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Deformations of Strictly Pseudoconvex Domains

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§ 1. Introduction

Suppose D_1 and D_2 are two open sets in $\mathbb{C}^n$. One of the oldest problems in complex analysis has been to determine geometric conditions which imply that D_1 and D_2 are biholomorphically equivalent. The first step in this direction is the Riemann mapping theorem and the well-known classification of simply-connected open subsets of $\mathbb{P}_1$ in the case $n=1$. For domains in $\mathbb{C}^n$, $n \geqq 2$, this problem has been studied by Poincaré [13], who assumed that D_1 and D_2 had smooth boundaries and introduced geometric invariants on the boundary to study this equivalence problem, assuming the biholomorphic mapping $F: D_1 \to D_2$ was holomorphic past the boundary. In particular, Poincaré studied perturbations of the unit ball B_2 in $\mathbb{C}^2$ of a particular kind, and found necessary and sufficient conditions on a first order perturbation that the perturbed domain be biholomorphically equivalent to B_2. Poincaré's paper was very influential and led to a series of developments by Segre [14, 15], E. Cartan [5], Tanaka [16, 17], and Chern-Moser [7], in which the equivalence problem for CR-structures on real hypersurfaces was carefully studied (the survey paper [4] gives a discussion of this problem). The fundamental paper of Fefferman [9] showed that the biholomorphic equivalence problem for bounded domains with strictly pseudoconvex boundaries can be reduced to the CR-hypersurface equivalence problem. Namely, a biholomorphic mapping between two strictly pseudoconvex domains is smooth up to the boundary, by Fefferman's main theorem, and the induced boundary mapping is a CR-equivalence on the boundary. In this paper we use this principle to study deformations of complex structure on bounded strictly pseudoconvex domains in $\mathbb{C}^n$.

Suppose D_0 is a relatively compact, strictly pseudoconvex domain in the complex manifold X, with smooth boundary M defined by $\{z \in U \mid F(z)=0\}$, for

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some neighborhood U of M and defining function F strictly plurisubharmonic on U. Our main result asserts:

(1.1) Given a closed ball $\bar{B} \subset \mathbb{R}^k$, k arbitrary, there exist functions $\tilde{F} = \tilde{F}(z,t)$, $(z,t) \in \bar{U} \times \bar{B}$, so that

1) $\tilde{F}_t(z) = \tilde{F}(z,t)$ is arbitrarily close to F in the C^∞-topology, uniformly in t.

If $D_{t,\delta}$ is the region in X bounded by $M_{t,\delta} = \{z \mid \tilde{F}(z,t) = \delta\}$, $t \in B$, $\delta \in \mathbb{R}$ small enough, then

2) $D_{t,\delta}$ is biholomorphic to $D_{t',\delta'}$ iff $t = t'$, $\delta = \delta'$.

Further, the only biholomorphic automorphism of $D_{t,\delta}$ is the identity.

For the slightly more precise statement of what we prove, see Theorem 4.1 below.

The result is, in principle, obvious "locally" because of all the local invariants attached to bounding hypersurfaces $M_{t,\delta}$. We have to tie these local invariants together globally, and do this by means of generically defined CR-invariant functions on the hypersurfaces, derived from the local normal form (equivalently, curvature) invariants. Transversality applied to these curvature functions allows us to globally distinguish large families of hypersurfaces, and by Fefferman's theorem, the regions they bound. We do not cover here the formalizing of "deformation of complex manifold with boundary" or the corresponding "number of moduli". (Cf. [10] or [12] for a discussion of this.) That k in (1.1) is arbitrary says that this number of moduli must be infinite. It would be interesting to see whether any reasonable structure could be imposed on the "moduli space" of a (perhaps generic) strictly pseudoconvex domain by means of pseudo-conformal curvature functions.

The breakdown of the paper is as follows: § 2 records the well-known equivalence of biholomorphic classification of smooth, strictly pseudoconvex domain in complex manifolds and the CR-equivalence of their boundaries, following from Fefferman's theorem. We also include here some tangential remarks on slight improvements in the differentiability requirements for such arguments. § 3 gives one method for generating scalar CR-invariants on a strictly pseudoconvex hypersurface M, directly from Moser's normal form. The point here is to give a class of restricted normal forms at a non-umbilic point on M which are ambiguous by only a unitary group action. Invariant polynomials applied to coefficients of the normal forms yield the desired invariants on M. § 4 uses these functions and transversality to distinguish generic perturbations of a given domain. The main technical point here is to calculate the critical points of the curvature functions of § 3 and verify their functional independence.

In § 5 we describe an alternative method to § 3, using the curvature of the pseudoconformal connection associated to M (M as above). We also give an extension to higher dimensions of some results of E. Cartan and Moser on finding a CR-invariant distinguished frame at a generic point of a hypersurface. In the appendix to § 5, we check that an algebraic condition we require on the curvature tensor is, indeed, generic. Finally, in § 6, we describe still another method for perturbing domains. This is to eliminate all non-trivial global symmetries in a family of domains, so that local invariants at a pre-chosen boundary point become global invariants. We carry out an example on the unit ball. This method leads most directly to explicit examples, our reason for including it here.

We wish to express our gratitude to the referee of the first version of this paper whose suggestion led simultaneously to extensions of our results and simplification in their presentation.

Some of the results here were announced in [3] and [21].

§ 2. Reduction to the Boundary

Let M be a real C^1 hypersurface in a complex manifold X, and let $H(M) = T(M) \cap JT(M)$ be the bundle of holomorphic tangent spaces to M, where J is the almost complex tensor given by the complex structure of X, and $T(M)$ is the real tangent bundle to M, i.e., for each $x \in M$, $H_x(M) \subset T_x(M)$ is the maximal complex subspace of $T_x(X)$ which is contained in $T_x(M)$. The pair $(M, H(M))$ is called a CR-*hypersurface*, and $H(M) \subset T(M)$ is called the CR-*structure* of M induced by the inclusion $M \subset X$. Suppose $M \subset X$, $M' \subset X'$ are two CR-hypersurfaces in complex manifolds X and X', then a C^1-mapping

$$f\colon M \to M'$$

is called a CR-*mapping* if $df|_{H(M)}\colon H(M) \to H(M')$ is well defined and $\mathbb{C}$-linear. If $f\colon M \to M'$ is a diffeomorphism, such that f and f^{-1} are CR-mappings, then f is called a CR-*equivalence*. If $f\colon M \to M$ is a CR-equivalence, then f is called a CR-*automorphism*. We speak of CR-equivalences or automorphisms of class C^κ, $1 \leqq \kappa \leqq \omega$ (where C^ω = real analytic). (Cf. Wells [21] for a discussion of CR-function theory.)

Our major interest in this paper is the biholomorphic equivalence of certain classes of open complex manifolds. The following theorem, based on well known arguments, allows us to reduce this question to one of CR-equivalence on the boundary for nice geometric situations.

Theorem 2.1. *Let D and D' be bounded domains in $\mathbb{C}^n$, $n \geqq 2$, with C^∞, strictly pseudoconvex boundaries, then D and D' are biholomorphically equivalent if and only if ∂D and $\partial D'$ are* CR-*equivalent of class C^∞.*

The only if statement follows immediately from C. Fefferman's extension theorem [8]. The converse follows from the Bochner-Hartogs' theorem as follows:

1) Extend f to $F\colon \tilde{D} \to \mathbb{C}^n$, F holomorphic, $F \in C^\infty(\bar{D})$.
2) Check dF is invertible on ∂D and, hence, on all of $\bar{D}$.
3) $F(\bar{D})$ is compact in $\mathbb{C}^n$, while F is an open map on D, so $F(D) \subset D'$.
4) Apply the same procedure to f^{-1} to find F^{-1}.

We note in passing the following slight improvement of the converse, using [11].

