(i)}(\mathfrak{b})\, S_{(j)}(\mathfrak{b}) \subset S_{(i+j)}(\mathfrak{b})$ so we get a grading of $S(\mathfrak{b})$. *The* x_0*-filtration of* $U(\mathfrak{b})$ is defined in a similar way.

Extend f to a character of $U(\mathfrak{n}_+)$ and let ρ_f: $U(\mathfrak{g}) \to U(\mathfrak{b})$ be the associated projection. Let p_f: $S(\mathfrak{g}) \to S(\mathfrak{b})$ be the restriction map

$$p_f u(x) = u(x+f), \qquad u \in S(\mathfrak{g}),\ x \in \mathfrak{b}_+.$$

Recall that the affine space $f+\mathfrak{b}_+$ is invariant under the adjoint action of N_+. We define the action $N_+ \times \mathfrak{b}_+ \to \mathfrak{b}_+$ by putting

$$n \cdot x = \mathrm{Ad}\, n(x+f) - f.$$

Let $S(\mathfrak{b})^{N_+}$ be the subalgebra of N_+-invariants with respect to the contragredient action of N_+ on $S(\mathfrak{b})$ (Recall that $\mathfrak{b}_+ \simeq \mathfrak{b}^*$ and so $S(\mathfrak{b}) \simeq P(\mathfrak{b}_+)$). Equip $S(\mathfrak{g})$ with the standard grading and $U(\mathfrak{g})$ with the standard filtration.

Theorem (D. Kazhdan, B. Kostant, [8]).

(i) $S(\mathfrak{b})^{N_+}$ *is a graded subalgebra with respect to the* x_0*-grading and* p_f: $S(\mathfrak{g}) \to S(\mathfrak{b})^{N_+}$ *is a graded algebra isomorphism.*

(ii) *Put* $W = \rho_f(Z(\mathfrak{g}))$. *Then* $W \subset U(\mathfrak{b})$ *is a filtered subalgebra with respect to the* x_0*-filtration and* ρ_f: $Z(\mathfrak{g}) \to W$ *is a filtered algebra isomorphism.*

(iii) *The following diagram with exact rows is commutative*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_{k-1} & \longrightarrow & Z_k & \longrightarrow & S_k(\mathfrak{g})^G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho_f} & & \downarrow{\scriptstyle \rho_f} & & \downarrow{\scriptstyle p_f} & & \\
0 & \longrightarrow & W_{(k-1)} & \longrightarrow & W_{(k)} & \longrightarrow & S_{(k)}(\mathfrak{b})^{N_+} & \longrightarrow & 0.
\end{array}$$

This diagram shows that classical Hamiltonians are principal symbols of the operators $\rho_f(z)$.

The correspondence between quantum and classical Hamiltonians may be inverted. To this end consider the direct sum decomposition $U(\mathfrak{g}) = U(\mathfrak{b}) + U(\mathfrak{g})\,\mathfrak{n}_+$ and let ρ_0: $U(\mathfrak{g}) \to U(\mathfrak{b})$ be the projection onto $U(\mathfrak{b})$ along $U(\mathfrak{g})\mathfrak{n}_+$. The restriction of ρ_0 to $Z(\mathfrak{g})$ is the well known Harish-Chandra homomorphism; its image lies in $U(\mathfrak{a})$. Let δ be half the sum of positive roots of $(\mathfrak{g}, \mathfrak{a})$, let α be the automorphism of $U(\mathfrak{a})$ given by $\alpha(x) = x - \delta(x)$ for $x \in \mathfrak{a}$. Put $\gamma_0 = \alpha \circ \rho_0$. The mapping γ_0 is an isomorphism of $Z(\mathfrak{g})$ onto the subalgebra $U(\mathfrak{a})^W$ of Weyl group invariants in $U(\mathfrak{a})$. Combining it with the Chevalley isomorphism $U(\mathfrak{a})^W \simeq S(\mathfrak{g})^G$ we get an algebra isomorphism Γ: $Z(\mathfrak{g}) \to S(\mathfrak{g})^G$. Put $\Delta = \Gamma^{-1}$ [3].

[3] For the mapping Δ there is an explicit formula [3]. Let β: $S(\mathfrak{g}) \to U(\mathfrak{g})$ be the symmetrization map. Define the formal power series $q \in \hat{S}(\mathfrak{g}^*)$ by $q = \det(\mathrm{sh\ ad}/\mathrm{ad})$. Let $D(q)$ be the natural differential operator on $S(\mathfrak{g})$ defined by q. Then $\Delta(u) = \beta(D(q)u)$

Let $q_f = \gamma_f \circ \Delta \circ p_f^{-1}$. The following diagram is commutative

$$\begin{array}{ccc} S(\mathfrak{g})^G & \xrightarrow{\Delta} & Z(\mathfrak{g}) \\ {\scriptstyle p_f}\downarrow & & \downarrow{\scriptstyle \gamma_f} \\ S(\mathfrak{b})^{N_+} & \xrightarrow{q_f} & W. \end{array}$$

Clearly, q_f is an algebra homomorphism. If $u \in S_{(k)}(\mathfrak{b})^{N_+}$, then $q_f(u) \in W_{(k)}$ and $\mathrm{gr}_{(k)} q_f(u) = u$. This permits to regard q_f as a "quantization map".

Let $c \in \mathfrak{b}^*$ let L_c be the corresponding Lagrangian subgroup and let π_c be the representation of $U(\mathfrak{b})$ in a Hilbert space H_c which corresponds to the orbit $(\mathcal{O}_c, L_c)$. The mapping $\gamma_{f,c} = \pi_c \circ \gamma_f$ is a quantum analogue of the two-sided reduction.

