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Ergodic Theorems for Convex Sets and Operator Algebras

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1. Introduction

In its simplest form, the classical individual (or pointwise) ergodic theorem of G.D. Birkhoff states that if $(X, \mathscr{S}, \mu)$ is a probability measure space, α is an invertible measure-preserving transformation of X and f is an integrable complex-valued function on X, then the averages

$$s_n(f) = \frac{1}{n}(f + \alpha f + \alpha^2 f + \cdots + \alpha^{n-1} f)$$

converge almost everywhere to an α-invariant function $\hat{f}$ (where αf is the function defined by $\alpha f(x) = f(\alpha(x))$). If we restrict attention to the case where f is bounded, we are dealing with an element of the commutative von Neumann algebra $\mathfrak{A} = L^\infty(X, \mu)$, and α gives rise to an automorphism of $\mathfrak{A}$. There is a standard way of expressing the almost everywhere convergence intrinsically in terms of $\mathfrak{A}$ (i.e. without explicit reference to the base space X), and that is to make use of Egorov's theorem. In fact, if $s_n(f) \to \hat{f}$ almost everywhere on X, then there is a "large" measurable subset of X on which $s_n(f) \to \hat{f}$ uniformly. If e is the characteristic function of this subset then $s_n(f)e \to \hat{f}e$ in the L^∞ norm. So we can state the ergodic theorem in terms of the algebra $\mathfrak{A}$ as follows.

Theorem. *Let $\mathfrak{A}$ be a (commutative) von Neumann algebra, α an automorphism of $\mathfrak{A}$ and μ a faithful α-invariant normal state of $\mathfrak{A}$. For each f in $\mathfrak{A}$ and $\varepsilon > 0$ there is a projection e in A with $\mu(e) > 1 - \varepsilon$ such that*

$$\|(s_n(f) - \hat{f})e\| \to 0 \quad \textit{as } n \to \infty.$$

The object of this paper is to show that in this form the ergodic theorem is still valid if we delete the hypothesis that $\mathfrak{A}$ be commutative (see Theorem 5.7). In other words, if α is an automorphism of an arbitrary von Neumann algebra $\mathfrak{A}$ and there is a faithful α-invariant normal state of $\mathfrak{A}$ then for any A in $\mathfrak{A}$ the averages $s_n(A)$ converge "almost uniformly" to an α-invariant element $\hat{A}$.

All the existing proofs of the (commutative) ergodic theorem seem to make use of a result known as the maximal ergodic theorem. This states that, if X, μ, α

are as above and f is real-valued, there is a measurable subset Y of X on which $s_n(f) \leqq 0$ for all n, such that $\int_{X \setminus Y} f d\mu \geqq 0$. Let K be the Choquet simplex of all probability measures on X. If Y and $X \setminus Y$ are not null then $\mu_1 = \mu(Y)^{-1} \mu|Y$ (where $\mu|Y$ is the restriction of μ to Y) and $\mu_2 = \mu(X \setminus Y)^{-1} \mu|(X \setminus Y)$ are in K and μ is a convex combination of them. Also $\int_X f d\mu_2 \geqq 0$ and $\int_X s_n(f) d\nu \leqq 0$ for all n and for all ν in the face of K generated by μ_1. In order to prove the non-commutative ergodic theorem we have to establish a result of this type for general compact convex sets, and this is done in Section 2. Looked at another way, Theorem 2.1 says that if the space X above is a compact convex set in some real locally convex space, α is a bicontinuous affine automorphism of X and μ is the point mass situated at some point z of X (so that $\alpha(z) = z$), then for a continuous affine function f we can replace μ by another probability measure with resultant z in such a way that the set Y above can be chosen to be a face of X. This result may possibly point the way to an ergodic theorem for affine automorphisms of compact convex sets, but we have not attempted to make any further investigations in this direction.

Returning to the non-commutative situation, let α be an automorphism of a von Neumann algebra with a faithful α-invariant normal state. For an element A in $\mathfrak{A}$, we wish to show that $s_n(A) \to \hat{A}$ almost uniformly, and this we do in three stages. In Section 3 we present the construction, due to Kovács and Szücs [11], of the map $A \mapsto \hat{A}$. In Section 4 we show that there is a large linear subspace $\mathfrak{A}_u$ of $\mathfrak{A}$ in which the convergence $s_n(A) \to \hat{A}$ is actually uniform. This result is used in Section 5, together with Theorem 2.1, in the proof of the main theorem.

We assume familiarity with the general theory of operator algebras, as contained in [4] and [17]. If $\mathfrak{A}$ is a unital C^*-algebra then its state space $S(\mathfrak{A})$ is a compact convex subset of (the selfadjoint part of) the dual space $\mathfrak{A}^*$ with the weak* topology, and every selfadjoint element of $\mathfrak{A}$ defines a continuous affine real-valued function on $S(\mathfrak{A})$. If α is an automorphism of $\mathfrak{A}$ (by which is meant an automorphism for the $*$-algebra structure) then α is isometric, and the adjoint mapping α^* on $\mathfrak{A}^*$ gives a bicontinuous affine automorphism of $S(\mathfrak{A})$. If $\mathfrak{A}$ is a von Neumann algebra, with predual $\mathfrak{A}_*$, then α is also bicontinuous for the ultraweak (or weak*) topology and is therefore the adjoint of an isometric linear automorphism α_* of $\mathfrak{A}_*$.

The second dual $\mathfrak{A}^{**}$ of a C^*-algebra $\mathfrak{A}$ is a von Neumann algebra (the enveloping von Neumann algebra of $\mathfrak{A}$), and every state ρ of $\mathfrak{A}$ is a normal state of $\mathfrak{A}^{**}$. By a slight abuse of notation, we identify $\mathfrak{A}$ with its canonical image in $\mathfrak{A}^{**}$ (and in the case of a von Neumann algebra we identify $\mathfrak{A}_*$ with its canonical image in $\mathfrak{A}^*$). If $\mathfrak{A}$ is a von Neumann algebra then the adjoint of the canonical inclusion of $\mathfrak{A}_*$ in $\mathfrak{A}^*$ is a normal retraction π from $\mathfrak{A}^{**}$ onto $\mathfrak{A}$. It follows from the definition of π that if ρ is a normal state of $\mathfrak{A}$ then the composition $\rho\pi$ is equal to ρ. The mapping π is order-preserving so, in particular, if $\mathfrak{S}$ is any subset of $\mathfrak{A}$ and A is an upper bound for $\mathfrak{S}$ in $\mathfrak{A}^{**}$ then $\pi(A)$ is an upper bound for $\mathfrak{S}$ in $\mathfrak{A}$.

