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Spline Collocation for Singular Integro-differential Equations over (0, 1)

Gunther Schmidt

Karl-Weierstraß-Institut für Mathematik der Akademie der Wissenschaften der DDR, Mohrenstraße 39, DDR-1086 Berlin, German Democratic Republic

Summary. This paper analyses the convergence of spline collocation methods for singular integro-differential equations over the interval (0, 1). As trial functions we utilize smooth polynomial splines the degree of which coincides with the order of the equation. Depending on the choice of collocation points we obtain sufficient and even necessary conditions for the convergence in Sobolev norms. We give asymptotic error estimates and some numerical results.

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1. Introduction

1.1. In this paper we consider the approximate solution by splines of singular integro-differential equations of the form

$$A u(x)=\sum_{k=0}^{m}\left(a_k(x)\,u^{(k)}(x)+\frac{b_k(x)}{\pi i}\int_0^1\frac{u^{(k)}(y)}{y-x}\,dy\right.$$
$$\left.+\int_0^1 K_k(x,y)\,u^{(k)}(y)\,dy\right)=f(x),\qquad x\in[0,1], \tag{1.1}$$
$$(B u)_j=\sum_{k=0}^{m-1}(p_{jk}\,u^{(k)}(0)+q_{jk}\,u^{(k)}(1))=v_j,\quad j=0,\dots,m-1.$$

Here the right-hand side f, the continuous functions a_k, b_k, K_k and the numbers p_{jk}, q_{jk}, v_j are given, u is the unknown function and the first integrals are to be interpreted as Cauchy principal values. There is a considerable engineering interest in solving such equations, which stems from the fact that a large number of boundary value problems in aerodynamics, elasticity, electromagnetics and many other fields of mechanics and engineering can be reduced to equations of the form (1.1). Here we mention only the famous Prandtl integro-differential equation of wing theory (cf. [9]).

It is well known (cf. [9, § 117]) that (1.1) is equivalent to a singular integral equation of the form

$$a_m(x)\,v(x)+\frac{b_m(x)}{\pi i}\int_0^1 \frac{v(y)}{y-x}\,dy+\int_0^1 \tilde{K}(x,y)\,v(y)\,dy=\tilde{f}(x), \tag{1.2}$$

where $v(x)=u^{(m)}(x)$. Approximation methods for (1.2), which are based on special polynomials, have been studied for many years. The probably most complete analysis of such methods was given in Junghanns and Silbermann [7]. In the last years other methods were proposed, which use splines and special finite elements at the endpoints as trial functions. Here we mention the papers of Washizu and Ikegawa [15], Dang and Norrie [1], Gerasoulis and Srivastav [4], Jen and Srivastav [6], Gerasoulis [3], which deal mainly with collocation methods for (1.2) with $a_m\equiv 0$ and $b_m\equiv 1$. The obtained numerical results show a high efficiency of these methods, but up to now no convergence results have been proved. A first rigorous analysis for the L^2-convergence of Galerkin methods with splines of arbitrary degree was given by Elschner [2]. He proved in particular that Galerkin's method for (1.2) converges in L^2 if the corresponding operator is invertible and strongly elliptic, i.e., the coefficients satisfy $a_m(x)+\lambda\, b_m(x)\neq 0$, $x\in[0,1]$, $\lambda\in[-1,1]$. Later on it was proved in Prößdorf and Rathsfeld [11] that these conditions are necessary and sufficient for the L^2-convergence of a collocation method, which seeks the approximate solution as a piecewise linear function vanishing at the endpoints and which collocates (1.2) at the uniformly distributed mesh points.

In this paper we shall analyse collocation methods for (1.1) using smooth polynomial splines of degree m on a quasiuniform mesh as trial functions. Depending on the choice of collocation knots we obtain sufficient conditions for the convergence in Sobolev norms. These conditions show in particular that collocation methods can converge when the function $a_m(x)$ vanishes inside the interval (0, 1).

1.2. We write (1.1) in the form

$$\mathscr{A}u=\begin{pmatrix} f\\ \bar{v}\end{pmatrix},\qquad \text{where } \mathscr{A}=\begin{pmatrix} A\\ B\end{pmatrix}:\ H^m\to \begin{matrix} L^2\\ \oplus\\ \mathbb{C}^m\end{matrix}. \tag{1.3}$$

Here H^m denotes the usual Sobolev space of order m on (0, 1) and $L^2=L^2(0,1)$. As usual we define the norm of

$$\begin{pmatrix} f\\ \bar{v}\end{pmatrix}\in \begin{matrix} L^2\\ \oplus\\ \mathbb{C}^m\end{matrix}\quad \text{by}\quad \left\|\begin{pmatrix} f\\ \bar{v}\end{pmatrix}\right\|_0^2:=\|f\|_{L^2}^2+\|\bar{v}\|_{\mathbb{C}^m}^2=\|f\|_{L^2}^2+\sum_{j=0}^{m-1}|v_j|^2.$$

Obviously, $\mathscr{A}$ is bounded. We associate with this operator the symbol

$$\sigma_{\mathscr{A}}(x,z)=a_m(x)+b_m(x)\,z,\qquad (x,z)\in\Gamma_0,$$

where Γ_0 is the oriented boundary of the rectangle $\{0\leqq x\leqq 1,\ -1\leqq z\leqq 1\}$ in the $x-z$-plane, and we denote by $\operatorname{ind}_{\Gamma_0}\sigma_{\mathscr{A}}$ the winding number of this closed oriented curve around the origin. With the assumptions $\sigma_{\mathscr{A}}(x,z)\neq 0$, $(x,z)\in\Gamma_0$,

$\mathrm{ind}_{\Gamma_0}\sigma_{\mathscr{A}}=0$ and $\dim\ker\mathscr{A}=0$ we have that for any $f\in L^2$, $\bar{v}\in\mathbb{C}^m$ the problem (1.1) has a unique solution $u\in H^m$. This follows from the theory of singular integral equations (cf. [5]) and the fact that the operator

$$Tu(x):=\int_0^1 K_m(x,y)\,u^{(m)}(y)\,dy+\sum_{k=0}^{m-1}\left(a_k(x)\,u^{(k)}(x)\right.$$
$$\left.+\frac{b_k(x)}{\pi i}\int_0^1\frac{u^{(k)}(y)}{y-x}\,dy+\int_0^1 K_k(x,y)\,u^{(k)}(y)\,dy\right) \tag{1.4}$$

maps H^m compactly into L^2.