Suppose that there is a B-invariant measure $d\mu$ on B/L_c. Let

$$W_{f,c}^2 = \{\varphi \in W_{f,c} \colon \|\varphi\|^2 = \int_{B/L_c} |\varphi(b)|^2 \, d\mu(b) < \infty\}$$

and let $\mathbf{W}_{f,c}$ be the completion of $W_{f,c}^2$ in this norm. The representation space H_c may be identified with $\mathbf{W}_{f,c}$. Consequently, the study of the spectrum of $\gamma_{f,c}(Z(\mathfrak{g}))$ in H_c is reduced to the study of the spectrum of $Z(\mathfrak{g})$ in $\mathbf{W}_{f,c}$.

For the generalized Toda lattice, i.e. for the orbit $\mathcal{O}_e^0$, we have a more definite result. To state it recall that there is a natural isomorphism $\gamma_0 \colon Z(\mathfrak{g}) \to S(\mathfrak{a})^W$. Let us consider $S(\mathfrak{a})^W$ as a commutative algebra of (unbounded) operators acting in $L_2(\mathfrak{a}^*)$ via multiplication.

Theorem. *The algebra $\gamma_{f,e}(Z(\mathfrak{g}))$ of (unbounded) operators in H_e is unitarily equivalent to $S(\mathfrak{a})^W$. Under this equivalence $\gamma_{f,e}(z)$ goes into $\gamma_0(z)$.*

The proof is lengthy and will not be presented here. We only mention that the eigenfunctions are related to the so-called Whittaker functions. The algebraic spectrum of $Z(\mathfrak{g})$ in $W_{f,e}$ is described in [8] (see also footnote 2).

Remark. The Iwasawa decomposition leads to another variant of quantum reduction for the generalized Toda lattice. Let $\tilde{p} \colon U(\mathfrak{g}) \to U(\mathfrak{b})$ be the projection along $U(\mathfrak{g})\mathfrak{k}$ and put $\tilde{\gamma}_e = \pi_e \circ \alpha \circ \tilde{p}$. Now using the co-ordinates p, c_α (see the end of n°4) we may regard $\pi_e(u)$, $u \in U(\mathfrak{b})$ as differential operators in the space of variables c_α.

Proposition 18. *Let T be a transformation which maps c_α into $1/4 c_\alpha^2$. Let $u \in S(\mathfrak{g})^G$ be homogeneous of degree d, let $\beta \colon S(\mathfrak{g}) \to U(\mathfrak{g})$ be the symmetrization map, $z = \beta(u)$. Then $\tilde{\gamma}_e(z) = 2^d T_* \gamma_{f,e}(z)$.*

The proof is straightforward and will be omitted.

6°. The most interesting finite-dimensional dynamical systems provided by the Adler-Kostant method are those associated with the Kac-Moody algebras [17]. As we show below, these systems may also be obtained by reduction. We recall briefly their construction for the simplest case of the so called affine Lie algebras. (or, Kac-Moody algebras of height 1) consisting of Laurent polynomials with coefficients in a finite dimensional Lie algebra.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{R}$, let $\tilde{\mathfrak{g}}=\mathfrak{g}\otimes\mathbf{R}[z,z^{-1}]$ be the Lie algebra of Laurent polynomials with coefficients in $\mathfrak{g}$ and the commutator

$$[x\otimes z^m, y\otimes z^n]=[x,y]\otimes z^{m+n}.$$

There are two important subalgebras $\mathfrak{g}_\pm\subset\tilde{\mathfrak{g}}$ defined by $\mathfrak{g}_\pm=\mathfrak{g}\otimes\mathbf{R}[z^{\pm 1}]$. The subalgebras $\mathfrak{g}_\pm^k=z^{\pm k}\mathfrak{g}_\pm$, $k>0$, form a decreasing sequence of ideals in $\mathfrak{g}_\pm$. Given a subalgebra $\mathfrak{q}\subset\mathfrak{g}$ define the subalgebras $\mathfrak{q}_\pm\subset\tilde{\mathfrak{g}}$ by putting $\mathfrak{q}_\pm=\mathfrak{q}+\mathfrak{g}_\pm^1$. If $\mathfrak{g}$ splits into a linear sum of two its subalgebras, $\mathfrak{g}=\mathfrak{a}+\mathfrak{b}$, then $\tilde{\mathfrak{g}}=\mathfrak{a}_++\mathfrak{b}_-$. Since $\mathfrak{g}_-^k$ is an ideal in $\mathfrak{b}_-$, one can set $\mathfrak{b}_k=\mathfrak{b}_-/\mathfrak{g}_-^k$.

Put $\tilde{\mathfrak{g}}'=\mathfrak{g}^*\otimes\mathbf{R}[[z,z^{-1}]]$. If $f\in\tilde{\mathfrak{g}}'$, $x\in\tilde{\mathfrak{g}}$, then $f(x)$ is a Laurent polynomial. Putting $\langle f,x\rangle=\operatorname{Res}(z^{-1}f(x))_{z=0}$ we may identify $\tilde{\mathfrak{g}}'$ with the dual of $\tilde{\mathfrak{g}}$. In this paper we shall only be interested in its subspace $\tilde{\mathfrak{g}}^*=\mathfrak{g}^*\otimes\mathbf{R}[z,z^{-1}]$ consisting of finite sums.

