Numer. Math. 50, 363-376 (1987)

Numerische Mathematik

Spline Finite Difference Methods for Singular Two Point Boundary Value Problems

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Summary. In this paper we discuss the construction of a spline function for a class of singular two-point boundary value problem $x^{-\alpha}(x^\alpha u')' = f(x, u)$, $u(0) = A$, $u(1) = B$, $0 < \alpha < 1$ or $\alpha = 1, 2$. The boundary conditions may also be of the form $u'(0) = 0$, $u(1) = B$. Three point finite difference methods, using the above splines, are obtained for the solution of the boundary value problem. These methods are of second order and are illustrated by four numerical examples.

Subject Classifications: AMS(MOS): 65L10; CR: Gl.7.

1. Introduction

Consider the class of singular two-point boundary value problem

$$x^{-\alpha}(x^\alpha u')' = f(x, u), \qquad 0 < x \leqq 1, \tag{1}$$

$$u(0) = A, \qquad u(1) = B \tag{1a}$$

or

$$u'(0) = 0, \qquad u(1) = B. \tag{1b}$$

Here, $\alpha \in (0, 1)$ or it may take values 1 or 2. If $\alpha = 1$ then (1) becomes a cylindrical problem and if $\alpha = 2$, then it becomes a spherical problem. A and B are finite constants. It is well known that (1) has a unique solution if (A) $f(x, u)$ is continuous, $\partial f/\partial u$ exists and is continuous and $\partial f/\partial u \geqq 0$. For $\alpha = 1$, Russell and Shampine [9] have shown that for linear $f(x, u) = au + g(x)$: $g \in C[0, 1]$, (1) possesses unique solution if $-\infty < a < J_0^2$, where J_0 is the smallest positive zero of the Bessel's function $J_0(x)$. For $\alpha \in (0, 1)$ and in the linear case Jamet [5] considered a standard three point finite difference scheme with uniform mesh, for the solution and has shown that the error is $O(h^{1-\alpha})$ in maximum norm. Ciarlet [2] considered the application of Rayleigh-Ritz-Galerkin method and have shown that the error is $O(h^{2-\alpha})$ in uniform norm.

Gustafsson [3] considered the linear problem in $(\delta, 1)$ instead of $(0,1)$ and constructed compact second order, fourth order and non-compact fourth order methods for its solution. Reddien [7] and Reddien and Schumaker [8] used certain projection methods and singular splines to solve the linear problem and studied the existence, uniqueness and convergence rates of these methods. Recently [1] have proposed three point difference methods of second order under appropriate conditions using the boundedness of f', f'' and f'''.

For $\alpha=1$, Russell and Shampine [9] wrote the differential equation in the form $(xu')'+xf(x,u)=0$ and considered the discretization

$$-x_{k-1/2}u_{k-1}+2x_k u_k-x_{k+1/2}u_{k+1}-h^2x_k f_k+T_k(h)=0$$

where $T_k(h)$ is $O(h^4)$. The case $\alpha=1$ is often encountered in electro hydrodynamics and the theory of thermal explosions.

In the present paper, in Sect. 2 we construct splines and the three point finite difference methods using these splines for the solution of (1) in the cases $\alpha\in(0,1)$ and $\alpha=1,2$. In Sect. 3, we show that these schemes are of $O(h^2)$ under appropriate conditions. The advantage of the spline approximation is that (1) may be solved with a particular steplength h and the intermediate values, if required, can be computed using the spline. It is shown through numerical computations that these spline solutions are of the same accuracy as the two neighbouring finite difference solutions. In Sect. 4, the three point methods are applied on four examples. Some solutions obtained by the spline approximation are also given. These numerical results show that the methods are robust and the spline gives good approximation at the intermediate points.

2. Splines and Finite Difference Methods

Consider first the case $\alpha\in(0,1)$. We consider a general non-uniform mesh $0=x_0<x_1<x_2<\ldots<x_N=1$. Denote $u_j=u(x_j)$ and $f_j=f(x_j,u_j)$ etc. We write

$$x^{-\alpha}\frac{d}{dx}\left(x^\alpha\frac{du}{dx}\right)=\frac{M_{j-1}}{h_j}(x_j-x)+\frac{M_j}{h_j}(x-x_{j-1}),\qquad x_{j-1}<x<x_j \tag{2}$$

where $h_j=x_j-x_{j-1}$. It is obvious that

$$[x^{-\alpha}(x^\alpha u')']_{x_j}=M_j=f_j$$

and

$$[x^{-\alpha}(x^\alpha u')']_{x_{j-1}}=M_{j-1}=f_{j-1}$$

Integrating (2) twice and setting the interpolating conditions $u(x_{j-1})=u_{j-1}$ and $u(x_j)=u_j$ we get the spline approximation as