We shall analyse the following collocation method for (1.1). Let $\gamma(x)\in C^\infty[0,1]$ with $\gamma(j)=j$, $j=0,1$, and $\gamma'(x)>0$. For $h=n^{-1}$, $n\in\mathbb{N}$, S_h^m denotes the space of $(m-1)$ times continuously differentiable polynomial splines of degree m subordinate to the mesh $\{\gamma(rh)\}_{r=0}^n$. In addition, we fix $\varepsilon\in(0,1)$ and set $x_r=\gamma((r+\varepsilon)\,h)$, $r=0,\dots,n-1$. The collocation method under consideration defines $u_h\in S_h^m$ by

$$\begin{aligned} A\,u_h(x_r)&=f(x_r), \qquad r=0,\dots,n-1,\\ (B\,u_h)_j&=v_j, \qquad j=0,\dots,m-1.\end{aligned} \tag{1.5}$$

Since $\dim S_h^m=n+m$, we may ask for conditions ensuring, for given ε, the existence of approximate solutions $u_h\in S_h^m$ for all n large enough and their convergence to the exact solution u in H^m as $n\to\infty$.

In order to formulate this condition, we introduce the function

$$\Phi_\varepsilon(\lambda)=\int_{-\varepsilon}^{1-\varepsilon}\frac{\sin\lambda\pi y}{\sin\pi y}\,dy-i\int_{\varepsilon}^{1-\varepsilon}\frac{\cos\lambda\pi y}{\sin\pi y}\,dy, \qquad \lambda\in[-1,1]. \tag{1.6}$$

We note that, for fixed $\varepsilon\in(0,1)$, we have $\Phi_\varepsilon\in C^\infty[-1,1]$ with $\Phi_\varepsilon(j)=j$, $j=-1,1$, and $\{\Phi_{1/2}(\lambda)\colon \lambda\in[-1,1]\}=[-1,1]$.

Theorem 1. *Assume that* $\mathscr{A}\colon H^m\to\begin{smallmatrix}L^2\\ \oplus\\ \mathbb{C}^m\end{smallmatrix}$ *is invertible and that*

$$\begin{aligned} a_m(x)+\Phi_\varepsilon(\lambda)\,b_m(x)&\neq 0,\\ a_m(j)+\Phi_\delta(\lambda)\,b_m(j)&\neq 0,\end{aligned} \tag{1.7}$$

where $x\in(0,1)$, $\lambda\in[-1,1]$, $j=0,1,\delta\in[\varepsilon,1/2]$. *Then the linear system* (1.5) *is uniquely solvable for any* n *large enough and for any bounded and Riemann integrable function* f *and* $\bar{v}\in\mathbb{C}^m$ *one has* $\|u-u_h\|_{H^m}\to 0$ *as* $n\to\infty$.

2. Preliminary Results

2.1. Theorem 1 can be proved by using recently obtained results on the convergence of spline collocation for pseudodifferential equations with piecewise continuous coefficients on a closed curve (cf. [13, 14]). In this section we give simple proofs of the required results.

We start with a simple closed C^∞-curve Γ, which contains the interval $[0,1]$ and is given by a parametrization $z=z(x)$, $x\in[0,2]$, such that $z'(x)\neq 0$, $z(0)=z(2)$ and $z(x)=\gamma(x)$, $x\in[0,1]$. On this curve we consider a singular integral operator L defined by

$$L\varphi(z)=a(z)\varphi(z)+\frac{b(z)}{\pi i}\int_\Gamma \frac{\varphi(\zeta)}{\zeta-z}d\zeta, \qquad z\in\Gamma, \tag{2.1}$$

with

$$a(z)=\begin{cases} a_m(\gamma(x)), & z=\gamma(x)\in[0,1];\\ 1, & z\in\Gamma\setminus[0,1];\end{cases}$$

$$b(z)=\begin{cases} b_m(\gamma(x)), & z=\gamma(x)\in[0,1];\\ 0, & z\in\Gamma\setminus[0,1].\end{cases}$$

Under the assumptions on a_m and b_m this operator is invertible in $L^2(\Gamma)$ ([5]) and we represent it in the form

$$L\varphi(z)=a(z)\varphi(z)+b(z)S\varphi(z)+b(z)K\varphi(z), \qquad z\in\Gamma, \tag{2.2}$$

where

$$S\varphi(z(x)):=\int_0^2 \frac{\varphi(z(y))}{1-\exp(\pi i(x-y))}dy, \qquad x\in[0,2], \tag{2.3}$$

and the operator

$$K\varphi(z(x)):=\frac{1}{\pi i}\int_0^2 \varphi(z(y))\left(\frac{z'(y)}{z(y)-z(x)}-\frac{\pi i}{1-\exp(\pi i(x-y))}\right)dy$$

maps $L^2(\Gamma)$ compactly into the space of continuous functions $C(\Gamma)$.

We consider the so-called ε-collocation for the equation

$$L\varphi=\psi. \tag{2.4}$$

For $h=n^{-1}$, $n\in\mathbb{N}$, X_h denotes the set of piecewise constant functions on Γ with the break points $z(kh)$, $k=0,\dots,2n-1$. We seek $\varphi_h\in X_h$ such that

$$L\varphi_h(z_r)=\psi(z_r), \qquad z_r=z((r+\varepsilon)h), \qquad r=0,\dots,2n-1. \tag{2.5}$$

It is well known (cf. [5]) that for any

$$\varphi(z(x))=\sum_{j=-\infty}^{+\infty}\hat\varphi_j\exp(\pi ijx)\in L^2(0,2) \quad\text{with}\quad \hat\varphi_j=\tfrac12\int_0^2\varphi(x)\exp(-\pi ijx)\,dx$$

one has

$$S\varphi(z(x))=\sum_{j=0}^{+\infty}\hat\varphi_j\exp(\pi ijx)-\sum_{j=-\infty}^{-1}\hat\varphi_j\exp(\pi ijx). \tag{2.6}$$

Simple facts on Fourier series imply that for

$$\theta_k(z(x))=\begin{cases}1, & x\in[kh,(k+1)h)\\ 0, & x\notin[kh,(k+1)h), \qquad k=0,\dots,2n-1,\end{cases} \tag{2.7}$$

$$S\theta_k(z_r)=h/2+\sum_{0\neq j\in\mathbb{Z}}\frac{\sin\pi jh/2}{\pi|j|}\exp(\pi ijh(r-k+\varepsilon-1/2)).$$

