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Extension of Whitney Fields from Subanalytic Sets

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1. Introduction

Whitney's extension theorem [14] provides a continuous linear extension operator from the space of $\mathscr{C}^m$ Whitney fields ($m<\infty$) on a closed subset X of $\mathbb{R}^n$, to the space of $\mathscr{C}^m$ functions on $\mathbb{R}^n$. Though $\mathscr{C}^\infty$ Whitney fields on X extend to $\mathscr{C}^\infty$ functions on $\mathbb{R}^n$, there does not exist a continuous linear extension operator for every closed subset X. Let $\mathscr{E}(X)$ be the Fréchet space of $\mathscr{C}^\infty$ Whitney fields on X. Then $\mathscr{E}(\mathbb{R}^n)$ identifies with the space of $\mathscr{C}^\infty$ functions on $\mathbb{R}^n$. The following extension problem arises: Under what conditions on X is there an extension operator E: $\mathscr{E}(X)\to\mathscr{E}(\mathbb{R}^n)$? An extension operator means, of course, a continuous linear operator E: $\mathscr{E}(X)\to\mathscr{E}(\mathbb{R}^n)$ such that $\mathscr{E}(F)|X=F$ for all $F\in\mathscr{E}(X)$. Seeley [11] and Mityagin [10] showed that an extension operator exists if X is a closed half-space $\mathbb{H}^n$. Stein [12] proved that an extension operator exists when X is a domain with boundary which is locally the graph of a function of Lipschitz class 1.

In this paper we prove two extension theorems. Our main theorem resolves the extension problem for subanalytic sets (hence, in particular, for semianalytic sets).

Theorem I. *Let N be a real analytic manifold and X a closed subanalytic subset of N. Then there exists an extension operator*

$$E\colon\ \mathscr{E}(X)\to\mathscr{E}(N)$$

if and only if the interior of X is dense in X.

The necessity of the hypothesis follows easily from the classical example X = point. Though the theorem is local in nature, we have stated it for a subanalytic subset of a real analytic manifold since the proof will involve working in this context (see Remark 2.3). We will prove the theorem using Hironaka's resolution of singularities, by induction on the lengths of the finite

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sequences of local blowings-up with smooth centers needed locally to rectilinearize the singularities on the boundary of X.

Our second theorem, which in fact will be used in the proof of Theorem I, is a generalization of Stein's extension theorem, for $\mathscr{C}^\infty$ Whitney fields, to the case of a domain with boundary which is Lipschitz of any order.

Let ϕ: $\mathbb{R}^{n-1} \to \mathbb{R}$ be a function which satisfies a Lipschitz condition of order γ, $0<\gamma\leqq 1$; i.e. there is a constant $M>0$ such that

$$|\phi(x)-\phi(x')|\leqq M|x-x'|^\gamma$$

for all $x, x' \in \mathbb{R}^{n-1}$. We consider points in $\mathbb{R}^n$ as pairs (x, y), $x\in\mathbb{R}^{n-1}$, $y\in\mathbb{R}$. The open subset

$$\{(x,y)\in\mathbb{R}^n \mid y>\phi(x)\}$$

is called a *special Lipschitz domain* of *class* Lip γ. A rotation of such a domain will also be called a special Lipschitz domain.

Let Ω be an open subset of $\mathbb{R}^n$, and $\partial\Omega$ its boundary. We say, more generally, that Ω is a *Lipschitz domain* if for each point a in $\partial\Omega$, there exists an open neighborhood U_a of a in $\mathbb{R}^n$, and a special Lipschitz domain Ω_a, such that $\Omega\cap U_a=\Omega_a\cap U_a$. If each Ω_a is of class Lip γ (independent of a), then we say Ω is a Lipschitz domain of *class* Lip γ.

Theorem II. *If X is the closure of a Lipschitz domain Ω, then there exists an extension operator*

$$E\colon \mathscr{E}(X)\to\mathscr{E}(\mathbb{R}^n).$$

If Ω is of class Lip $1/k$, *for some positive integer k, then E may be chosen so that for every compact subset L of $\mathbb{R}^n$, there exists a compact subset K of X such that E satisfies the following estimates. For each $m\in\mathbb{N}$, there is a positive constant C such that*

$$|E(F)|_m^L \leqq C|F|_{km}^K$$

for all $F\in\mathscr{E}(X)$ (the seminorms will be defined in Section 2).

For a Lipschitz domain Ω of class Lip 1, Stein actually defines an extension operator which maps the Sobolov spaces $L_k^p(\Omega)$ into $L_k^p(\mathbb{R}^n)$ for all $k\in\mathbb{N}$ and $1\leqq p\leqq\infty$. Theorem II illustrates an important distinction between the extension problems for the spaces of smooth functions, and for the Sobolov spaces. In the latter case, the Lipschitz condition of order 1 for the boundary of the domain is in the nature of the best possible [12, p. 182].

We will actually prove more than the assertions of Theorems I and II. The formulas of both Seeley and Stein simultaneously extend all classes of differentiability. We will show that in each of our theorems we may also choose an extension operator E which is universal in this sense; E may be induced (locally) by a sequence of extension operators on the Whitney fields of finite differentiability, but with a certain loss of differentiability depending on the singularities of the closed set X. These more precise formulations of Theorems I and II will be stated in Section 3 as Theorem I′ and II′. The loss of differentiability in extending from a Lipschitz domain of class Lip $1/k$, for example, is exactly that indicated in the estimates of Theorem II.

The proofs of both theorems use an extension lemma from [1], generalized in Section 4 to handle all classes of differentiability. In Section 5 we recall the definitions of semianalytic and subanalytic sets, and some lemmas of Łojasiewicz that will be used in the sequel. Thèorem II′ will be proved in Section 7, using the "averaging" Proposition 6.1. Theorem I′ will be proved in Section 9, but in Section 8 we will give a short elementary proof in the 2-dimensional semianalytic case, which motivated the general approach of the following section. These two sections are independent (we will see, in fact, that Proposition 6.1, which is used also in Section 8, is not needed for the special case of Theorem II′ which is used in the proof of Theorem I′).

In an earlier manuscript of this paper, "Extension of $\mathscr{C}^\infty$ Whitney fields from semianalytic sets", our extension operators were defined only on the spaces of $\mathscr{C}^\infty$ Whitney fields. The author is indebted to Jean-Jacques Risler for suggesting that the universal nature of the operators be made explicit. The author profitted also from conversations with Bernard Tessier (on subanalytic sets) and with Hugh Miller and Pierre Milman (the latter, in particular, pointed out an error in the proof of Theorem II in the earlier version).

2. Preliminaries

If $k=(k_1, \dots, k_n)\in\mathbb{N}^n$, $x=(x_1, \dots, x_n)\in\mathbb{R}^n$, then we write $|k|=k_1+\dots+k_n$, $k!=k_1!\dots k_n!$, $x^k=x_1^{k_1}\dots x_n^{k_n}$. $\mathbb{N}^n$ is partially ordered by the relation: $k\leqq l$ if and only if $k_j\leqq l_j$, $j=1,\dots,n$. We write $\binom{l}{k}=\frac{l!}{k!(l-k)!}$ if $k\leqq l$, $\binom{l}{k}=0$ otherwise. $|x|$ denotes the Euclidean norm $|x|=(x_1^2+\dots+x_n^2)^{\frac{1}{2}}$, and $d(x,y)$ the Euclidean distance $d(x,y)=|x-y|$.

If $m\in\mathbb{N}$ and U is an open subset of $\mathbb{R}^n$, then $\mathscr{E}^m(U)$ denotes the space of $\mathscr{C}^m$ functions on U. $\mathscr{E}^m(U)$ is a Fréchet space; its topology is defined by the seminorms

$$|f|_m^K=\sup_{\substack{x\in K\\|k|\leqq m}}\left|\frac{\partial^{|k|}f}{\partial x^k}(x)\right|,$$

where $K\subset U$ is compact.

Let X be a closed subset of U. A *jet* of *order* m on X is a sequence of continuous functions $F=(F^k)_{|k|\leqq m}$ on X. $J^m(X)$ denotes the space of such jets. We write $|F|_m^K=\sup_{\substack{x\in K\\|k|\leqq m}}|F^k(x)|$ if $K\subset X$ is compact, and $F(x)=F^0(x)$, $x\in X$.

There is a linear map J^m: $\mathscr{E}^m(U)\to J^m(X)$, associating to each $f\in\mathscr{E}^m(U)$ the jet $J^m(f)=\left(\frac{\partial^{|k|}f}{\partial x^k}\bigg|X\right)_{|k|\leqq m}$. For each $|k|\leqq m$, there is a linear map D^k: $J^m(X)\to J^{m-|k|}(X)$, defined by $D^kF=(F^{k+l})_{|l|\leqq m-|k|}$. We also denote by D^k the map of $\mathscr{E}^m(U)$ into $\mathscr{E}^{m-|k|}(U)$ given by $D^kf=\frac{\partial^{|k|}f}{\partial x^k}$. This should cause no confusion since $D^k\circ J^m=J^{m-|k|}\circ D^k$.

If $a \in X$ and $F \in J^m(X)$, then the *Taylor polynomial* (of *order m*) of F at a is the polynomial

$$T_a^m F(x) = \sum_{|k| \leqq m} \frac{F^k(a)}{k!} \cdot (x-a)^k$$

of degree $\leqq m$. We define $R_a^m F = F - J^m(T_a^m F)$, so that

$$(R_a^m F)^k(x) = F^k(x) - \sum_{|l| \leqq m-|k|} \frac{F^{k+l}(a)}{l!} \cdot (x-a)^l$$

if $|k| \leqq m$.

We say that $F \in J^m(X)$ is a *Whitney field* of *class* $\mathscr{C}^m$ on X if for each $|k| \leqq m$,

$$(R_x^m F)^k(y) = o(|x-y|^{m-|k|})$$

as $|x-y| \to 0$, $x, y \in X$. $\mathscr{E}^m(X) \subset J^m(X)$ denotes the subspace of Whitney fields of class $\mathscr{C}^m$. $\mathscr{E}^m(X)$ is a Fréchet space, with the seminorms

$$\|F\|_m^K = |F|_m^K + \sup_{\substack{x, y \in K \\ x \neq y \\ |k| \leqq m}} \frac{|(R_x^m F)^k(y)|}{|x-y|^{m-|k|}},$$

where $K \subset X$ is compact. If r is a non-negative rational number, then $\mathscr{E}^r(X)$ will mean $\mathscr{E}^{[r]}(X)$, where $[r]$ is the greatest integer $\leqq r$. Likewise $|\cdot|_r^K$ will denote the seminorm $|\cdot|_{[r]}^K$.

Remark 2.1. If $F \in J^m(U)$ and for all $x \in U$, $|k| \leqq m$ we have

$$\lim_{y \to x} \frac{|(R_x^m F)^k(y)|}{|x-y|^{m-|k|}} = 0,$$

then there exists $f \in \mathscr{E}^m(U)$ such that $F = J^m(f)$. This simple converse of Taylor's theorem shows, in particular, that the two spaces we have denoted $\mathscr{E}^m(U)$ are equivalent. On $\mathscr{E}^m(U)$, the topologies defined by the seminorms $|\cdot|_m^K$, $\|\cdot\|_m^K$ are equivalent (by the open mapping theorem).

Let $\pi_{m,m+1}: J^{m+1}(X) \to J^m(X)$ be the projection which associates to each jet $(F^k)_{|k| \leqq m+1}$ the jet $(F^k)_{|k| \leqq m}$. Clearly $\pi_{m,m+1}(\mathscr{E}^{m+1}(X)) \subset \mathscr{E}^m(X)$. The projective limit $J(X) = \varprojlim J^m(X)$ is the space of *jets* of *infinite order* on X. The projective limit $\mathscr{E}(X) = \varprojlim \mathscr{E}^m(X)$ identifies with a subspace of $J(X)$. An element F of $\mathscr{E}(X)$ is a *Whitney field* of *class* $\mathscr{C}^\infty$ on X. $\mathscr{E}(X)$ has the structure of a Fréchet space defined by the seminorms $\|F\|_m^K$, where $m \in \mathbb{N}$ and $K \subset X$ is compact.