Proposition 2.2. *Let D and D' be bounded domains in $\mathbb{C}^n$, $n \geqq 2$, each with a connected C^1 boundary, and suppose that $f\colon \partial D \to \partial D'$ is a* CR-*equivalence of class C^1, then D and D' are biholomorphically equivalent.*

(The usual statement in this context "$\partial D, \partial D' \in C^2$" and "$f \in C^2(\bar{D})$".)

Proof. The relevant fact from [11] is that f extends to $F: \bar{D} \to \mathbb{C}^n$, $F \in C^1(\bar{D})$. The rest follows the outline above.

The Harvey-Lawson result also permits one to generalize to the case where D, D' are bounded normal subvarities of $\mathbb{C}^n$ with C^1-boundaries ∂D, $\partial D'$.

§ 3. Some Scalar CR Invariants

Let M be a smooth strictly pseudoconvex real hypersurface through $0 \in \mathbb{C}^{n+1}$, $n \geqq 1$. We use the formal theory of [7, § 2] to derive functions on the set of "non-degenerate" points of M which are CR invariants of M. We will assume $n \geqq 2$ since the reduction for $n=1$ is carried out in § 3 of [7], where a reduced normal form is determined up to an action of $\{\pm 1\}$. Given this reduction the coefficients become scalar invariants on M (with possible $\pm$ indeterminacy). Cartan previously had derived scalar invariants by reducing to $\{\pm 1\}$ the group of the structure bundle he had defined. On the structure bundle there exists a Cartan connection which when pulled back to the reduced bundle gives rise to a set of dependency relations. The coefficients in these equations give Cartan his 9 scalar invariants. Using Moser's reduced normal form at a non-umbilic point one can explicitly compute these scalar invariants as functions of the coefficients. The result is that if M is described near the nonumbilic point p by $v - \Phi(z,u) = 0$ where

$$\begin{aligned}\Phi(z,u) = {} & z\bar{z} + \frac{1}{8} z^4 \bar{z}^2 + \frac{1}{8} z^2 \bar{z}^4 + \frac{j}{5!2!} z^5 \bar{z}^2 \\ & + \frac{\bar{j}}{5!2!} z^2 \bar{z}^5 + \frac{i\kappa}{4!2!} u z^4 \bar{z}^2 - \frac{i\kappa}{4!2!} u z^2 \bar{z}^4 \\ & + \frac{c_{26}}{2!6!} z^2 \bar{z}^6 + \frac{c_{35}}{3!5!} z^3 \bar{z}^5 + \frac{c_{44}}{4!4!} z^4 \bar{z}^4 + \frac{c_{53}}{5!3!} z^5 \bar{z}^3 \\ & + \frac{c_{62}}{6!2!} z^6 \bar{z}^2 + \frac{d_{25}}{2!5!} u z^2 \bar{z}^5 + \frac{d_{34}}{3!4!} u z^3 \bar{z}^4 \\ & + \frac{d_{43}}{3!4!} u z^4 \bar{z}^3 + \frac{d_{52}}{2!5!} u z^5 \bar{z}^2 + \frac{e_{24}}{2\cdot 2!4!} u^2 z^2 \bar{z}^4 \\ & + \frac{e_{42}}{2\cdot 4!2!} u^2 z^4 \bar{z}^2 + O(|(u,z)|^9)\end{aligned}$$

where κ and c_{44} are real, then at $p=(0,0)$ the 9 Cartan invariants $\{\alpha, \bar{\alpha}, \beta, \theta, \bar{\theta}, \gamma, \eta, \bar{\eta}, \zeta\}$ are given by (letting $\varepsilon = \pm 1$):

$$\alpha = \frac{\varepsilon i}{6\cdot 8} j,$$

$$\beta = -\frac{\kappa}{2} - \frac{1}{2^7 3^2} |j|^2,$$

$$\theta = -\frac{i}{4\cdot 6} c_{26} + \frac{i}{4\cdot 6^2} (\bar{j})^2 + \frac{i}{4\cdot 6} (\bar{j})^2,$$

$$\gamma = -\frac{1}{2\cdot 4}\kappa + \frac{1}{4\cdot 3}\operatorname{Re}(c_{35}) + \frac{1}{2\cdot 4^2 6^2}|j|^2,$$

$$\eta = \varepsilon\left\{\frac{1}{3\cdot 4}d_{25} + \frac{i}{3\cdot 4\cdot 6}\bar{j}\kappa + \frac{1}{3\cdot 4}\bar{d}_{34} + \frac{i}{4^3 6^3}\bar{j}|j|^2\right\},$$

$$\zeta = -\frac{1}{6}\operatorname{Re}(e_{24}) - \frac{1}{3^2 4}\kappa^2 - \frac{1}{3\cdot 4\cdot 6^2 8}|j|^2 + \frac{1}{2\cdot 3\cdot 6\cdot 8}\operatorname{Im}(\bar{j}\, d_{34})$$

$$+\frac{1}{2\cdot 3\cdot 6\cdot 8}\operatorname{Im}(\bar{j}\,\overline{d_{25}}) - \frac{1}{3\cdot 4\cdot 6^2\cdot 8}|j|^2\kappa - \frac{3}{4^5 6^4}|j|^4.$$

See Cartan [5, I pp. 76–86], in particular formulas (36), (42) and (43).

We proceed to the case $n \geqq 2$. By [7, §2] after a formal biholomorphic change of coordinates (i.e. by a formal power series transformation involving (z, w), $z = (z^1 \dots z^n) \in \mathbb{C}^n$, and $w = u + iv \in \mathbb{C}$) the Taylor series of a defining function F of M may be written

$$F = v - \langle z, z\rangle - N$$

where we take $\langle z, z\rangle = \sum_{\alpha=1}^{n} |z^\alpha|^2$, and $N = \sum_{p,q\geqq 2} N_{p,q}$. Here $N_{p,q}$ is a polynomial of type (p, q) in z, with coefficients formal power series in the variable u:

$$N_{p,q} = \sum N_{\alpha_1\dots\alpha_p,\,\bar\beta_1\dots\bar\beta_q}(u)\, z^{\alpha_1}\dots z^{\alpha_p}\,\overline{z^{\beta_1}}\dots\overline{z^{\beta_q}}.$$

The $N_{\alpha_1\dots\alpha_p,\,\bar\beta_1\dots\bar\beta_q}$ are symmetric in the α's and β's and each may be expanded in powers of u:

$$N_{\alpha_1\dots\alpha_p\,\bar\beta_1\dots\bar\beta_q} = \sum_{l\geqq 0} N^{(l)}_{\alpha_1\dots\alpha_p\,\bar\beta_1\dots\bar\beta_q} u^l.$$

There are further normalizations on N:

$$\operatorname{tr}(N_{2,2}) = \sum_{\beta} N_{\alpha\beta\bar\gamma\bar\beta} = 0,$$

$$\operatorname{tr}^2(N_{3,2}) = \sum_{\beta,\gamma} N_{\alpha\beta\gamma,\bar\beta\bar\gamma} = 0,$$

$$\operatorname{tr}^3(N_{3,3}) = \sum_{\alpha\beta\gamma} N_{\alpha\beta\gamma,\bar\alpha\bar\beta\bar\gamma} = 0. \tag{3.1}$$

Furthermore, given any two such formal coordinate systems and normalized F for a given M, say (z, w) and F, (z^*, w^*) and F^*, then there exists a unique formal biholomorphic transformation

$$\Phi\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0),$$

such that

$$F^* \circ \Phi = h \cdot F \tag{3.2}$$

where h is a real formal power series with $h(0)\neq 0$ (i.e., non-zero constant term), and $\Phi=(f,g)$,

$$z^* = f(z,w),$$
$$w^* = g(z,w)$$

is uniquely specified by:

$$\frac{\partial f}{\partial z^\alpha}, \frac{\partial f}{\partial w}, \frac{\partial g}{\partial w}, \quad \text{and} \quad \operatorname{Re}\left(\frac{\partial^2 g}{\partial w^2}\right). \tag{3.3}$$

The point 0 is called umbilic if $N^{(0)}_{\alpha\beta\bar\gamma\bar\delta}=0$, and this is independent of the normal form F and normal coordinates (z,w). We assume in this § that $\sum |N^{(0)}_{\alpha\beta\bar\gamma\bar\delta}|^2 \neq 0$, i.e., 0 is *non-umbilic*. We wish to impose further restrictions on the normal form at a non-umbilic point.