Then the matrix $S_n := \|S\,\theta_k(z_r)\|_{r,k=0}^{2n-1}$ is a circulant and, by the known formula for its eigenvalues $\{\lambda_l\}_{l=0}^{2n-1}$, we obtain

$$\lambda_l = \sum_{k=0}^{2n-1} \exp(\pi i l k h) S\theta_k(z_0) = \delta_{l0}$$
$$+ \sum_{0 \neq j \in \mathbb{Z}} \frac{\sin \pi j h/2}{\pi |j|} \exp(\pi i j h(\varepsilon - 1/2)) \sum_{k=0}^{2n-1} \exp(\pi i j k(l-j))$$
$$= \delta_{l0} + 2n \sum_{j \in \mathbb{Z}} \frac{\sin \pi h(l+2jn)/2}{\pi |l+2jn|} \exp(\pi i h(l+2jn)(\varepsilon - 1/2)),$$

i.e.

$$\lambda_0 = 1,$$
$$\lambda_l = \frac{\sin \pi l h/2}{\pi} \exp(\pi i l h(\varepsilon - 1/2) \sum_{j \in \mathbb{Z}} \frac{\exp(2\pi i j \varepsilon)}{|j + l h/2|}, \qquad l = 1, \ldots, 2n-1.$$

Since

$$\frac{\tau}{\sin \pi \tau} \exp(-2\pi i \tau(\varepsilon - 1/2) = \sum_{j \in \mathbb{Z}} \frac{\exp(2\pi i j \varepsilon)}{j + \tau}, \qquad \tau \in (0, 1),$$

by (2.6) we derive

$$\lambda_l = 2 \int_0^1 \frac{\exp(\pi i l h(\varepsilon - y))}{1 - \exp(2\pi i(\varepsilon - y))} dy, \qquad l = 0, \ldots, 2n-1.$$

Setting $lh/2 = \tau$ and remarking that

$$2 \int_0^1 \frac{\exp(2\pi i \tau(\varepsilon - y))}{1 - \exp(2\pi i(\varepsilon - y))} dy = -i \int_{-\varepsilon}^{1-\varepsilon} \frac{\exp(\pi i(1 - 2\tau) y)}{\sin \pi y} dy,$$

by (1.6) we obtain

Lemma 1. *The eigenvalues of* $S_n = \|S\,\theta_k(z_r)\|_{r,k=0}^{2n-1}$ *are*

$$\lambda_l = \Phi_\varepsilon(1 - l h), \qquad l = 0, \ldots, 2n-1.$$

Moreover, $S_n = V_n \Phi_{n,\varepsilon} V_n^*$, *where* $\Phi_{n,\varepsilon} = \|\delta_{kl} \Phi_\varepsilon(1 - l h)\|_{k,l=0}^{2n-1}$ *and* $V_n = (2n)^{-1/2} \|\exp(\pi i k l h)\|_{k,l=0}^{2n-1}$.

2.2. By using the theory of projection methods, we can now analyse the convergence of ε-collocation. To this end we introduce the orthogonal projections P_h: $L^2(\Gamma) \to X_h$ and the interpolation projections $Q_h \psi(z) = \sum_{r=0}^{2n-1} \psi(z_r) \theta_r(z)$. Then the collocation Eqs. (2.5) can be written as

$$Q_h L \varphi_h = Q_h \psi, \qquad \varphi_h \in X_h. \tag{2.8}$$

Let us state some properties of the operators $Q_h L|_{X_h}$: $X_h \to X_h$.

Lemma 2. 1) $P_h \to I$ *as* $h \to 0$ (*strong convergence in* $L^2(\Gamma)$).

2) $\|(I - Q_h)\psi\|_{L^2(\Gamma)} \to 0$ *for any bounded and Riemann integrable function* ψ *on* Γ ($\psi \in R(\Gamma)$).

3) $Q_h L P_h \to L$ *as* $h \to 0$.

Proof. 1) follows from the density of step functions.

2) is an immediate consequence of[1]

$$\|(I-Q_h)\psi\|^2_{L^2(\Gamma)} = \sum_{k=0}^{2n-1} \int_{kh}^{(k+1)h} |\psi(z(x))-\psi(z_k)|^2 |z'(x)|\,dx$$
$$\leqq c \sup_{z\in\Gamma}|\psi(z)| \sum_{k=0}^{2n-1} h \sup_{kh\leqq x,y\leqq(k+1)h} |\psi(z(x))-\psi(z(y))|.$$

To prove 3) we remark that

$$c_1 h \sum_{k=0}^{2n-1} |\varphi_k|^2 \leqq \left\| \sum_{k=0}^{2n-1} \varphi_k \theta_k(z) \right\|^2_{L^2(\Gamma)} \leqq c_2 h \sum_{k=0}^{2n-1} |\varphi_k|^2, \tag{2.9}$$

where c_1, c_2 do not depend on n and $\varphi_h \in X_h$. Hence, for $\psi \in R(\Gamma)$ we have $\|(I-Q_h)\psi P_h\|_{L^2(\Gamma)} \leqq c$ and

$$\|(I-Q_h)\psi Q_h \exp(\pi i j x)\|^2_{L^2(\Gamma)} = \sum_{k=0}^{2n-1} \int_{kh}^{(k+1)h} |\psi(z(x))-\psi(z_k)|^2 |z'(x)|\,dx,$$

which prove that $(I-Q_h)\psi P_h \to 0$. Moreover, from Lemma 1 and (2.9) we know that $\|Q_h S P_h\|_{L^2(\Gamma)} \leqq c \max_{\lambda\in[-1,1]} |\Phi_\varepsilon(\lambda)|$ and that

$$Q_h S Q_h \exp(\pi i j x) = \begin{cases} \Phi_\varepsilon(1-jh)\, Q_h \exp(\pi i j x), & 0 \leqq j < 2n; \\ \Phi_\varepsilon(-1-jh)\, Q_h \exp(\pi i j x), & -2n < j \leqq -1. \end{cases}$$

Thus $Q_h S P_h \to S$, which together with $\|(I-Q_h)K\|_{L^2(\Gamma)} \to 0$ proves the third assertion. □

It is well known that Lemma 2 together with the stability of the operators $Q_h L P_h$ proves the L^2-convergence of the ε-collocation when the right-hand side $\psi \in R(\Gamma)$. Here *stability* means that

$$\|Q_h L P_h \varphi\|_{L^2(\Gamma)} \geqq c\, \|P_h \varphi\|_{L^2(\Gamma)}$$

for all n large enough and all $\varphi \in L^2(\Gamma)$ with a constant c independent of n and φ.