Remark 2.2. Neither $\mathscr{E}^m(X)$ nor $\mathscr{E}(X)$ is in general complete in the topology defined by the seminorms $|\cdot|_m^K$. Let p be a positive integer. A compact subset K of $\mathbb{R}^n$ is called *p-regular* if it is connected by rectifiable arcs, and there exists a positive constant C such that

$$|x-y| \geqq C\delta(x,y)^p$$

or all $x, y \in K$ (δ denotes the geodesic distance on K). If K is p-regular, then for each $m \in \mathbb{N}$, there exists a constant C_m such that $\|F\|_m^K \leqq C_m |F|_{pm}^K$ for all $F \in \mathscr{E}^{pm}(K)$ [13, IV, 3.11]. In particular if K is 1-regular, then the norms $|\cdot|_m^K$ and $\|\cdot\|_m^K$ are equivalent on $\mathscr{E}^m(K)$.

We say that X is *regular* if it is connected, and for all $a \in X$ there exists a positive integer p and a p-regular compact neighborhood of a in X (X is *p-regular* if p is independent of a). In this case the topology of $\mathscr{E}(X)$ is defined by the seminorms $|\cdot|_m^K$, where $m \in \mathbb{N}$ and $K \subset X$ is compact.

A closed set which satisfies the hypotheses of either Theorem I or Theorem II is regular. In fact any closed subanalytic set is regular (cf. [6, Section 18]; the regularity of a closed subset of a $\mathscr{C}^\infty$ manifold may, of course, be expressed locally using a coordinate system). It is easy to check that a Lipschitz domain with boundary of class Lip $1/p$, where p is a positive integer, is p-regular.

Remark 2.3. We will also work with Whitney fields defined on closed subsets of a $\mathscr{C}^\infty$ manifold. Let N be a $\mathscr{C}^\infty$ manifold, and X a closed subset of N. We denote by $J^m(X)$ the space of continuous sections over X of the bundle of jets of order m of $\mathscr{C}^m$ functions on N. The subspace $\mathscr{E}^m(X) \subset J^m(X)$ of Whitney fields of class $\mathscr{C}^m$ may be defined as above, using a local coordinate system. $\mathscr{E}^m(X)$ has the structure of a Fréchet space, with the seminorms defined as before, using compact subsets of coordinate charts. If $X = N$, then $\mathscr{E}^m(N)$ identifies with the space of $\mathscr{C}^m$ functions on N. By Whitney's extension theorem, we may identify $\mathscr{E}^m(X)$ with the quotient of $\mathscr{E}^m(N)$ by the ideal $\mathscr{I}^m(X; N)$ of $\mathscr{C}^m$ functions which are m-flat on X.

For each $m \in \mathbb{N}$, there is a canonical projection $J^{m+1}(X) \to J^m(X)$. Let $J(X)$ (respectively $\mathscr{E}(X)$) be the projective limit $\varprojlim J^m(X)$ (respectively $\varprojlim \mathscr{E}^m(X)$). $\mathscr{E}(X)$ is the space of Whitney fields of class $\mathscr{C}^\infty$ on X.

Let U, U' be open subsets of $\mathbb{R}^n$, $\mathbb{R}^{n'}$, and X, X' closed subsets of U, U' (respectively). If $\pi = (\pi_1, \ldots, \pi_n)$: $U' \to U$ is a $\mathscr{C}^\infty$ mapping such that $\pi(X') \subset X$, then π induces an $\mathbb{R}$-algebra homomorphism $\pi^*(F) = F \circ (\pi | X')$ from $\mathscr{E}(X)$ to $\mathscr{E}(X')$. Suppose $a \in X'$ and $b = \pi(a)$, $b = (b_1, \ldots, b_n)$. If $T_b F \in \mathbb{R}[[y_1 - b_1, \ldots, y_n - b_n]]$ is the formal Taylor series of F at b; i.e.

$$T_b F = \sum_{k \in \mathbb{N}^n} \frac{F^k(b)}{k!} \cdot (y - b)^k,$$

then $T_a(\pi^*(F))$ is obtained by substituting for each y_j in $T_b F$, the formal Taylor series at a of the function π_j. We likewise define π^*: $\mathscr{E}^m(X) \to \mathscr{E}^m(X')$. If $F \in \mathscr{E}^m(X)$, then $T_a^m(\pi^*(F))$ is obtained by substituting the Taylor polynomial $T_a^m \pi_j(x)$ for each y_j in the Taylor polynomial $T_b^m F(y)$, and dropping terms of degree $> m$ in $x - a$.

It will sometimes be convenient to use the index m for either a natural number or $+\infty$, and to write $\mathscr{E}(X) = \mathscr{E}^\infty(X)$, $T_a = T_a^\infty$, etc.

3. Simultaneous Extension of all Classes of Differentiability

Theorem I′. *Let N be a real analytic manifold, and X a compact subanalytic subset of N. Assume that the interior of X is dense in X. Then there exists a positive* *integer k, and for each* $m\in\mathbb{N}$ *an extension operator*

$$E^m\colon \mathcal{E}^m(X)\to\mathcal{E}^{m/k}(N)$$

such that the following diagram commutes for all $m\leqq m'$:

$$\begin{array}{ccc} \mathcal{E}^m(X) & \xrightarrow{E^m} & \mathcal{E}^{m/k}(N) \\ \cup & & \cup \\ \mathcal{E}^{m'}(X) & \xrightarrow{E^{m'}} & \mathcal{E}^{m'/k}(N) \end{array}$$

In particular, an extension operator $E\colon\mathcal{E}(X)\to\mathcal{E}(N)$ *is obtained as the projective limit of the operators* E^m.

Remark 3.1. Theorem I′ will be proved by induction on the lengths of the finite sequences of local blowings-up with smooth centers needed locally to rectilinearize the singularities on the boundary of X. The order of differentiability will be divided by 8 for each local blowing-up (Proposition 9.1).

Theorem II′. *Let* $\Omega\subset\mathbb{R}^n$ *be a Lipschitz domain of class* Lip $1/k$, *and* $X=\Omega\cup\partial\Omega$. *Then for each* $m\in\mathbb{N}$ *there exists an extension operator*

$$E^m\colon \mathcal{E}^m(X)\to\mathcal{E}^{m/k}(\mathbb{R}^n)$$

such that

(1) *the following diagram commutes for all* $m\leqq m'$:

$$\begin{array}{ccc} \mathcal{E}^m(X) & \xrightarrow{E^m} & \mathcal{E}^{m/k}(\mathbb{R}^n) \\ \cup & & \cup \\ \mathcal{E}^{m'}(X) & \xrightarrow{E^{m'}} & \mathcal{E}^{m'/k}(\mathbb{R}^n); \end{array}$$

(2) *for every compact subset* L *of* $\mathbb{R}^n$, *there exists a compact subset* K *of* X *such that* E^m *satisfies the following estimates for each* $m\in\mathbb{N}$. *There exists a positive constant* C *such that*

$$|E^m(F)|^L_{m/k}\leqq C|F|^K_m$$

for all $F\in\mathcal{E}^m(X)$.

In particular, the extension operator $E\colon\mathcal{E}(X)\to\mathcal{E}(\mathbb{R}^n)$ *of Theorem II may be realized as the projective limit of the operators* E^m.

Remark 3.2. We will abbreviate notation throughout the paper by saying that the operators E^m form a *projective system*, rather than giving the commutative diagrams.

4. Technical Extension Lemmas

Lemmas 4.1 and 4.2 of this section, which will be used in the proofs of Theorems I′ and II′ (respectively), are variants of the lifting theorem of [1] and one of its corollaries. The argument in [1] applies directly to all classes of differentiability, so that we refer to [1] for the proofs of these lemmas. The precise estimates in Lemma 4.2 follow from the proof in [1].

Lemma 4.1. *Let X, Y be closed subsets of $\mathbb{R}^n$, $\mathbb{R}^p$ (respectively), and k a positive integer. Let G^m: $\mathscr{E}^m(X) \to \mathscr{E}^{m/k}(Y)$, $m \in \mathbb{N}$, be a projective system of continuous linear mappings. Suppose that for each $b \in Y$, there is a projective system of continuous linear mappings G_b^m: $\mathscr{E}^m(X) \to \mathscr{E}^{m/k}(\mathbb{R}^p)$ such that*

(1) *$G_b^m(F)^l(b) = G^m(F)^l(b)$ for all $m \in \mathbb{N}$, $l \in \mathbb{N}^p$ with $|l| \leqq m/k$, and $F \in \mathscr{E}^m(X)$;*

(2) *for each $m \in \mathbb{N}$ and $L \subset \mathbb{R}^p$ compact, there exists $K = K(m, L) \subset X$ compact and a constant $c = c(m, L)$ such that*

$$|G_b^m(F)|_{m/k}^L \leqq c \, \|F\|_m^K$$

for all $F \in \mathscr{E}^m(X)$ and $b \in Y$.

Then there exists a projective system of continuous linear mappings $\tilde{G}^m$: $\mathscr{E}^m(X) \to \mathscr{E}^{m/k}(\mathbb{R}^p)$ such that $\tilde{G}^m(F) | Y = G^m(F)$ for all $m \in \mathbb{N}$ and $F \in \mathscr{E}^m(X)$.

Lemma 4.2. *Let k be a positive integer, and X a k-regular closed subset of $\mathbb{R}^n$. Suppose that for each point a in the frontier of X, there is a projective system W_a^m: $\mathscr{E}^m(X) \to \mathscr{E}^{m/k}(\mathbb{R}^n)$, $m \in \mathbb{N}$, of continuous linear mappings such that*

(1) *$W_a^m(F)^l(a) = F^l(a)$ for all $m \in \mathbb{N}$, $l \in \mathbb{N}^n$ with $|l| \leqq m/k$, and $F \in \mathscr{E}^m(X)$;*

(2) *for every $L \subset \mathbb{R}^n$ compact, there exists $K \subset K$ compact such that the following uniformity condition is satisfied. For each $m \in \mathbb{N}$, there exists a constant $c = c(m)$ such that*

$$|W_a^m(F)|_{m/k}^L \leqq c \, |F|_m^K$$

for all $F \in \mathscr{E}^m(X)$ and a in the frontier of X.

Then there exists a projective system of extension operators W^m: $\mathscr{E}^m(X) \to \mathscr{E}^{m/k}(\mathbb{R}^n)$ such that W^m satisfies the following estimates for all $m \in \mathbb{N}$. For every $L \subset \mathbb{R}^n$ compact, there exists $K = K(L) \subset X$ compact and a constant $c' = c'(m)$ such that

$$|W^m(F)|_{m/k}^L \leqq c' \, |F|_m^K$$

for all $F \in \mathscr{E}^m(X)$.

Remark 4.3. The operators $\tilde{G}^m$ and W^m of Lemmas 4.1 and 4.2 induce continuous linear mappings on the spaces of $\mathscr{C}^\infty$ Whitney fields. If we work directly in the $\mathscr{C}^\infty$ case, however, the statements of the lemmas may be strengthened by replacing each projective system of continuous linear mappings by a continuous linear mapping between the corresponding spaces of $\mathscr{C}^\infty$ Whitney fields [1].

5. Lemmas of Łojasiewicz

In this section we recall some results of Łojasiewicz [6–8] which will be used in the rest of the paper.

Let U be an open subset of $\mathbb{R}^n$, and X, Y closed subsets of U.