Let $\mathfrak{a}$, $\mathfrak{b}$ be as above. In a similar may, the finite duals $\mathfrak{a}_+^*$, $\mathfrak{b}_-^*$ are defined. Using the decomposition $\tilde{\mathfrak{g}}=\mathfrak{a}_++\mathfrak{b}_-$, we identify $\mathfrak{b}_-^*$ with the subspace $\mathfrak{a}_+^\perp\subset\tilde{\mathfrak{g}}^*$, $\mathfrak{a}_+^\perp=\{\sum_{k\geq 0} f_k z^k,\ f_0\in\mathfrak{a}^\perp\}$. The finite-dimensional space $\mathfrak{b}_n^*=\left\{\sum_{k=0}^{n-1} f_k z^k,\ f_0\in\mathfrak{a}^\perp\right\}$ is naturally isomorphic to the dual of the factor-algebra $\mathfrak{b}_n$ and hence is invariant under $\operatorname{ad}^*\mathfrak{b}_-$. Effectively, the action of $\mathfrak{b}_-$ on $\mathfrak{b}_n^*$ reduces to that of $\mathfrak{b}_n$, which accounts for the validity of the finite-dimensional scheme of n° 4 as applied to $\tilde{\mathfrak{g}}=\mathfrak{a}_++\mathfrak{b}_-$.

With a slight abuse of language we shall speak of the orbits of $\mathfrak{b}_-$ in $\mathfrak{b}_n^*$ meaning the orbits of the finite-dimensional Lie group which corresponds to $\mathfrak{b}_n$.

Let $p\in I(\mathfrak{g}^*)$. Define a sequence of polynomials on $\tilde{\mathfrak{g}}^*$ by putting

$$p_n(\xi)=\operatorname{Res}[z^{-n-1}p(\xi(z))]_{z=0},\qquad n\in\mathbf{Z}.$$

Clearly, $p_n\in I(\tilde{\mathfrak{g}}^*)$. Now we are able to restate Theorem 5.

Theorem 5′. *Let $\mathfrak{g}=\mathfrak{a}+\mathfrak{b}$, $\tilde{\mathfrak{g}}=\mathfrak{a}_++\mathfrak{b}_-$ be as above. Let $f\in\mathfrak{b}_-^\perp$ be a character of $\mathfrak{a}_+$. The polynomials $p_{n,f}(\xi)=p_n(\xi+f)$, $p\in I(\mathfrak{g}^*)$, $n\in\mathbf{Z}$, Poisson commute on $\mathfrak{b}_-^*$. If there is a nondegenerate invariant bilinear form on $\mathfrak{g}$, the corresponding equations of motion are of Lax type.*

The last condition holds if $\mathfrak{g}$ is semisimple; then $\tilde{\mathfrak{g}}$ is an affine Lie algebra, or Kac-Moody algebra of height 1 ([5, 6, 14]).

In [17] some quotient dynamical systems related to these algebras were described, in particular, the non-abelian Toda lattices. Other systems, notably, the Euler equations of the n-dimensional top are connected with Kac-Moody algebras of height 2. As stated, Theorem 5 does not apply directly to this case, however, it may easily be generalized. The same is true of the reduction scheme presented below. For the sake of brevity we do not dwell upon this. Some more examples of dynamical systems related to height 2 algebras are discussed in Appendix 1.

Our aim is to give another realization for dynamical systems of Theorem 5′ in terms of reduced Hamiltonian systems. To this end we replace $\tilde{\mathfrak{g}}$ by its suitable completion. Here we consider the Banach-Lie algebra $\mathfrak{g}_{\mathscr{W}}$ consisting of absolutely convergent Fourier series with coefficients in $\mathfrak{g}$.

Let $\mathfrak{b}_{\mathscr{W}}$ be the closure of $\mathfrak{b}_- \subset \mathfrak{g}_{\mathscr{W}}$ in the topology of absolute convergence. Clearly, the finite-dimensional subspaces $\mathfrak{b}_n^* \subset \mathfrak{b}_{\mathscr{W}}^*$ are invariant under $\mathrm{ad}^*\,\mathfrak{b}_{\mathscr{W}}$ and the orbits of $\mathfrak{b}_{\mathscr{W}}$ in $\mathfrak{b}_n^*$ coincide with those of $\mathfrak{b}_-$.

Now we define Lie groups which correspond to $\mathfrak{g}_{\mathscr{W}}$ and its subalgebras. The aim of our definitions is to get an analogue of the Bruhat decomposition for these infinite-dimensional groups. Note, that the group of "all" functions on the unit circle with values in G is too large for our purposes.

For simplicity, we restrict ourselves to functions with values in special matrix groups, i.e. in the subgroups of $SL(N)$.

First let $G=SL(N,\mathbf{R})$. Let $\mathscr{W}$ denote the real algebra of absolutely convergent Fourier series with coefficients in $\mathrm{Mat}(N,\mathbf{R})$. Let $\mathscr{W}_\pm$ be its subalgebras whose elements are analytic (antianalytic) in the unit disc.

Put

$$G_{\mathscr{W}}^{\pm}=\{g\in\mathscr{W}_\pm : g(z)\in G \text{ for } z\in[-1,1]\},$$
$$N_{\mathscr{W}}^{\pm}=\{g\in G_{\mathscr{W}}^{\pm} : g(0)=1\}.$$

Clearly, $G_{\mathscr{W}}^{\pm}$, $N_{\mathscr{W}}^{\pm}$ are groups under pointwise multiplication. Put ${}^0G_{\mathscr{W}}=G_{\mathscr{W}}^{+}N_{\mathscr{W}}^{-}$.