In order to study the stability of $Q_h L P_h$, for any $\zeta \in \Gamma$ we relate with L an operator L_ζ defined by

$$L_\zeta \varphi(z) := a(\zeta)\varphi(z) + b(\zeta) S\varphi(z), \qquad \zeta \neq 0, 1,$$

and

$$L_j \varphi(z) := (a_m(j) p(z) + q(z))\varphi(z) + b_m(j) p(z) S\varphi(z), \qquad j = 0, 1,$$

where $p(z)=1$, $z\in[0,1]$, $p(z)=0$, $z\in\Gamma\setminus[0,1]$ and $q(z)=1-p(z)$, and analyse the stability of $Q_h L_\zeta P_h$, $\zeta\in\Gamma$.

Seeking the solution of $Q_h L_\zeta \varphi_h = Q_h \psi$ in the form $\varphi_h(z) = \sum_{k=0}^{2n-1} \varphi_k \theta_k(z)$, for $\zeta \neq 0, 1$ we get a linear system with the matrix

$$L_{\zeta,n} = a(\zeta) I_n + b(\zeta) S_n = V_n \|\delta_{kl}(a(\zeta) + b(\zeta)\Phi_\varepsilon(1-lh)\|_{k,l=0}^{2n-1} V_n^*,$$

[1] Here and in the following $c, c_1, \ldots$ denote generic constants having different values at different places

I_n denoting the $2n\times 2n$ identity matrix. By (2.9), $Q_h L_\zeta P_h$, $\zeta \neq 0,1$, is stable if and only if $a(\zeta)+b(\zeta)\Phi_\varepsilon(\lambda)\neq 0$, $\lambda\in[-1,1]$. If $\zeta=j=0,1$, then the matrix of the linear system is

$$L_{j,n}=(a_m(j)\,p_n+q_n)\,I_n+b_m(j)\,p_n S_n,$$

where $p_n=\|\delta_{kr}\,p(z_r)\|_{k,r=0}^{2n-1}$, $q_n=I_n-p_n$. After multiplying the transposed matrix $L'_{j,n}$ by V_n^* we obtain

$$V_n^* L'_{j,n}=(a_m(j)+b_m(j)\,\Phi_{n,\varepsilon})\,V_n^*\,p_n+V_n^*\,q_n.$$

Therefore the k-th coordinate of the vector $V_n^* L'_{j,n}\bar{\varphi}$, $\bar{\varphi}=(\varphi_0,\ldots,\varphi_{2n-1})\in\mathbb{C}^{2n}$, equals

$$\begin{aligned}(V_n^* L'_{j,n}\bar{\varphi})_k&=(2n)^{-1/2}\left\{[a_m(j)+b_m(j)\,\Phi_\varepsilon(1-kh)]\sum_{r=0}^{n-1}\varphi_r\exp(-\pi i k r h)\right.\\&\quad\left.+\sum_{r=n}^{2n-1}\varphi_r\exp(-\pi i k r h)\right\}=(2n)^{-1/2}\exp(\pi i k(n-1)h)\\&\quad\times\left\{[a_m(j)+b_m(j)\,\Phi_\varepsilon(1-kh)]\sum_{r=0}^{n-1}\varphi_{n-r-1}\exp(\pi i k r h)\right.\\&\quad\left.+\sum_{r=-n}^{-1}\varphi_{n-r-1}\exp(\pi i k r h)\right\}.\end{aligned}$$

At this point we utilize a nice result on collocation methods via trigonometric polynomials for singular integral equations on the unit circle. Consider a singular integral equation with piecewise continuous coefficients

$$\begin{aligned}M_{j,\varepsilon}\,\psi(\exp(2\pi i\tau))&=(a_m(j)+b_m(j)\,g_\varepsilon(\exp(2\pi i\tau))\,P\psi(\exp(2\pi i\tau))\\&\quad+Q\psi(\exp(2\pi i\tau))=\chi(\exp(2\pi i\tau)),\qquad 0\leqq\tau<1,\end{aligned}$$

with $P=(I+S)/2$, $Q=I-P$ (cf. (2.6)) and $g_\varepsilon(\exp(2\pi i\tau))=\Phi_\varepsilon(1-2\tau)$. By seeking an approximate solution in the form

$$\psi_n(\exp(2\pi i\tau))=\sum_{r=-n}^{n-1}\varphi_{n-r-1}\exp(2\pi i r\tau)$$

such that

$$M_{j,\varepsilon}\,\psi_n(\exp(\pi i k h))=\chi(\exp(\pi i k h)),\qquad k=0,\ldots,2n-1,$$

we get a linear system, whose matrix coincides with $(2n)^{1/2}D_n^{-1}V_n^* L'_{j,n}$, where $D_n=\|\delta_{kr}\exp(\pi i k h(n-1))\|_{k,r=0}^{2n-1}$. It was proved in Junghanns and Silbermann [8] that

$$\sum_{k=0}^{2n-1}|M_{j,\varepsilon}\,\psi_n(\exp(\pi i k h))|^2\geqq c(2n)\sum_{k=0}^{2n-1}|\varphi_k|^2$$

for all n large enough, where c does not depend on n and ψ_n, if and only if the operator $M_{j,\varepsilon}$ is invertible in L^2, i.e. (cf. [5])

$$\begin{aligned}&a_m(j)+b_m(j)\,\Phi_\varepsilon(\lambda)\neq 0,\\&a_m(j)+\lambda\, b_m(j)\neq 0,\qquad \lambda\in[-1,1],\end{aligned}$$

and the winding number of the curve consisting of these two pieces around the origin equals zero. Since the matrices D_n and V_n are unitary, we conclude that $Q_h L_j P_h$, $j=0,1$, are stable if and only if the numbers $a_m(j)$ and $b_m(j)$, $j=0,1$, satisfy (1.7). By standard perturbation theorems for projection methods we derive

Lemma 3. *The operators* $Q_h(L_\zeta+K)P_h$, $\zeta\in\Gamma$, *are stable if and only if the conditions* (1.7) *are satisfied.*

Now we are in position to establish the stability of $Q_h L P_h$. Obviously, for any $\delta>0$ and any $\zeta\in\Gamma$ there exists a nonnegative C^∞ function $g_\zeta(z)$ with small support and $g_\zeta(\zeta)=1$ such that

$$\|Q_h g_\zeta(L-L_\zeta-K)P_h\|_{L^2(\Gamma)}<\delta$$

for all n large enough. Hence, if (1.7) holds, then there exists a sequence of uniformly bounded operators $D_{\zeta,h}$: $X_h\to X_h$ so that

$$Q_h g_\zeta L P_h D_{\zeta,h} P_h = Q_h g_\zeta P_h. \tag{2.10}$$

After choosing a finite number of points $\zeta_1,\dots,\zeta_N$ such that $\sum_{j=1}^{N} g_{\zeta_j}(z)>0$ we introduce

$$G_h=\sum_{j=1}^{N} Q_h g_j D_{j,h} P_h \quad \text{with } g_j=g_{\zeta_j}, \quad D_{j,h}=D_{\zeta_j,h}.$$