Lemma 3.1. *There exist normal coordinate systems* (z,w) *for* M, *non-umbilic at* 0, *and corresponding normal forms satisfying*[1]*:*

1) $\sum |N^{(0)}_{\alpha\beta\bar\gamma\bar\delta}|^2 = 1$,
2) $\sum N^{(0)}_{\alpha\beta\gamma,\bar\delta\bar\varepsilon}\, \overline{N^{(0)}_{\beta\gamma,\bar\delta\bar\varepsilon}} = 0$,
3) $\sum N^{(1)}_{\alpha\beta\bar\gamma\bar\delta} \cdot \overline{N^{(0)}_{\alpha\beta,\bar\gamma\bar\delta}} = 0$.

Any two such coordinate systems are related by a (unique) transformation Φ of the form $w^*=w$, $z^*=U\cdot z$, where U is a unitary $n\times n$ matrix.

Remark. The above conditions were derived from trying to set

$$\Psi = \sum \left| \frac{\partial^4 N}{\partial z^\alpha\, \partial z^\beta\, \partial \overline{z^\gamma}\, \partial \overline{z^\delta}} \right|^2 = \text{constant}.$$

Conditions 2), 3) simply say grad $\Psi=0$ at 0.

Proof. It is easy to satisfy condition 1), by dilating a given normal coordinate system (z,w) by $z^*=\rho z$, $w^*=\rho^2 w$, $\rho>0$. Then

$$\sum |N^{*(0)}_{\alpha\beta\bar\gamma\bar\delta}|^2 = \rho^{-2} \sum |N^{(0)}_{\alpha\beta\bar\gamma\bar\delta}|^2,$$

and the proper choice of ρ guarantees (z^*,w^*) satisfy 1). For the rest, it is clear that $\phi(z,w)=(U\cdot z,w)$ preserves 1), 2), 3), so we are done if we show, given a normal coordinate system (z,w) and normal form $F=v-\langle z,z\rangle - N$, that there is a *unique* transformation Φ of the form

$$z^* = z + aw + \text{higher order terms}$$
$$w^* = w + \text{higher order terms}, \tag{3.4}$$

where we are free to specify $\operatorname{Re}\left(\frac{\partial^2 w^*}{\partial w^2}\right) \in \mathbb{R}$, and $a=(a^1,\ldots,a^2)\in\mathbb{C}^n$, such that

[1] This normalization also occurs in [19]

$F^* \circ \Phi = h \cdot F$, as in (3.1), with F^* satisfying conditions 1)–3) of the lemma, assuming F already satisfies 1).

Solving (3.1) with Φ as in (3.4), we find:

A) $N^{*(0)}_{\alpha\beta\bar\gamma\bar\delta} = N^{(0)}_{\alpha\beta\bar\gamma\bar\delta}$.

B) Assuming 1) of the lemma for $N^{(0)}_{\alpha\beta\bar\gamma\bar\delta}$, then

$$\sum N^{*(0)}_{\alpha\beta\gamma\bar\delta\bar\varepsilon} \cdot \overline{N^{*(0)}_{\beta\gamma\bar\delta\bar\varepsilon}} = \sum N^{(0)}_{\alpha\beta\gamma,\bar\delta\bar\varepsilon} \cdot \overline{N_{\beta\gamma\bar\delta\bar\varepsilon}} - \frac{2i}{3}\overline{a^\alpha}.$$

Hence, there is a unique choice of vector a guaranteeing 2) for N^*.

C) Assuming 1) and 2) of the lemma for N, then

$$\sum N^{*(1)}_{\alpha\beta\bar\gamma\bar\delta} \overline{N^{*(0)}_{\alpha\beta\bar\gamma\bar\delta}} = \sum N^{(1)}_{\alpha\beta\bar\gamma\bar\delta} \overline{N^{(0)}_{\alpha\beta\bar\gamma\bar\delta}} - 2t,$$

where $t = \mathrm{Re}\left(\frac{\partial^2 g}{\partial w^2}\right)$. Hence, $\mathrm{Re}\left(\frac{\partial^2 g}{\partial w^2}\right)$ is also uniquely determined to guarantee 3).

We omit the explicit computation of A)–C). It simply requires solving (3.1) for relevant terms. This requires examination of (3.2) through terms of degree $\leqq 6$, and weight with respect to (z, w) (i.e., z^α = weight 1, w = weight 2) $\leqq 6$.

Call a normal form as in the lemma a *restricted normal form*. Since they differ from one another only by a $U(n)$ action on the z-variable, any $U(n)$-invariant polynomial on the tensor algebra applied to the coefficients $N^{(l)}_{\alpha_1\ldots\alpha_p,\bar\beta_1\ldots\bar\beta_q}$ will give a CR-invariant number attached to M at 0. It is clear from the construction in [7, § 2], and the arguments above, that such invariants will vary smoothly with the (non-umbilic) point $p \in M$.

Finally, we remark that such "$U(n)$-invariants" obviously do not give all possible scalar invariants of a hypersurface M (cf., § 5).

§ 4. Deformations of Pseudoconvex Domains

In this section M denotes a compact connected strictly pseudoconvex smooth real hypersurface in a complex manifold X of dimension $n+1$. Let φ be a smooth strictly plurisubharmonic function defined near M with $\varphi = 0$ and $d\varphi \neq 0$ on M. Let $U = \{x \in X \mid -\varepsilon < \varphi(x) < \varepsilon\}$ for small ε so that $\bar U$ is compact and smoothly bounded in X.

Let $\mathscr{P}(\bar U)$ denote the open set in $C^\infty(\bar U)$ of strictly plurisubharmonic functions ψ with $d\psi \wedge d^c\psi \wedge (dd^c\psi)^n \neq 0$ on $\bar U$. Letting $B \subset \mathbb{R}^k$ be a small ball around 0, we set $\mathscr{P}(\bar U \times \bar B)$ = the set of $\psi \in C^\infty(\bar U \times \bar B)$ such that $\psi(x,t) = \psi_t(x) \in \mathscr{P}(\bar U)$ for all $t \in \bar B$. For $\psi \in \mathscr{P}(\bar U \times \bar B)$ set $M_{t,\delta} = \{X \in U \mid \psi_t(x) = \delta\}$. Note that $\mathscr{P}(\bar U) \subset \mathscr{P}(\bar U \times \bar B)$ as functions independent of t.

Theorem 4.1. *There exists an open dense set $\mathscr{V} \subset \mathscr{P}(\bar U \times \bar B)$ with $\varphi \in \mathscr{V}$ and a set of second category $\mathscr{R} \subset \mathscr{V}$ such that for every $\psi \in \mathscr{R}$, $t_i \in \bar B$, and $\delta_i \in \mathbb{R}$ small enough, $i = 1, 2$,*

1) M_{t_1,δ_1} *is* CR-*equivalent to* M_{t_2,δ_2} *iff* $t_1 = t_2$, $\delta_1 = \delta_2$.

2) *The group of* CR-*automorphisms of* M_{t_1,δ_2} *reduces to the identity transformation.*

For $\psi\in\mathscr{P}(\bar{U}\times\bar{B})$, taking $t\in B$, $\delta\in\mathbb{R}$, small enough, $M_{t,\delta}$ is a compact, connected, strictly pseudoconvex hypersurface in X. If M bounds the relatively compact region D in X, then such an $M_{t,\delta}$ also bounds a relatively compact region $D_{t,\delta}$.