We make essential use of the correspondence (due to Effros [7] and Prosser [15]) between the ideal structure of an operator algebra and the facial structure of its state space. If $\mathfrak{A}$ is a unital C^*-algebra and $\mathscr{F}$ is a face of $S(\mathfrak{A})$ then the norm closure $\bar{\mathscr{F}}$ is also a face of $S(\mathfrak{A})$ (Theorem 4.6 of [7]; the weak* closure of $\mathscr{F}$ is not in general a face – see §6 of [7]). The ultraweakly closed left ideal

$$\{A\colon \rho(A^*A)=0\ (\rho\in\bar{\mathscr{F}})\}$$

in $\mathfrak{A}^{**}$ is equal to $\mathfrak{A}^{**}F$ for some projection F in $\mathfrak{A}^{**}$ (by Proposition 1.10.1 of [17]). Let $E=I-F$, then

$$\bar{\mathscr{F}}=\{\rho\in S(\mathfrak{A})\colon \rho(E)=1\}.$$

Finally, $\bar{\mathscr{F}}$ is identified in an obvious way with the normal state space of the von Neumann algebra $E\mathfrak{A}^{**}E$, so that if $A\in\mathfrak{A}^{**}$ and $\rho(A)\geqq 0$ for all ρ in $\bar{\mathscr{F}}$ then it follows that $EAE\geqq 0$.

The main results of this paper are announced in [12].

2. An Ergodic Theorem for Convex Sets

Let K be a compact convex subset of a real locally convex Hausdorff linear topological space V. For x in K let $\mathscr{F}(x)$ be the face of K generated by x, so that $u\in\mathscr{F}(x)$ if and only if $x=\lambda u+(1-\lambda)v$ for some v in K and some λ with $0<\lambda<1$.

Let α be a bicontinuous affine automorphism of K and suppose there is a point z in K such that $\alpha(z)=z$. Let f be a continuous affine real valued function on K and n a positive integer. Define a continuous convex function h on K by

$$h=\max\{0, f, f+\alpha f, \dots, f+\alpha f+\dots+\alpha^{n-1}f\},$$

where αf is the function given by $\alpha f(x)=f(\alpha(x))$ $(x\in K)$. Let $Q=\{x\in K\colon h(x)=0\}$, so Q is a closed convex subset of K. As in Section 3 of [14] we define $\bar{h}$ to be the infimum of all the continuous affine functions on K which majorize h, so that $\bar{h}$ is a concave bounded upper semicontinuous function on K with $\bar{h}\geqq h\geqq 0$.

(2.1) **Theorem.** *Either $f(z)\geqq 0$, or there are points z_1, z_2 in K such that z is a convex combination $\lambda z_1+(1-\lambda)z_2$ with $\mathscr{F}(z_1)\subseteq Q$ and $(1-\lambda)f(z_2)\geqq 0$.*

Proof. We need only consider the case $f(z)<0$. By Proposition 3.1 of [14], $\bar{h}(z)=\sup\int_K h\,d\mu$, where the supremum is taken over all probability measures on K with resultant z. This supremum is moreover attained, because of the weak*-compactness of the set of all such measures. So choose μ so that $\int_K h\,d\mu=\bar{h}(z)$.

For $0\leqq k\leqq n$ it follows from the definition of h that

$$f+\alpha h\geqq f+\alpha f+\dots+\alpha^k f.$$

At each point of $K\setminus Q$, there must be at least one value of k for which the function on the right hand side of this inequality is positive, and so $f+\alpha h\geqq 0$ on $K\setminus Q$. Thus $f+\alpha h\geqq h$ on $K\setminus Q$, hence

$$\int_{K\setminus Q} f\,d\mu+\int_{K\setminus Q}\alpha h\,d\mu\geqq\int_{K\setminus Q} h\,d\mu.$$

On the set Q, we have $h=0$ and $\alpha h \geqq 0$. Therefore

$$\int_{K\setminus Q} f\,d\mu \geqq \int_K h\,d\mu - \int_K \alpha h\,d\mu$$
$$= \int_K h\,d\mu - \int_K h\,d\mu^\alpha,$$

where μ^α is the composition of μ with α. But μ^α is a probability measure on K with resultant z (since $\alpha(z)=z$) and it follows from the maximality of $\int_K h\,d\mu$ among such measures that the right hand side of the above inequality must be non-negative. So $\int_{K\setminus Q} f\,d\mu \geqq 0$.

By assumption, $\int_K f\,d\mu = f(z) < 0$, from which it follows that $\int_Q f\,d\mu < 0$, so that $\lambda = \mu(Q) > 0$. Thus $\lambda^{-1}\mu|Q$ is a probability measure on Q, with resultant z_1 say. Suppose $\bar{h}(z_1) > 0$. From the way in which $\bar{h}$ is defined, there is a continuous affine function g on K such that $g \geqq \bar{h}$ and $g(z) < \bar{h}(z) + \lambda \bar{h}(z_1)$. Then

$$\bar{h}(z) = \int_K h\,d\mu = \int_{K\setminus Q} h\,d\mu$$
$$\leqq \int_{K\setminus Q} g\,d\mu$$
$$= \int_K g\,d\mu - \int_Q g\,d\mu$$
$$= g(z) - \lambda g(z_1)$$
$$< \bar{h}(z) + \lambda(\bar{h}(z_1) - g(z_1)) \leqq \bar{h}(z).$$

This contradiction shows that $\bar{h}(z_1)=0$. Since $\bar{h}$ is concave and nonnegative it follows that $\bar{h}=0$ on $\mathscr{F}(z_1)$, so that $\mathscr{F}(z_1) \subseteq Q$.

Then either $\lambda=1$ and $z_1=z$, so that the theorem holds for any choice of z_2; or $0<\lambda<1$ in which case $(1-\lambda)^{-1}\mu|(K\setminus Q)$ is a probability measure, with resultant z_2 say, and

$$f(z_2) = (1-\lambda)^{-1} \int_{K\setminus Q} f\,d\mu \geqq 0.$$

It is then clear that $z = \lambda z_1 + (1-\lambda) z_2$.