Lemma 19. *Suppose $u\in\mathscr{W}$ admits a factorization $u=g_+n_-$, $g_+\in G_{\mathscr{W}}^{+}$, $n_-\in N_{\mathscr{W}}^{-}$. Then it may also be factorized as $u=n'_-g'_+$, $n'_-\in N_{\mathscr{W}}^{-}$, $g'_+\in G_{\mathscr{W}}^{+}$, and the factors g_+, g'_+, n_-, n'_- are unique.*

Proof. By a theorem of Gohberg and Krein [4], for each $u\in\mathscr{W}$ there exist decompositions

$$u=\varphi_+ d\varphi_- =\varphi'_- d'\varphi'_+$$

(right and left factorization, respectively) with $\varphi_\pm$, $\varphi'_\pm\in\mathscr{W}_\pm$, $d=\mathrm{diag}(z^{r_1},\dots,{}^{r_N})$, $d'=\mathrm{diag}(z^{l_1},\dots,z^{l_N})$, l_i, $r_i\in\mathbf{Z}$ (by definition, $z^l=\bar{z}^{-l}$ for $l<0$). The integers $r_1,\dots,r_N$, $l_1,\dots,l_N$ are characteristics of u and are called the right and left indices of u. Let $P_\pm:\mathscr{W}\to\mathscr{W}_\pm$ be the natural projections. For $u\in\mathscr{W}$ define a Toeplitz operator T_u in $\mathscr{W}_+$ by

$$T_u\varphi=P_+(u\varphi).$$

Then

$$\sum_{r_i>0} r_i=\sum_{l_i>0} l_i=\dim\operatorname{coker} T_u,$$
$$-\sum_{r_i<0} r_i=-\sum_{l_i<0} l_i=\dim\ker T_u,$$
$$\operatorname{ind} T_u=\operatorname{ind}(\det u).$$

Thus $d=1$ implies $d'=1$ and vice versa. In this case $\varphi_\pm$ and $\varphi'_\pm$ are defined uniquely up to a constant matrix factor. We normalize then by putting $n_-=\varphi_-(0)^{-1}\varphi_-$, $n'_-=\varphi'_-\varphi'_-(0)^{-1}$, $g_+=\varphi_+\varphi_-(0)$, $g'_+=\varphi'_-(0)\varphi'_+$. Hence the existence and uniqueness of the left factorization is proved. If $u=n'_-g'_+$ then $1=\det u=\det n'_-\det g'_+$ whence $\det n'_-=\text{const}$, $\det g'_+=\text{const}$ (both $\det n'_-$ and $\det g'_+$ being outer functions). Besides, $n'_-(0)=1$ so that $n'_-\in N^-_{\mathscr{W}}$, $g'_+\in G^+_{\mathscr{W}}$. ∎

Corollary. *${}^0G_{\mathscr{W}}$ is a group.*

Proposition 20. *Let u_t, $t\in\mathbf{R}$ be a smooth curve in ${}^0G_{\mathscr{W}}$, let $u_t=g_+(t)n_-(t)$ be its right factorization. Then the curves $g_+(t)$, $n_-(t)$ are smooth.*

Proof. Since $g_+^{-1}\dot{u}n_-^{-1}=g_+^{-1}\dot{g}_++\dot{n}_-n_-^{-1}$, one gets $g_+^{-1}\dot{g}_+=P_+(g_+^{-1}\dot{u}n_-^{-1})$, $\dot{n}_-n_-^{-1}=(1-P_+)(g_+^{-1}\dot{u}n_-^{-1})$. By assumption, $\dot{u}\in\mathscr{W}$. Since P_+ is bounded, $\dot{g}_+\in\mathscr{W}$, $\dot{n}_-\in\mathscr{W}$. The existence of higher derivatives is established in the same way.

Corollary. *The Lie algebra of ${}^0G_{\mathscr{W}}$ is*

$$\mathfrak{g}_{\mathscr{W}}=\{u\in\mathscr{W}:\operatorname{tr}u=0\}.$$

Proposition 21. *The exponential mapping maps $\mathfrak{g}_{\mathscr{W}}$ into ${}^0G_{\mathscr{W}}$.*

Proof. Let $\mathscr{W}^{-1}$ denote the group of invertible elements of $\mathscr{W}$. The Wiener-Lévy theorem implies that for $u\in\mathfrak{g}_{\mathscr{W}}$ $\exp u\in\mathscr{W}^{-1}$. Thus it remains to prove that $\exp u$ has zero indices. For any integer n, $\exp u=\left(\exp\frac{1}{n}u\right)^n$. Choose n sufficiently large so that $\exp\frac{1}{n}u$ be close to identity. Then from [4, Lemma 1.51] it follows that $\exp\frac{1}{n}u\in{}^0G_{\mathscr{W}}$. ∎

Now let $G\subset SL(N,\mathbf{R})$ be an arbitrary closed connected subgroup. The definitions of $G^\pm_{\mathscr{W}}$, $N^\pm_{\mathscr{W}}$ remain intact. Put ${}^0G_{\mathscr{W}}=G^+_{\mathscr{W}}N^-_{\mathscr{W}}$.

Lemma 22. *${}^0G_{\mathscr{W}}$ is arcwise connected, i.e. any $u\in{}^0G_{\mathscr{W}}$ may be connected with the identity by a smooth curve.*

The proof of this lemma will be omitted.