Corollary 4.2. *ψ, t_i, δ_i as in the theorem,*

1) *D_{t_1,δ_1} is not biholomorphic to D_{t_2,δ_2}, unless $t_1=t_2$, $\delta_1=\delta_2$.*

2) $\mathrm{Aut}(D_{t,\delta})=\{\mathrm{id}\}$.

The corollary follows directly from the theorem and Fefferman's theorem (§ 2).

Note that since $k=\dim B$ is arbitrary, the number of moduli of D is indeed infinite.

Proof of the Theorem. The argument is based on Thom transversality. $J^N=J^N(\bar{U})$ denotes the bundle of N-jets of real-valued functions on $\bar{U}$. $J_0^N\subset J^N$ is the open set of jets $j^N(\varphi)$, where $d\varphi\wedge d^c\varphi\wedge(dd^c\varphi)^n\neq 0$ and where φ is plurisubharmonic. (Here, and in what follows, N will be assumed large enough for the context, e.g., here $N\geqq 2$ to allow for the formal computation of $dd^c\varphi$ from N-jet data.) Let $J_1^N\subset J_0^N$ denote the jets $j^N(\varphi)$ such that the hypersurface $\{\varphi=\varphi(p)\}$ is non-umbilic at $p\in\bar{U}$. For φ with $j^N(\varphi)\in J_1^N$, we may form the first several (depending on N) terms of restricted normal form for φ at p, and use the method of § 3 to assign scalar curvature invariants to $j^N(\varphi)$. For the sake of definiteness, let us consider smooth functions $K_{p,q}^{(l)}$, for integers $l\geqq 0$, $p\geqq q\geqq 2$, $5\leqq p+q+2l\leqq N$,

$$K_{p,q}^{(l)}: J_1^N\to\mathbb{R},$$

defined by

$$K_{p,q}^{(l)}(j^N(\varphi))=\sum |N_{\alpha_1\dots\alpha_p,\beta_1,\dots,\beta_q}^{(l)}|^2.$$

Here, $N_{\alpha_1,\dots,\alpha_p,\beta_1,\dots,\beta_q}^{(l)}$ are the coefficients of a restricted normal form at p. Thus $K_{2,2}^{(0)}\equiv 1$ on J_1^N. These $K_{p,q}^{(l)}$ are along each fiber of J_2^N given by real algebraic functions in the natural jet coordinates. Let $S_{p,q}^{(l)}\subset J_1^N$ denote the zero locus of $K_{p,q}^{(l)}$, and set $S_{2,2}^{(0)}=J_0^N-J_1^N$.

Lemma 4.3. A) *$S_{2,2}^{(0)}$ is a smooth submanifold of J_0^N of codimension* $=\left[\frac{n(n+1)}{2}\right]^2-n^2$.

B) *$S_{p,q}^{(l)}$, for $l+p+q\geqq 5$, is a smooth submanifold of J_1^N of*

$$\textit{codimension}=\begin{cases} 2\binom{n+p-1}{n-1}\binom{n+q-1}{n-1}, & q\neq p>3\\ \binom{n+p-1}{n-1}^2, & p=q>3\\ 2\binom{n+2}{n-1}\binom{n+1}{n-1}-n, & p=3,\ q=2\\ \binom{n+2}{n-1}^2-1, & p=q=3.\end{cases}$$

Proof. First, note that it suffices to prove $S^{(0)}_{2,2}\cap J^N_{0,x}$ is smooth, of the given codimension, for any $x\in\bar{U}$, and similarly for the other $S^{(l)}_{p,q}$. ($J^N_{0,x}$ = fiber of J^N_0 over x.) Next, given a fixed set of reference coordinates (z,w) at x, and a fixed reference positive hermitian form $\langle z,z\rangle=\sum_{\alpha=1}^{n}|z^\alpha|^2$, we denote by $\mathcal{N}^N_{(k)}(z,w)\subset J^N_{0,x}$ the space of jets whose (z,w)-Taylor expansions are in Moser normal form through terms of weight $k\leqq N$, weight measured with respect to (z,w) (z^α = weight 1, w = weight 2).

Now $\mathcal{N}^N_{(N)}(z,w)\cap S^{(0)}_{2,2}$ is clearly smooth, of the required codimension, since it is given by

$$N^{(0)}_{\alpha\beta\bar{\gamma}\bar{\delta}}=0,\qquad \text{all } \alpha,\beta,\gamma,\delta,$$

($N^{(l)}_{\alpha\beta\bar{\gamma}\bar{\delta}}$ and the like represent coefficients in (z,w)-Taylor expansions). The codimension given is the number of independent equations here, given the normalization condition $\sum_\beta N^{(0)}_{\alpha\beta\bar{\gamma}\bar{\beta}}=0$, and the reality condition. (Similar comments explain the other codimensions given.)

The lemma will follow from this fact, and the following elementary observation:

Suppose $V\supset W$ are two closed smooth submanifolds of $J^N_{0,x}$ and H a smooth manifold of transformations of $J^N_{0,x}$, leaving $S^{(0)}_{2,2}$ stable, with dimension H = codimension W in V. If $W\times H\ni(\varphi,g)\to\varphi\cdot g$ gives a differmorphism onto a neighborhood of W in V, then $S^{(0)}_{2,2}\cap V$ is smooth at $\varphi\in S^{(0)}_{2,2}\cap W$ of codimension d in V if and only if $S^{(0)}_{2,2}\cap W$ is smooth at φ of codimension d in W.

To apply this, we first take $J^N_{0,x}=V_1$, $W_1=\{\varphi\in J^N_{0,x}\,|\,\varphi(x)=0\}$, and $H=\mathbb{R}$ acting on W by $\varphi\to\varphi+t$. Next we take $V_2=W_1$, and note that we have a group G acting in $V_2\subset J^N_{0,x}$, $G=\{(h,\Phi)\,|\,h=N\text{-jet of a real function with } h(x)\neq 0$, and $\Phi=N$-jet of a biholomorphic transformation, $\Phi(x)=x\}$. The pair (h,Φ) in G acts on V_2 on the right by taking $\varphi\in V_2$ to $(1/h)\cdot\varphi(\Phi)$. $S^{(0)}_{2,2}\cap V_2$ (as well as the other $S^{(l)}_{p,q}\cap V_2$) is G-stable. Note that by Moser's theorem, $V_2=\bigcup_{g\in G}\mathcal{N}^N_{(N)}\cdot g$, and hence we have only to show $S^{(0)}_{2,2}\cap V_2$ is smooth near any $\varphi\in\mathcal{N}^N_{(N)}\cap S^{(0)}_{2,2}$. With that in mind, we apply the observation above repeatedly about such a φ.

1) Take $W_2\subset V_2$ as $\{\varphi\in V_2\,|\,d\varphi=\text{positive real multiple of } dv\}$,

$$H=\{(1,\Phi)\in G\,|\,\Phi(z,w)=(z+aw,w-\langle z,a\rangle+itw^2),\ a\in\mathbb{C}^n,\ t\in\mathbb{R}\}.$$

2) $V_3=W_2$, $\quad W_3=\{\varphi\in V_3\,|\,d\varphi=dv\}$, $\quad H=\{(r,\mathrm{Id})\in G\,|\,r\in\mathbb{R}^+\}$,

3) $V_4=W_3$, $\quad W_4=\left\{\varphi\in V_4\,\middle|\,\dfrac{\partial^2\varphi}{\partial z^\alpha\,\partial z^\beta}=0,\alpha,\beta=1,\dots,n\right\}$,

$H=\{(1,\Phi)\in G\,|\,\Phi(z,w)=(z,w+Q(z)),Q(z)$ a (holomorphic) quadratic polynomial in $z\}$.

4) $V_5=W_4$, $W_4=\mathcal{N}^N_{(2)}(z,w)$, $H=\{(1,\Phi)\in G\,|\,\Phi(z,w)=(Az,w)$, A a linear transformation with $\langle Az,z'\rangle=\langle z,Az'\rangle$ and $\langle Az,z\rangle>0\}$.