Definition 5.1. X, Y are *regularly situated* if for all $a \in X \cap Y$, there exists a neighborhood V of a, and constants $C > 0$, $\alpha \geqq 0$ such that for all $x \in V$,

$$d(x, X) + d(x, Y) \geqq C d(x, X \cap Y)^{\alpha}.$$

Definition 5.2. Let A be a subset of a real analytic manifold N. We say A is *semianalytic* in N if for each point $x \in N$, there exists an open neighborhood U of x in N and a finite number of real analytic functions g_{ij}, f_{ij} on U such that

$$A \cap U = \bigcup_i \{g_{ij} = 0, f_{ij} > 0 \text{ for all } j\}.$$

Lemma 5.3 [6, Section 18]. *Let U be an open subset of $\mathbb{R}^n$. Then any pair of closed semianalytic subsets of U is regularly situated.*

We will also need Hironaka's generalization of Lemma 5.3 to subanalytic sets [3, Section 9].

Definition 5.4. A subset A of a real analytic manifold N is *subanalytic* if for each $x \in N$, there exists an open neighborhood U of x in N, and a finite system of proper real analytic maps $f_{ij}: N_{ij} \to U$ $(j = 1, 2)$, such that

$$A \cap U = \bigcup_i (\operatorname{Im} f_{i1} - \operatorname{Im} f_{i2}).$$

A semianalytic set is subanalytic [3, 5]. The interior, closure and frontier of a semianalytic (respectively subanalytic) set are semianalytic (respectively subanalytic).

The following lemma will be used for glueing together Whitney fields defined on closed sets.

Lemma 5.5 (cf. [13, IV, Section 4]). *Let U be an open subset of $\mathbb{R}^n$, and X, Y closed subsets of U. Suppose that for all $a \in X \cap Y$, there exists a neighborhood V of a, and a constant $C > 0$ such that for all $x \in V$,*

$$d(x, X) + d(x, Y) \geqq C d(x, X \cap Y).$$

Then:

(1) *There exists a function $\phi \in \mathscr{E}(U - (X \cap Y))$ which has the following properties:*

(a) *$\phi = 0$ in a neighborhood of $X - (X \cap Y)$;*

(b) *$\phi = 1$ in a neighborhood of $Y - (X \cap Y)$;*

(c) *for all $l \in \mathbb{N}^n$ and $K \subset U$ compact, there exists a constant $C' > 0$ such that*

$$|D^l \phi(x)| \leqq C' d(x, X \cap Y)^{-|l|}$$

when $x \in K - (X \cap Y)$.

(2) *Let $m \in \mathbb{N}$ or $m = +\infty$. If $f \in \mathscr{I}^m(X \cap Y; U)$, then the function $\phi \cdot f$ extends uniquely to a function (also denoted $\phi \cdot f$) which is $\mathscr{E}^m$ on U and m-flat on $X \cap Y$. The mapping $f \mapsto \phi \cdot f$ of $\mathscr{I}^m(X \cap Y; U)$ to itself is continuous and linear.*

(3) *Let* ρ *be the epimorphism* $\rho(f,g)=f|X\cap Y-g|X\cap Y$ *of* $\mathscr{E}^m(U)\oplus\mathscr{E}^m(U)$ *onto* $\mathscr{E}^m(X\cap Y)$. *Then the mapping* $\sigma\colon \ker\rho\to\mathscr{E}^m(U)$ *defined by* $\sigma(f,g)=f-\phi\cdot(g-f)$ *is a continuous linear mapping such that* $\sigma(f,g)-f$ *(respectively* $\sigma(f,g)-g$*) is m-flat on* X *(respectively* Y*) for all* $(f,g)\in\ker\rho$.

(4) *Let* δ *be the diagonal injection* $\delta(F)=(F|X,F|Y)$ *of* $\mathscr{E}^m(X\cup Y)$ *into* $\mathscr{E}^m(X)\oplus\mathscr{E}^m(Y)$. *Let* π *be the epimorphism* $\pi(F,G)=F|X\cap Y-G|X\cap Y$ *of* $\mathscr{E}^m(X)\oplus\mathscr{E}^m(Y)$ *onto* $\mathscr{E}^m(X\cap Y)$. *Then the following sequence is exact:*

$$0\to\mathscr{E}^m(X\cup Y)\xrightarrow{\delta}\mathscr{E}^m(X)\oplus\mathscr{E}^m(Y)\xrightarrow{\pi}\mathscr{E}^m(X\cap Y)\to 0.$$

Remark 5.6. Suppose that the closed subsets X, Y in Lemma 5.5 are merely regularly situated, but that $m=+\infty$. Then there exists $\phi\in\mathscr{E}(U-(X\cap Y))$ satisfying (1)(a), (b) and (2); hence (3), (4) are valid. If $m\in\mathbb{N}$, however, then (3), (4) involve a loss of differentiability depending on α in Definition 5.1.

The remainder of this section will be used only in the proof of Proposition 6.1. The relationship of Proposition 6.1 with the proofs of our main theorems will be discussed in Remark 6.2.

Let X be a closed subset of $\mathbb{R}^n$, and $\mathscr{E}^m(X,\mathbb{C})$ the space of complex valued $\mathscr{C}^m$ Whitney fields ($m\in\mathbb{N}$) on X; i.e. the set of $F=F_1+iF_2$, where $F_1,F_2\in\mathscr{E}^m(X)$. We give $\mathscr{E}^m(X,\mathbb{C})$ the structure of a Fréchet space defined by the seminorms $\|F\|_m^K=\|F_1\|_m^K+\|F_2\|_m^K$, where $K\subset X$ is compact.

We denote by $(z,t)=(z_1,\dots,z_n,t_1,\dots,t_p)$ $(z_j=x_j+iy_j)$ a point in $\mathbb{C}^n\times\mathbb{R}^p$. Let X be a closed subset of $\mathbb{C}^n\times\mathbb{R}^p$ and $F\in\mathscr{E}^m(X,\mathbb{C})$. To each point $a\in X$, we associate the Taylor expansion $T_a^mF\in\mathbb{C}[x,y,t]$ (of order m) of F at a:

$$T_a^mF=\sum_{\substack{j,k\in\mathbb{N}^n,\ l\in\mathbb{N}^p\\ |j|+|k|+|l|\le m}}\frac{F^{j,k,l}(a)}{j!\,k!\,l!}\cdot x^jy^kt^l.$$

Definition 5.7. $F\in\mathscr{E}^m(X,\mathbb{C})$ is *formally holomorphic* (in the variables z) if F satisfies the Cauchy-Riemann equations

$$i\frac{\partial F}{\partial x_j}=\frac{\partial F}{\partial y_j},\qquad j=1,\dots,n;$$

i.e. the Taylor expansion of F at each point of X belongs to $\mathbb{C}[z,t]$.

Let $\mathscr{H}^m(X)$ be the set of formally holomorphic $\mathscr{C}^m$ Whitney fields on X. $\mathscr{H}^m(X)$ is a closed subalgebra of $\mathscr{E}^m(X,\mathbb{C})$, hence a Fréchet space in the induced topology. We also define $\mathscr{E}(X,\mathbb{C})=\varprojlim\mathscr{E}^m(X,\mathbb{C})$ and $\mathscr{H}(X)=\varprojlim\mathscr{H}^m(X)$.

Now let $m\in\mathbb{N}$ or $m=+\infty$. Let X be a closed subset of $\mathbb{R}^n\times\mathbb{R}^p\subset\mathbb{C}^n\times\mathbb{R}^p$. We associate to each $F\in\mathscr{E}^m(X\subset\mathbb{R}^n\times\mathbb{R}^p,\mathbb{C})$ its *formal complexification* $\tilde{F}\in\mathscr{H}^m(X)$. At each $a\in X$, $T_a^m\tilde{F}$ is obtained by substituting the variables z_j for the variables x_j in T_a^mF; i.e.

$$T_a^m\tilde{F}=\sum_{\substack{k\in\mathbb{N}^n,\ l\in\mathbb{N}^p\\ |k|+|l|\le m}}\frac{F^{k,l}(a)}{k!\,l!}\cdot(x+iy)^kt^l.$$

The mapping $F \mapsto \tilde{F}$ of $\mathscr{E}^m(X \subset \mathbb{R}^n \times \mathbb{R}^p, \mathbb{C})$ to $\mathscr{H}^m(X)$ is an isomorphism of topological algebras with derivation.

Let L be a real linear automorphism of $\mathbb{C}^n \times \mathbb{R}^p$ which is holomorphic with respect to the variables z_j, $j=1, \ldots, n$. Then L defines a Fréchet space isomorphism $L^*(F) = F \circ L$ from $\mathscr{H}^m(X)$ to $\mathscr{H}^m(L^{-1}(X))$; i.e. $T^m_{L^{-1}(a)}(L^*(F))(z,t) = T^m_a F(L(z,t))$.

We say that a subspace Π of $\mathbb{C}^n \times \mathbb{R}^p$ is *real situated* if Π is the inverse image of $\mathbb{R}^n \times \mathbb{R}^p$ by a real linear automorphism L which is holomorphic with respect to $z \in \mathbb{C}^n$. The following lemma is a simple consequence of Lemma 5.5.

Lemma 5.8 (cf. [13, IV, 5.4, 5.5]). *Let Π be a real situated subspace of $\mathbb{C}^n \times \mathbb{R}^n$, and X, Y closed subsets of Π which satisfy the hypothesis of Lemma 5.5. Then for $m \in \mathbb{N}$ or $m = +\infty$ we have:*

(1) *Let ρ: $\mathscr{H}^m(\Pi) \oplus \mathscr{H}^m(\Pi) \to \mathscr{H}^m(X \cap Y)$ be the surjection $\rho(F,G) = F|X \cap Y - G|X \cap Y$. Then there exists a continuous linear map σ: $\ker \rho \to \mathscr{H}^m(\Pi)$ such that $\sigma(F,G) = F$ on X, $\sigma(F,G) = G$ on Y, for all $(F,G) \in \ker \rho$. As in Lemma 5.5 (3), the mappings σ are defined simultaneously for all m.*

(2) *The following sequence is exact:*

$$0 \to \mathscr{H}^m(X \cup Y) \to \mathscr{H}^m(X) \oplus \mathscr{H}^m(Y) \to \mathscr{H}^m(X \cap Y) \to 0.$$

Lemma 5.9 (cf. [13, IV, 5.6]). *Let $\Pi_1, \ldots, \Pi_r, \ldots, \Pi_s$ be real situated subspaces of $\mathbb{C}^n \times \mathbb{R}^p$. Then there exists a projective system of extension operators $\mathscr{H}^m\left(\bigcup_{i=1}^{r} \Pi_i\right) \to \mathscr{H}^m\left(\bigcup_{i=1}^{s} \Pi_i\right)$.*

6. Averaging a Real Function Over the k'th Roots of Unity

Let k be a positive integer, and π: $\mathbb{R}^{n+p} \to \mathbb{R}^{n+p}$ the mapping defined by $\pi(x,t) = (x_1^k, \ldots, x_n^k, t_1, \ldots, t_p)$, where $(x,t) = (x_1, \ldots, x_n, t_1, \ldots, t_p)$ denotes a point of the source. Let U be an open neighborhood of the origin in the target, $U' = \pi^{-1}(U)$, and X, X' closed subsets of U, U' such that $X' = \pi^{-1}(X)$. For each $m \in \mathbb{N}$ or $m = +\infty$, we let $(\pi^* \mathscr{E}^m(X))^\wedge$ be the subalgebra of $\mathscr{E}^m(X')$ of Whitney fields F with the property that for all $b \in X$, there exists $G \in \mathscr{E}^m(X)$ such that $F - \pi^*(G)$ is m-flat on $\pi^{-1}(b)$. $(\pi^* \mathscr{E}^m(X))^\wedge$ is closed in $\mathscr{E}^m(X')$ since it may be characterized as the subalgebra of Whitney fields F such that

(1) if $l = (l_1, \ldots, l_{n+p}) \in \mathbb{N}^{n+p}$, $|l| \leqq m$, and l_i is *not* divisible by k for some $i = 1, \ldots, n$, then $D^l F(x,t) = 0$ for all (x,t) such that $x_i = 0$;

(2) if k is even, then F is even in each coordinate x_i.