$$\begin{aligned}u(x)=&-\frac{S_j}{a}[(u_jx^a_{j-1}-u_{j-1}x^a_j)-x^a(u_j-u_{j-1})]\\&+\left[S^*_jx^2\{2x(1+\alpha)-3(2+\alpha)x_{j-1}\}+\frac{a_j}{a}x^a+a^*_j\right]M_j\\&+\left[S^*_jx^2\{3(2+\alpha)x_j-2x(1+\alpha)\}+\frac{b_j}{a}x^a+b^*_j\right]M_{j-1},\qquad x_{j-1}<x<x_j\end{aligned} \tag{3}$$

where

$$a=1-\alpha$$

$$S_j=\frac{a}{x_j^a-x_{j-1}^a},\qquad S_j^*=\frac{1}{6h_j(1+\alpha)(2+\alpha)}$$

$$a_j=-S_jS_j^*[\alpha x_j^2(2x_j-3x_{j-1})+2x_j^2(x_j-3x_{j-1})+x_{j-1}^3(\alpha+4)]$$

$$b_j=-S_jS_j^*[(\alpha+4)x_j^3-\alpha x_{j-1}^2(3x_j-2x_{j-1})-2x_{j-1}^2(3x_j-x_{j-1})]$$

$$a_j^*=\frac{S_jS_j^*}{a}[x_j^2x_{j-1}^a\{2(1+\alpha)x_j-3(2+\alpha)x_{j-1}\}+(\alpha+4)x_{j-1}^3x_j^a]$$

$$b_j^*=\frac{S_jS_j^*}{a}[(\alpha+4)x_j^3x_{j-1}^a-x_j^ax_{j-1}^2\{3(2+\alpha)x_j-2(1+\alpha)x_{j-1}\}].$$

Setting $j=j+1$ in (3) we get the spline valid in the interval (x_j, x_{j+1}). If we now require that $u'(x)$ be continuous at the node x_j we get

$$-S_ju_{j-1}+(S_j+S_{j+1})u_j-S_{j+1}u_{j+1}=A_jM_{j+1}+B_jM_j+C_jM_{j-1},\qquad j=1,2,\dots,N-1 \tag{4}$$

where

$$A_j=-6S_{j+1}^*x_j^{2+\alpha}+a_{j+1},\qquad C_j=-6S_j^*x_j^{2+\alpha}-b_j,$$

$$B_j=6x_j^{1+\alpha}[(2+\alpha)(x_{j+1}S_{j+1}^*+x_{j-1}S_j^*)-(1+\alpha)x_j(S_{j+1}^*+S_j^*)]+(b_{j+1}-a_j).$$

Setting $M_{j-1}=f_{j-1}$ etc., we have the three-point finite difference approximation

$$-S_ju_{j-1}+(S_j+S_{j+1})u_j-S_{j+1}u_{j+1}=A_jf_{j+1}+B_jf_j+C_jf_{j-1},\qquad j=1,2,\dots,N-1. \tag{5}$$

Consider now a non-uniform mesh by considering the mesh ratio parameter $\sigma_j=h_{j+1}/h_j$. We find $x_{j-1}=x_j-h_j$, $x_{j+1}=x_j+h_{j+1}=x_j+\sigma_jh_j$. When $\sigma_j=1$, it reduces to the uniform mesh case. This non-uniform mesh has been successfully used by Jain, Iyengar and Subramanyam [4] in solving two point singular perturbation boundary value problems. Substituting the expressions for x_{j-1}, x_{j+1} and expanding in Taylor's series, we get

$$S_j=\frac{x_j^\alpha}{h_j}[1+p_1x'+p_2(x')^2+p_3(x')^3+p_4(x')^4+\dots]$$

$$S_{j+1}=\frac{x_j^\alpha}{h_j}[1-p_1x^*+p_2(x^*)^2-p_3(x^*)^3+p_4(x^*)^4-\dots]$$

$$A_j=-\frac{1}{6}x_j^\alpha\sigma_jh_j\left[1+\frac{\alpha}{4}x^*-\frac{\alpha(2-\alpha)}{120}(x^*)^2+\dots\right] \tag{6a}$$

$$C_j=-\frac{1}{6}x_j^\alpha h_j\left[1-\frac{\alpha}{4}x'-\frac{\alpha(2-\alpha)}{120}(x')^2+\dots\right]$$

$$B_j=-\frac{1}{6}x_j^\alpha(1+\sigma_j)h_j\left[2-\frac{\alpha}{4}(1-\sigma_j)x'+\frac{(1-\sigma_j+\sigma_j^2)}{120}\alpha(2-\alpha)(x')^2+\dots\right]$$

where $p_1=-\frac{\alpha}{2}$, $p_2=-\alpha(2-\alpha)/12$, $p_3=-\alpha(2-\alpha)/24$, $p_4=-\alpha(2-\alpha)(18+2\alpha-\alpha^2)/720$, $x'=h_j/x_j$, $x^*=\sigma_j h_j/x_j$.