Proposition 23. *Let $u\in{}^0G_{\mathscr{W}}$, let $u=n_-g_+$ be its left factorization in $\mathscr{W}$. Then $n_-\in N^-_{\mathscr{W}}$, $g_+\in G^+_{\mathscr{W}}$.*

Corollary. *${}^0G_{\mathscr{W}}$ is a group.*

Proof of Proposition 23. Let u_t be a smooth curve connecting u with the identity, $t\in[0,1]$. Let $u_t=n_-(t)g_+(t)$ be its factorization in $\mathscr{W}$. Then $n_-(0)=g_+(0)=1$. After differentiation one gets

$$\dot{g}_+g_+^{-1}=P_+(n_-^{-1}\dot{u}g_+^{-1})$$
$$n_-^{-1}\dot{n}_-=(1-P_+)(n_-^{-1}\dot{u}g_+^{-1}).$$

The curves $n_-(t)$, $g_+(t)$ are the integral curves for $L_t=n_-^{-1}\dot{n}_-$, $R_t=\dot{g}_+g_+^{-1}$. To check that $n_-(t,z)$, $g_+(t,z)$ take values in G for all $t\in[0,1]$, $z\in[-1,1]$ it suffices to verify that $L_t(z)$, $R_t(z)$ take values in $\mathfrak{g}$. This is clear for $t=0$. For all $t\in[0,1]$ the fact follows from the standard uniqueness theorem for differential equations. ∎

Now let A, $B \subset G$ be two subgroups such that AB is open in G and the mapping $A \times B \to AB$ is one-to-one. Define the subgroups $A_{\mathscr{W}}^{+}$, $B_{\mathscr{W}}^{-}$ of ${}^{0}G_{\mathscr{W}}$ by

$$A_{\mathscr{W}}^{+} = \{g \in G_{\mathscr{W}}^{+}, g(0) \in A\}, \qquad B_{\mathscr{W}}^{-} = \{g \in G_{\mathscr{W}}^{-}, g(0) \in B\}.$$

Proposition 24. *The mapping* $A_{\mathscr{W}}^{+} \times B_{\mathscr{W}}^{-} \to$ *is one-to-one and* $A_{\mathscr{W}}^{+} B_{\mathscr{W}}^{-}$ *is open in* ${}^{0}G_{\mathscr{W}}$.

If G is semisimple, there is an analogue of Bruhat decomposition for ${}^{0}G_{\mathscr{W}}$. Let B be the Borel subgroup, W the corresponding Weyl group. For any $w \in W$ choose its representative in G, denote it by the same letter and consider it as a constant function. Put

$$B_{\mathscr{W}}^{\pm} = \{g \in G_{\mathscr{W}}^{\pm}, g(0) \in B\}.$$

Proposition 25. *Let G be semisimple. Then for ${}^{0}G_{\mathscr{W}}$ holds the decomposition* ${}^{0}G_{\mathscr{W}} = \bigcup_{w \in W} B_{\mathscr{W}}^{+} w B_{\mathscr{W}}^{-}$.

Proof. ${}^{0}G_{\mathscr{W}} = N_{\mathscr{W}}^{+} G N_{\mathscr{W}}^{-} = \bigcup_{w \in W} N_{\mathscr{W}}^{+} B w B N_{\mathscr{W}}^{-} = \bigcup_{w \in W} B_{\mathscr{W}}^{+} w B_{\mathscr{W}}^{-}$. ■

Now we apply the scheme of n° 3 to ${}^{0}G_{\mathscr{W}}$ regarding $\tilde{\mathfrak{g}}^{*}$ as a subspace of $\mathfrak{g}_{\mathscr{W}}^{*}$. Under the assumptions of Proposition 25 consider the reduction of $T^{*}\,{}^{0}G_{\mathscr{W}}$ with respect to the left action of $A_{\mathscr{W}}^{+}$ and the right action of $B_{\mathscr{W}}^{-}$ over the point $(f, c) \in \mathfrak{a}_{+}^{*} \oplus \mathfrak{b}_{-}^{*}$. As usually, we suppose f to be a character of $\mathfrak{a}_{+}$. The quotient space $\overline{M}_{f,c}$ contains an open subset isomorphic to the $B_{\mathscr{W}}^{-}$-orbit of c in $\mathfrak{b}_{\mathscr{W}}^{*}$. Let $\mathcal{O}_{c}$ be this orbit. Let F be an invariant function on $\mathfrak{g}_{\mathscr{W}}^{*}$. Hamiltonian flow on $\overline{M}_{f,c}$ generated by the Hamiltonian $\overline{F}$ coincides with the reduced flow generated by F (Proposition 12).

Proposition 26. *Fix a point $\xi \in \mathcal{O}_{c}$ and let*

$$\exp t\, dF(\xi + f) = a(t)\, b(t),$$

$a(t) \in A_{\mathscr{W}}^{+}$, $b(t) \in B_{\mathscr{W}}^{-}$ be the factorization of $\exp t\, dF(\xi + f)$. *The trajectory with a starting point ξ of the reduced dynamical system on $\mathcal{O}_{c}$ generated by the Hamiltonian F is given by any of the two equivalent formulae*

$$\xi(t) = \mathrm{Ad}_{B_{\mathscr{W}}^{-}}^{*} b(t)\, \xi,$$

$$\xi(t) + f = \mathrm{Ad}_{{}^{0}G_{\mathscr{W}}}^{*} a(t)^{-1} (\xi + f).$$

Proof. It follows from Propositions 21, 24 that for t sufficiently small the factorization is well defined. The trajectory formulae were obtained in Proposition 12(i).

In general, there is no reason for the reduced flows on $\mathcal{O}_{c}$ to be complete. We note two particular cases.