Then

$$\begin{aligned} Q_h L G_h P_h &= \sum_{j=1}^{N} Q_h L Q_h g_j D_{j,h} P_h = \sum_{j=1}^{N} Q_h g_j L P_h D_{j,h} P_h \\ &\quad + \sum_{j=1}^{N} Q_h(LQ_h g_j - g_j L) P_h D_{j,h} P_h. \end{aligned}$$

By (2.10), the first sum equals $Q_h \sum_{j=1}^{N} g_j P_h$, which is invertible in X_h. To handle the second term we note that

$$\|Q_h S(Q_h-I) g_j P_h\|_{L^2(\Gamma)} \leqq c\, h\, |\ln h| \quad \text{(cf. [12])}$$

and that Lg_j-g_jL maps $L^2(\Gamma)$ compactly into $C(\Gamma)$. Hence,

$$\sum_{j=1}^{N} Q_h(LQ_h g_j - g_j L) P_h D_{j,h} P_h = Q_h M' P_h + M_h$$

with compact M': $L^2(\Gamma)\to C(\Gamma)$ and $\|M_h\|_{L^2(\Gamma)}\to 0$ as $h\to 0$. Thus we have shown that, if (1.7) holds, then there exist uniformly bounded operators G'_h: $X_h\to X_h$ such that, for all n large enough,

$$Q_h L G'_h P_h = P_h + Q_h M P_h,$$

where M: $L^2(\Gamma)\to C(\Gamma)$ is compact. Setting $F_h=G'_h-P_h L^{-1} M P_h$ we obtain

$$\begin{aligned} Q_h L F_h P_h &= Q_h L G'_h P_h - Q_h L P_h L^{-1} M P_h \\ &= P_h + (Q_h - Q_h L P_h L^{-1}) M P_h. \end{aligned}$$

By Lemma 2, $Q_h L P_h L^{-1} \to I$ and, therefore,

$$\|(Q_h - Q_h L P_h L^{-1}) M P_h\|_{L^2(\Gamma)} \to 0 \quad \text{as } h \to 0.$$

Consequently, for all sufficiently large n there exist uniformly bounded operators F'_h: $X_h \to X_h$ such that $Q_h L P_h F'_h P_h = P_h$. Thus, we have proved

Lemma 4. *The operators $Q_h L P_h$ are stable if* (1.7) *holds.*

Remark. This proof uses a local principle developed in [10] for a more general situation. By this method one can also show that the conditions (1.7) are even necessary for the stability of $Q_h L P_h$ (cf. [14]).

Now we consider the matrix of the system (2.5) when φ_h is sought in the form $\varphi_h(z) = \sum_{k=0}^{2n-1} \varphi_k \theta_k(z)$. By (2.1), we have $L\theta_k(z_r) = 0$, $0 \leqq k < n \leqq r \leqq 2n-1$, and $L\theta_k(z_r) = \delta_{kr}$, $n \leqq k$, $r \leqq 2n-1$. Hence

$$\|L\theta_k(z_r)\|_{r,k=0}^{2n-1} = \begin{pmatrix} A_n & 0_n \\ L_n & I_n \end{pmatrix}$$

with the $n \times n$ matrices $A_n = \|L\theta_k(z_r)\|_{r,k=0}^{n-1}$, the identity matrix I_n and the zero matrix 0_n. From Lemma 4 and (2.9) we conclude

Corollary 1. *If the conditions* (1.7) *are satisfied, then the matrices A_n are invertible for all n large enough and*

$$\sum_{k=0}^{n-1} |(A_n \bar{\varphi})_k|^2 \geqq c \sum_{k=0}^{n-1} |\varphi_k|^2$$

for all $\bar{\varphi} = (\varphi_0, \ldots, \varphi_{n-1}) \in \mathbb{C}^n$ with a constant c independent of n and $\bar{\varphi}$.

3. Proof of Theorem 1

3.1. In this section we shall prove Theorem 1 and make some remarks concerning error estimates.

We begin by analysing the equations

$$\begin{aligned} A_o u_h(x_r) &= f(x_r), \qquad r = 0, \ldots, n-1, \\ (B u_h)_j &= v_j, \qquad j = 0, \ldots, m-1, \end{aligned} \tag{3.1}$$

where $A_o u(x) := (A - T) u(x) = a_m(x) u^{(m)}(x) + \frac{b_m(x)}{\pi i} \int_0^1 \frac{u^{(m)}(y)}{y-x} dy$. Letting $Q_h f(x) = \sum_{k=0}^{n-1} f(x_k) \theta_k(\gamma(x))$ (cf. (2.7)), (3.1) can be written in the form (cf. (1.3))

$$\mathcal{Q}_h \mathcal{A}_o u_h = \mathcal{Q}_h \begin{pmatrix} f \\ \bar{v} \end{pmatrix}, \tag{3.2}$$

where $\mathcal{Q}_h = \begin{pmatrix} Q_h & 0 \\ 0 & I_m \end{pmatrix}$, I_m denoting the identity mapping in $\mathbb{C}^m$. Hence, (3.1) can be considered as a projection method and

$$\mathcal{Q}_h \mathcal{A}_o \colon S_h^m \to \begin{matrix} S_h^0 \\ \oplus \\ \mathbb{C}^m \end{matrix}.$$

By using the results of Sect. 2 it is easy to establish the properties of the operators $\mathcal{Q}_h \mathcal{A}_o|_{S_h^m}$, which are needed for the convergence analysis of (3.1).

First, for any $u \in H^m$, $u_h \in S_h^m$ with $\|u_h - u\|_{H^m} \to 0$ we have

$$\|\mathcal{Q}_h \mathcal{A}_o u_h - \mathcal{A}_o u\|_0 \to 0 \quad \text{as } h \to 0. \tag{3.3}$$

Indeed, the third assertion of Lemma 2 ensures that for $\varphi \in L^2 (\subset L^2(\Gamma))$ and $\varphi_h \in S_h^0 (\subset X_h)$ with $\|\varphi - \varphi_h\|_{L^2} \to 0$ one has $\|Q_h L \varphi_h - L\varphi\|_{L^2} \to 0$. Setting $\varphi = u^{(m)}$, $\varphi_h = u_h^{(m)}$ we obtain $\|Q_h A_o u_h - A_o u\|_{L^2} \to 0$, which together with $Bu_h \to Bu$ establishes (3.3). This together with the Banach-Steinhaus-Theorem yields, in particular, that the mappings $\mathcal{Q}_h \mathcal{A}_o|_{S_h^m}$: $H^m \to \underset{\mathbb{C}^m}{\overset{L^2}{\oplus}}$ are uniformly bounded.