Substituting (6a) in (5) and simplifying, we find the truncation error in the difference method (5) to be

$$t_3^{(1)}(h_j)=\frac{1}{24}x_j^{\alpha}(1+\sigma_j^3)h_j^3\left[u_j^{(4)}+\frac{\alpha}{x_j}\left(u_j'''-\frac{2}{x_j}u_j''+\frac{2}{x_j^2}u_j'\right)\right]+\dots$$
$$=\frac{1}{24}x_j^{\alpha}(1+\sigma_j^3)h_j^3 f_j''+\dots. \tag{6b}$$

Substituting $\sigma_j=1$ in (6b), we get the truncation error in (5), in the case of *uniform mesh*, as

$$t_3^{(1)}(h)=\frac{1}{12}x_j^{\alpha}h^3 f_j''+\dots. \tag{6c}$$

When $\alpha=0$, the method (5) reduces to

$$u_{j-1}-2u_j+u_{j+1}=\frac{h^2}{6}(f_{j-1}+4f_j+f_{j+1}) \tag{7}$$

which is same as the scheme obtained by a cubic spline for $u''=f(x,u)$.

Cylindrical case. For $\alpha=1$, we write (1) in the form

$$\frac{1}{r}(ru')'=f(r,u). \tag{8}$$

Write

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)=\frac{M_{j-1}}{h_j}(r_j-r)+\frac{M_j}{h_j}(r-r_{j-1}),\qquad r_{j-1}<r<r_j$$

where $h_j=r_j-r_{j-1}$. Integrating twice, we get

$$u(r)=\frac{M_{j-1}}{h_j}\left(\frac{1}{4}r^2r_j-\frac{r^3}{9}\right)+\frac{M_j}{h_j}\left(\frac{r^3}{9}-\frac{1}{4}r^2r_{j-1}\right)$$
$$+C_j\log r+D_j \tag{9}$$

where C_j and D_j are arbitrary constants to be determined. In the interval $r_0<r<r_1$, finiteness at the origin requires $C_1=0$ and $D_1=u(0)=u_0$. For the remaining intervals, using the interpolating conditions $u(r_{j-1})=u_{j-1}$ and $u(r_j)=u_j$ we get

$$u(r)=S_j[(u_j-u_{j-1})\log r+u_{j-1}\log r_j-u_j\log r_{j-1}]$$
$$+\frac{M_{j-1}}{36h_j}[r^2(9r_j-4r)-S_j\log r(r_j-r_{j-1})(5r_j^2+5r_jr_{j-1}-4r_{j-1}^2)$$
$$-S_j\{r_{j-1}^2(9r_j-4r_{j-1})\log r_j-5r_j^3\log r_{j-1}\}]$$
$$+\frac{M_j}{36h_j}[r^2(4r-9r_{j-1})-S_j\log r(r_j-r_{j-1})(4r_j^2-5r_jr_{j-1}-5r_{j-1}^2)$$
$$+S_j\{5r_{j-1}^3\log r_j+r_j^2(4r_j-9r_{j-1})\log r_{j-1}\}],\qquad r_{j-1}<r<r_j \tag{10}$$

where $S_j = 1/\log(r_j/r_{j-1})$. Setting $j=j+1$ in (10) we get the spline valid in (r_j, r_{j+1}). If we now require that $u'(r)$ be continuous at the node r_j we obtain

$$-S_j u_{j-1} + (S_j + S_{j+1})u_j - S_{j+1}u_{j+1} = A_j^* M_{j+1} + B_j^* M_j + C_j^* M_{j-1},$$
$$j = 2, 3, \ldots, N-1 \tag{11}$$

where

$$C_j^* = -\frac{1}{6h_j}\left[r_j^3 - \frac{S_j}{6}(5r_j^3 - 9r_j r_{j-1}^2 + 4r_{j-1}^3)\right],$$

$$A_j^* = -\frac{1}{6h_{j+1}}\left[r_j^3 + \frac{S_{j+1}}{6}(4r_{j+1}^3 - 9r_{j+1}^2 r_j + 5r_j^3)\right],$$

$$B_j^* = \frac{1}{6h_{j+1}}\left[r_j^2(3r_{j+1} - 2r_j) - \frac{S_{j+1}}{6}(5r_{j+1}^3 - 9r_{j+1}r_j^2 + 4r_j^3)\right]$$
$$-\frac{1}{6h_j}\left[r_j^2(2r_j - 3r_{j-1}) - \frac{S_j}{6}(4r_j^3 - 9r_j^2 r_{j-1} + 5r_{j-1}^3)\right].$$