5) Finally, to pass from $V=\mathcal{N}^N_{(k)}$ to $W=\mathcal{N}^N_{(k+1)}$, use the formulation of Moser's algorithm given in [4, §3]. Let H = set of (h, Φ) in G with

$$h=1+h_{(k-1)},$$
$$\Phi(z,w)=(z+f_{(k)}, w+g_{(k+1)})$$

where $h_{(k-1)}, f_{(k)}, g_{(k+1)}$ are homogeneous of weights $k-1$, k, $k+1$, respectively. Then the action $W\times H\to V$ is a diffeomorphism. (Note that for $k=2$, we must take $\frac{\partial f}{\partial w}(0)=0$; and for $k=3$, we must take $\frac{\partial^2 g(0)}{\partial w^2}=0$ for this diffeomorphism to hold.)

Finally, since $S^{(0)}_{2,2}\cap\mathcal{N}^N_{(N)}$ is smooth, we conclude that $S^0_{2,2}\cap J^N_{0,x}$ is smooth and of correct codimension. The same arguments clearly apply as well to the other $S^{(l)}_{p,q}$ proving the lemma.

Proof of Theorem 4.1. We will assume that $n\geqq 2$. Let $\mathcal{V}\in\mathcal{P}(\bar U\times\bar B)$ be the set of all ψ for which $j^N(\psi_t)\pitchfork S^{(0)}_{2,2}$ for all $t\in\bar B$. $\mathcal{V}$ is of second category in $\mathcal{P}(\bar U\times\bar B)$ by Thom transversality, and moreover since $\bar U\times\bar B$ is compact, and $S^{(0)}_{2,2}$ is closed in J^N_0, $\mathcal{V}$ is open.

Given a collection $\{K_1,\ldots,K_m\}$ of m distinct curvature functions $K^i=K^{(h)}_{p_i,q_i}$ defined on J^N_1 earlier, the mapping

$$K=(K_1,\ldots,K_m)\colon \hat J^N_1=J^N_1-\bigcup_{i=1\ldots m} S^{(h)}_{p_i,q_i}\to\mathbb{R}^m$$

is of maximal rank. This follows because it is so when restricted to $\hat J^N_1\cap\mathcal{N}^N_{(N)}(z,w)$ in each fiber, and such a set goes through any point of $\hat J^N_1$. Consider the map

$$K\times K\colon \hat J^N_1\times\hat J^N_1\to\mathbb{R}^m\times\mathbb{R}^m$$

and the smooth manifold $K\times K^{-1}(\Delta)$ of codimension m, where Δ is the diagonal in $\mathbb{R}^m\times\mathbb{R}^m$. By Mather's extension of Thom transversality (cf. [9]), there is a set $\mathcal{R}\subset\mathcal{V}$ of second category, such that for $\psi\in\mathcal{R}$, $[j^N(\psi_t)\times j^N(\psi_t)]\pitchfork(K\times K)^{-1}(\Delta)$, intersections taken in $J^N_0\times J^N_0$. By taking $m>2\dim(\bar U\times\bar B)$, this guarantees that $[j^N(\psi_t)\times j^N(\psi_t)]\cap(K\times K)^{-1}(\Delta)$ is empty.

For $\psi\in\mathcal{V}$, let $\Sigma\subset\bar U\times\bar B$ be the umbilic locus, i.e., those (x,t) for which $j^N(\psi_t)_x\in S^{(0)}_{2,2}$. Σ is a smooth submanifold of $\bar U\times\bar B$, as is $\Sigma\cap M_{t,\delta}$, of codimension $\geqq\left[\frac{n(n+1)}{2}\right]^2-n^2$. For $\psi\in\mathcal{V}$, we can define $K(j^N(\psi))\colon\bar U\times\bar B-\Sigma\to\mathbb{R}^m$.

Suppose $\psi\in\mathcal{R}$, and that $T\colon M_{t_1,\delta_1}\to M_{t_2,\delta_2}$ is a CR-equivalence, for $t_i\in\bar B$, $\delta_i\in\mathbb{R}$ (we assume $M_{t_i,\delta_i}\neq\emptyset$). On $M_{t_i,\delta_i}-(M_{t_i,\delta_i}\cap\Sigma)$, $K(j^N(\psi_{t_i}))$ is well-defined, and we have $K(j^N(\psi_{t_2}))\circ T=K(j^N(\psi_{t_1}))$. Since K separates points on $\bar U\times\bar B-\Sigma$, this implies $t_1=t_2$, $M_{t_1,\delta_1}=M_{t_2,\delta_2}$, $\delta_1=\delta_2$. The same argument shows that if $T\colon M_{t_1,\delta_1}\xrightarrow{\sim}M_{t_1,\delta_1}$, then T must be the identity on $M_{t_1,\delta_1}-(\Sigma\cap M_{t_1,\delta_1})$, and thus on all of M_{t_1,δ_1}.

Remarks. 1) Lemma 4.3 shows that for $n=2$, the umbilic locus is generically a set of isolated points, or empty, and is generically empty for $n\geqq 3$. (For $n=1$, it is generically a smooth curve, or empty.)

2) If $n=1$, one can use the 9 scalar invariants of E. Cartan as $K_1, \ldots, K_9$ (cf. the beginning of § 3). The expression of these scalar invariants in terms of Moser normal coordinates gives easily the required functional independence. Alternatively, one can modify the construction given above using the higher order terms in order to obtain a suitable set of scalar invariants. Details in this case are omitted.

§ 5. A Complete set of Scalar Invariants

Given further assumptions beyond $\|N\|^2 \neq 0$ for $n \geqq 2$ on the value at p of the tensor $N = N^{(0)}_{\alpha\beta\bar{\gamma}\bar{\delta}}$ it is possible to reduce the group transforming the normal forms at p from $U(n)$ to the trivial group consisting of the identity alone. (Note: The curvature term $S_{\alpha\beta\bar{\gamma}\bar{\delta}}$ in [7, § 4] is a nonzero multiple of $N_{\alpha\beta\bar{\gamma}\bar{\delta}}$.) Once this reduction is achieved the normal form at p is uniquely determined and when the reduction is carried out on the subset M_0 of M consisting of "non-degenerate" points the coefficients of the normal form give a complete set of scalar invariants on M_0.

The reduction is based on the observation that $N_{\alpha\beta\bar{\gamma}\bar{\delta}}$ defines an endomorphism of the real vector space $\mathscr{H}_0$ of hermitian $n \times n$ matrices of trace 0.

$$(A_{\gamma\bar{\alpha}}) \mapsto (N_{\alpha\beta\bar{\gamma}\bar{\delta}} A_{\gamma\bar{\alpha}}) = (B_{\beta\bar{\delta}}).$$

As a linear transformation N has trace 0 and is symmetric with respect to the inner product on $\mathscr{H}_0$ given by $(A, B) = \operatorname{tr}(AB)$. Letting $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_{n^2-1}$ be the eigenvalues of N acting on $\mathscr{H}_0$ and $E_1, E_2, \ldots, E_{n^2-1}$ the corresponding eigenvectors, the following conditions are generic (see Proposition 5.1 below):

a) the λ_i are distinct,

b) if $\mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_n$ are the eigenvalues of E_1 with corresponding eigenvectors $e_1, e_2, \ldots, e_n$ then the μ_i are distinct and $(E_2 e_i, e_{i+1}) \neq 0$.

For $n>2$ the same proof that the μ_i are generically distinct shows that the $|\mu_i|$ are also. Since initially E_1 is determined only up to ± 1 we can determine E_1 uniquely by requiring $|\mu_1| > |\mu_n|$. Then we can reduce to a subgroup of the group $U(n)$ or equivalently restrict the permissible unitary frames $\{e_1, \ldots, e_n\}$ for the complex tangent space to M at p by requiring that the frame be an eigenbasis as in b) above. In this way we reduce to the action of an n-torus $T(n)$. If we make the further assumption $(E_2 e_i, e_{i+1}) > 0$ the group $T(n)$ is reduced to a circle group S^1. Reducing the circle group requires a higher order assumption since the surface given by $v = \|z\|^2 + N_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\gamma z^\delta \bar{z}^\alpha \bar{z}^\beta$ admits a circle action for arbitrary $N_{\alpha\beta\bar{\gamma}\bar{\delta}}$. Such an assumption is given by $N^{(0)}_{11\alpha\bar{1}\bar{\alpha}} \neq 0$ in which case we can require that the value be positive real.