Furthermore, these mappings are stable if the conditions (1.7) are satisfied. To prove this assertion, we introduce the norm

$$\|u\|_m^2 := \sum_{j=0}^{m-1} |(Bu)_j|^2 + \int_0^1 |u^{(m)}(x)|^2 \, dx,$$

which is, in view of $\dim \ker \mathcal{A}_o = 0$, equivalent to the usual Sobolev norm $\|\cdot\|_{H^m}$. Besides this, we construct a special basis of S_h^m. Since the m-th derivative of u_h is a step function on $[0,1]$ with break points at $\gamma(kh)$, we may find $s_k \in S_h^m$, $k = 0, \ldots, n-1$, such that $s_k^{(m)} = \theta_k$. Obviously, these functions are linearly independent and form together with the functions $s_k(x) = x^{k-n}$, $k = n, \ldots, n+m-1$, a basis of S_h^m. On seeking the solution of (3.1) in the form $u_h(x) = \sum_{k=0}^{n+m-1} u_k s_k(x)$, we get a linear system with the matrix

$$A_n = \begin{pmatrix} A_n & 0_{n,m} \\ C_{m,n} & B_m \end{pmatrix}$$

where

$$A_n = \left\| a_m(x_r)\,\delta_{kr} + \frac{b_m(x_r)}{\pi i} \int_0^1 \frac{\theta_k(y)}{y - x_r} \, dy \right\|_{r,k=0}^{n-1},$$

$B_m = \|(Bx^k)_j\|_{j,k=0}^{m-1}$, $C_{m,n} = \|(Bs_k)_j\|_{j=0,k=0}^{m-1,n-1}$ and $0_{n,m}$ denotes the $n \times m$ zero matrix, since $A_o s_k(x) \equiv 0$, $k = n, \ldots, n+m-1$.

The notation A_n for the $n \times n$ matrix is justified since

$$\begin{aligned} \|L\theta_k(z_r)\|_{r,k=0}^{n-1} &= \left\| a_m(\gamma((r+\varepsilon)h))\,\delta_{kr} + \frac{b_m(\gamma((r+\varepsilon)h))}{\pi i} \int_{\gamma(kh)}^{\gamma((k+1)h)} \frac{d\zeta}{\zeta - \gamma((r+\varepsilon)h)} \right\|_{r,k=0}^{n-1} \\ &= \left\| a_m(x_r)\,\delta_{kr} + \frac{b_m(x_r)}{\pi i} \int_0^1 \frac{\theta_k(y)}{y - x_r} \, dy \right\|_{r,k=0}^{n-1}. \end{aligned}$$

From Corollary 1 we know that A_n is invertible for any sufficiently large n if (1.7) holds. Since $\dim \ker \mathcal{A}_o = 0$, the matrix B_m is nonsingular. Hence, for n large enough (3.2) is uniquely solvable. Then Corollary 1 and (2.9) lead to

$$\|\mathcal{Q}_h \mathcal{A}_o u_h\|_0^2 = \int_0^1 |Q_h A_o u_h|^2 \, dx + \sum_{j=0}^{m-1} |(B u_h)_j|^2 \geqq c_1 h \sum_{k=0}^{n-1} |(A_n \bar{u})_k|^2$$
$$+ \sum_{j=0}^{m-1} |(B u_h)_j|^2 \geqq c_2 h \sum_{k=0}^{n-1} |u_k|^2 + \sum_{j=0}^{m-1} |(B u_h)_j|^2$$
$$\geqq c_3 \sum_{k=0}^{n-1} \int_0^1 |u_k s_k^{(m)}(x)|^2 \, dx + \sum_{j=0}^{m-1} |(B u_h)_j|^2 \geqq c \, \|u_h\|_m^2,$$

which proves the stability of $\mathcal{Q}_h \mathcal{A}_o|_{S_h^m}$.

Now we denote

$$u_h = (\mathcal{Q}_h \mathcal{A}_o|_{S_h^m})^{-1} \mathcal{Q}_h \begin{pmatrix} f \\ \bar{v} \end{pmatrix}, \qquad u = \mathcal{A}_o^{-1} \begin{pmatrix} f \\ \bar{v} \end{pmatrix}.$$

Since

$$\|u - u_h\|_m \leqq \inf_{w_h \in S_h^m} (\|u - w_h\|_m + \|u_h - w_h\|_m)$$
$$\leqq \inf_{w_h \in S_h^m} (\|u - w_h\|_m + c \, \|\mathcal{Q}_h \mathcal{A}_o u - \mathcal{Q}_h \mathcal{A}_o w_h\|_0)$$
$$\leqq \inf_{w_h \in S_h^m} (\|u - w_h\|_m + c \, \|\mathcal{A}_o u - \mathcal{Q}_h \mathcal{A}_o w_h\|_0) + c \, \|(I - Q_h) f\|_{L^2},$$

by (3.3) and the well-known approximation property of splines (cf. [2]) we have

$$\|u - u_h\|_m \leqq c \, \|(I - Q_h) f\|_{L^2} + o(1).$$

Thus we have proved

Lemma 5. *Under the hypotheses of Theorem* 1 *the Eqs.* (3.1) *are uniquely solvable for all n large enough. Moreover, for any* $f \in L^2$ *with* $\|Q_h f - f\|_{L^2} \to 0$ *and any* $\bar{v} \in \mathbb{C}^m$ *we have* $\|u - u_h\|_{H^m} \to 0$ *as* $n \to \infty$.

Now Theorem 1 is an immediate consequence of Lemma 5 and standard perturbation technique for projection methods. Indeed, from (1.4) we obtain

$$Tu(x) = \frac{1}{\pi i} \sum_{k=0}^{m-1} [b_k(x)(u^{(k)}(1) \ln(1-x) - u^{(k)}(0) \ln x)] + T'' u(x), \tag{3.4}$$

where

$$T'' u(x) = \sum_{k=0}^{m-1} \left(\frac{b_k(x)}{\pi i} \int_0^1 u^{(k+1)}(y) \ln|x-y| \, dy + a_k(x) \, u^{(k)}(x) \right)$$
$$+ \sum_{k=0}^{m} \int_0^1 K_k(x,y) \, u^{(k)}(y) \, dy. \tag{3.5}$$

Obviously, T'' maps H^m compactly into C.