The second paragraph of the above proof is a modification of Garsia's proof (see [8]) of the maximal ergodic theorem. In fact, if K is a Choquet simplex then, as explained in the Introduction, Theorem 2.1 yields a result which looks very like the maximal ergodic theorem. We shall see in Section 5 that, in the presence of some additional structure, this theorem can be used to prove results which are more recognizably like ergodic theorems. In the general case, it is not easy to see how to proceed. One obvious question is whether Theorem 2.1 holds with $\mathscr{F}(z_1)$ replaced by the *closed* face generated by z_1 (which is not in general the closure of $\mathscr{F}(z_1)$).

Notice that if $s_m(f) = \frac{1}{m}(f + \alpha f + \cdots + \alpha^{m-1} f)$ then

$$h = \max\{0, s_1(f), 2s_2(f), \ldots, n s_n(f)\},$$

so that $s_m(f) \leqq 0$ on Q $(1 \leqq m \leqq n)$. In particular, $s_m(f) \leqq 0$ on $\mathscr{F}(z_1)$ $(1 \leqq m \leqq n)$, and it is this fact that we shall need in Section 5.

3. Mean Ergodic Theorems for von Neumann Algebras

Since the appearance of the original noncommutative ergodic theorem of Kovács and Szücs [11], mean ergodicity in von Neumann algebras has been extensively studied (see especially [2, 3, 5, 18]), and the results presented in this section are not essentially new. However, in keeping with the theme of the rest of the paper, they are formulated in such a way as to draw attention to the way in which the averaging operators converge to the conditional expectation, rather than just demonstrating the existence of the expectation.

Let α be an automorphism of a von Neumann algebra $\mathfrak{A}$. For A in $\mathfrak{A}$, define

$$s_n(A) = \frac{1}{n}[A + \alpha(A) + \alpha^2(A) + \cdots + \alpha^{n-1}(A)] \quad (n \geqq 1),$$

and for ϕ in $\mathfrak{A}_*$, define

$$s_n(\phi) = \frac{1}{n}[\phi + \alpha_*(\phi) + \alpha_*^2(\phi) + \cdots + \alpha_*^{n-1}(\phi)] \quad (n \geqq 1).$$

Let ρ be a faithful α-invariant semifinite normal weight on $\mathfrak{A}$ (later in the paper we shall require ρ to be a state, but the results of this section are proved nearly as easily for a weight).

We now establish some notation and recall some of the basic properties of weights from [1] and [9]. Let

$$\mathfrak{N}_\rho = \{A \in \mathfrak{A} : \rho(A^* A) < \infty\}$$

and let $\mathfrak{A}_0 = \mathfrak{N}_\rho \cap \mathfrak{N}_\rho^*$. Then $\mathfrak{A}_0$ is a prehilbert space under the inner product

$$\langle A, B \rangle = \rho(B^* A).$$

We denote by H_ρ the Hilbert space completion of $\mathfrak{A}_0$, by Λ_ρ the canonical injection of $\mathfrak{A}_0$ in H_ρ and by π_ρ the faithful representation of $\mathfrak{A}$ on H_ρ given by

$$\pi_\rho(A)\Lambda_\rho(B) = \Lambda_\rho(AB) \quad (A \in \mathfrak{A}, B \in \mathfrak{A}_0).$$

Then $\Lambda_\rho(\mathfrak{A})$ is a full left Hilbert algebra with associated left von Neumann algebra $\pi_\rho(\mathfrak{A})$. Denote by $\mathfrak{A}_0'$ the associated right Hilbert algebra. Define

$$\mathfrak{A}^\alpha = \{A \in \mathfrak{A} : \alpha(A) = A\}$$

and let $\hat{\mathfrak{A}}$ be the weak closure of $\mathfrak{A}^\alpha \cap \mathfrak{A}_0$.

(3.1) **Theorem.** *Let $\mathfrak{A}, \alpha, \rho, \mathfrak{A}_0, \hat{\mathfrak{A}}$ be as above.*

(i) *There is a normal retraction $A \mapsto \hat{A}$ from $\mathfrak{A}$ onto $\hat{\mathfrak{A}}$.*

(ii) *For A in $\mathfrak{A}_0$, $s_n(A) \to \hat{A}$ ultrastrongly as $n \to \infty$.*

(iii) *For ϕ in $\mathfrak{A}_*$, there is an α_*-invariant $\hat{\phi}$ in $\mathfrak{A}_*$ such that*

$$s_n(\phi)(A) \to \hat{\phi}(A) \quad (A \in \mathfrak{A}_0).$$

(iv) *If ρ is bounded then $s_n(\phi) \to \hat{\phi}$ in norm.*

Proof. Since ρ is α-invariant, the map

$$\Lambda_\rho(A) \mapsto \Lambda_\rho(\alpha(A)) \qquad (A \in \mathfrak{A}_0)$$

extends to a unitary operator U on H_ρ which implements the automorphism $\pi_\rho \alpha \pi_\rho^{-1}$ of $\pi_\rho(\mathfrak{A})$. For η_1, η_2 in $\mathfrak{A}'_0$ we have

$$\begin{aligned}\langle \pi_\rho(s_n(A))\eta_1, \eta_2\rangle &= \langle \Lambda_\rho(s_n(A)), \eta_2\eta_1^\flat\rangle \\ &= \left\langle \frac{1}{n}\sum_{j=0}^{n-1} U^j \Lambda_\rho(A), \eta_2\eta_1^\flat\right\rangle \to \langle E\Lambda_\rho(A), \eta_2\eta_1^\flat\rangle \qquad (A \in \mathfrak{A}_0)\end{aligned}$$

(where E is the strong limit of the sequence $\frac{1}{n}\sum_{j=0}^{n-1} U^j$, which exists by von Neumann's ergodic theorem). Thus if ϕ in $\mathfrak{A}_*$ is defined by

$$\phi(A) = \langle \pi_\rho(A)\eta_1, \eta_2\rangle \qquad (A \in \mathfrak{A})$$

and $A \in \mathfrak{A}_0$ then $\phi(s_n(A))$ converges to a limit $f(A, \phi)$ as $n \to \infty$, and $|f(A, \phi)| \leqq \|A\| \, \|\phi\|$ since $\|s_n(A)\| \leqq \|A\|$. Since elements of this form are norm-dense in $\mathfrak{A}_*$ it follows by a simple approximation argument that $\phi(s_n(A))$ converges to a limit, which we again call $f(A, \phi)$, for any ϕ in $\mathfrak{A}_*$ and A in $\mathfrak{A}_0$.