Setting $M_{j-1} = f_{j-1}$ etc., we get the three-point finite difference approximation

$$-S_j u_{j-1} + (S_j + S_{j+1})u_j - S_{j+1}u_{j+1} = A_j^* f_{j-1} + B_j^* f_j + C_j^* f_{j-1},$$
$$j = 2, 3, \ldots, N-1. \tag{12}$$

Expanding in Taylor series, the truncation error in the case of uniform mesh, is obtained as

$$t_j^{(2)}(h) = \frac{h^3 r_j}{12}\left[u_j^{(4)} + \frac{u_j'''}{r_j} - \frac{2u_j''}{r_j^2} + \frac{2u_j'}{r_j^3}\right] + \ldots$$
$$= \frac{h^3}{12} r_j f_j'' + \ldots. \tag{13}$$

The difference scheme (12) cannot be used at $j=1$ as S_1 is not defined. In the case of uniform mesh, the following interpolating approximation may be used.

$$-\frac{19}{68}u_0 + u_1 - \frac{49}{68}u_2 = -\frac{h^2}{17}(7f_1 + f_2). \tag{14}$$

The truncation error at $j=1$ is

$$t_1^{(2)}(h) = -\frac{1}{136}h^4 u^{(4)}(r_1) + \ldots. \tag{15}$$

Spherical case. For $\alpha = 2$, we write (1) in the form

$$\frac{1}{r^2}(r^2 u')' = f(r, u). \tag{16}$$

Following the above procedure, we get

$$u(r) = \frac{M_{j-1}}{h_j}\left(\frac{r^2}{6}r_j - \frac{r^3}{12}\right) + \frac{M_j}{h_j}\left(\frac{r^3}{12} - \frac{r^2}{6}r_{j+1}\right) - \frac{C_j}{r} + D_j, \quad r_{j-1} < r < r_j \tag{17}$$

where C_j and D_j are arbitrary constants to be determined from the interpolatory conditions. In the interval $r_0<r<r_1$, finiteness at the origin requires $C_1=0$, $D_1=u(0)=u_0$. For the remaining intervals, interpolating conditions give the spline as

$$
\begin{aligned}
u(r)=&\frac{1}{h_j}(u_jr_j-u_{j-1}r_{j-1})-\frac{S_j}{r}(u_j-u_{j-1})+\frac{M_{j-1}}{12h_j}\Big[r^2(2r_j-r)\\
&+\frac{S_j}{r}(r_j-r_{j-1})(r_j^2+r_jr_{j-1}-r_{j-1}^2)-\frac{1}{h_j}(r_j^4-2r_jr_{j-1}^3+r_{j-1}^4)\Big]\\
&+\frac{M_j}{12h_j}\Big[r^2(r-2r_{j-1})+\frac{S_j}{r}(r_j-r_{j-1})(r_j^2-r_jr_{j-1}-r_{j-1}^2)\\
&-\frac{1}{h_j}(r_j^4-2r_j^3r_{j-1}+r_{j-1}^4)\Big],\qquad r_{j-1}<r<r_j
\end{aligned}
\tag{18}
$$

where

$$S_j=\frac{r_jr_{j-1}}{h_j}\quad\text{and}\quad h_j=r_j-r_{j-1}.$$

Setting $j=j+1$ in (18) we get the spline valid in the interval (r_j, r_{j+1}). Requiring that $u'(r)$ be continuous at r_j we get the difference method

$$
\begin{aligned}
r_{j-1}u_{j-1}-2r_ju_j+r_{j+1}u_{j+1}=&\frac{h^2}{12}[(r_j+r_{j-1})f_{j-1}+8r_jf_j\\
&+(r_j+r_{j+1})f_{j+1}],\qquad j=1,2,\ldots,N-1.
\end{aligned}
\tag{19}
$$

In the case of uniform mesh, the truncation error in (19) is obtained as

$$t_j^{(3)}(h)=-\frac{h^4}{12}r_jf_j''+\ldots. \tag{20}$$

3. Convergence of the Spline Difference Methods

All the three difference schemes (5), (12) and (19) can be written in the form

$$\mathbf{SU}+\mathbf{Mf}+\mathbf{T}=\mathbf{R} \tag{21}$$

where $\mathbf{S}$ and $\mathbf{M}$ are suitably defined and

$$
\begin{aligned}
\mathbf{U}&=[u_1u_2\ldots u_{N-1}]^T, & \mathbf{T}&=[t_1t_2\ldots t_{N-1}]^T,\\
\mathbf{f}&=[f_1f_2\ldots f_{N-1}]^T, & \mathbf{R}&=[(S_1A+C_1f_0)\,0\ldots0\,(S_NB+A_Nf_N)]^T.
\end{aligned}
$$