Now let $d(x) \in C$. Then

$$\|(I-Q_h)\,d(x)\ln x\|_{L^2}^2=\sum_{k=0}^{n-1}\int_{\gamma(kh)}^{\gamma((k+1)h)}|d(x)\ln x-d(x_k)\ln x_k|^2\,dx$$

$$\leqq 2\sum_{k=0}^{n-1}\int_{\gamma(kh)}^{\gamma((k+1)h)}(|d(x)-d(x_k)|^2(\ln x)^2+|d(x_k)|^2(\ln x-\ln x_k)^2)\,dx$$

$$\leqq c\left(\omega^2(\gamma_h,d)+\sum_{k=0}^{n-1}\int_{kh}^{(k+1)h}\left(\ln\frac{x}{(k+\varepsilon)\,h}\right)^2dx\right),$$

where $\omega(\gamma_h,d):=\sup_{|x-y|\leqq h}|d(\gamma(x))-d(\gamma(y))|\to 0$ as $h\to 0$, and

$$\sum_{k=0}^{n-1}\int_{kh}^{(k+1)h}\left(\ln\frac{x}{(k+\varepsilon)\,h}\right)^2dx=c\,h\sum_{k=0}^{n-1}d_k$$

with

$$d_k=(k+1)\left(1+\ln\frac{k+\varepsilon}{k+1}\right)^2-k\left(1+\ln\frac{k+\varepsilon}{k}\right)^2+1=\frac{(1-\varepsilon)^3}{3(k+1)^2}+\frac{\varepsilon^3}{3k^2}+O(k^3).$$

Therefore $\|(I-Q_h)\,d(x)\ln x\|_{L^2}^2\to 0$ as $h\to 0$. Hence, $\|(I-Q_h)\,T\|_{H^m\to L^2}\to 0$, which implies

$$\|\mathscr{A}u-\mathscr{Q}_h\mathscr{A}u_h\|_0\to 0 \text{ for } u\in H^m,\quad u_h\in S_h^m \text{ with } \|u-u_h\|_m\to 0. \tag{3.6}$$

From Lemma 5 we conclude that

$$\|(\mathscr{A}_o^{-1}-(\mathscr{Q}_h\mathscr{A}_o|_{S_h^m})^{-1}\mathscr{Q}_h)\,\mathscr{T}\|_{H^m\to H^m}\to 0,$$

where $\mathscr{T}=\begin{pmatrix}T\\0\end{pmatrix}:\ H^m\to\underset{\mathbb{C}^m}{\overset{L^2}{\oplus}}$. Since $\mathscr{A}$ is assumed to be invertible, the mappings $I+(\mathscr{Q}_h\mathscr{A}_o|_{S_h^m})^{-1}\mathscr{Q}_h\mathscr{T}:\ H^m\to H^m$ are invertible for all n large enough. Thus we derive

$$\|\mathscr{Q}_h\mathscr{A}u_h\|_0=\|\mathscr{Q}_h(\mathscr{A}_o+\mathscr{T})\,u_h\|_0=\|\mathscr{Q}_h\mathscr{A}_o(I+(\mathscr{Q}_h\mathscr{A}_o|_{S_h^m})^{-1}\mathscr{Q}_h\mathscr{T}u_h)\|_0\geqq c\,\|u_h\|_{H^m}$$

for all n large enough and all $u_h\in S_h^m$. This together with (3.6) and Lemma 2 proves Theorem 1.

We note that conditions (1.7) are also necessary for the stability of $\mathscr{Q}_h\mathscr{A}|_{S_h^m}$.

3.2. In these concluding remarks we shall assume that $a_m,\ b_m\in H^1$ and that the operator T'' defined by (3.5) maps H^m boundedly into H^1. Then by [2, Corollary 2.3] we deduce that if the conditions of Theorem 1 are satisfied, then $\mathscr{A}_o$ maps H^{m+s} isomorphically onto $\underset{\mathbb{C}^m}{\overset{H^s}{\oplus}}$, where $0\leqq s<\min\{\operatorname{Re}\kappa_0,\operatorname{Re}\kappa_1\}+1/2$, $\kappa_0=\theta(0)$, $\kappa_1=-\theta(1)$,

$$\theta(x)=\frac{1}{2\pi i}\ln\frac{a_m(x)+b_m(x)}{a_m(x)-b_m(x)}$$

and ln denotes the continuous branch of the logarithm in $\mathbb{C}\setminus(-\infty,0]$ which takes real values on the positive real axis. Notice that, by (1.7), $-1/2<\operatorname{Re}\kappa_j<1/2$, $j=0,1$. Since the operator T defined by (3.4) maps H^m

compactly into H^t, $0 \leqq t < 1/2$, we obtain that $\mathscr{A}: H^{s+m} \to \overset{H^s}{\underset{\mathbb{C}^m}{\oplus}}$ is an isomorphism for $0 \leqq s < \min\{\operatorname{Re}\kappa_0, \operatorname{Re}\kappa_1, 0\} + 1/2$. Hence, for $f \in H^1$ the solution of (1.1) $u \in H^{m+s}$, in general. Using the approximation properties of splines (cf. [2]) we obtain

Theorem 2. *Suppose the conditions of Theorem 1 to be satisfied and let* $f \in H^1$. *Then the approximate solutions* $u_h \in S_h^m$ *of* (1.5) *converge to the exact solution* u *with the rate*

$$\|u - u_h\|_{H^m} \leqq c\, h^s \, \|f\|_{H^1} \tag{3.7}$$

with $0 \leqq s < \min\{\operatorname{Re}\kappa_0, \operatorname{Re}\kappa_1, 0\} + 1/2$.

If the coefficients $b_k(x)$, $k = 0, \ldots, m-1$, *satisfy* $b_k(j) = 0$, $j = 0, 1$, *the estimation* (3.7) *holds with* $0 \leqq s < \min\{\operatorname{Re}\kappa_0, \operatorname{Re}\kappa_1\} + 1/2$. *If, additionally,* $b_m(j) = 0$, $j = 0, 1$, *then the approximate solutions* u_h *converge with optimal order to the exact solution*

$$\|u - u_h\|_{H^m} \leqq c\, h \, \|f\|_{H^1}.$$

Proof. By $P_h u \in S_h^m$ we denote the orthogonal projection of u with respect to the scalar product $\langle \cdot, \cdot \rangle_m$ given by $\langle v, v \rangle_m = \|v\|_m^2$, $v \in H^m$. Analogously to the proof of Lemma 5 we obtain

$$\begin{aligned}\|u - u_h\|_m &\leqq \|u - P_h u\|_m + c\, \|\mathscr{Q}_h \mathscr{A} u - \mathscr{Q}_h \mathscr{A} P_h u\|_0 \\ &= \|u - P_h u\|_m + c\, \|Q_h A u - Q_h A P_h u\|_{L^2},\end{aligned}$$

since $B P_h u = B u$.