For fixed A in $\mathfrak{A}_0$, the map $\phi \mapsto f(A, \phi)$ defines a bounded linear functional on $\mathfrak{A}_*$, i.e. an element of $\mathfrak{A}$, which we call $\hat{A}$. Thus

$$\phi(s_n(A)) \to \phi(\hat{A}) \qquad (\phi \in \mathfrak{A}_*, A \in \mathfrak{A}_0).$$

Clearly $\hat{A} \in \mathfrak{A}^\alpha$ and $\|\hat{A}\| \leqq \|A\|$. It follows from the calculations above that

$$\begin{aligned}\langle \pi_\rho(\hat{A})\eta_1, \eta_2\rangle &= \langle E\Lambda_\rho(A), \eta_2\eta_1^\flat\rangle \\ &= \langle \pi'(\eta_1)E\Lambda_\rho(A), \eta_2\rangle \qquad (\eta_1, \eta_2 \in \mathfrak{A}'_0),\end{aligned}$$

where π' is the right regular representation associated with $\mathfrak{A}'_0$. Since this holds for all η_2 in $\mathfrak{A}'_0$, we have

$$\pi_\rho(\hat{A})\eta_1 = \pi'(\eta_1)E\Lambda_\rho(A) \qquad (\eta_1 \in \mathfrak{A}'_0, A \in \mathfrak{A}_0).$$

Since it is easily seen that $\widehat{A^*} = (\hat{A})^*$, we also have

$$\pi_\rho(\hat{A})^*\eta_1 = \pi'(\eta_1)E\Lambda_\rho(A^*).$$

Since $\Lambda_\rho(\mathfrak{A}_0)$ is full, it follows from Lemma 2.3 of [19] that $E\Lambda_\rho(A) \in \Lambda_\rho(\mathfrak{A}_0)$ and

$$\pi_\rho(\Lambda_\rho^{-1}E\Lambda_\rho(A)) = \pi_\rho(\hat{A}),$$

so that $\hat{A} \in \mathfrak{A}_0$ and $\Lambda_\rho(\hat{A}) = E\Lambda_\rho(A)$.

Next, we show that the map $A \mapsto \hat{A}$ is weakly continuous at 0 on the unit ball of $\mathfrak{A}_0$. For this, suppose that (A_γ) is a directed net in the unit ball of $\mathfrak{A}_0$ converging weakly to zero. Let $B, C \in \mathfrak{A}^\alpha \cap \mathfrak{A}_0$. Then $U\Lambda_\rho(B) = \Lambda_\rho(B)$, $U\Lambda_\rho(C) = \Lambda_\rho(C)$ and $\pi_\rho(A_\gamma) \to 0$ weakly. Hence

$$\begin{aligned}\langle \pi_\rho(\alpha(A_\gamma))\Lambda_\rho(B), \Lambda_\rho(C)\rangle &= \langle U\pi_\rho(A_\gamma)U^*\Lambda_\rho(B), \Lambda_\rho(C)\rangle \\ &= \langle \pi_\rho(A_\gamma)\Lambda_\rho(B), \Lambda_\rho(C)\rangle\end{aligned}$$

and so

$$\langle \pi_\rho(s_n(A_\gamma))\Lambda_\rho(B), \Lambda_\rho(C)\rangle = \langle \pi_\rho(A_\gamma)\Lambda_\rho(B), \Lambda_\rho(C)\rangle \qquad (\text{all } n).$$

Hence

$$\langle \pi_\rho(\hat{A}_\gamma)\Lambda_\rho(B), \Lambda_\rho(C)\rangle = \langle \pi_\rho(A_\gamma)\Lambda_\rho(B), \Lambda_\rho(C)\rangle \xrightarrow[\gamma]{} 0.$$

Thus $\rho(C^*\hat{A}_\gamma B) \to 0$ $(B, C \in \mathfrak{A}^\alpha \cap \mathfrak{A}_0)$. Notice also that $\hat{A}_\gamma \in \mathfrak{A}^\alpha \cap \mathfrak{A}_0 \subseteq \hat{\mathfrak{A}}$. Consider $\psi = \rho | \hat{\mathfrak{A}}$. It is a faithful semifinite normal weight on $\hat{\mathfrak{A}}$. Hence the associated representation π_ψ is faithful, and we know that

$$\psi(C^* A_\gamma B) = \langle \pi_\psi(\hat{A}_\gamma)\Lambda_\psi(B), \Lambda_\psi(C)\rangle \xrightarrow[\gamma]{} 0.$$

Since $\|\pi_\psi(\hat{A}_\gamma)\| \leqq 1$ for all γ it follows that $\pi_\psi(A_\gamma) \xrightarrow[\gamma]{} 0$ weakly and thus (since π_ψ is faithful) $\hat{A}_\gamma \to 0$ weakly in $\hat{\mathfrak{A}}$ and hence in $\mathfrak{A}$.

It now follows (by Remark 2.2.3 of [10]) that the maping $A \mapsto \hat{A}$ extends to an ultraweakly continuous mapping from $\mathfrak{A}$ onto $\hat{\mathfrak{A}}$, which is evidently a retraction. We show next that, for A in $\mathfrak{A}_0$, $s_n(A) \to \hat{A}$ ultrastrongly. In fact, given η in $\mathfrak{A}_0'$, we have

$$\begin{aligned}
\pi_\rho(s_n(A))\eta &= \pi'(\eta)\Lambda_\rho(s_n(A)) \\
&= \pi'(\eta)\frac{1}{n}\sum_{j=0}^{n-1} U^j \Lambda_\rho(A) \\
&\to \pi'(\eta) E \Lambda_\rho(A) \\
&= \pi'(\eta)\Lambda_\rho(\hat{A}) = \pi_\rho(\hat{A})\eta.
\end{aligned}$$

Since $\|\pi_\rho(s_n(A))\| \leqq \|A\|$ for all n it follows that $\pi_\rho(s_n(A)) \to \pi_\rho(\hat{A})$ strongly and hence ultrastrongly. Therefore $s_n(A) \to \hat{A}$ ultrastrongly.

For ϕ in $\mathfrak{A}_*$ and A in $\mathfrak{A}_0$ define $\hat{\phi}(A) = \phi(\hat{A})$. If $A_\gamma \to 0$ ultraweakly then so does $\hat{A}_\gamma$ and so $\hat{\phi}(A_\gamma) \to 0$. Thus $\hat{\phi}$ is an ultraweakly continuous linear functional on $\mathfrak{A}_0$ and extends by continuity to an element $\hat{\phi}$ of $\mathfrak{A}_*$. Since $\hat{\phi}$ is bounded by $\|\phi\|$ on $\mathfrak{A}_0$, it follows from Kaplansky's density theorem that $\|\hat{\phi}\| \leqq \|\phi\|$. Clearly $\hat{\phi}$ is α_*-invariant.