For the scheme (5) we have

$$\mathbf{S}=\begin{bmatrix} S_1+S_2 & -S_2 & & \mathbf{0} \\ -S_2 & S_2+S_3 & -S_3 & \\ & -S_3 & S_3+S_4 & -S_4 \\ & \vdots & \vdots & \vdots \\ \mathbf{0} & & -S_{N-1} & S_{N-1}+S_N \end{bmatrix},$$

$$\mathbf{M}=\begin{bmatrix} -B_1 & -A_1 & & \mathbf{0} \\ -C_2 & -B_2 & -A_2 & \\ & -C_3 & -B_3 & -A_3 \\ & \vdots & \vdots & \vdots \\ \mathbf{0} & & -C_{N-1} & -B_{N-1} \end{bmatrix},$$

while for the scheme (19) we have

$$\mathbf{S}=\begin{bmatrix} 2r_1 & -r_2 & & \mathbf{0} \\ -r_1 & 2r_2 & -r_3 & \\ & -r_2 & 2r_3 & -r_4 \\ & \vdots & \vdots & \vdots \\ \mathbf{0} & & -r_{N-2} & 2r_{N-1} \end{bmatrix},$$

$$\mathbf{M}=\frac{h^2}{12}\begin{bmatrix} 8r_1 & r_1+r_2 & & \mathbf{0} \\ r_1+r_2 & 8r_2 & r_2+r_3 & \\ & r_2+r_3 & 8r_3 & r_3+r_4 \\ & \vdots & \vdots & \vdots \\ \mathbf{0} & & r_{N-2}+r_{N-1} & 8r_{N-1} \end{bmatrix}.$$

For the difference scheme (12), $\mathbf{M}$ is defined by the starred quantities. We note that

$$S_j>0, \quad A_j<0, \quad B_j<0, \quad C_j<0,$$
$$A_j^*<0, \quad B_j^*<0, \quad C_j^*<0.$$

Hence we have, $\mathbf{M}>\mathbf{0}$ in all the cases. Dropping the truncation error in (21) we get

$$\mathbf{S}\tilde{\mathbf{U}}+\mathbf{M}\mathbf{f}(\tilde{\mathbf{U}})=\mathbf{R} \tag{22}$$

where $\tilde{\mathbf{U}}$ denotes the numerical solution.

Subtracting (21) from (22), we have

$$(\mathbf{S}+\mathbf{MF})\mathbf{E}=\mathbf{T} \tag{23}$$

where $\mathbf{E}=\tilde{\mathbf{U}}-\mathbf{U}$ and $\mathbf{FE}=\mathbf{f}(\tilde{\mathbf{U}})-\mathbf{f}(\mathbf{U})$ and $\mathbf{F}=\operatorname{diag}\left[\frac{\partial f_1}{\partial u_1}\frac{\partial f_2}{\partial u_2}\cdots\frac{\partial f_{N-1}}{\partial u_{N-1}}\right]$. Note that we have assumed $\frac{\partial f}{\partial u}\geqq 0$. Hence $\mathbf{F}\geqq\mathbf{0}$ and $\mathbf{MF}\geqq\mathbf{0}$. We have

$$\mathbf{S}+\mathbf{MF}\geqq\mathbf{S}. \tag{24}$$

$\mathbf{S}$ is irreducible and monotone and $\mathbf{MF} \geqq \mathbf{0}$. Therefore,

$$(\mathbf{S}+\mathbf{MF})^{-1} \leqq \mathbf{S}^{-1}.$$

From (23), we find

$$\|\mathbf{E}\| = \|(\mathbf{S}+\mathbf{MF})^{-1}\mathbf{T}\| \leqq \|\mathbf{S}^{-1}\| \, \|\mathbf{T}\|. \tag{25}$$

Consider now the spline difference scheme (5). Denote $\mathbf{S}^{-1}=(S^*_{i,j})$, $a=1-\alpha$. We find for $j=1,2,\dots,i-1$

$$\begin{aligned}
S^*_{i,j} &= \left(\frac{x_i}{x_1}\right)^a S^*_{1,j} - \frac{1}{a}(x_i^a - x_j^a), \qquad i=1(1)N-1,\ i \geqq j \\
S^*_{1,j} &= -\frac{1}{aD}\left[(x_N^a - x_{N-1}^a)(x_{N-2}^a - x_j^a) - (x_N^a - x_{N-2}^a)(x_{N-1}^a - x_j^a)\right] \\
D &= \left(\frac{x_{N-1}}{x_1}\right)^a (x_N^a - x_{N-2}^a) - \left(\frac{x_{N-2}}{x_1}\right)^a (x_N^a - x_{N-1}^a).
\end{aligned} \tag{26}$$