For $n=2$ alter the argument as follows. Suppose $\lambda_1 > \lambda_2 > \lambda_3$ are the eigenvalues of N with eigenvectors E_1, E_2, E_3 determined up to a factor of ± 1 then there is a basis (e_1, e_2) for $H_p(M)$ such that

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

The basis is determined up to the action

$$A\cdot(e_1,e_2)=(e_2,e_1),$$
$$B\cdot(e_1,e_2)=(e_1,-e_2),$$
$$C_\vartheta(e_1,e_2)=(e^{i\vartheta}e_1,e^{i\vartheta}e_2).$$

The matrices A, B, C_ϑ generate a subgroup G of $U(2)$.

$$S^1\longrightarrow G\xrightarrow{p}\mathbb{Z}_2\times\mathbb{Z}_2$$

where p is the restriction of the adjoint map on $U(2)$ to G. Let $f_1=N^{(0)}_{11\alpha 1\bar{\alpha}}$ and $f_2=N^{(0)}_{22\alpha\bar{2}\bar{\alpha}}$. Assuming f_1 and f_2 not both 0 the ratio $\frac{f_1}{f_2}$ takes values in the extended complex plane and assuming $\frac{f_1}{f_2}\neq\pm1,\ \pm i$ the conditions $\left|\frac{f_1}{f_2}\right|\leqq 1$ $\operatorname{Re}\left(\frac{f_1}{f_2}\right)\geqq 0$ and $f_2\in\mathbb{R}^+$ reduce G to the identity. These give generic conditions reducing the group acting on normal forms to the identity.

§5a. Appendix

Let $\mathbb{C}^n$ be given a positive definite hermitian form $\langle z,z\rangle=\sum_{\alpha=1}^{n}|z^\alpha|^2$. Let $\mathscr{H}$ be the space of $n\times n$ hermitian matrices, $\mathscr{H}_0$ the subspace of those with trace 0. Let $\mathscr{N}$ be the space of tensors $N_{\alpha\beta\bar{\gamma}\bar{\delta}}$ symmetric in α, β and also in γ, δ and satisfying the equation $\overline{N_{\alpha\beta\bar{\gamma}\bar{\delta}}}=N_{\gamma\delta\bar{\alpha}\bar{\beta}}$. Each $N\in\mathscr{N}$ defines an endomorphism of $\mathscr{H}$

$$A_{\alpha\bar{\beta}}\to B_{\gamma\bar{\delta}}=\Sigma N_{\alpha\gamma\bar{\beta}\bar{\delta}}A_{\beta\bar{\alpha}}.$$

Define on $\mathscr{H}$ the real inner product $(A,B)=\operatorname{tr}(AB)$. For $N\in\mathscr{N}$, $N:\mathscr{N}\to\mathscr{H}$ is symmetric with respect to this inner product. $\mathscr{N}_0$ is the subspace of $N\in\mathscr{N}$ with trace 0 (equivalently $N(I_n)=0$). Since $\mathscr{H}_0$ is orthogonal to I_n, $N\in\mathscr{N}_0$ maps $\mathscr{H}_0$ into itself.

Proposition 5.1. *For generic $N\in\mathscr{N}_0$, $N:\mathscr{H}_0\to\mathscr{H}_0$ has distinct eigenvalues $\lambda_1>\lambda_2>\dots>\lambda_{n^2-1}$. If $A_i\in\mathscr{H}$ is the unit λ_i-eigenvector (determined up to ±1), then all eigenvalues of A_1, as an hermitian matrix on $\mathbb{C}^n$, are generically distinct, say $\mu_1>\dots>\mu_n$. If $e_1,\dots,e_n$ is an orthonormal basis of eigenvectors in $\mathbb{C}^n$ for A_1, $A_1(e_i)=\mu_i e_i$, then generically, $\langle A_2(e_i),e_{i+1}\rangle\neq 0$.*

Proof. The main point is the following lemma.

Lemma 5.2. *Given $E, F\in\mathscr{H}_0$ linearly independent and $A, B\in\mathscr{H}_0$ with $(A,F)=(E,B)$, then there exists $N\in\mathscr{N}_0$ with $\mathscr{N}(E)=A$, $N(F)=B$.*

Proof of the Lemma. Let $v_1,\dots,v_n$ be an orthonormal basis of $\mathbb{C}^n$, eigenvectors for E, and let $D\subset\mathscr{H}$ be the subspace of all those $C\in\mathscr{H}_0$ for which $v_1,\dots,v_n$ are all eigenvectors. Take $E_0=I_n$, $E_1=E$, $E_2,\dots,E_{n-1}$ an orthonormal basis of D.

Define inductively:

$$\mathcal{N}_i = \{N \in \mathcal{N}_{i-1} \mid N(E_i) = 0\},$$
$$\mathcal{H}_i = \{G \in \mathcal{H}_{i-1} \mid (G, E_i) = 0\},$$

with $\mathcal{N}_0$, $\mathcal{H}_0$ as above. Note that $\mathcal{N}_i = \ker\{E_i: \mathcal{N}_{i-1} \to \mathcal{H}_{i-1}\}$.

Claim. The evaluation maps $E_i: \mathcal{N}_{i-1} \to \mathcal{H}_{i-1}$, sending N to $\mathcal{N}(E_i)$, are onto, $i = 1, \ldots, n$. If $F \notin D$, $F: \mathcal{N}_{n-1} \to \mathcal{H}_{n-1}$ is onto.

The first part of the claim is equivalent to showing that $\operatorname{cod}\{\mathcal{N}_i \subset \mathcal{N}_{i-1}\} = \dim \mathcal{H}_{i-1} = n^2 - i$, $i = 1, \ldots, n$. Since $\operatorname{cod}\{\mathcal{N}_i \subset \mathcal{N}_{i-1}\} \leqq n^2 - i$, for each i, it suffices to show $\operatorname{cod}\{\mathcal{N}_{n-1} \subset \mathcal{N}\} = n^2 + (n^2 - 1) + \cdots + (n^2 - n + 1)$. But $\mathcal{N}_{n-1} = \{N \in \mathcal{N} \mid N(C) = 0, \text{ any } C \in D\}$, and thus $N \in \mathcal{N}_{n-1}$ iff $N_{\alpha\gamma\bar\beta\bar\delta}$ satisfies the n systems of equations ($\gamma = 1, \ldots, n$)

$$(L_\gamma): N_{\alpha\gamma\bar\beta\bar\gamma} = 0.$$

To compute the number of independent equations here, note that by the reality condition, each L_γ consists of n^2 independent equations, of which exactly $\gamma - 1$ are dependent on the system of equations L_δ, $\delta < \gamma$, namely:

$$N_{\delta\gamma\bar\delta\bar\gamma} = 0, \qquad \delta = 1, \ldots, \gamma - 1.$$

Thus, $\operatorname{cod}\{\mathcal{N}_{n-1} \subset \mathcal{N}\}$ is $n^2 + (n^2 - 1) + \cdots + (n^2 - n + 1)$.

For the second part of the claim, write $F = F_{\alpha\bar\beta}$, with at least one $\alpha \neq \beta$ such that $F_{\alpha\bar\beta} \neq 0$. We get equations

$$\sum_{\delta, \gamma} N_{\alpha\gamma\bar\beta\bar\delta} F_{\delta\bar\gamma} = 0.$$

Adjoined to the previous equations $(L_1), \ldots, (L_{n-1})$, we get

$$\sum_{\delta \neq \gamma} N_{\alpha\gamma\bar\beta\bar\delta} F_{\delta\bar\gamma} = 0.$$

We see then that exactly the n independent equations

$$\sum_{\gamma \neq \delta} N_{\alpha\gamma\bar\alpha\bar\delta} F_{\delta\bar\gamma} = 0, \qquad \alpha = 1, \ldots, n$$

are dependent on the previous equations, proving the claim.