Finally, let F be a projection in $\mathfrak{A}_0$ and define ϕ in $\mathfrak{A}_*$ by

$$\phi(A) = \langle \pi_\rho(A)\eta_1, \eta_2\rangle \qquad (A \in \mathfrak{A}),$$

where $\eta_1, \eta_2 \in \mathfrak{A}_0'$. Then, for A in $\mathfrak{A}$,

$$\begin{aligned}
s_n(\phi)(AF) &= \langle \pi_\rho(s_n(AF))\eta_1, \eta_2\rangle \\
&= \langle \Lambda_\rho(s_n(AF)), \eta_2\eta_1^\flat\rangle \\
&= \left\langle \frac{1}{n}\sum_{j=0}^{n-1} U^j \Lambda_\rho(AF), \eta_2\eta_1^\flat \right\rangle \\
&= \left\langle \pi_\rho(A)\Lambda_\rho(F), \frac{1}{n}\sum_{j=0}^{n-1} U^{-j}(\eta_2\eta_1^\flat) \right\rangle \\
&\to \langle \pi_\rho(A)\Lambda_\rho(F), E(\eta_2\eta_1^\flat)\rangle = \hat{\phi}(AF),
\end{aligned}$$

and the convergence is uniform on the unit ball $\mathfrak{A}_1 F$ of $\mathfrak{A} F$. A simple approximation argument shows that $s_n(\phi)\to\hat{\phi}$ uniformly on $\mathfrak{A}_1 F$ for any ϕ in $\mathfrak{A}_*$. In particular, if ρ is bounded then $\mathfrak{A}_0=\mathfrak{A}$ so we can take F to be the identity and deduce that $s_n(\phi)\to\hat{\phi}$ in norm.

Remark. If $\mathfrak{A}$ has no α-invariant normal states then it is easy to see that $\hat{\mathfrak{A}}=(0)$. Notice that in this case it is possible for $s_n(A)$ to converge to a limit different from $\hat{A}$, if $A\notin\mathfrak{A}_0$. (For example, take A to be the identity.) From a probabilistic point of view it is thus rather misleading, in this case, to think of the retraction $A\mapsto\hat{A}$ as a "conditional expectation". However, for the remainder of this paper, we shall assume that ρ is bounded, so this situation will not arise.

4. Uniform Ergodicity

Let α be an automorphism of a von Neumann algebra $\mathfrak{A}$ as before, but this time let ρ be a faithful α-invariant normal state. We wish to investigate whether the result of the previous section, that $s_n(A)\to\hat{A}$ ultrastrongly for A in $\mathfrak{A}$, can be strengthened. In this section we show that any A in $\mathfrak{A}$ is the strong* limit of a bounded sequence (A_k) in $\mathfrak{A}$ such that $s_n(A_k)\to\hat{A}_k$ in norm, as $n\to\infty$. If (A_k) were not required to be bounded, this would be an easy result to prove, but in fact the boundedness is essential for the application of the result in the next section.

We need to use the following lemma, which is concerned with the convergence to zero (in the l^1-norm) of the averages of an element x in $l^1(\mathbb{Z})$ under the shift operator on $l^1(\mathbb{Z})$.

(4.1) **Lemma.** *If* $x\in l^1(\mathbb{Z})$ *and* $\sum_{k\in\mathbb{Z}} x_k=0$ *then*

$$\sum_{r\in\mathbb{Z}}\left|\frac{1}{n}\sum_{j=0}^{n-1} x_{r-j}\right|\to 0 \quad \text{as } n\to\infty.$$

Proof. Let $\varepsilon>0$ and choose n so that $\sum_{|k|>n}|x_k|<\frac{1}{4}\varepsilon$. Then also $|\sum_{|k|\leq n} x_k|<\frac{1}{4}\varepsilon$. For r in $\mathbb{Z}$, consider $a_N=\left|\frac{1}{N}\sum_{j=0}^{N-1} x_{r-j}\right|$, where $N>16n\varepsilon^{-1}\|x\|_1$. If $r\geq N+n$ or $r<-n$ then $a_N\leq\frac{1}{N}\sum_{j=0}^{N-1}|x_{r-j}|$; if $n\leq r<N-n$ then

$$a_N\leq\frac{1}{N}\Big(\Big|\sum_{|k|\leq n} x_k\Big|+\sum_{\substack{0\leq j\leq N-1\\ |r-j|>n}}|x_{r-j}|\Big)<\tfrac{1}{2}\varepsilon N^{-1};$$

while if $-n\leq r<n$ or $N-n\leq r<N+n$ then $a_N\leq N^{-1}\|x\|_1$. Keeping N fixed, we now sum over r to obtain the estimate

$$\sum_{r\in\mathbb{Z}}\frac{1}{N}\left|\sum_{j=0}^{N-1} x_{r-j}\right|\leq\frac{1}{N}\sum|x_{r-j}|+\tfrac{1}{2}\varepsilon N^{-1}(N-2n)+4nN^{-1}\|x\|_1,$$

where in the first term on the right hand side the summation is taken over all values of r and j such that $0 \leqq j \leqq N-1$ and $|r-j|>n$. Summing over r first and then j, we see that this sum cannot exceed $\frac{1}{4}\varepsilon$. Thus

$$\sum_{r\in\mathbb{Z}} \frac{1}{N}\left|\sum_{j=0}^{N-1} x_{r-j}\right| < \varepsilon \quad \text{whenever } N > 16n\varepsilon^{-1}\|x\|_1 .$$

This completes the proof.