For $j=i, i+1, \dots, N-1$, we have

$$\begin{aligned}
S^*_{i,j} &= \left(\frac{x_i}{x_1}\right)^a S^*_{1,j}, \qquad i \leqq j \\
S^*_{1,j} &= \frac{1}{aD_1}(x_N^a - x_j^a)(x_j^a - x_{j-1}^a) \\
D_1 &= \left(\frac{x_j}{x_1}\right)^a (x_N^a - x_{j-1}^a) - \left(\frac{x_{j-1}}{x_1}\right)^a (x_N^a - x_j^a).
\end{aligned} \tag{27}$$

We find

$$\sum_{j=1}^{i-1} S^*_{1,j} = -\frac{1}{a}\left(\frac{x_1}{x_N}\right)^a \left[(x_1^a + x_2^a + \dots + x_{i-1}^a) - (i-1)\, x_N^a\right]$$

and

$$\sum_{j=i}^{N-1} S^*_{1,j} = \frac{1}{a}\left(\frac{x_1}{x_N}\right)^a \sum_{j=1}^{N-1} (x_N^a - x_j^a).$$

The row sum is obtained as

$$\sum_{j=1}^{N-1} S^*_{i,j} = \frac{1}{a}\left[(N-i)x_i^a + \sum_{j=1}^{i-1} x_j^a - \left(\frac{x_i}{x_N}\right)^a \sum_{j=1}^{N-1} x_j^a\right] \tag{28}$$

where we have used $x_N=1$.

Consider now a uniform mesh $x_j=jh$. Then from (28), we have

$$\sum_{j=1}^{N-1} S^*_{i,j} = \frac{1}{a}\left[(N-i+1)x_i^a + \sum_{j=1}^{i-1} x_j^a - \left(\frac{x_i}{x_N}\right)^a \sum_{j=1}^{N} x_j^a\right]. \tag{29}$$

It is known that

$$1^a+2^a+\ldots+(i-1)^a<\frac{i^{a+1}}{a+1} \tag{30}$$

and

$$1^a+2^a+\ldots+N^a>\frac{N^{a+1}}{a+1}. \tag{31}$$

Hence from (29)

$$\sum_{j=1}^{N-1} S^*_{i,j}<\frac{1}{a}\left[(N-i+1)x_i^a+\frac{h^a i^{a+1}}{a+1}-\left(\frac{x_i}{x_N}\right)^a\frac{h^a N^{a+1}}{a+1}\right] \tag{32}$$

$$<\frac{t^{a+1}}{a(a+1)h} \tag{33}$$

where t is the point at which maximum occurs for the expression on the right hand side of (32) and is given by

$$t=[a+(a+1)h]/a+1.$$

Hence

$$\begin{aligned}\|E\| &<\frac{t^{a+1}}{a(a+1)h}\|t_j\| \\ &\leqq\frac{t^{a+1}}{a(a+1)h}\frac{h^3x_j^\alpha}{12}|f''| \\ &\leqq\left(\frac{a}{a+1}\right)^{a+1}\frac{h^2M}{a(a+1)}\end{aligned} \tag{34}$$

where $x^\alpha|f''|\leqq M$. We have

Theorem 1. *Assume that f satisfies (A) and $x^\alpha|f''|\leqq M$. Then for the spline difference scheme (5) with $x_j=jh$, we have $\|E\|=0(h^2)$ for sufficiently small h.*

Consider now the spline difference scheme (19) for uniform mesh. Write $\mathbf{S}=h\mathbf{S}^*$ *where*

$$\mathbf{S}^*=\begin{bmatrix} 2 & -2 & & & & \mathbf{0} \\ -1 & 4 & -3 & & & \\ & -2 & 6 & -4 & & \\ & \ddots & \ddots & \ddots & & \\ & -(j-1) & 2j & & -(j+1) & \\ & \ddots & \ddots & & \ddots & \\ \mathbf{0} & & & -(N-2) & & 2(N-1)\end{bmatrix}. \tag{35}$$

The inverse $\mathbf{S}^{*-1}$ of $\mathbf{S}^*$ is obtained to be

$$\begin{aligned}S^{-1}_{i,j}&=\frac{j(N-i)}{Ni}, \quad i\geqq j\\ &=\frac{N-j}{N}, \quad i\leqq j.\end{aligned} \tag{36}$$

The ith row sum is

$$\sum_{m=1}^{N-1} S_{i,m}^{-1} = \sum_{m=1}^{i} S_{i,m}^{-1} + \sum_{m=i+1}^{N-1} S_{i,m}^{-1}$$
$$= \frac{(N-i)(i+1)}{2N} + \frac{(N-i)(N-i-1)}{2N} = \frac{N-i}{2}. \tag{37}$$