Let $\mathscr{E}^m(U'; X, \pi)$ be the closed subspace of $\mathscr{E}^m(U')$ of functions f such that $f|X' \in (\pi^* \mathscr{E}^m(X))^\wedge$. We ask whether there exists a continuous linear mapping of $\mathscr{E}^m(U'; X, \pi)$ to $(\pi^* \mathscr{E}^m(U))^\wedge$ which induces the identity on $(\pi^* \mathscr{E}^m(X))^\wedge$. If $k=2$, then such a mapping may be defined simply by taking the even part with respect to each coordinate x_i, $i=1, \ldots, n$. In general, however, there does not exist a continuous linear projection operator from functions of (x,t) to functions of $(x_1^k, \ldots, x_n^k, t)$ [9]. Instead, the required mapping will be constructed (under suitable conditions on X') by extension to a formally holomorphic Whitney field on a union of real situated subspaces of $\mathbb{C}^n \times \mathbb{R}^p$, followed by averaging over the k'th roots of unity in each copy of $\mathbb{C}$.

Proposition 6.1. *Suppose that for each $i=1, \dots, n$, X' and the intersection with U' of the hyperplane $\{x_i=0\}$ satisfy the hypothesis of Lemma 5.5 in U'. Then there exists a projective system of continuous linear operators*

$$A^m = A^m(X, \pi): \mathscr{E}^m(U'; X, \pi) \to \widehat{\pi^* \mathscr{E}^m(U)}$$

such that $A^m(f)\,|X' = f\,|X'$ for all $f \in \mathscr{E}^m(U'; X, \pi)$.

Proof. For convenience of notation we assume $U=\mathbb{R}^{n+p}$. Let Z_k be the group of k'th roots of unity $e^{2\pi i l/k}$, $l=0, \dots, k-1$. Then $(Z_k)^n = Z_k \times \cdots \times Z_k$ (n copies) acts on $\mathbb{C}^n \times \mathbb{R}^p$ by the real linear automorphisms L_γ defined by

$$L_\gamma(z, t) = (e^{2\pi i l_1/k} z_1, \dots, e^{2\pi i l_n/k} z_n, t_1, \dots, t_p),$$

where $\gamma = (e^{2\pi i l_1/k}, \dots, e^{2\pi i l_n/k})$ and $(z, t) = (x+iy, t) \in \mathbb{C}^n \times \mathbb{R}^p$. Each L_γ is holomorphic in $(z_1, \dots, z_n)$.

We order the elements $\gamma = (e^{2\pi i l_1/k}, \dots, e^{2\pi i l_n/k})$ of $(Z_k)^n$ lexicographically with respect to $(l_1, \dots, l_n)$, and write $\gamma = \gamma_j$, $j=0, \dots, k^n-1$, for the elements in this order. We also write $L_j = L_{\gamma_j}$, $j=0, \dots, k^n-1$. Let $\Pi_j = L_j^{-1}(\mathbb{R}^n \times \mathbb{R}^p)$ and $X_j = L_j^{-1}(X')$, $j=0, \dots, k^n-1$ (so that $\Pi_0 = \mathbb{R}^n \times \mathbb{R}^p$, $X_0 = X'$).

Given $f \in \mathscr{E}^m(\mathbb{R}^{n+p}; X, \pi)$, let $\tilde{f} \in \mathscr{H}^m(\Pi_0)$ be the formal complexification of f. For each $j=0, \dots, k^n-1$, let $F_j = L_j^*(\tilde{f}) \in \mathscr{H}^m(\Pi_j)$ (so that $F_0 = \tilde{f}$). Since $f\,|X' \in (\pi^* \mathscr{E}^m(X))^\wedge$, then

$$F_j\,|X_j \cap X_l = F_l\,|\,X_j \cap X_l, \qquad j, l = 0, \dots, k^n-1.$$

By Lemma 5.8(2), there exists a unique $F \in \mathscr{H}^m(\Pi_0 \cup \bigcup_j X_j)$ such that $F\,|\,\Pi_0 = \tilde{f}$ and $F\,|X_j = F_j|\,X_j$, $j=0, \dots, k^n-1$. It is clear that $F\,|\bigcup_j X_j$ is invariant under substitution of the elements of $(Z_k)^n$.

We now extend F to $F' \in \mathscr{H}^m(\bigcup_j \Pi_j)$. By induction on $l=1, \dots, k^n-1$, we assume that F has been extended to $F^{l-1} \in \mathscr{H}^m(\bigcup_j X_j \cup \bigcup_{j \le l-1} \Pi_j)$. By Lemma 5.9, there exists an extension operator

$$\alpha_l: \mathscr{H}^m(\bigcup_{j \le l-1} \Pi_j) \to \mathscr{H}^m(\bigcup_{j \le l} \Pi_j).$$

Let $\rho_l: \mathscr{H}^m(\Pi_l) \oplus \mathscr{H}^m(\Pi_l) \to \mathscr{H}^m(X_l \cap \bigcup_{j \le l-1} \Pi_j)$ be the surjection $\rho_l(G, H) = G\,|(X_l \cap \bigcup_{j \le l-1} \Pi_j) - H\,|(X_l \cap \bigcup_{j \le l-1} \Pi_j)$, $(G, H) \in \mathscr{H}^m(\Pi_l) \oplus \mathscr{H}^m(\Pi_l)$. Then there exists a continuous linear map $\sigma_l: \ker \rho_l \to \mathscr{H}^m(\Pi_l)$ such that $\sigma_l(G, H) = G$ on X_l and $\sigma_l(G, H) = H$ on $\Pi_l \cap \bigcup_{j \le l-1} \Pi_j$, for all $(G, H) \in \ker \rho_l$, by Lemma 5.8(1). Let H_l be the restriction of $\alpha_l(F^{l-1}\,|\bigcup_{j \le l-1} \Pi_j)$ to Π_l. Since $F^{l-1}\,|\bigcup_j X_j$ is invariant under the action of $(Z_k)^n$, then $(F_l, H_l)\in\ker\rho_l$. Define F^l on $\bigcup_j X_j \cup \bigcup_{j\leq l} \Pi_j$ by

$$F^l = F^{l-1} \quad \text{on } \bigcup_j X_j \cup \bigcup_{j\leq l-1} \Pi_j$$

$$F^l = \sigma_l(F_l, H_l) \quad \text{on } \Pi_l.$$

Then $F^l\in\mathscr{H}^m(\bigcup_j X_j \cup \bigcup_{j\leq l}\Pi_j)$ by Lemma 5.8(2). $F' = F^{k^n-1}\in\mathscr{H}^m(\bigcup_j \Pi_j)$ is a continuous linear extension of F.

Define $F'' = \frac{1}{k^n}\Sigma_j L_j^*(F')$. Since $F' \mid \bigcup_j X_j = F \mid \bigcup_j X_j$ is $(Z_k)^n$-invariant, then $F'' \mid \bigcup_j X_j = F \mid \bigcup_j X_j$. $F'' \mid \Pi_0$ is the formal complexification of a unique Whitney field $F'''\in\mathscr{E}^m(\mathbb{R}^n\times\mathbb{R}^p, \mathbb{C})$ such that $\operatorname{Re} F''' \mid X' = f \mid X'$. Since F'' is $(Z_k)^n$-invariant, then $\operatorname{Re} F'''\in(\pi^*\mathscr{E}^m(U))^\wedge$. The required operator is $A^m(f) = \operatorname{Re} F'''$. The operators A^m clearly form a projective system.

Remark 6.2. Proposition 6.1 will be used in the proof of Theorem II′ and in Section 8. It is not needed, however, for the special case $(k=2)$ of Theorem II′ which will be used in the proof of Theorem I′. In fact Theorem II′ with $k=2^l$, $l\in\mathbb{N}$, may be proved by induction on l, replacing the argument of Proposition 6.1 by merely projecting onto the even part with respect to each coordinate x_i, $i=1,\dots,n$. From the cases $k=2^l$ of Theorem II′, we can deduce Theorem II′ for any positive integer k, but with less precise estimates on the loss of differentiability when $k\neq 2^l$, $l\in\mathbb{N}$.

Proposition 6.1 remains valid in the $\mathscr{C}^\infty$ case under the weaker assumption that X' is regularly situated with respect to the hyperplanes $\{x_i=0\}$, $i=1,\dots,n$. Our stronger assumption is needed in order to obtain the precise estimates of Theorem II.

7. Proof of Theorem II′

It suffices to prove Theorem II′ in the case of a special Lipschitz domain. The general case follows using a partition of unity.

It will be convenient to work in $\mathbb{R}^{n+1}$ instead of $\mathbb{R}^n$. We consider points in $\mathbb{R}^{n+1}$ as pairs (x, y), where $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ and $y\in\mathbb{R}$. Let $\phi:\mathbb{R}^n\to\mathbb{R}$ be a function which satisfies a Lipschitz condition of order $1/k$, where k is a positive integer; i.e. there is a positive constant M such that

$$|\phi(x)-\phi(x')| \leq M\,|x-x'|^{1/k} \qquad (*)$$

for all $x, x'\in\mathbb{R}^n$. Let Ω be the special Lipschitz domain

$$\Omega = \{(x, y)\in\mathbb{R}^{n+1} \mid y > \phi(x)\},$$

and let $X = \Omega\cup\partial\Omega$. We will prove the following theorem.

Theorem 7.1. *There exists a projective system of extension operators*

$$E^m: \mathscr{E}^m(X) \to \mathscr{E}^{m/k}(\mathbb{R}^{n+1})$$

which satisfy the following estimates for all $m \in \mathbb{N}$. For every compact subset L of $\mathbb{R}^{n+1}$, there exists a compact subset $K=K(L)$ of X and a positive constant C (depending only on n, k, m and the bound M in the Lipschitz inequality (*)) such that

$$|E^m(F)|^L_{m/k} \leqq C\, |F|^K_m$$

for all $F \in \mathscr{E}^m(X)$.

Remark 7.2. In the following proof we may, in fact, take $m=+\infty$ directly.

Proof of Theorem 7.1. Let Γ be the (upper) half cone with vertex at the origin of $\mathbb{R}^{n+1}$ (with coordinates $(t, y)=(t_1, \dots, t_n, y)$) defined by

$$\Gamma=\{(t, y)\in\mathbb{R}^{n+1} \mid y \geqq M\,|t|\}.$$

We define a mapping $\pi\colon \mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ by $(x_1, \dots, x_n, y)=\pi(t_1, \dots, t_n, y)$ $=(t_1^k, \dots, t_n^k, y)$. The Lipschitz condition (*) implies that if $a=(x', \phi(x'))$ is any point in $\partial\Omega$, then

$$\{(x, y)\in a+\pi(\Gamma) \mid |x-x'| \leqq 1\} \subset X$$

($a+\pi(\Gamma)$ denotes $\pi(\Gamma)$ translated so that its vertex is a).

Lemma 7.3. *There exists a projective system of extension operators*

$$E_0^m\colon \mathscr{E}^m(\pi(\Gamma))\to\mathscr{E}^{m/k}(\mathbb{R}^{n+1})$$

which satisfy the following estimates for all $m\in\mathbb{N}$. For every $L\subset\mathbb{R}^{n+1}$ compact, there exists $K=K(L)\subset\pi(\Gamma)$ compact and a constant C' (depending only on n, k, m and M) such that

$$|E_0^m(F)|^L_{m/k} \leqq C'\, |F|^K_m$$

for all $F\in\mathscr{E}^m(\pi(\Gamma))$.