(4.2) **Theorem.** *Let α be an automorphism of a von Neumann algebra $\mathfrak{A}$, let ρ be a faithful α-invariant normal state of $\mathfrak{A}$ and let*

$$\mathfrak{A}_u = \{A \in \mathfrak{A} : \|s_n(A) - \hat{A}\| \to 0 \text{ as } n \to \infty\}.$$

For any A in $\mathfrak{A}$ there is a bounded sequence in $\mathfrak{A}_u$ which converges to A in the strong topology.*

Proof. For $x=(x_r)$ in $l^1(\mathbb{Z})$, define $x \cdot \alpha = \sum_{r\in\mathbb{Z}} x_r \alpha^r$ (the sum converges to give a bounded linear operator on $\mathfrak{A}$). If $\sum x_r = 0$ and $A \in \mathfrak{A}$ then $s_n((x\cdot\alpha)(A))$ converges to zero in norm, since

$$\begin{aligned}\frac{1}{n}\sum_{j=0}^{n-1} \alpha^j\Big(\sum_{r\in\mathbb{Z}} x_r \alpha^r(A)\Big) &= \frac{1}{n}\sum_{j=0}^{n-1}\sum_{r\in\mathbb{Z}} x_r \alpha^{r+j}(A)\\ &= \sum_{k\in\mathbb{Z}}\left(\frac{1}{n}\sum_{j=0}^{n-1} x_{k-j}\right)\alpha^k(A),\end{aligned}$$

so that

$$\|s_n((x\cdot\alpha)(A))\| \leqq \sum_{k\in\mathbb{Z}}\left|\frac{1}{n}\sum_{j=0}^{n-1} x_{k-j}\right| \|A\|,$$

and this converges to zero as $n\to\infty$ by Lemma 4.1.

Define a sequence (f_k) of functions on the unit circle by

$$f_k(e^{i\theta}) = \begin{cases} 1 & (|\theta| \leqq \pi/k) \\ 0 & (\pi/k < |\theta| \leqq \pi). \end{cases}$$

Then the Fourier coefficients of f_k are given by

$$\hat{f}_k(r) = \begin{cases} 1/k & (r=0) \\ \dfrac{1}{r\pi}\sin\dfrac{r\pi}{k} & (r \neq 0). \end{cases}$$

Let $g_k = 1 - k(f_k * f_k)$ (where the star denotes the convolution product), and let $x^{(k)} = (x_r^{(k)})$ be the Fourier series of g_k, so that

$$x_r^{(k)} = \begin{cases} 1 - 1/k & (r=0) \\ \dfrac{-k}{r^2\pi^2}\sin^2\dfrac{r\pi}{k} & (r \neq 0). \end{cases}$$

Then g_k takes the value 0 at 1, 1 at $e^{i\theta}$ whenever $2\pi/k \leqq |\theta| \leqq \pi$ and is linear (in θ) on the complementary arcs. So (g_k) is uniformly bounded and $1-g_k$ converges pointwise to the characteristic function h of $\{1\}$. Also, $x^{(k)}$ is in $l^1(\mathbb{Z})$, with

$$\|x^{(k)}\|_1 \leqq 1 + k\,\|\hat{f}_k\|_2^2 = 1 + k\,\|f_k\|_2^2 = 2$$

(by the Parseval relation: we use the normalized Haar measure on the circle, so that $\|f_k\|_2 = k^{-\frac{1}{2}}$); and $\sum_{r\in\mathbb{Z}} x_r^{(k)} = 0$ since $g_k(1) = 0$.

Now fix A in $\mathfrak{A}$ and let $A_k = \hat{A} + (x^{(k)} \cdot \alpha)(A)$. It follows from the first part of the proof that $s_n(A_k) \to \hat{A}$ in norm as $n \to \infty$, so $A_k \in \mathfrak{A}_u$. Also, from the above calculations, $\|A_k\| \leqq \|x^{(k)}\|_1\, \|A\| \leqq 2\,\|A\|$, so the sequence (A_k) is bounded. The next step in the proof is to show that $A_k \to A$ strongly. Let π_ρ be the (faithful) GNS representation of $\mathfrak{A}$ on H_ρ, with cyclic vector ξ_ρ, associated with ρ. Since ρ is α-invariant, there is a unitary U on H_ρ which implements $\pi_\rho \alpha \pi_\rho^{-1}$, and

$$U\pi_\rho(B)\,\xi_\rho = \pi_\rho(\alpha(B))\,\xi_\rho \qquad (B \in \mathfrak{A}).$$

Let $U = \int \lambda E(d\lambda)$ be the spectral representation of U, then for x in $l^1(\mathbb{Z})$ it is easily seen that

$$\pi_\rho((x \cdot \alpha)(A))\,\xi_\rho = \int \check{x}(\lambda)\, E(d\lambda)\, \pi_\rho(A)\,\xi_\rho$$

(where $\check{x}$ is the inverse Fourier transform of x, so that $\check{x}(\lambda) = \sum_{r\in\mathbb{Z}} x_r \lambda^r$). Also, it is a consequence of von Neumann's mean ergodic theorem that $\pi_\rho(\hat{A})\,\xi_\rho = E\pi_\rho(A)\,\xi_\rho$, where E is the projection onto the eigenspace of U corresponding to the eigenvalue 1, so that $E = \int h(\lambda)\, E(d\lambda)$ with h as above. Therefore,

$$\begin{aligned}\pi_\rho(A_k)\,\xi_\rho &= \pi_\rho(\hat{A} + (x^{(k)} \cdot \alpha)(A))\,\xi_\rho \\ &= \int (h(\lambda) + g_k(\lambda))\, E(d\lambda)\, \pi_\rho(A)\,\xi_\rho .\end{aligned}$$

Since the functions $h + g_k$ are uniformly bounded and converge pointwise to 1 it follows from Corollary X.2.8 of [6] that $\int (h(\lambda) + g_k(\lambda))\, E(d\lambda)$ converges strongly to the identity, so in particular $\pi_\rho(A_k)\,\xi_\rho \to \pi_\rho(A)\,\xi_\rho$. Since ξ_ρ is separating for $\pi_\rho(\mathfrak{A})$ and the operators $\pi_\rho(A_k)$ form a bounded sequence, we conclude that $\pi_\rho(A_k)$ converges strongly to $\pi_\rho(A)$ and therefore that $A_k \to A$ strongly, as required.

Finally, since all the coefficients $x_r^{(k)}$ are real, we have $A_k^* = A^* + (x^{(k)} \cdot \alpha)(A^*)$, and the argument of the previous paragraph applied to A^* in place of A shows that $A_k^* \to A^*$ strongly. Thus $A_k \to A$ in the strong* topology, and the proof is complete.

Notice that $\mathfrak{A}_u$ is a norm-closed selfadjoint linear subspace of $\mathfrak{A}$ (the norm-closure follows from Theorem VIII.5.1 of [6]), but there appears to be no reason to suppose that it is closed under multiplication. It would be interesting to know whether $\mathfrak{A}_u$ contains a C^*-algebra which is weakly dense in $\mathfrak{A}$.