Proposition 27. (i) *Let G be an arbitrary Lie group. Let f be a character of $\mathfrak{n}_{\mathscr{W}}^{+}$. Reduced Hamiltonian flows on orbits of $G_{\mathscr{W}}^{-}$ generated by Hamiltonians $p_{n,f}$, $p \in I(\mathfrak{g})$, $n \in \mathbf{Z}$, are always complete.*

(ii) *For the generalized periodic Toda lattice all the flows of its integrals of motion are complete.*

Proof. (i) is evident because ${}^0G_{\mathscr{W}} = N^+_{\mathscr{W}} G^-_{\mathscr{W}}$ and hence the reduced phase space $\bar{M}_{f,c}$ is isomorphic to $\mathcal{O}_c$. (Here, as usual, c is a point in $(\mathfrak{g}^-_{\mathscr{W}})^*$, $M_{f,c}$ is the inverse image of (f, c) under the momentum map). A realization of periodic Toda lattice in terms of affine Lie algebras is given in [17]. Let G be a real split semisimple Lie group, B its Borel subgroup, $B^0 = AN$ its identity component. Recall that the periodic Toda lattice is associated with the orbit $\mathcal{O}_T$ of $(B^0_{\mathscr{W}})^-$ which passes through the sum of root vectors corresponding to the simple roots of $\tilde{\mathfrak{g}}$. Let Q be the Killing form on $\mathfrak{g}$; for $x \in \tilde{\mathfrak{g}}$ put $I(x) = \frac{1}{2}\operatorname{Res}(z^{-1} Q(x, x))_{z=0}$. The Hamiltonian of the Toda lattice is the restriction of I to $\mathcal{O}_T$. It is not hard to see that all the surfaces of constant energy on $\mathcal{O}_T$ are compact (This distinguishes $\mathcal{O}_T$ from other $(B^0_{\mathscr{W}})^-$-orbits in the same graded subspace of $\mathfrak{b}^*_{\mathscr{W}}$.) ■

Remark. The proof of Proposition 13 does not extend to the present case because a direct analogue of B. Kostant's theorem [9, Th. 1.2] fails here.

The factorization problem which solves the reduced equations of motion is a direct analogue of the matrix Riemann problem used by V.E. Zakharov and A.B. Shabat in their study of "Lax equations with spectral parameter" [19]. It would be interesting to obtain explicit formulae for the trajectories by solving this problem directly (cf. [11, 12])[4].

Appendix 1

Here we consider Kac-Moody algebras of height 2 which give rise to some interesting integrable Hamiltonian systems. In particular, we discuss the rotation of an n-dimensional top and the movement of a point on various flag-manifolds in a quadratic potential. We use the notation of n° 6.

Let θ be Cartan automorphism of a semisimple Lie algebra $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding Cartan decomposition. Extend θ to $\tilde{\mathfrak{g}}$ by putting $\tilde{\theta}(x \otimes z^n) = (-1)^n \theta(x) \otimes z^n$ and let $\mathfrak{g}_\theta$ denote the fixed subalgebra of the involution $\tilde{\theta}$:

$$\mathfrak{g}_\theta = \{\textstyle\sum x_k z^k : x_{2k} \in \mathfrak{k}, x_{2k+1} \in \mathfrak{p}\}.$$

Put

$$\mathfrak{g}^\pm_\theta = \mathfrak{g}_\pm \cap \mathfrak{g}_\theta, \qquad \mathfrak{g}^{\pm n}_\theta = \mathfrak{g}^n_\pm \cap \mathfrak{g}_\theta, \quad n > 0.$$

The restriction to $\mathfrak{g}_\theta$ of the invariant bilinear form on $\tilde{\mathfrak{g}}$ being nondegenerate, one may identify $\mathfrak{g}^*_\theta$ with $\mathfrak{g}_\theta$ and $(\mathfrak{g}^-_\theta)^*$ with $\mathfrak{g}^+_\theta$ (Here again we are dealing with the finite duals of our Lie algebras.) We are interested in dynamical systems on the orbits of $\mathfrak{g}^-_\theta$. As a shift vector (Theorem 5′) we take $f = J z^{-1}$, $J \in \mathfrak{p}$. We consider some particular cases.

1. Orbits in the subspace $\mathfrak{k} \subset \mathfrak{g}^+_\theta$ coincide with those of K. Theorem 5′ provides a family of Poisson commuting functions on $\mathfrak{k}$:

$$p_{k,f}(\pi) = \operatorname{Res}[z^{k-1} p(\pi + J z^{-1})]_{z=0}, \qquad p \in I(\mathfrak{g}), \ \pi \in \mathfrak{k}.$$

[4] An exhaustive treatment of the finite-dimensional factorization problem for the case of (nonperiodic) generalized Toda lattice is given by B. Kostant (see footnote 2)

This family was pointed out by Mishchenko and Fomenko in a different setting [13]. The corresponding equations of motion may be written in the Lax form $\frac{d}{dt}L=[L,M]$, where $L=\pi+Jz^{-1}$, $M=[z^k \operatorname{grad} p(\pi+Jz^{-1})]_+$ ($+$ denotes projection on $\mathfrak{g}_\theta^1$ along $\mathfrak{g}_\theta^-$).

It turns out ([13]) that the Euler equations of the n-dimensional top fit into this framework. For the sake of completeness we write down the corresponding formulae.

Assume $\mathfrak{g}=sl(n,\mathbf{R})$, $\mathfrak{k}=so(n)$. Let $I\in\mathfrak{p}$ be a symmetric positively definite matrix (the inertia tensor of the top). Let Ω: $\mathfrak{k}\to\mathfrak{k}$ be the inverse of $\pi\mapsto I\pi+\pi I$.