We will prove Lemma 7.3 shortly. To obtain Theorem 7.1 from the lemma, let $\rho\in\mathscr{E}(\mathbb{R}^{n+1})$ be a function with support in the unit ball centered at the origin, such that $\rho=1$ in a neighborhood of 0. For each $a\in\partial\Omega$, define $\rho_a(u)=\rho(u-a)$, $u\in\mathbb{R}^{n+1}$. For each $m\in\mathbb{N}$, let $E_a^m\colon \mathscr{E}^m(a+\pi(\Gamma))\to\mathscr{E}^{m/k}(\mathbb{R}^{n+1})$ be the operator obtained by translating E_0^m to a. If $F\in\mathscr{E}^m(X)$, then $\operatorname{supp}\rho_a\cdot F$ lies in the unit ball with center a, so we may assume that $\rho_a\cdot F$ restricts to a $\mathscr{C}^m$ Whitney field on $a+\pi(\Gamma)$. The operators $W_a^m\colon \mathscr{E}^m(X)\to\mathscr{E}^{m/k}(\mathbb{R}^{n+1})$ defined by $W_a^m(F)$ $=\rho_a\cdot E_a^m(\rho_a\cdot F)$ satisfy the hypotheses of Lemma 4.2, so that Theorem 7.1 follows.

The following two elementary lemmas will be used in the proof of Lemma 7.3.

Lemma 7.4. *Let U be an open subset of $\mathbb{R}^{p+1}$ (with coordinates (t,y) $=(t,y_1,\dots,y_p)$). Given $q,m\in\mathbb{N}$ with $q\leqq m$, let $\mathscr{I}_q^m$ be the ideal in $\mathscr{E}^m(U)$ of functions f such that $\dfrac{\partial^l f}{\partial t^l}(0, y)=0$ whenever $(0, y)\in U$ and $l<q$. Then $\mathscr{I}_q^m$ is closed in*

$\mathcal{E}^m(U)$, and for each $\phi \in \mathcal{I}_q^m$, there exists a unique $\psi = \iota_q^m(\phi) \in \mathcal{E}^{m-q}(U)$ such that $\phi(t, y) = t^q \psi(t, y)$ for all $(t, y) \in U$. The mapping $\iota_q^m: \mathcal{I}_q^m \to \mathcal{E}^{m-q}(U)$ is continuous and linear.

Proof. In any open ball in U with center on the hyperplane $\{t=0\}$, the mapping ι_q^m may be defined by Hadamard's formula $\iota_q^m(\phi)(t, y) = \xi_q(t, y)$, where $\xi_0(t, y) = \phi(t, y)$ and

$$\xi_i(t, y) = \int_0^1 \frac{\partial \xi_{i-1}}{\partial t}(st, x)\, ds, \qquad 1 \leqq i \leqq q.$$

Lemma 7.5. *Let k be a positive integer, and $\pi: \mathbb{R}^{n+p} \to \mathbb{R}^{n+p}$ the mapping defined by $\pi(t, y) = (t_1^k, \ldots, t_n^k, y_1, \ldots, y_p)$, where $(t, y) = (t_1, \ldots, t_n, y_1, \ldots, y_p)$ denotes a point of the source. Then for each $m \in \mathbb{N}$ and $f \in (\pi^* \mathcal{E}^m(\pi(\mathbb{R}^{n+p})))^\wedge$, there exists a unique $g = \gamma^m(f) \in \mathcal{E}^{m/k}(\pi(\mathbb{R}^{n+p}))$ such that $f(t, y) = g(t_1^k, \ldots, t_n^k, y)$ for all $(t, y) \in \mathbb{R}^{n+p}$. The mappings $\gamma^m: (\pi^* \mathcal{E}^m(\pi(\mathbb{R}^{n+p})))^\wedge \to \mathcal{E}^{m/k}(\pi(\mathbb{R}^{n+p}))$, $m \in \mathbb{N}$, are continuous linear operators which form a projective system.*

Proof. For each $l \in \mathbb{N}^{n+p}$ with $|l| \leqq m/k$, we may use Lemma 7.4 to define $h_l(t, y) \in (\pi^* \mathcal{E}^{m-|l|k}(\pi(\mathbb{R}^{n+p})))^\wedge$ as follows:

$$h_0(t, y) = f(t, y),$$

$$\frac{\partial h_l(t, y)}{\partial t_i} = k t_i^{k-1} h_{l+(i)}(t, y), \qquad 1 \leqq i \leqq n,$$

$$\frac{\partial h_l(t, y)}{\partial y_j} = h_{l+(n+j)}(t, y), \qquad 1 \leqq j \leqq p$$

((i) denotes the multiindex whose i'th component is 1 and whose other components are 0). On the other hand, there exists a unique $g \in \mathcal{E}^0(\pi(\mathbb{R}^{n+p}))$ such that g is $\mathcal{C}^m$ outside the images of the hyperplanes $\{t_i = 0\}$, $i = 1, \ldots, n$, and $f(t, y) = g(t_1^k, \ldots, t_n^k, y)$ outside the hyperplanes $\{t_i = 0\}$. Hence for each $|l| \leqq m/k$, $h_l(t, y) = (D^l g)(t_1^k, \ldots, t_n^k, y)$ outside the hyperplanes $\{t_i = 0\}$, so that $D^l g$ may be continued up to the boundary of $\pi(\mathbb{R}^{n+p})$. It follows that $g \in \mathcal{E}^{m/k}(\pi(\mathbb{R}^{n+p}))$. The remaining assertions also follow from Lemma 7.4.

Proof of Lemma 7.3. By Stein's extension theorem (or by Seeley's extension theorem together with Lemma 4.2), there exists a projective system of extension operators $S^m: \mathcal{E}^m(\Gamma) \to \mathcal{E}^m(\mathbb{R}^{n+1})$ which satisfy the following estimates for all $m \in \mathbb{N}$. There exists a constant C'' (depending only on n, m and M), and for every compact subset L of $\mathbb{R}^{n+1}$ a compact subset $K = K(L)$ of Γ, such that $|S^m(F)|_m^L \leqq C'' |F|_m^K$ for all $F \in \mathcal{E}^m(\Gamma)$.

Given $F \in \mathcal{E}^m(\pi(\Gamma))$, let $F' = S^m(\pi^*(F)) \in \mathcal{E}^m(\mathbb{R}^{n+1})$. Then $F' | \Gamma = \pi^*(F)$, so that F' lies in the closed subspace $\mathcal{E}^m(\mathbb{R}^{n+1}; \pi(\Gamma), \pi)$ of $\mathcal{E}^m(\mathbb{R}^{n+1})$ of functions f such that $f | \Gamma \in (\pi^* \mathcal{E}^m(\pi(\Gamma)))^\wedge$. We let $G = \gamma^m(A^m(F')) \in \mathcal{E}^{m/k}(\pi(\mathbb{R}^{n+1}))$, where

$$A^m: \mathcal{E}^m(\mathbb{R}^{n+1}; \pi(\Gamma), \pi) \to (\pi^* \mathcal{E}^m(\pi(\mathbb{R}^{n+1})))^\wedge,$$

$$\gamma^m: (\pi^* \mathcal{E}^m(\pi(\mathbb{R}^{n+1})))^\wedge \to \mathcal{E}^{m/k}(\pi(\mathbb{R}^{n+1}))$$

are the operators given by Proposition 6.1 and Lemma 7.5 (respectively). If k is odd, then $\pi(\mathbb{R}^{n+1})=\mathbb{R}^{n+1}$, and we define $E_0^m(F)=G$. If k is even, then

$$\pi(\mathbb{R}^{n+1})=\{(x,y)\in\mathbb{R}^{n+1}\mid x_i\geqq 0, i=1,\ldots,n\}.$$

In this case $E_0^m(F)$ is obtained by extending G to $\mathbb{R}^{n+1}$ using Seeley's or Stein's theorem. In either case E_0^m is a continuous linear operator which satisfies the conditions of Lemma 7.3.

8. Theorem I in the 2-Dimensional Semianalytic Case

Let $X\subset\mathbb{R}^2$ be a semianalytic set which is the closure of an open set. To prove that there exists an extension operator $\mathscr{E}(X)\to\mathscr{E}(\mathbb{R}^2)$, it suffices to show that for each point a in the frontier of X, there exists an open neighborhood U_a of a in $\mathbb{R}^2$, and an extension operator $E_a\colon\mathscr{E}(X\cap U_a)\to\mathscr{E}(U_a)$ (the following argument actually applies simultaneously to the extension of all classes of differentiability).

Suppose the origin 0 lies in the frontier of X. There exists an open neighborhood U'' of 0 in $\mathbb{R}^2$, and a real analytic curve Y in U'' such that $X\cap U''$ is the closure in U'' of a union of components of the complement of Y. By Remark 4.3 (or Remark 5.6), it suffices to assume that $X\cap U''$ is the closure in U'' of a single component Ω of the complement of Y.

After a linear change of coordinates, we can find open neighborhoods U_1, U_2 of 0 in $\mathbb{R}$ such that

(1) Y is the zero set of a Weierstrass polynomial

$$g(y,t)=t^n+\sum_{i=1}^{n} c_i(y)\,t^{n-i}$$

in $V=U_1\times U_2$, with coefficients $c_i(y)$ which are analytic in U_1 and vanish at 0;

(2) for all $y_0\in U_1$, all real roots of $g(y_0,t)=0$ lie in U_2;

(3) the discriminant D of g vanishes at most at 0.

By adding some extra branches to Y, we may in fact assume that $X\cap V\subset\{y\geqq 0\}$, so that D vanishes exactly at 0, and $\Omega\cap V$ is the region between two adjacent analytic sheets Σ_1, Σ_2 of the Weierstrass polynomial g.

Let $k=n!$. There exists $\varepsilon>0$ and real power series $\gamma_1(x)$, $\gamma_2(x)$ which converge in the interval $(-\varepsilon,\varepsilon)$ such that

(1) $(-\varepsilon^k,\varepsilon^k)\subset U_1$;

(2) Σ_1, Σ_2 are given by the Puiseux expansions $t=\gamma_1(y^{1/k})$, $t=\gamma_2(y^{1/k})$ for $y\in(0,\varepsilon^k)$.

Define $\pi\colon(-\varepsilon,\varepsilon)\times U_2\to U_1\times U_2$ by $\pi(x,t)=(x^k,t)$. After the analytic coordinate change

$$x'=x,$$
$$t'=t-\tfrac{1}{2}(\gamma_1(x)+\gamma_2(x)),$$

there is an open neighborhood U' of 0 in $(-\varepsilon, \varepsilon) \times U_2$, such that $X' = \pi^{-1}(X) \cap U'$ is the closure of the region above the graph of a continuous function $x' = \phi(t')$. The function ϕ satisfies a Lipschitz condition locally, so that by Theorem II there is an extension operator $E': \mathscr{E}(X') \to \mathscr{E}(U')$.

We may choose U' such that $U' = \pi^{-1}(U)$, where U is an open neighborhood of 0 in $\mathbb{R}^2$. We can now deduce the existence of an extension operator $E: \mathscr{E}(X \cap U) \to \mathscr{E}(U)$, using Proposition 6.1 and Lemma 7.5.

Remark 8.1. The method of this section may be used in any dimension to prove Theorem I, provided that X is a semianalytic set which may be expressed locally as a finite union $\bigcup_j X_j$ of sets of the following type. Each X_j is the closure of a component of the complement of a hypersurface which has only *quasi-ordinary* singularities. This means that the hypersurface is locally the zero set of a Weierstrass polynomial such that the discriminant variety of its complexification has only normal crossings (cf. [15]).

9. Proof of Theorem I′

Let N be a real analytic manifold of dimension n, and $\mathscr{A}_N$ the sheaf of real analytic functions on N. Let X be a compact subanalytic subset of N such that the interior of X is dense in X. We will show there exists a positive integer k, and for each $m \in \mathbb{N}$ (or $m = +\infty$) an extension operator $E^m: \mathscr{E}^m(X) \to \mathscr{E}^{m/k}(N)$ (defined universally for all m). It suffices to show that for each $x \in X$, there exists an open neighborhood U of x in N, and a (universal) extension operator $\mathscr{E}^m(X \cap U) \to \mathscr{E}^{m/k}(U)$.