As mentioned before in the first case we have $u \in H^{m+s}$, $0 \leqq s < \min\{\operatorname{Re}\kappa_0, \operatorname{Re}\kappa_1, 0\} + 1/2$, such that

$$\|u - P_h u\|_m \leqq c\, h^s \, \|u\|_{H^{m+s}}.$$

Furthermore by (3.4)

$$\|Q_h A(u - P_h u)\|_{L^2} \leqq \|Q_h A_0(u - P_h u)\|_{L^2} + \|Q_h T'(u - P_h u)\|_{L^2} + \|Q_h T''(u - P_h u)\|_{L^2}.$$

In [14] we have proved that

$$\|Q_h A_o(u - P_h u)\|_{L^2} \leqq c\, \|u^{(m)} - (P_h u)^{(m)}\|_{L^2} \leqq c\, h^s \, \|u\|_{H^{m+s}}.$$

Since $b_k \in H^1$, $k = 0, \ldots, m-1$, from (3.6) we obtain $\|(I - Q_h)\, T'\|^2_{H^m \to L^2} \leqq c\, h$, such that

$$\begin{aligned}\|Q_h T'(u - P_h u)\|_{L^2} &\leqq \|(I - Q_h)\, T'(u - P_h u)\|_{L^2} + \|T'(u - P_h u)\|_{L^2} \\ &\leqq c_1 h^{1/2} \, \|u - P_h u\|_m + c_2 \, \|u - P_h u\|_m \leqq c\, h^s \, \|u\|_{H^{m+s}}.\end{aligned}$$

Since $T'' \in L(H^m, H^1)$ we obtain

$$\|Q_h T''(u - P_h u)\|_{L^2} \leqq c\, h^{1+s} \, \|u\|_{H^{m+s}}.$$

Finally

$$\|u - u_h\|_{H^m} \leqq c_1 h^s \, \|u\|_{H^{m+s}} \leqq c\, h^s \, \|f\|_{H^1}.$$

In the second case we have $T'u(j)=0$, $j=0,1$, such that $\mathscr{A}$ maps H^{m+s} isomorphically onto $\overset{H^s}{\underset{\mathbb{C}^m}{\oplus}}$, where $0 \leqq s < \min\{\operatorname{Re}\kappa_0, \operatorname{Re}\kappa_1\} + 1/2$. Hence

$$\|u-u_h\|_{H^m} \leqq c h^s \|f\|_{H^1} \quad \text{with } 0 \leqq s < \min\{\operatorname{Re}\kappa_0, \operatorname{Re}\kappa_1\} + 1/2.$$

If in addition $b_m(j)=0$, $j=0,1$, then (1.1) can be considered as the restriction on $(0,1)$ of a singular integro-differential equation with sufficiently smooth coefficients on the closed curve Γ. Hence, $\mathscr{A}$ is an isomorphism from H^{m+1} onto $\overset{H^1}{\underset{\mathbb{C}^m}{\oplus}}$, therefore the exact solution $u \in H^{m+1}$ and $\|u-u_h\|_{H^m} \leqq c h \|f\|_{H^1}$. □

Remark. The presence of singularities of the solution u for smooth right-hand sides f indicates that estimate (3.7) cannot be improved even if higher degree splines on quasiuniform meshes are used. An improvement of (3.7) for spline collocation methods is possible by using special nonuniform meshes or by adding special finite elements representing the singularities of the solution to the spline base. But up to now the stability analysis of such collocation methods is an open problem.

3.3. Numerical Results

The spline collocation method has been used to solve a number of singular integral and integro-differential equations, including the following:

$$\begin{aligned} &(a(x^2-1)-1)\,u'(x) + \frac{1}{\pi}\int_{-1}^{1} \frac{u'(y)}{y-x}\,dy = -a(1-x^2)^{3/4}(1-x)^{1/2} - \sqrt{2} \\ &u(1)=0, \qquad a \in \mathbb{R}, \end{aligned} \tag{3.8}$$

which have the exact solution

$$u(x) = \int_{-1}^{x} \left(\frac{1-x}{1+x}\right)^{1/4} dx - \pi/\sqrt{2}.$$

From Theorem 1 we conclude that ε-collocation with piecewise linear trial functions converges in the norm of H^1 iff

$$\varepsilon > \begin{cases} \dfrac{2}{\pi}\arctan\exp(-(1+a)\pi/2) & \text{for } a<0, \\[2ex] \dfrac{2}{\pi}\arctan\exp(-\pi/2) & \text{for } a \geqq 0. \end{cases}$$

Note that for $a \leqq -1$ Eq. (3.8) is not strongly elliptic, such that Galerkin methods do not converge, in general.

The obtained numerical results confirm the statements of Theorem 1 and, for uniform partitions, of Theorem 2, too. The tables below collect some results for different values of a, ε, and x with $n=80$, Δ denotes the maximale difference between the exact and approximate solution in mesh points.

$a=0$; Theorem 1 proves convergence in H^1 if $\varepsilon>0{,}1305$

	$\varepsilon=0.25$	$\varepsilon=0.5$	$\varepsilon=0.75$	exact
−0.8	−1.7592	−1.7583	−1.7503	−1.7524
−0.2	−0.9473	−0.9472	−0.9434	−0.9444
0.2	−0.5447	−0.5442	−0.5432	−0.5437
0.8	−0.0915	−0.0914	−0.0910	−0.0913
Δ	0.0092	0.0075	0.0027	

$a=-1$; Theorem 1 proves convergence in H^1 if $\varepsilon>0.5$

	$\varepsilon=0.25$	$\varepsilon=0.5$	$\varepsilon=0.75$	exact
−0.8	− 1.7607	−1.7653	−1.7549	−1.7524
−0.2	− 0.9466	−0.9447	−0.9437	−0.9444
0.2	− 0.5844	−0.5437	−0.5429	−0.5437
0.8	−19.4668	−0.0913	−0.0912	−0.0913
Δ	66.3646	0.0228	0.0048	

$a=-2$; Theorem 1 proves convergence in H^1 if $\varepsilon>0.87$

	$\varepsilon=0.5$	$\varepsilon=0.75$	$\varepsilon=0.9$	exact
−0.8	−1.7491	−1.7629	−1.7582	−1.7524
−0.2	−0.9499	−0.9329	−0.9414	−0.9444
0.2	−0.5283	−0.5333	−0.5435	−0.5437
0.8	78.8829	6.7821	−0.0951	−0.0913
Δ	324.2297	22.8431	0.0068	

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