The Euler equations of the top on the orbits in are generated by the Hamiltonian

$$H(\pi)=-\tfrac{1}{2}\operatorname{tr}\pi\Omega(\pi).$$

Now put $f=I^2z^{-1}$, $p^m(x)=-\frac{1}{m}\operatorname{tr}x^m$, $m>0$. Functions $p^m_{k,f}$ Poisson commute; it turns out that H is a linear combination of the functions $H_k=p^{k+1}_{k-1,f}$ which are quadratic in π. To check this we may assume I to be diagonal, $I=\operatorname{diag}(a_1,\dots,a_n)$. One finds

$$H(\pi)=\sum_{i<j}\frac{\pi_{ij}^2}{a_i+a_j},\qquad H_k(\pi)=\sum_{i<j}\frac{a_i^{2k}-a_j^{2k}}{a_i^2-a_j^2}\pi_{ij}^2.$$

Setting $H=\sum_{k=1}^n c_k H_k$ one gets for c_k the following linear system

$$\sum_{k=1}^n c_k a_i^{2k}=a_i \qquad i=1,\dots,n$$

which is always solvable. The Lax operator M is

$$M=-\left[\sum_{k=1}^n c_k z^{k-1}(\pi+I^2z^{-1})^k\right]_+.$$

2. Now consider the orbits of $\mathfrak{g}_\theta^-$ in the space $\mathfrak{k}+\mathfrak{p}z$. Actually these are the orbits of $\mathfrak{g}_1=\mathfrak{g}_\theta^-/\mathfrak{g}_\theta^{-2}$. Denote by K_J the stationary subgroup of J with respect to the action of K in $\mathfrak{p}$. Let r: $\mathfrak{k}^*\to\mathfrak{k}_J^*$ be the natural projection. Consider the reduction of T^*K with respect to the left action of K_J. Fix $\pi\in\mathfrak{k}$ and let M_π be the inverse image of $\bar{\pi}=r(\pi)$ under the momentum map, Φ_J: $T^*K\to\mathfrak{k}_J^*$. Let $\overline{M}_\pi$ be the quotient manifold, $\overline{M}_\pi=(K_J)_{\bar{\pi}}\backslash M_\pi$.

Lemma. *The orbit $\mathcal{O}_{\pi+Jz}$ passing through $\pi+Jz$ in $\mathfrak{k}+\mathfrak{p}z$ is isomorphic to $\overline{M}_\pi$ as a symplectic K-manifold.*

Proof. Consider the mapping Φ: $T^*K\to\mathfrak{g}_1^*$, $\Phi(k,\xi)=\operatorname{Ad}k^{-1}(\xi+Jz)$. It is not hard to see that Φ may be regarded as the momentum map of a Hamiltonian action of $\mathfrak{g}_1$ on T^*K. The left action of K_J leaves Φ invariant, so Φ is well defined on the quotient manifold $\overline{M}_\pi$ and $\Phi(M_\pi)$ is G_1-invariant. Clearly, $M_\pi = K \times (\pi + \mathfrak{k}_J^\perp)$ and hence $\Phi(M_\pi) = \operatorname{Ad} K(\pi + \mathfrak{k}_J^\perp + Jz)$. But $\pi + \mathfrak{k}_J^\perp + Jz \subset \mathcal{O}_{\pi + Jz}$ whence $\Phi(M_\pi) = \mathcal{O}_{\pi + Jz}$. Moreover, the fibers of the mapping Φ: $M_\pi \to \mathcal{O}_{\pi + Jz}$ are precisely the $(K_J)_{\bar{\pi}}$-orbits, which accomplishes the proof.

Corollary. *The orbit $\mathcal{O}_{Jz}$ passing through Jz is isomorphic to $T^*(K/K_J)$ as a symplectic K-manifold.*

The orbits $\mathcal{O}_{\pi + Jz}$ provide another realization for the n-dimensional top related to its Lagrange description. We keep to the previous notation.

Let $\mathfrak{g} = sl(n, \mathbf{R})$, let the shift vector f be zero. On $\mathfrak{k} + \mathfrak{p} z$ we have a family of Poisson commuting functions

$$p_k^m(\pi + sz) = -\frac{1}{m} \operatorname{Res}[z^{-k-1} \operatorname{tr}(\pi + sz)^m]_{z=0}.$$

The Hamiltonian of the top on $\mathcal{O}_{\pi + Jz}$ is

$$H = \sum_{k=1}^{n} c_k(I)\, p_{k-1}^{k+1}.$$

The Lax representation is $\frac{d}{dt} L = [L, M]$, $L = \pi + sz \in \mathcal{O}_{\pi + Jz}$, $M = (\operatorname{grad} H)_+$.

Clearly, $\operatorname{grad} p_{k-1}^{k+1} = z^{-k-1}(\pi + sz)^k$, so that $(\operatorname{grad} p_{k-1}^{k+1})_+ = s^k z$ and $\sum_{k=1}^{n} c_k s^k = s^{1/2}$. Finally, the top equation takes the form

$$\frac{d}{dt}(\pi + sz) = [\pi + sz, s^{1/2} z].$$

Let us point out the connection between Lagrange's and Euler's description of the top. Both are obtained by reduction of the same dynamical system on T^*K, the former with respect to the left K_J-action and the latter with respect to the right K-action. Using this relation, one may transform the above Lax representation into one which refers to Euler equations:

$$\frac{d}{dt}(\pi + I^2 z) = [\pi + I^2 z, \Omega(\pi) + Iz].$$

This representation was found by Manakov [12].