We first assume that X is semianalytic. Let $x \in X$. We can find an open neighborhood V of x in N, and a finite system of real analytic functions f_{ij} on V such that

$$X \cap V = \bigcup_i \{f_{ij} \geqq 0 \text{ for all } j\}.$$

We may assume that none of the f_{ij} is identically zero. We then apply Hironaka's desingularization theorem [4, main theorem II] (see also [3, 5.11 and 7.2]) to the ideal sheaf $J = (\Pi_{i,j} f_{ij}) \mathscr{A}_N | V$. According to the desingularization theorem, there exists an open neighborhood U of x in V, and a real analytic mapping $\pi: N' \to U$ which has the following properties:

(1) N' is smooth.

(2) π is surjective and proper. In fact π is obtained by composing a finite sequence of blowings-up with non-singular centers.

(3) If Y is the closed real analytic subspace of V defined by J, then $N' - \pi^{-1}(Y)$ is dense in N', and π induces an isomorphism $N' - \pi^{-1}(Y) \to U - Y$.

(4) $\pi^{-1}(Y)$ has only normal crossings; i.e. for each $x' \in N'$, there exists a coordinate system $(z_1, \ldots, z_n)$ of N' centered at x', such that $J \mathscr{A}_{N', x'}$ is generated by a monomial $z_1^{a_1} \ldots z_n^{a_n}$ with non-negative integers a_i.

It follows that there exists a closed semianalytic subset X' of N' such that

(1) $\pi(X') = X \cap U$;

(2) the interior of X' is dense in X';

(3) X' may be defined locally by linear inequalities.

Hence by Seeley's extension theorem and Łojasiewicz's glueing lemma 5.5(3) (or Lemma 4.1), there exists a (universal) extension operator $\mathscr{E}^m(X') \to \mathscr{E}^m(N')$. Theorem I' (in the semianalytic case) will follow from Proposition 9.1 below, by induction on the number of blowings-up of which the mapping $\pi: N' \to U$ is composed.

Now we suppose X is subanalytic. Let $x \in X$. There exists a finite number of real analytic mappings $\pi_i: N_i \to N$ with the following properties:

(1) each N_i is smooth.

(2) There exists a compact subset K_i of N_i, for each i, such that $\bigcup_i \pi_i(K_i)$ is a neighborhood of x in N.

(3) For each i, π_i is obtained by composing a finite sequence of local blowings-up with non-singular centers.

(4) For each i, $\pi_i^{-1}(X)$ is semianalytic in N_i.

We recall that a local blowing-up over a real analytic space Z is the composition of a blowing-up over an open subset U of Z, with the inclusion $U \hookrightarrow Z$. Hironaka [3, 7.3] states the above result with each π_i obtained by a finite sequence of local blowings-up whose centers are nowhere dense in their respective ambient spaces, but perhaps singular. Given such mappings π_i, we may obtain the stronger statement using the desingularization theorem cited above (each π_i is dominated over $\pi_i(K_i)$ by a finite number of perhaps longer finite sequences of local blowings-up with smooth centers).

Suppose we have a finite number of real analytic mappings $\pi_i: N_i \to N$ with the properties (1)–(4). For each i, let B_i be a compact semianalytic neighborhood of K_i in N_i, and let X_i be the closure in N_i of the interior of $\pi_i^{-1}(X) \cap B_i$. Then X_i is semianalytic, and $\bigcup_i \pi_i(X_i)$ is a neighborhood of x in X. Hence Theorem I' may be reduced to the semianalytic case, using Lemma 4.1 and Proposition 9.1 below.

Local Blowing-up. Let $\mathbb{P}^r$ denote real projective space of dimension r. There is a natural mapping $p_0: \mathbb{R}^n - \{0\} \to \mathbb{P}^{n-1}$ such that for each $\xi \in \mathbb{P}^{n-1}$, $p_0^{-1}(\xi) \cup \{0\}$ is a line through the origin in $\mathbb{R}^n$. By assigning to each $\xi \in \mathbb{P}^{n-1}$ the line obtained in this way, we get a real line bundle $p: L \to \mathbb{P}^{n-1}$, and a natural mapping $\pi_0: L \to \mathbb{R}^n$ which is isomorphic outside the zero section of p, and such that the zero section is mapped to the origin of $\mathbb{R}^n$.

With the coordinate system $(y_1, \ldots, y_n)$ for $\mathbb{R}^n$, the real analytic manifold L is constructed as follows: $L = \bigcup_{i=1}^{n} L_i$, where $L_i = \mathbb{R}^n$ with coordinate system $(t_{i1}, \ldots, t_{in})$, and $\pi_0 | L_i$ is defined by

$$y_j \circ \pi_0 = \begin{cases} t_{ii} & \text{if } j = i \\ t_{ii}\, t_{ij} & \text{if } j \neq i. \end{cases}$$

The mapping $\pi_0: L \to \mathbb{R}^n$ is the *blowing-up* of $\mathbb{R}^n$ with *center* 0.

Let $\mathbb{R}^p$ be a linear subspace of $\mathbb{R}^q$. Write $\mathbb{R}^q = \mathbb{R}^p \times \mathbb{R}^n$. With π_0 as above, the mapping $\pi = id_{\mathbb{R}^p} \times \pi_0$ is the *blowing-up* of $\mathbb{R}^q$ with *center* $\mathbb{R}^p$.

Though the $\mathscr{C}^\infty$ case of our main theorem follows from the cases of finite differentiability class, we may, in fact, prove the $\mathscr{C}^\infty$ case directly, so we make it explicit in the following proposition.

Proposition 9.1. *Let X' be a non-empty compact subanalytic subset of $\mathbb{R}^p \times L$, and let $X = \pi(X')$. Suppose there exists a positive integer k and a projective system of extension operators $\mathscr{E}^m(X') \to \mathscr{E}^{m/k}(\mathbb{R}^p \times L)$, $m \in \mathbb{N}$ (respectively an extension operator $\mathscr{E}(X') \to \mathscr{E}(\mathbb{R}^p \times L)$). Then there exists a projective system of extension operators $\mathscr{E}^m(X) \to \mathscr{E}^{m/8k}(\mathbb{R}^q)$ (respectively an extension operator $\mathscr{E}(X) \to \mathscr{E}(\mathbb{R}^q)$).*

Remarks 9.2. The definition of our extension operators will be universal, so we will neglect to say explicitly that we are working with projective systems at every stage of the proof. We will allow m to denote either a natural number or $+\infty$. We also adopt the following convention. Suppose that Z is a closed subset of a real analytic manifold, and that $F \in \mathscr{E}^m(Z)$, $G \in \mathscr{E}^{m'}(Z)$, where $m \leqq m'$. If the image of G in $\mathscr{E}^m(Z)$ coïncides with F, we will write $F = G$.

Proof of Proposition 9.1. For each $i = 1, \ldots, n$, we let X'_i be the intersection of X' with the closed subset of $\mathbb{R}^p \times L_i$ defined by $|t_{ij}| \leqq 1$ for all $j \neq i$. Let $X_i = \pi(X'_i)$. Then X_i, X'_i are compact subanalytic subsets of X, X' (respectively), such that $X = \bigcup_{i=1}^n X_i$, $X' = \bigcup_{i=1}^n X'_i$. We also let X'_{i+} (respectively X'_{i-}) be the subset of X'_i of points such that $t_{ii} \geqq 0$ (respectively $t_{ii} \leqq 0$), and let $X_{i+} = \pi(X'_{i+})$, $X_{i-} = \pi(X'_{i-})$. Then $X_i = X_{i+} \cup X_{i-}$ and $X'_i = X'_{i+} \cup X'_{i-}$, $i = 1, \ldots, n$. By Lemma 4.1, it suffices to show that for each $i = 1, \ldots, n$, there exist (universal) continuous linear operators

$$E^m_{i+}, E^m_{i-}: \mathscr{E}^m(X) \to \mathscr{E}^{m/8k}(\mathbb{R}^q)$$

such that $E^m_{i+}(F) | X_{i+} = F | X_{i+}$ and $E^m_{i-}(F) | X_{i-} = F | X_{i-}$ for all $F \in \mathscr{E}^m(X)$.

We fix i, and show there exists a (universal) continuous linear operator $E^m: \mathscr{E}^m(X) \to \mathscr{E}^{m/8k}(\mathbb{R}^q)$ such that $E^m(F) | X_{i+} = F | X_{i+}$ for all $F \in \mathscr{E}^m(X)$ (the same argument works for X_{i-} by symmetry).

For convenience we relabel the coordinates $(x_1, \ldots, x_p, t_{i1}, \ldots, t_{in})$ of $\mathbb{R}^p \times L_i$ by $x = (x_1, \ldots, x_{p+n})$ as follows:

$$\begin{aligned} x_{p+1} &= t_{ii}, & & \\ x_{p+j} &= t_{i,j-1}, & & 1 < j \leqq i, \\ x_{p+j} &= t_{ij}, & & i < j \leqq n. \end{aligned}$$

Coordinates $y = (y_1, \ldots, y_q)$ for $\mathbb{R}^q$ are chosen so that $\pi_i = \pi | \mathbb{R}^p \times L_i$ is given by

$$(y_1, \ldots, y_q) = (x_1, \ldots, x_p, x_{p+1}, x_{p+1} x_{p+2}, \ldots, x_{p+1} x_{p+n}).$$

In the new coordinates, any point of X'_{i+} satisfies the inequalities $x_{p+1} \geqq 0$ and $|x_j| \leqq 1$, $j = p+2, \ldots, q$.

Let C' be the truncated half-cone defined by $(x_1^2+\cdots+x_p^2+x_{p+2}^2+\cdots+x_{p+n}^2)^{\frac{1}{2}}\leqq x_{p+1}\leqq 1$. For any point x with $x_{p+1}\geqq 0$, we denote by $C'(x)$ the truncated half-cone with vertex x obtained from C' by translating the origin to the point x:

$$C'(x)=x+C'.$$

Any line parallel to the x_{p+1}-axis is mapped by π_i to a straight line in $\mathbb{R}^q$. Let C be the compact semialgebraic subset of $\mathbb{R}^q$ defined by

$$(y_1^2+\cdots+y_p^2+y_{p+2}^2+\cdots+y_{p+n}^2)^{\frac{1}{2}}\leqq\tfrac{1}{5}y_{p+1}^2,\qquad 0\leqq y_{p+1}\leqq\tfrac{1}{2}.$$

For any point x with $x_{p+1}\geqq 0$, we denote by $C(x)$ the unique compact semialgebraic subset of $\mathbb{R}^q$ which is obtained from C by a rigid motion of $\mathbb{R}^q$ taking the positive y_{p+1}-axis to the image under π_i of the half-line $\{(x_1,\ldots,x_p,x_{p+1}+\xi,x_{p+2},\ldots,x_{p+n})\,|\,\xi\geqq 0\}$. A straightforward calculation shows that if $0\leqq x_{p+1}\leqq 1$ and $|x_j|\leqq 1$ for $p+2\leqq j\leqq p+n$, then

$$C(x)\subset\pi_i(C'(x)).$$

Now let $Z\subset\mathbb{R}^q$ (with coordinates $\xi=(\xi_1,\ldots,\xi_q)$) be the truncated solid cylinder defined by $0\leqq\xi_{p+1}\leqq 1$ and $\xi_1^2+\cdots+\xi_p^2+\xi_{p+2}^2+\cdots+\xi_{p+n}^2\leqq 1$. Let $Y'\subset\mathbb{R}^p\times L_i$ be the image of $\{x\in X'_{i+}\,|\,0\leqq x_{p+1}\leqq 1\}$ by the mapping

$$(x,\xi)\mapsto(x_1+\xi_{p+1}\xi_1,\ldots,x_p+\xi_{p+1}\xi_p,x_{p+1}+\xi_{p+1},x_{p+2}+\xi_{p+1}\xi_{p+2},\ldots,x_{p+n}+\xi_{p+1}\xi_{p+n})$$

of $(\mathbb{R}^p\times L_i)\times Z$ into $\mathbb{R}^p\times L_i$ (Y' is the union of the truncated cones $C'(x)$ for each $x\in X'_{i+}$ with $x_{p+1}\leqq 1$). Then Y' is a compact subanalytic subset of $\mathbb{R}^p\times L_i$ (if X'_{i+} is semialgebraic, then Y' is semialgebraic, but Y' needn't be semianalytic if X'_{i+} is semianalytic). Note that Y' intersects the hyperplane $\{x_{p+1}=0\}$ only in points of X'_{i+}.