Hence,

$$\|\mathbf{S}^{*-1}\| \leqq \frac{1}{2h} \tag{38}$$

and

$$\|\mathbf{S}^{-1}\| \leqq \frac{1}{2h^2}. \tag{39}$$

From (25), we get

$$\|\mathbf{E}\| \leqq \frac{1}{2h^2} \|t_j\| \leqq \frac{h^2}{24} N \tag{40}$$

where $|r_j f''| \leqq N$. We have

Theorem 2. *Assume that f satisfies (A) and let $f'' \in C\{(0,1] \times \mathbb{R}\}$ and $r_j|f''| \leqq N$. Then, for the spline difference scheme* (19) *with $r_j = jh$, we have, $\|\mathbf{E}\| = O(h^2)$, for sufficiently small h.*

4. Computational Experiments

If the left boundary condition is $u'(0)=0$, then we need an extra difference equation valid at $j=0$. In the limit, at $j=0$, the differential Eq. (1) may be written as

$$(1+\alpha)u'' = f(0, u). \tag{41}$$

A suitable approximation to (41) along with an approximation to $u'(0)=0$ may be combined to get the difference equation at $j=0$. Alternately, $j=0$ may be avoided and a suitable approximation may be written at $j=1$. For example, in the cylindrical case, to go along with the difference scheme (12), we may write in the case of uniform mesh

$$u_1 - u_2 + \frac{h^2}{4}(5f_1 - 2f_2) = 0 \tag{42}$$

with the truncation error

$$t_1^{(2)}(h) = -\frac{19}{24} h^3 u_1''' + \dots. \tag{43}$$

Equations (12) along with (42) give a $(N-1) \times (N-1)$ system of equations for the unknowns $u_1, u_2, \dots, u_{N-1}$. The solution at $r=0$ may be determined by using any difference approximation to

$$2\frac{\partial^2 u}{\partial r^2}=f(0,u) \tag{44}$$

and the computed solutions.

The difference Eq. (19) is valid for $j=1,2,\ldots,N-1$. At $j=0$, we have in the spherical case

$$3\frac{\partial^2 u}{\partial r^2}=f(0,u). \tag{45}$$

Again a suitable $O(h^2)$ approximation to (45) may be written to combine with (19). Application of the above difference schemes to (1) generally produces a nonlinear tridiagonal system of equations. This nonlinear system may be solved by Newton's iteration. The Jacobian in this case is again a tridiagonal matrix, so that one tridiagonal system is to be solved for each iteration.

We illustrate the above methods on the following boundary value problems.

Example 1. (Gustafsson [3]).

$$u''+\frac{\sigma}{x}u'=-x^{1-\sigma}\cos x-(2-\sigma)x^{-\sigma}\sin x,$$

$$u(0)=0, \qquad u(1)=\cos 1.$$

The exact solution is $u(x)=x^{1-\sigma}\cos x$.

Gustafsson [3] considered the above problem in $(\delta,1)$ instead of $(0,1)$ and constructed compact second order, compact fourth order and noncompact fourth order methods for its solution. We have solved the above problem using the method (5) for $\sigma=1/2$. The results are tabulated in Table 1. Our results are superior to the results obtained by the compact second order method of Gustafsson. The results also verify the second order convergence of (5).
The computational implementation of the non-uniform mesh using $\sigma_j=h_{j+1}/h_j$ was given in Jain [4]. We have also implemented this procedure for this problem using $N=20$, 40, 80, 160 for $\sigma_j=\text{constant}=\sigma_0$ and various values of $\sigma_0>1$. We found that these results are accurate and show the second order convergence of (5).

Example 2 (Chawla and Katti [1]).

$$(x^\alpha u')'=\beta x^{\alpha+\beta-2}[(\alpha+\beta-1)+\beta x^\beta]u, \qquad u(0)=1,\ u(1)=e.$$

The exact solution is $u(x)=\exp(x^\beta)$.