3. Now consider a family of Poisson commuting polynomials on $\mathcal{O}_{Jz} \simeq T^*(K/K_J)$

$$p_f(\pi + sz) = p(\pi + sz - Az^{-1}), \qquad p \in I(\tilde{\mathfrak{g}}).$$

This family contains the Hamiltonian

$$H = -\tfrac{1}{2} \operatorname{tr} \pi^2 + \operatorname{tr} sA$$

which describes a particle on the Riemannian manifold K/K_J moving in the potential $Q=\operatorname{tr} sA$. Let A, J be diagonal, $A=\operatorname{diag}(a_1,\ldots,a_n)$, $J=\operatorname{diag}(b_1,\ldots,b_n)$. If $s=kJk^{-1}$, then $s_{ij}=\sum_m b_m k_{im} k_{jm}$. In the co-ordinates k_{ij} the potential Q is quadratic

$$Q=\sum_{i,j} a_i b_j k_{ij}^2.$$

Thus we have obtained conservation laws for motion on various flag-manifolds in a quadratic potential.

In particular, if $J=\operatorname{diag}(1,0,\ldots,0)$, then $K/K_J=RP^{n-1}$. Consider the polynomials

$$\begin{aligned} H_k &= \frac{(-1)^k}{k+1} \operatorname{Res}[z^{-k} \operatorname{tr}(\pi+sz-Az^{-1})^{k+1}]_{z=0} \\ &= H_k(\pi, A) + \operatorname{tr} sA^k. \end{aligned}$$

Suppose all the eigenvalues of A are distinct. Choose c_{ik} so that $\sum_{k=1}^{n} c_{ik} a_j^k = \delta_{ij}$. Then $F_i = \sum_{k=1}^{n} c_{ik} H_k$ are the integrals discussed by Moser [15]:

$$F_i = \sum_{j \neq i} \frac{\pi_{ij}^2}{a_i - a_j} + k_{1i}^2.$$

Appendix 2

We suggest a definition of the quantum integrals of motion for quotient systems associated with Kac-Moody algebras. Unfortunately, we failed to prove their commutativity, though there is some evidence in favour of such a conjecture.

The main difficulty stems from the fact that "central elements" corresponding to invariant polynomials $p_n \in I(\tilde{\mathfrak{g}}^*)$ are not contained in $U(\tilde{\mathfrak{g}})$. To make the definition more clear we first rewrite classical integrals of motion in terms of the symmetric algebra of $\tilde{\mathfrak{g}}$.

Let $S(\tilde{\mathfrak{g}})$ be the symmetric algebra of $\tilde{\mathfrak{g}}$. Let S^N be a subspace of $S(\tilde{\mathfrak{g}})$ spanned by those $u=x_1 z^{k_1} \ldots x_s z^{k_s} \in S(\tilde{\mathfrak{g}})$ for which for some i $|k_i|>N$. Put $\hat{S}(\tilde{\mathfrak{g}}) = \lim \operatorname{proj} S(\tilde{\mathfrak{g}})/S^k$. The elements of $\hat{S}(\tilde{\mathfrak{g}})$ define polynomial functions on $\tilde{\mathfrak{g}}^*$. Let the mapping i_n: $S(\mathfrak{g}) \to \hat{S}(\tilde{\mathfrak{g}})$ be defined by

$$i_n(x_1 \ldots x_s) = \sum_{k_1+\ldots+k_s=n} x_1 z^{k_1} \ldots x_s z^{k_s}.$$

Recall that $p_n \in \hat{S}(\tilde{\mathfrak{g}})$ is defined by $p_n(\xi) = \operatorname{Res}[z^{-n} p(\xi(z))]_{z=0}$, $p \in S(\mathfrak{g})$, $\xi \in \tilde{\mathfrak{g}}^*$.

Lemma. $i_n p = p_n$.

Suppose that $\mathfrak{g}$ is split into a linear sum of two its subalgebras $\mathfrak{g}=\mathfrak{a}+\mathfrak{b}$ and let $\tilde{\mathfrak{g}}=\mathfrak{a}_+ + \mathfrak{b}_-$ be the decomposition defined in n° 6. Let of be a character of $\mathfrak{a}_+$

and let $\rho_f\colon U(\tilde{\mathfrak{g}})\to U(\mathfrak{b}_-)$ be defined as in n° 5. Let $x(k_1,\dots,k_s)=\sum\limits_{\sigma} x_{\sigma_1} z^{k_1}\dots x_{\sigma_s} z^{k_s}$ be the sum over all permutations regarded as an element of $U(\tilde{\mathfrak{g}})$.

Lemma. *Suppose that* $f|_{\mathfrak{g}_-^N}=0$ *for some* $N>0$. *If for some* i $k_i<-N$, *then* $\rho_f(x(k_1,\dots,k_s))=0$.

We set $i_n^N(x_1\dots x_s)=\sum\limits_{\substack{k_1+\dots+k_s=n\\ |k_i|<N}} x(k_1,\dots,k_s)$ and regard i_n^N as a map of $S(\mathfrak{g})$ into $U(\tilde{\mathfrak{g}})$. Due to the lemma the following definition makes sense.

Definition. $\rho_{f,n}(u)=\lim\limits_{N\to\infty}\rho_f(i_n^N(u))$, $u\in S(\mathfrak{g})$.

It is easy to prove that $[\rho_{f,0}(u_1),\rho_{f,0}(u_2)]=0$ for all u_1, $u_2\in S(\mathfrak{g})^G$. The authors suppose that also

$$[\rho_{f,m}(u_1),\rho_{f,n}(u_2)]=0$$

for all m, $n\in\mathbf{Z}$.

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