To prove Lemma 5.2, suppose first that F is in the span of $\{E_1, \ldots, E_k\}$ but not $\{E_1, \ldots, E_{k-1}\}$ for $k \geqq 2$. Then there exists $G \in \mathcal{H}_0$ with $(G, E_i) = 0$, $i = 1, \ldots, k-1$, and $(G, F) = 1$. By the claim, there is an $N_1 \in \mathcal{N}_0$ with $N_1(E) = N_1(E_1) = A$. Let $B_1 = \mathcal{N}_1(F)$. Then $(B - B_1, E_1) = (B, E_1) - (F, A) = 0$, by assumption. Let $A_2 = (B - B_1, E_2) \cdot G$, which is perpendicular to E_0, E_1. By the claim, there exists $N_2 \in \mathcal{N}_1$, $N_2(E_2) = A_2$, and $N_2(F) = B_2$. By construction $B_2 \in \mathcal{H}_2$, so $B - B_1 - B_2$ is orthogonal to E_0, E_1, and $(B - B_1 - B_2, E_2) = (B - B_1, E_2) - (B_2, E_2) = (N_2(E_2), F) - (N_2(F), E_2) = 0$. Continue in this way until we have $B - B_1 - \cdots - B_{k-1}$ orthogonal to $E_1, \ldots, E_{k-1}$. Write $F = F_1 + F_2$, where F_2 is in the span of $\{E_1, \ldots E_{k-1}\}$, and $F_2 \in \mathcal{H}_{k-1}$. By the claim, since F_2 is a non-zero multiple of E_k, there is an $N_k \in \mathcal{N}_{k-1}$, with $N_k(F)=N_k(F_2)=B-B_2-\cdots-B_{k-1}$. Hence, $N=N_1+\cdots+N_k$ satisfies the requirements of the lemma. The case $F \notin D$ is similar, using the second part of the claim.

Returning to the proof of the proposition, we treat first the case $n=2$. Here $\dim \mathcal{H}_0=3$, the space of symmetric endomorphisms of $\mathcal{H}_0$ is 6, $\dim \mathcal{N}_0=5$, and it is easy to verify that $\mathcal{N}_0 \subset$ space of symmetric endomorphisms of trace 0. Equal dimensions implies equality and the proposition is immediate from this.

For $n>2$, we first consider special $N \in \mathcal{N}_0$ of the following form. Suppose $A=\lambda_\alpha \delta_{\alpha\beta}$ is a diagonal matrix in $\mathcal{H}_0$. Define $N \in \mathcal{N}_0$ by the formula

$$N_{\alpha\beta\bar{\gamma}\bar{\delta}}=\lambda_\alpha \lambda_\beta(\delta_{\alpha\bar{\gamma}}\delta_{\beta\bar{\gamma}}+\delta_{\alpha\bar{\delta}}\delta_{\beta\bar{\gamma}})-\lambda_\mu^2 \delta_{\alpha\bar{\mu}}\delta_{\beta\bar{\mu}}\delta_{\mu\bar{\gamma}}\delta_{\mu}.$$

If $B \in \mathcal{H}_0$ is also diagonal, $N(B)=\operatorname{tr}(B \cdot A)\cdot A$. For each $i<j$, $1 \leqq i,j \leqq n$, $E_{ij}=\delta_{\alpha\bar{i}}\delta_{j\beta}+\delta_{\beta\bar{i}}\delta_{j\bar{\alpha}}$, $F_{ij}=\sqrt{-1}(\delta_{\alpha\bar{i}}\delta_{j\beta}-\delta_{\beta\bar{i}}\delta_{j\bar{\alpha}})$. Then $N(E_{i,j})=\lambda_i\lambda_j E_{i,j}$, $N(F_{ij})=\lambda_i\lambda_j F_{ij}$. Now, by taking a suitable sum $N=N_1+\cdots+N_{n-1}$ of such operators associated to generic independent diagonal $A_1,\ldots,A_{n-1}$, we have an $N' \in \mathcal{N}_0$ such that $N': D_0=D\cap\mathcal{H}_0 \to D_0$ and has distinct eigenvalues there. Furthermore, $N': D_0^\perp \to D_0^\perp$, where it has two eigenvectors E_{ij}, F_{ij}, which span $D_0^\perp$. By what we already know for the case $n=2$, for fixed i,j, we may find $N_{ij} \in \mathcal{N}_0$ such that $N_{ij}(E_{ij})=\mu_{ij}E_{ij}$, $N_{ij}(F_{ij})=-\mu_{ij}F_{ij}$, and N_{ij} is 0 on the space orthogonal to E_{ij}, F_{ij}. Here μ_{ij} is arbitrary. By adding sums of such N_{ij}, we can split the multiple eigenvalues of N'. Hence, there is a proper real algebraic subvariety $\mathcal{S}_1 \subset \mathcal{N}_0$, such that for $N \in \mathcal{N}_0-\mathcal{S}_1$, $N: \mathcal{H}_0 \to \mathcal{H}_0$ has distinct eigenvalues.

For $N_0 \in \mathcal{N}_0-\mathcal{S}_1$, the highest unit eigenvector $\pm E_1$ has eigenvalues $\mu_1,\ldots,\mu_n$, and $\prod_{i \neq j}(\mu_i-\mu_j)^2$ is a real-analytic function of N_0. We show its zero variety is nowhere dense in $\mathcal{N}_0-\mathcal{S}_1$. Consider a linear perturbation $N(t)=N_0+tN_1$, $t \in \mathbb{R}$, small, and $N_1 \in \mathcal{N}_0$ to be chosen so that for $t \neq 0$, $\prod_{i \neq j}(\mu_i(t)-\mu_j(t))^2 \neq 0$. Let $E_1^{(1)}$ be any trace-zero hermitian endomorphism commuting with E_1, and with distinct eigenvalues $\mu_1^{(1)},\ldots,\mu_n^{(1)}$. Let $e_i,\ldots,e_n$ be a orthonormal basis of (mutual) eigenvectors for E_1 and $E_1^{(1)}$. There are analytic functions $\lambda_1(t)$, $E_1(t)$ satisfying

$$N(t)(E_1(t))=\lambda_1(t)E_1(t) \tag{5.1}$$

with $\lambda_1(0)=\lambda_1$, $E_1(0)=E_1$. Differentiating (5.1) at $t=0$ gives

$$N_1(E_1)+N_0(E_1')=\lambda_1 E_1'+\lambda_1' E_1. \tag{5.2}$$

If we chose $E_1'=E_1^{(1)}$, $\lambda_1'=0$, by the lemma, we may choose N_1 so that (5.2) is verified, i.e.,

$$N_1(E_1)=(-N_0\lambda_1)(E_1^{(1)}).$$

With this choice of N_1, we have

$$E_1(t)=E_1+tE_1^{(1)}+\text{h.o.t.}$$

with eigenvalues $\mu_i(t)=\mu_i+t\mu_i^{(1)}+\text{h.o.t.}$

Now let $\mathcal{S}_2=$ analytic subvariety of $\mathcal{N}_0-\mathcal{S}_1$ given by $\prod_{i \neq j}(\mu_i-\mu_j)^2=0$, and assume that $N_0 \in \mathcal{N}_0 - \mathcal{S}_1 - \mathcal{S}_2$. Let $e_1, \ldots, e_n$ be an eigen-basis for E_1, and consider $\prod_{i \neq j} (E_2 e_i, e_j)^2$, which is an analytic function on $\mathcal{N}_0 - \mathcal{S}_1 - \mathcal{S}_2$. With $N_1 \in \mathcal{N}_0$ again to be chosen, set $N(t) = N_0 + tN_1$, and consider showing

A) $N(t)(E_1) = \lambda E_1$,

B) $N(t)(E_2(t)) = \lambda_2(t) E_2(t)$.