Let $Y=\pi_i(Y')\subset\mathbb{R}^q$. Then Y is a compact subanalytic subset of $\mathbb{R}^q$ containing $C(x)$ for each point $x\in X'_{i+}$ with $x_{p+1}\leqq 1$ (in particular Y contains $X_{i+}\cap\{y_{p+1}\leqq 1\}$).

Let $\Phi^m\colon\mathscr{E}^m(X)\to\mathscr{E}^{m/k}(Y')$ be the continuous linear mapping obtained by composing the algebra homomorphism $\pi^*\colon\mathscr{E}^m(X)\to\mathscr{E}^m(X')$, the extension operator $\mathscr{E}^m(X')\to\mathscr{E}^{m/k}(\mathbb{R}^p\times L)$ of the hypothesis of Proposition 9.1, and the restriction operator $\mathscr{E}^{m/k}(\mathbb{R}^p\times L)\to\mathscr{E}^{m/k}(Y')$. The mapping π_i induces an algebra homomorphism $\Psi^m=\pi_i^*\colon\mathscr{E}^m(Y)\to\mathscr{E}^m(Y')$.

Proposition 9.3. *For all $F\in\mathscr{E}^m(X)$, there exists a unique $G\in\mathscr{E}^{m/4k}(Y)$ such that $\Phi^m(F)=\Psi^{m/4k}(G)$. The mapping $\Theta^m\colon\mathscr{E}^m(X)\to\mathscr{E}^{m/4k}(Y)$ thus defined is continuous and linear.*

We will prove Proposition 9.3 below. To obtain Proposition 9.1 from Proposition 9.3, we first note that $\Theta^m(F)$ coincides with (the $(m/4k)$-jet of) F on $X_{i+}\cap\{y_{p+1}\leqq 1\}$ for all $F\in\mathscr{E}^m(X)$.

By Theorem II′ there exists a (universal) extension operator $E_0^m\colon\mathscr{E}^m(C)\to\mathscr{E}^{m/2}(\mathbb{R}^q)$. By a rigid motion of $\mathbb{R}^q$, E_0^m induces an extension operator $E^m(x)\colon \mathscr{E}^m(C(x)) \to \mathscr{E}^{m/2}(\mathbb{R}^q)$ for any point x. For each point $x \in X'_{i+} \cap \{x_{p+1} \leq 1\}$, let $G^m(x)\colon \mathscr{E}^m(X) \to \mathscr{E}^{m/8k}(\mathbb{R}^q)$ be the operator obtained by following Θ^m with the restriction to $C(x)$, and then with $E^{m/4k}(x)$. By Lemma 4.1 there exists a continuous linear operator $\tilde{G}^m\colon \mathscr{E}^m(X) \to \mathscr{E}^{m/8k}(\mathbb{R}^q)$ such that $\tilde{G}^m(F)$ coincides with F on $X_{i+} \cap \{y_{p+1} \leq 1\}$ for all $F \in \mathscr{E}^m(X)$. Since π_i is isomorphic outside $\{x_{p+1} = 0\}$, then there exists a continuous linear operator $E^m\colon \mathscr{E}^m(X) \to \mathscr{E}^{m/8k}(\mathbb{R}^q)$ such that $E^m(F) \mid X_{i+} = F \mid X_{i+}$ for all $F \in \mathscr{E}^m(X)$.

Remark 9.4. In the $\mathscr{C}^\infty$ case, we can do better than Proposition 9.3. We denote by $\Psi(\mathscr{E}(Y))^\wedge$ the set of $F \in \mathscr{E}(Y')$ such that for all $b \in Y$, there exists $G \in \mathscr{E}(Y)$ with the property that $\Psi(G) - F$ is flat on $\pi_i^{-1}(b) \cap Y'$; i.e. $\Psi(\mathscr{E}(Y))^\wedge$ is the subalgebra of $\mathscr{E}(Y')$ of Whitney fields which are formally in $\Psi(\mathscr{E}(Y))$.

Proposition 9.5. *Let $Y = \pi_i(Y')$, where Y' is any closed subanalytic subset of $\mathbb{R}^p \times L_i$ such that $\pi_i \mid Y'$ is proper. Then*

$$\Psi(\mathscr{E}(Y)) = \overline{\Psi(\mathscr{E}(Y))} = \widehat{\Psi(\mathscr{E}(Y))}.$$

Hence in Proposition 9.1 in the $\mathscr{C}^\infty$ case, $\Psi(\mathscr{E}(Y))$ is closed and $\Psi\colon \mathscr{E}(Y) \to \mathscr{E}(Y')$ induces a Fréchet space isomorphism onto $\Psi(\mathscr{E}(Y))$ by the open mapping theorem. Since $\Phi(\mathscr{E}(X)) \subset \Psi(\mathscr{E}(Y))^\wedge = \Psi(\mathscr{E}(Y))$, then $\Theta = \Psi^{-1} \circ \Phi$.

The following lemma (as well as its proof) will be used in the proofs of Propositions 9.3 and 9.5 (cf. [2], [13, IX, Section 1]). For convenience we now write π for π_i. If $b \in Y$, then $\mathscr{F}_b$ denotes the $\mathbb{R}$-algebra of formal Taylor series at b of elements in $\mathscr{E}(Y)$, and $\hat{\mathfrak{m}}_b$ its maximal ideal. $\mathscr{F}_b$ is isomorphic to $\mathbb{R}[[y_1, \ldots, y_q]]$. Let $T_b\colon \mathscr{E}(Y) \to \mathscr{F}_b$ be the projection associating to each Whitney field its formal Taylor series at b. We write $\hat{G}_b = T_b G$ for $G \in \mathscr{E}(Y)$. If $a \in Y'$ and $b = \pi(a)$, then π induces a homomorphism $\hat{\pi}_a^*\colon \mathscr{F}_b \to \mathscr{F}_a$ such that $T_a \circ \Psi = \hat{\pi}_a^* \circ T_b$.

Lemma 9.6. *For all $r \in \mathbb{N}$, $\hat{\pi}_a^{*-1}(\hat{\mathfrak{m}}_a^{2r}) \subset \hat{\mathfrak{m}}_b^r$. In particular, the homomorphism $\hat{\pi}_a^*$ is injective.*

Proof. We consider $a = (a_1, \ldots, a_q)$ with $a_{p+1} = 0$ (the assertion of the lemma is obvious at other points $a \in Y'$). Then $b = \pi(a) = (a_1, \ldots, a_p, 0, \ldots, 0)$. In coordinate systems $(x_1, \ldots, x_q)$ and $(y_1, \ldots, y_q)$ translated to the points a and b (respectively), the mapping π is given by

$$(y_1, \ldots, y_q) = (x_1, \ldots, x_p, x_{p+1}, x_{p+1}(a_{p+2} + x_{p+2}), \ldots, x_{p+1}(a_{p+n} + x_{p+n})).$$

Since $\hat{\pi}_a^*\colon \mathscr{F}_b \to \mathscr{F}_a$ is a local homomorphism, then $\hat{\pi}_a^{*-1}(\hat{\mathfrak{m}}_a^2) \subset \hat{\mathfrak{m}}_b$. We suppose $r > 1$ and argue by induction on r. Let $S \in \mathscr{F}_b$ such that $S \circ \hat{\pi}_a \in \hat{\mathfrak{m}}_a^{2r}$. We differentiate $S \circ \hat{\pi}_a$ with respect to $x_1, \ldots, x_q$:

$$\left(\frac{\partial S}{\partial y_1} \circ \hat{\pi}_a, \ldots, \frac{\partial S}{\partial y_q} \circ \hat{\pi}_a\right) \cdot \left(\begin{array}{c|cccc} I_p & & & & \\ \hline & 1 & 0 & \cdots & 0 \\ & a_{p+2} + x_{p+2} & x_{p+1} & \cdots & 0 \\ & \vdots & \vdots & \ddots & \vdots \\ & a_{p+n} + x_{p+n} & 0 & \cdots & x_{p+1} \end{array}\right) \in \bigoplus_{j=1}^{q} \hat{\mathfrak{m}}_a^{2r-1} \cdot \mathscr{F}_a.$$

(Here I_p denotes the $p\times p$ identity matrix.) Multiplying on the right by the matrix

$$\begin{pmatrix} I_p & \\ & \begin{matrix} x_{p+1} & 0 \dots 0 \\ -a_{p+2}-x_{p+2} & 1 \dots 0 \\ \vdots & \vdots \ddots \vdots \\ -a_{p+n}-x_{p+n} & 0 \dots 1 \end{matrix} \end{pmatrix},$$

we see that $x_{p+1}\cdot\left(\frac{\partial S}{\partial y_j}\circ\hat{\pi}_a\right)\in\hat{\mathfrak{m}}_a^{2r-1}$, $j=1,\dots,q$. Hence $\frac{\partial S}{\partial y_j}\circ\hat{\pi}_a\in\hat{\mathfrak{m}}_a^{2r-2}$, so that $\frac{\partial S}{\partial y_j}\in\hat{\mathfrak{m}}_b^{r-1}$ by induction. Since $S(0)=0$, then $S\in\hat{\mathfrak{m}}_b^r$.

Proof of Proposition 9.5. It is easy to show that $\overline{\Psi(\mathscr{E}(Y))}\subset\Psi(\mathscr{E}(Y))^\wedge$ (cf. [13, IX, 1.3]). We will prove that $\Psi(\mathscr{E}(Y))^\wedge\subset\Psi(\mathscr{E}(Y))$. Let $F\in\Psi(\mathscr{E}(Y))^\wedge$. By Lemma 9.6 there exists a unique field of formal series G on Y such that $G\circ\pi=F$ (to simplify the notation, we write merely π for $\pi\,|\,Y'$).

We will show that for each $l\in\mathbb{N}^q$, $(D^l G)\circ\pi\in\mathscr{E}(Y')$. By hypothesis $G\circ\pi=F\in\mathscr{E}(Y')$. It suffices to prove the following lemma.

Lemma 9.7. *Let ψ be a field of formal series on Y, such that $\psi\circ\pi\in\mathscr{E}(Y')$. Then for each $j=1,\dots,q$, $\frac{\partial\psi}{\partial y_j}\circ\pi\in\mathscr{E}(Y')$.*

Proof. Let $\gamma_j=\frac{\partial\psi}{\partial y_j}\circ\pi$, $j=1,\dots,q$. By hypothesis there exists $\xi\in\mathscr{E}(Y')$ such that $\psi_{\pi(x)}\circ\hat{\pi}_x=\hat{\xi}_x$ for all $x\in Y'$. We differentiate this equation with respect to $x_1,\dots,x_q$, and argue as in the proof of Lemma 9.6 to obtain $\hat{\delta}_x\cdot\hat{\gamma}_{j,x}=\hat{\xi}_{j,x}$, where $\delta(x)=x_{p+1}$ and $\xi_j\in\mathscr{E}(Y')$. In other words, ξ_j belongs pointwise to the ideal in $\mathscr{E}(Y')$ generated by the function $\delta(x)=x_{p+1}$. By Lemma 9.8 below it follows that $\xi_j=\delta\cdot\gamma_j'$, for some $\gamma_j'\in\mathscr{E}(Y')$. We necessarily have $\gamma_j=\gamma_j'$, so that $\frac{\partial\psi}{\partial y_j}\circ\pi\in\mathscr{E}(Y')$ as required.