5. The Almost Uniform Ergodic Theorem

Let $\mathfrak{A}$ be a C^*-algebra with identity and let $S(\mathfrak{A})$ denote the state space of $\mathfrak{A}$ with the weak* topology. Then $S(\mathfrak{A})$ is compact and convex, and each selfadjoint element of $\mathfrak{A}$ corresponds in a natural way to a continuous affine real-valued function on $S(\mathfrak{A})$. If α is an automorphism of $\mathfrak{A}$ then α^* gives a continuous affine automorphism of $S(\mathfrak{A})$.

(5.1) **Lemma.** *Let α be an automorphism of a unital C^*-algebra $\mathfrak{A}$ and let ρ be a faithful α-invariant state of $\mathfrak{A}$. Suppose that B is a positive element in the unit ball of $\mathfrak{A}$ with $\rho(B)=\varepsilon$ and that n is a positive integer. Then there is a projection E in $\mathfrak{A}^{**}$ such that $\rho(E)\geqq 1-\varepsilon^{\frac{1}{2}}$ and*

$$s_m(B)\leqq 2\varepsilon^{\frac{1}{2}}E+2(I-E) \quad (1\leqq m\leqq n).$$

Proof. If $\varepsilon=0$ then $B=0$ (since ρ is faithful) and we can take $E=I$. If $\varepsilon=1$ then $B=I$ and we can take $E=0$. So we may assume $0<\varepsilon<1$.

Let $A=B-\varepsilon^{\frac{1}{2}}I$. Applying Theorem 2.1 with $K=S(\mathfrak{A})$, $z=\rho$ and f equal to (the affine function corresponding to) A, we have $\rho(A)=\varepsilon-\varepsilon^{\frac{1}{2}}<0$ and so there are states σ_1, σ_2 of $\mathfrak{A}$ such that ρ is a convex combination $\lambda\sigma_1+(1-\lambda)\sigma_2$, $(1-\lambda)\sigma_2(A)\geqq 0$ and $\sigma(s_m(A))\leqq 0$ for $1\leqq m\leqq n$ whenever σ is in the face $\mathscr{F}$ of $S(\mathfrak{A})$ generated by σ_1. Let $\bar{\mathscr{F}}$ be the norm closure of $\mathscr{F}$ in $S(\mathfrak{A})$, then $\bar{\mathscr{F}}$ is a face of $S(\mathfrak{A})$ by Corollary 4.3 of [7] and so (see Section 1) there is a projection E in $\mathfrak{A}^{**}$ such that $\sigma_1(E)=1$ and $E s_m(A) E\leqq 0$ for $1\leqq m\leqq n$. Let $F=I-E$.

Since $A\geqq -\varepsilon^{\frac{1}{2}}I$, we have $\sigma_1(A)\geqq -\varepsilon^{\frac{1}{2}}$ and so

$$\varepsilon-\varepsilon^{\frac{1}{2}}=\rho(A)=\lambda\sigma_1(A)+(1-\lambda)\sigma_2(A)\geqq -\lambda\varepsilon^{\frac{1}{2}}.$$

Hence $\lambda\geqq 1-\varepsilon^{\frac{1}{2}}$, so that $\rho(E)\geqq\lambda\sigma_1(E)\geqq 1-\varepsilon^{\frac{1}{2}}$.

For any positive T in $\mathfrak{A}$ we have

$$T\leqq 2ETE+2FTF$$

(since $(E-F)\,T(E-F)\geqq 0$). Thus, for $1\leqq m\leqq n$,

$$\begin{aligned} s_m(B)&\leqq 2E s_m(B)\,E+2F s_m(B) F\\ &\leqq 2\varepsilon^{\frac{1}{2}}E+2F. \end{aligned}$$

(5.2) **Lemma.** *Let $\mathfrak{A}$, α, ρ, B, ε be as in Lemma 5.1. There is a positive element C of $\mathfrak{A}^{**}$ with $\|C\|\leqq 2$ and $\rho(C)\leqq 4\varepsilon^{\frac{1}{2}}$ such that $s_n(B)\leqq C$ for all n.*

Proof. For each n, let E_n be the projection obtained in Lemma 5.1. If $C_n=2\varepsilon^{\frac{1}{2}}E_n+2(I-E_n)$ then $\|C_n\|\leqq 2$, $\rho(C_n)\leqq 4\varepsilon^{\frac{1}{2}}$ and $s_m(B)\leqq C_n$ for $1\leqq m\leqq n$. The sequence (C_n) in $\mathfrak{A}^{**}$ has a weakly convergent subnet with limit C satisfying all the conclusions of the lemma.

(5.3) **Corollary.** *If $\mathfrak{A}$ is a von Neumann algebra and ρ is normal then we can choose C (in Lemma 5.2) in $\mathfrak{A}$.*

Proof. Let C be an element of $\mathfrak{A}^{**}$ which satisfies the conclusions of Lemma 5.2, then so does the element $\pi(C)$ of $\mathfrak{A}$ (see Section 1), where π is the natural projection from $\mathfrak{A}^{**}$ onto $\mathfrak{A}$.

Corollary 5.3 tells us that if B is very small (as measured by ρ) then all of its averages $s_n(B)$ are fairly small. This (in the commutative case) is intuitively the content of the maximal ergodic theorem, and in fact one can obtain Corollary 5.3 for commutative $\mathfrak{A}$ is an easy consequence of the maximal ergodic theorem (one can also take C in the unit ball and replace $4\varepsilon^{\frac{1}{2}}$ by the sharper estimate $\varepsilon(1-\log\varepsilon)$, in this case).

The next main step in the argument is Lemma 5.6, in the proof of which we shall require the following two minor technical results.