We solved this example using the method (5) for two sets of values $\alpha=0.5$, $\beta=4.0$; $\alpha=0.75$, $\beta=3.75$ with $h=2^{-k}$, $k=4(1)7$. The maximum absolute errors in the results along with the maximum absolute errors in the results obtained by the methods M_1 and M_2 of [1] are given in Table 2. The results obtained

Table 1. Maximum absolute errors. Example 1

N	Spline Scheme (5)	Gustafsson [3], $\sigma=1/2$, $\delta=0.1$	
	$\sigma=1/2$	compact second order	compact fourth order
20	2.7 (−4)	–	–
40	6.6 (−5)	7.7 (−4)	1.5 (−5)
80	1.6 (−5)	1.7 (−4)	7.2 (−5)
160	3.9 (−6)	4.0 (−5)	4.0 (−8)

Table 2. Maximum absolute errors in Example 2

N	Spline scheme (5) $\alpha=0.5$, $\beta=4.0$	Chawla and Katti [1] $\alpha=0.5$, $\beta=4.0$	
		Method M_1	Method M_3
16	1.0 (−2)	4.3 (−2)	1.2 (−2)
32	2.5 (−3)	1.1 (−2)	3.0 (−3)
64	6.2 (−4)	2.9 (−3)	7.3 (−4)
128	1.6 (−4)	7.2 (−4)	1.8 (−4)
	$\alpha=0.75$, $\beta=3.75$	$\alpha=0.75$, $\beta=3.75$	
		Method M_1	Method M_3
16	8.9 (−3)	1.4 (−1)	1.2 (−2)
32	2.2 (−3)	4.1 (−2)	2.9 (−3)
64	5.5 (−4)	1.1 (−2)	7.2 (−4)
128	1.4 (−4)	2.7 (−3)	1.8 (−4)

Table 3. Maximum absolute errors in the solution of the Bessel's equation. Example 3

N	10	20	40	80
	7.2 (−5)	2.6 (−5)	8.5 (−6)	2.3 (−6)

by our method are superior as compared to the results of [1]. The results also show the second order convergence of the method (5).

Example 3. Bessel's equation of order zero

$$(xu')'+xu=0, \qquad u'(0)=0, \qquad u(1)=1.$$

The exact solution is $u(x)=J_0(x)/J_0(1)$.

This example is solved by the method (12) and (42) with $h=1/10$, $1/20$, $1/40$ and $1/80$. The maximum absolute errors are given in Table 3.

Table 4. Maximum absolute errors in Example 4. Error tolerance $= 1.0 \times 10^{-8}$

δ \ N	8	16	32	64
-1.0	8.6 (-5)	2.0 (-5)	6.4 (-6)	1.7 (-6)
0.5	6.7 (-5)	7.1 (-6)	2.7 (-6)	7.6 (-7)
1.0	4.7 (-4)	3.1 (-5)	1.4 (-5)	4.0 (-6)

Table 5. Errors in spline solution. Example 4. $\delta = -1.0$, $h = 1/8$

$\frac{8}{32}$ [a]	$\frac{9}{32}$	$\frac{10}{32}$	$\frac{11}{32}$	$\frac{12}{32}$ [a]
1.9 (-5)	8.2 (-6)	-6.5 (-7)	-8.1 (-6)	-1.4 (-5)
$\frac{24}{32}$ [a]	$\frac{25}{32}$	$\frac{26}{32}$	$\frac{27}{32}$	$\frac{28}{32}$ [a]
-3.5 (-5)	-3.3 (-5)	-3.0 (-5)	-2.6 (-5)	-2.2 (-5)

[a] Errors in finite difference solution

Example 4 (Kubíček and Hlaváček [6]).

$$u'' + \frac{u'}{x} = -\delta e^{u}, \qquad u'(0) = 0, \qquad u(1) = 0.$$

The exact solution is given by

$$u(x) = \ln\left[\frac{8B/\delta}{(Bx^2+1)^2}\right] \quad \text{where} \quad \frac{8B/\delta}{(B+1)^2} = 1.$$

This problem has no solution for $\delta > 2$. For $\delta = 2$ it has a unique solution. For $\delta < 2$ it has two solutions. The numerical methods approximate smaller of the two solutions in this case. This example is solved by the method (12) and (42) with $h = 2^{-k}$, $k = 3(1)6$. The resulting nonlinear equations are solved by the Newton's method. The starting values for the solution are arbitrarily taken as $u_i = 1 - ih$, $i = 1, 2, \ldots, N$ and the iteration is stopped when the tolerance 10^{-8} is achieved. Maximum absolute errors for $\delta = -1.0$, 0.5, 1.0, are given in Table 4. We have also tested the efficiency of the spline functions to find the intermediate solutions using the computed numerical solutions. We have used the spline (10) to find solutions at three equidistant points inside the intervals used to find the numerical solution. Errors in the spline solution for $\delta = -1.0$, $h = 1/8$ for the intervals (1/4, 3/8) and (3/4, 7/8) are given in Table 5. The results show that the spline solution at intermediate points are atleast of the same accuracy as the neighbouring numerical solutions. The same behaviour is seen in all the other intervals and for other values of δ.

Acknowledgements. We wish to express our sincere thanks to Professor M.K. Jain for his comments and suggestions.

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Received December 3, 1985/June 13, 1986