Lemma 9.8. *Let Z be a closed subset of $\mathbb{R}^{k+1}$ (with coordinates $(t,x)=(t,x_1,\dots,x_k)$), which is regularly situated with respect to the hyperplane $\{t=0\}$. Let $\phi\in\mathscr{E}(Z)$ and q be a positive integer. If ϕ belongs pointwise to the ideal generated by the monomial t^q in $\mathscr{E}(Z)$, then ϕ belongs to this ideal.*

Proof. Let $W=Z\cap\{t=0\}$. There is a unique field of formal series ψ on W such that $\phi\,|\,W=t^q\psi$.

We will show first that $\psi\in\mathscr{E}(W)$. Let $(0,a)=(0,a_1,\dots,a_k)\in W$. Then $T_{(0,a)}\phi(t,x)=t^q\,T_{(0,a)}\psi(t,x)$ and $T_{(0,a)}^{m+q}\phi(t,x)=t^q\,T_{(0,a)}^m\psi(t,x)$ for all $m\in\mathbb{N}$. On the other hand if $m\in\mathbb{N}$ and $(0,a)$, $(0,b)$ are points in W, then

$$\begin{aligned} &T^{m+q}_{(0,a)}\phi(t,x)-T^{m+q}_{(0,b)}\phi(t,x)\\ &=\sum_{j+|l|\le m}\frac{t^{q+j}(x-a)^l}{(q+j)!\,l!}\cdot D^{q+j,l}\circ(T^{m+q}_{(0,a)}\phi-T^{m+q}_{(0,b)}\phi)(0,a)\\ &=t^q\sum_{j+|l|\le m}\frac{t^j(x-a)^l}{(q+j)!\,l!}\cdot D^{q+j,l}\circ(R^{m+q}_{(0,b)}\phi-R^{m+q}_{(0,a)}\phi)(0,a)\\ &=t^q\sum_{j+|l|\le m}\frac{t^j(x-a)^l}{(q+j)!\,l!}\cdot(R^{m+q}_{(0,b)}\phi)^{q+j,l}(0,a). \end{aligned}$$

Let K be a compact subset of W. There exists a modulus of continuity α_1 such that if $(0,a)$, $(0,b)\in K$ and $j+|l|\le m$, then

$$|(R^{m+q}_{(0,b)}\phi)^{q+j,l}(0,a)|\le|a-b|^{m-(j+|l|)}\alpha_1(|a-b|).$$

(We recall that a modulus of continuity is a continuous increasing function β: $[0,\infty)\to[0,\infty)$ such that β is concave downwards and $\beta(0)=0$.) Hence if $(0,a)$, $(0,b)\in K$ and $(t,x)\in\mathbb{R}^{k+1}$,

$$\begin{aligned} &|T^{m+q}_{(0,a)}\phi(t,x)-T^{m+q}_{(0,b)}\phi(t,x)|\\ &\le|t|^q\sum_{j+|l|\le m}\frac{|t|^j|x-a|^l}{(q+j)!\,l!}\cdot|a-b|^{m-(j+|l|)}\cdot\alpha_1(|a-b|)\\ &\le c_1|t|^q\sum_{j\le m}|t|^j|x-b|^{m-j}\cdot\alpha_1(|a-b|) \end{aligned}$$

for some constant c_1, provided that $|x-a|\le|x-b|$. If $|x-b|\le|x-a|$, then there is a constant c_2 such that

$$\begin{aligned} &|T^{m+q}_{(0,a)}\phi(t,x)-T^{m+q}_{(0,b)}\phi(t,x)|\\ &\le c_2|t|^q\sum_{j\le m}|t|^j|x-a|^{m-j}\cdot\alpha_1(|a-b|). \end{aligned}$$

Hence there is a modulus of continuity α such that

$$\begin{aligned} &|T^{m+q}_{(0,a)}\phi(t,x)-T^{m+q}_{(0,b)}\phi(t,x)|\\ &\le|t|^q\alpha(|a-b|)\cdot(|(t,x)-(0,a)|^m+|(t,x)-(0,b)|^m) \end{aligned}$$

for all $(0,a)$, $(0,b)\in K$, $(t,x)\in\mathbb{R}^{k+1}$. It follows that

$$\begin{aligned} &|T^m_{(0,a)}\psi(t,x)-T^m_{(0,b)}\psi(t,x)|\\ &\le\alpha(|a-b|)\cdot(|(t,x)-(0,a)|^m+|(t,x)-(0,b)|^m) \end{aligned}$$

for all $(0,a)$, $(0,b)\in K$, $(t,x)\in\mathbb{R}^{k+1}$. Hence $\psi\in\mathscr{E}(W)$.

Now let ξ be a Whitney field on $\{t=0\}$ such that $\xi|W=\psi$. Let $\phi'=t^q\xi$. Then ϕ' is a Whitney field on $\{t=0\}$ such that $\phi'|W=\phi|W$. Since Z and $\{t=0\}$ are regularly situated, then by Remark 5.6 there exists $f\in\mathscr{E}(\mathbb{R}^{k+1})$ such that $f|\{t=0\}=\phi'$ and $f|Z=\phi$. By Hadamard's lemma, f belongs to the ideal in $\mathscr{E}(\mathbb{R}^{k+1})$

generated by t^q. Hence ϕ belongs to the ideal generated by t^q in $\mathscr{E}(Z)$. This proves Lemma 9.8, and hence completes the proof of Lemma 9.7.

Returning to the proof of Proposition 9.5, we have shown that $(D^l G)\circ\pi\in\mathscr{E}(Y')$ for all $l\in\mathbb{N}^q$. Since $\pi|Y'$ is proper, it follows that the mapping G^l: $Y\to\mathbb{R}$ is continuous for each $l\in\mathbb{N}^q$. We may now proceed as in [2] to show that G is a Whitney field on Y. Here we use the fact that $Y'\times Y'$ is a subanalytic subset of $\mathbb{R}^q\times\mathbb{R}^q$, and Hironaka's result that two closed subanalytic sets are regularly situated. This completes the proof of Proposition 9.5, and hence of Theorem I.

Proof of Proposition 9.3. From now on $m\in\mathbb{N}$. Let $F\in\mathscr{E}^m(X)$. By Lemma 9.6 there exists a unique field G of Taylor polynomials of order $m/2k$ on Y, such that $\Phi^m(F)=G\circ\pi$ (in $\mathscr{E}^{m/2k}(Y')$). We will first show that

(1) $(D^l G)\circ\pi\in\mathscr{E}^{m/2k-|l|}(Y')$ for all $|l|\leqq m/2k$ (in particular G^l is continuous);

(2) for all $|l|\leqq m/2k$, the mapping $F\mapsto(D^l G)\circ\pi$ of $\mathscr{E}^m(X)$ into $\mathscr{E}^{m/2k-|l|}(Y')$ is continuous and linear.

These assertions must be proved by an argument somewhat different from that used in Proposition 9.5, in order to avoid an unnecessary loss of differentiability involved in applying Łojasiewicz's glueing theorem as in Lemma 9.8.

Lemma 9.9. *For all* $x'\in Y'$,

$$d(x', X'_{i+}\cup\{x_{p+1}=0\})\geqq\frac{1}{\sqrt{2}}d(x', X'_{i+}).$$

Proof. Let $x''\in X'_{i+}\cup\{x_{p+1}=0\}$ such that $|x'-x''|=d(x',X'_{i+}\cup\{x_{p+1}=0\})$. If $x''\in X'_{i+}$, then $d(x',X'_{i+})=d(x',X'_{i+}\cup\{x_{p+1}=0\})$. If $x''\in\{x_{p+1}=0\}$, then $|x'-x''|=x'_{p+1}$, where $x'=(x'_1\ \dots, x'_q)$. But $x'\in C'(a)$, for some point $a=(a_1,\dots,a_q)\in X'_{i+}$. Hence $d(x', X'_{i+})\leqq|x'-a|\leqq\sqrt{2}(x'_{p+1}-a_{p+1})\leqq\sqrt{2}x'_{p+1}$.

Now let W' be a cube with sides parallel to the coordinate axes in the hyperplane $\{x_{p+1}=0\}$, big enough so that Y' lies in the product Z' of W' with the interval $0\leqq x_{p+1}\leqq 2$ in the x_{p+1}-axis.

To prove assertions (1) and (2), we first use Whitney's extension theorem to extend F to $\tilde{F}\in\mathscr{E}^m(\mathbb{R}^q)$. (The mappings Θ^m are defined in a universal way; we are allowed to use Whitney's theorem to prove they are continuous.) Note that $\pi^*(\tilde{F})|X'_{i+}=\Phi^m(F)|X'_{i+}$, so that by Lemma 9.9 and Lemma 5.5(4), $\Phi^m(F)$ and $\pi^*(\tilde{F})|W'\cup X'_{i+}$ together define a unique element of $\mathscr{E}^{m/k}(W'\cup Y')$ which depends in a continuous linear way on F. Using Whitney's extension theorem again, we obtain $H'\in\mathscr{E}^{m/k}(Z')$ such that $H'|Y'=\Phi^m(F)$, $H'|W'=\pi^*(\tilde{F})|W'$, and $F\mapsto H'$ is a continuous linear mapping of $\mathscr{E}^m(X)$ into $\mathscr{E}^{m/k}(Z')$. By Lemma 9.6 there exists a unique field H of Taylor polynomials of order $m/2k$ on $Z=\pi(Z')$ such that $H'=H\circ\pi$ (in particular $H|Y=G$). To prove assertions (1) and (2) it suffices to show that

(1′) $(D^l H)\circ\pi\in\mathscr{E}^{m/2k-|l|}(Z')$ for all $|l|\leqq m/2k$;

(2′) for all $|l|\leqq m/2k$, the mapping $H'\mapsto(D^l H)\circ\pi$ of $(\pi^*\mathscr{E}^{m/k}(Z))^\wedge$ into $\mathscr{E}^{m/2k-|l|}(Z')$ is continuous and linear, with respect to the subspace topology of $(\pi^*\mathscr{E}^{m/k}(Z))^\wedge\subset\mathscr{E}^{m/k}(Z')$.

Assertions (1′) and (2′) are consequences of the following lemma.

Lemma 9.10. *Let ψ be a field of Taylor polynomials of order m on Z, such that $\psi \circ \pi \in \mathscr{E}^m(Z')$. Then $\frac{\partial \psi}{\partial y_j} \circ \pi \in \mathscr{E}^{m-2}(Z')$, $j=1,\ldots,q$. If $\{\psi_i\}$ is a sequence of fields of Taylor polynomials of order m on Z, such that $\{\psi_i \circ \pi\}$ is a Cauchy sequence in $\mathscr{E}^m(Z')$, then $\left\{\frac{\partial \psi_i}{\partial y_j} \circ \pi\right\}$ is a Cauchy sequence in $\mathscr{E}^{m-2}(Z')$.*

We may prove Lemma 9.10 using the argument of Lemma 9.7, replacing Lemma 9.8 by the analogue of Lemma 7.4 for $\mathscr{E}^m(Z')$, with $q=1$.

We have now verified assertions (1) and (2), and may proceed as in [2] or [13, IX, Section 1] to show that the $(m/4k)$-jet of G is a Whitney field on Y. The same computation also shows that the mapping Θ^m is continuous. The following lemma, which may be established by a straightforward calculation, is used in the argument and accounts for the further loss of differentiability.

Lemma 9.11. *There exists a positive constant c such that for every pair of points a, $x \in Y'$, there exist points a', $x' \in Y'$ such that $\pi(a') = \pi(x')$ and*

$$|\pi(a) - \pi(x)|^{1/2} \geqq c(|a-a'| + |x-x'|).$$

This completes the proof of Proposition 9.3, and hence of Theorem I'.

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