(5.4) **Lemma.** *Suppose A, B are Hilbert space operators with $0 \leqq A \leqq B \leqq I$, E is a projection and $\varepsilon > 0$. Then*

$$\|BE\| < \varepsilon \Rightarrow \|AE\| < \varepsilon^{\frac{1}{4}}.$$

Proof. By Lemma 2 in §1.1 of [4], there is an operator S with $\|S\| \leqq 1$ such that $A^{\frac{1}{2}} = SB^{\frac{1}{2}}$. Thus, if $\|BE\| < \varepsilon$, we have

$$\|B^{\frac{1}{2}} E\| = \|EBE\|^{\frac{1}{2}} \leqq \|BE\|^{\frac{1}{2}} < \varepsilon^{\frac{1}{2}},$$

so that

$$\|AE\| = \|A^{\frac{1}{2}} SB^{\frac{1}{2}} E\| \leqq \|B^{\frac{1}{2}} E\| < \varepsilon^{\frac{1}{2}}.$$

For the next lemma, suppose A is an element of a von Neumann algebra. By expressing A in terms of its selfadjoint and skew-adjoint parts and then writing each of these as the difference of its positive and negative parts, we obtain

$$A = A_1 - A_2 + iA_3 - iA_4,$$

where $A_i \geqq 0$ and $\|A_i\| \leqq \|A\|$. Define

$$A^{++} = A_1 + A_2 + A_3 + A_4.$$

(5.5) **Lemma.** *Suppose (B_k) is a bounded sequence in the von Neumann algebra $\mathfrak{A}$ which converges to zero in the strong* topology, then so is (B_k^{++}).*

Proof. If $B_k = H_k + iK_k$, so that $H_k = B_{k1} - B_{k2}$ and $K_k = B_{k3} - B_{k4}$, then it is clear that $H_k \to 0$ and $K_k \to 0$ (strong*). It is therefore sufficient to consider the case where each B_k is selfadjoint and to show that $B_{k1} \to 0$. However if $B_k \to 0$ then $|B_k| = (B_k^2)^{\frac{1}{2}} \to 0$ and so $B_{k1} = \frac{1}{2}(B_k + |B_k|) \to 0$ (the convergence in each case being strong*).

(5.6) **Lemma.** *Let α be an automorphism of a von Neumann algebra $\mathfrak{A}$ and let ρ be a faithful α-invariant normal state. Suppose that (B_k) is a bounded sequence in $\mathfrak{A}$ which converges to zero in the strong* topology and that $\varepsilon > 0$. Then there is a subsequence (B_{k_j}) of (B_k) and a projection E in $\mathfrak{A}$ such that $\rho(E) > 1 - \varepsilon$ and $s_n(B_{k_j}) E \to 0$ in norm as $j \to \infty$, uniformly in n.*

Proof. Consider first the case where each B_k is positive. We may assume that $0 \leqq B_k \leqq I$. Let $\varepsilon_k = \rho(B_k)$, then $\varepsilon_k \to 0$ as $k \to \infty$. We apply Corollary 5.3 with B_k in place of B and write C_k for the operator C. Then $s_n(B_k) \leqq C_k$ for all n (and all k), (C_k) is a bounded sequence in $\mathfrak{A}$ and $\rho(C_k) \leqq 4\varepsilon_k^{\frac{1}{2}} \to 0$ as $k \to \infty$. Since ρ is faithful it follows that $C_k \to 0$ strongly. Thus we can apply the non-commutative Egorov theorem of Saito and Pedersen ([13], [16]), from which we deduce that there is a subsequence (C_{k_j}) of (C_k) and a projection E in A such that $\rho(E) > 1 - \varepsilon$ and $\|C_{k_j} E\| \to 0$ as $j \to \infty$. It follows from Lemma 5.4 that $\|s_n(B_{k_j}) E\| \to 0$ uniformly in n, as required.

To deal with the general case, we observe that by Lemma 5.5 we can apply the first part of the proof to the sequence (B_k^{++}) to find a subsequence and a projection such that $\|s_n(B_{k_j}^{++}) E\| \to 0$ as $j \to \infty$, uniformly in n. Since

$$s_n(B_{k_j i}) \leqq s_n(B_{k_j}^{++}) \quad (i = 1, 2, 3, 4),$$

we deduce from Lemma 5.4 that $\|s_n(B_{k_{ji}})E\| \to 0$ and so $\|s_n(B_{k_j})E\| \to 0$ as $j \to \infty$, uniformly in n.

We are now in a position to prove the main result of this paper, the non-commutative version of the pointwise ergodic theorem.

(5.7) **Theorem.** *Let α be an automorphism of a von Neumann algebra $\mathfrak{A}$ and let ρ be a faithful α-invariant normal state. For each A in $\mathfrak{A}$ and $\varepsilon>0$ there is a projection E in $\mathfrak{A}$ with $\rho(E)>1-\varepsilon$ such that*

$$\|(s_n(A)-\hat{A})E\| \to 0 \quad \text{as } n \to \infty.$$

Proof. By Theorem 4.2 there is a bounded sequence (A_k) in $\mathfrak{A}_u$ which converges to A in the strong* topology. Since $\hat{A}_k \to \hat{A}$ (strong*), we can replace A by $A-\hat{A}$, A_k by $A_k-\hat{A}_k$ and suppose that $\hat{A}_k=\hat{A}=0$. Write $B_k=A-A_k$. Then (B_k) is a bounded sequence with strong* limit zero. By Lemma 5.6 there exists a projection E in A with $\rho(E)>1-\varepsilon$ such that, given any $\delta>0$, we can find an index k for which $\|s_n(B_k)E\|<\frac{1}{2}\delta$ for all n. Since $A_k \in \mathfrak{A}_u$ and $\hat{A}_k=0$ there is an integer N such that $\|s_n(A_k)\|<\frac{1}{2}\delta$ whenever $n \geqq N$. Thus, for $n \geqq N$,

$$\|s_n(A)E\| \leqq \|s_n(A_k)E\| + \|s_n(B_k)E\| < \delta.$$

This holds for any $\delta>0$, so the proof is complete.

Remark. Given a finite number of elements $A^{(1)}, \ldots, A^{(r)}$ of $\mathfrak{A}$ and $\varepsilon>0$, it is possible to find a single projection E in $\mathfrak{A}$ with $\rho(E)>1-\varepsilon$ such that

$$\|(s_n(A^{(i)})-\hat{A}^{(i)})E\| \to 0 \quad \text{as } n \to \infty$$

for each $i=1, 2, \ldots, r$. The reason for this is that, following the proof of Theorem 5.7, we obtain bounded sequences $(B_k^{(i)})$ $(1 \leqq i \leqq r)$ which converge strong* to zero. We then apply Lemma 5.6 to the sequence (D_k), where $D_k=\sum_{i=1}^{r} B_k^{(i)++}$, to obtain a projection E with $\rho(E)>1-\varepsilon$ such that a subsequence of $s_n(D_k)E$ tends to zero in norm, uniformly in n. The argument of the last paragraph in the proof of Lemma 5.6 then shows that the same subsequence of $s_n(B_k^{(i)})E$ tends to zero in norm, uniformly in n, for each i. With this choice of E, we may resume the proof of Theorem 5.7 to obtain the desired conclusion.

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