A) holds if $N_1(E_1) = 0$. B) implies

B′) $N_0(E_2') + (N_1(E_2)) = \lambda_2 E_2' + \lambda_2' E_2$.

Similar to the preceding, take $\lambda_2' = 0$, and $E_2' = E_2^{(1)}$ with $E_2^{(1)} \perp E_1$, and $(E_2^{(1)} e_i, e_j) \neq 0$ for every $i \neq j$. We want to choose N_1 so that

$$N_1(E_2) = -(N_0 - \lambda_2)(E_2^{(1)}). \tag{5.3}$$

But $((N_0 - \lambda_2)(E_2^{(1)}), E_1) = (\lambda_1 - \lambda_2)(E_2^{(1)}, E_1) = 0$, so by the lemma, we may choose N_1 so that A) and (5.3) hold. Then $E_2(t) = E_2 + tE_2^{(1)} + \text{h.o.t.}$, and

$$(E_2(t) e_i, e_j) = (E_2 e_i, e_j) + t(E_2^{(1)} e_i, e_j) + \text{h.o.t.}$$

This completes the proof of the proposition.

§6. "Rigidified" Domains and an Example

In this § we give another, more or less *ad hoc*, method for constructing large families of distinct perturbations of a given domain. The idea is simply to take a bounded, strictly pseudocenvex domain $D \subset \mathbb{C}^{n+1}$, with real-analytic boundary M, and a chosen point p on M, and to perturb $M \subset \mathbb{C}^{n+1}$ only in a small neighborhood of p. We want to have p lie in each of the perturbed hypersurfaces, and the geometry is suitably "rigidified" so that any possible global equivalence between domains of this restricted type must preserve the point p. In this case, local invariants at p become global invariants for the domains. Though this could, in principle, be carried through for more general domains we will describe briefly here how it works for $D = \mathbb{B}^{n+1} \subset \mathbb{C}^{n+1}$, the unit ball.

First, realize $\mathbb{B}^{n+1}$ as $\mathcal{U}_{n+1} = \{(z, w) \in \mathbb{C}^{n+1} \mid \operatorname{Im} w > |z|\}$ with boundary $\partial \mathcal{U}_{n+1}$ defined by $F = v - |z|^2 = 0$. It is easy to see (Cayley transform back to $\mathbb{B}^{n+1}$) that perturbing $\mathcal{U}_{n+1}$ near $(0, 0)$ is equivalent to perturbing $\mathbb{B}^{n+1}$ near a fixed point in its boundary.

Now the automorphism group of $\mathcal{U}_{n+1}$ is the group of fractional linear transformations $SU(n+1, 1)/(\text{center})$. Consider the function $f(z, w) = u^4 + |z|^4$, and let $\mathcal{W}$ denote the region $\{f < 1\}$. We claim that the only automorphism of $\mathcal{U}_{n+1}$ whose boundary values preserve $\mathcal{W} \cap \partial \mathcal{U}_{n+1}$ are those of the form $T(z, w) = (U \cdot z, w)$, $U \in U(n)$. To see this note that such a T fixes the points $(0, \pm 1) \in \partial \mathcal{U}_{n+1}$, because these are the only two points on $\{f = 1\} \cap \partial \mathcal{U}_{n+1}$ where the tangent space is a complex subspace of $\mathbb{C}^{n+1}$ and they cannot be interchanged because of orientation considerations. Since T is a complex projective transformation, it preserves the complex line through $(0, 1)$ and $(0, -1)$. This shows with a little computation that $T(z, w) = \left(\frac{U \cdot z}{sw + c}, \frac{cw + s}{sw + c}\right)$, with $U \in U(n)$, $c, s \in \mathbb{R}$, $c^2 - s^2$

$=1$. But the condition that $T\colon \{f=1\}\cap\partial\mathcal{U}_{n+1}\to\{f=1\}\cap\partial\mathcal{U}_{n+1}$ shows (after another little computation) $c=1$, $s=0$.

Define the cut-off function

$$\varphi(z,w)=\begin{cases}\exp(-f/1-f), & \text{on } \mathcal{W},\\ 0, & \text{on } \mathbb{C}^{n+1}-\mathcal{W}.\end{cases}$$

It is C^∞, real analytic when $f\neq 1$. Note also that if $N=N(z,\bar{z},u)$ is a real-analytic function, convergent and sufficiently small on $\overline{\mathcal{W}}$, then the function $F=v-|z|^2-\varphi\cdot N$ is in Moser normal form at 0, and the hypersurface $F=0$ will be a strictly pseudoconvex perturbation of $\partial\mathcal{U}_{n+1}$. This hypersurface is real-analytic when $f\neq 1$.

Suppose, then, that N_1 and N_2 are two such functions with $N_1\not\equiv 0$, F_1, F_2 the corresponding functions as above, and $D_i=\{(z,w)\mid F_i(z,w)>0\}$. Suppose we have an equivalence $T\colon D_1\to D_2$. By Fefferman's theorem, it extends smoothly to the boundary. Then $T(\partial D_1\cap\mathcal{W})\cap\partial D_2\cap(\mathbb{C}^{n+1}-\overline{\mathcal{W}})$ must be empty, for otherwise $\partial D_i\cap\mathcal{W}$ would have zero pseudoconformal curvature on an open set, and thus everywhere, by analytic continuation, forcing $N_1\equiv 0$. Similar reasoning applied to $(\mathbb{C}^{n+1}-\overline{\mathcal{W}})\cap\partial D_1$ and $\partial D_2\cap\mathcal{W}$ shows that $T(\partial D_1\cap W)=\partial D_2\cap\mathcal{W}$, and $T(\partial D_1\cap(\mathbb{C}^{n+1}-\overline{\mathcal{W}}))\cap(\partial D_2\cap(\mathbb{C}^{n+1}-\overline{\mathcal{W}}))$ is non-empty. By a theorem of Alexander [1], T must be a fractional-linear transformation, and by the observations above, $T(z,w)=(U\cdot z,w)$. At the origin, we have $F_2\circ T=h\cdot F_1$, $h(0)\neq 0$. Comparison of terms containing v in Taylor expansions of both sides shows $h\equiv 1$. Hence, $F_2\circ T=F_1$, and then $N_2\circ T=N_1$. This forces $T=I_n$, if e.g., we take the weight 4 and 5 terms of N_1 strongly generic as in § 5. An even simpler family is given by the set of all N of the form

$$N(z,\bar{z},u)=|z|^8\,\mathcal{P}(z)\,Q(z,\bar{z},u)$$

where $\mathcal{P}(z,\bar{z})=1+$fixed odd polynomial of degree d, such that $T\in U(n)$, $\mathcal{P}(Tz,\overline{Tz})=\mathcal{P}(z,\bar{z})$ implies $T=I_n\left(\text{e.g., } \mathcal{P}(z,\bar{z})=1+\sum_{j=1}^{n}\operatorname{Re}(z_j)^{2j-1}\right)$, and Q is any even power series in $z,\bar{z},u$, convergent on $\overline{W}$, and sufficiently small there to guarantee $\{v-|z|^2-N=0\}$ is non-singular and strictly pseudoconvex. This is a linear space parametrizing perturbations of $\mathbb{B}^{n+1}$. For N_1 and N_2 from this family, $D_1\simeq D_2$ iff there is $T\in U(n)$ such that $N_2\circ T=N_1$. Comparing terms of even and odd degree shows $\mathcal{P}(Tz,\overline{Tz})=\mathcal{P}(z,\bar{z})$ and $Q_2(Tz,\overline{Tz},u)=Q_1(z,\bar{z},u)$. Hence, $T=I_n$, and $Q_1=Q_2$, unless $Q_1=Q_2\equiv 0$.

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