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Generalized Cohomological Index Theories for Lie Group Actions with an Application to Bifurcation Questions for Hamiltonian Systems

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1. Introduction

In another work [1] the authors employed a cohomological index (see also Yang [2, 3], Conner-Floyd [4] and Holm-Spanier [5]) in place of the usual notion of genus [6, 7] which is useful in symmetric situations with the group of symmetry being $\mathbb{Z}_2$, e.g., in the consideration of (odd) maps f such that $f(-x)=-f(x)$. Replacing genus by this cohomological index was dictated by the need of additional property – the piercing property (Proposition (3.9)). In this paper we extend this idea of cohomological index to the general situation where the symmetry group is an arbitrary compact Lie group G. It turns out that any cohomology class $\alpha \in H^*(B_G)$, where B_G is the universal classifying space for G, gives rise to an integer, $\text{index}_\alpha X$, where X is an arbitrary paracompact free G-space, and this index enjoys (§ 3), quite generally, the usual notions required of such a theory, including the piercing property. Section 4 is devoted to three important special cases, namely when α is specialized to the generator of the cohomology of $\mathbb{F}P^\infty$, infinite projective space, where $\mathbb{F}$ is either the reals $\mathbb{R}$, the complex numbers $\mathbb{C}$, or the quaternions $\mathbb{H}$, and the group G is the unit sphere in $\mathbb{F}$. We use the notation $\text{index}_\mathbb{R} X$, $\text{index}_\mathbb{C} X$, $\text{index}_\mathbb{H} X$, for these three cases. The first, $\text{index}_\mathbb{R} X$, is equivalent in a restricted category, to the cohomological index of Yang [2, 3]. This is the index employed in [1] and it is designated in Conner-Floyd [4] by $\text{co-index}_{\mathbb{Z}_2} X$. In Section 5 we reformulate the theory in the setting of a normed linear space $\mathscr{B}$ over $\mathbb{F}$ using the notion $\text{Index}_\mathbb{F} X = \text{index}_\mathbb{F} X + 1$.

In applications of interest, where the underlying group of symmetry is S^1, the resulting action may not be free due to the presence of isotropy subgroups of arbitrary order. Accordingly, in Section 6, we employ the index theory developed for *free* G-spaces, to define index theories in the general situation, namely the category of paracompact G-spaces without the assumption of a free action. The

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basic idea here is to use the equivariant cohomology $H_G^*(X)$ of Borel [8,9] rather than the cohomology $H^*(X/G)$ of the orbit space which is used in the free case. Section 7 is devoted to the special cases of non-free actions which arise in our applications.

In our earlier paper [1], the index theory given there (for a free $\mathbb{Z}_2$ action) was used to help obtain lower bounds for the number of zeroes an odd potential operator possesses near a bifurcation point as a function of an eigenvalue parameter. In § 8 (Theorem (8.4)) we shall show how the constructions of [1] in conjunction with the index theories developed here (for a non-free S^1 action) give similar lower bounds in problems involving the bifurcation of time periodic solutions from an equilibrium point for Hamiltonian systems of ordinary differential equations. Bifurcation questions for Hamiltonian systems have been studied recently by Weinstein [10] and Moser [11] from another point of view. While completing the final draft of this paper, we learned of the work of Chow and Mallet-Paret [12] who also observed that the methods of [1] can be applied as we do in § 8. In particular they obtain a special case of Theorem (8.4) corresponding to a free S^1 action.

2. Preliminaries

Let G denote a compact Lie Group and $\mathscr{F}$ the category of paracompact free G-spaces. More precisely, an object in $\mathscr{F}$ is a paracompact (Hausdorff) space X together with a continuous (left) action $\mu: G\times X\to X$ (where $\mu(g,x)$ is written gx) such that $gx=x$, $g\in G$, $x\in X$, implies $g=1$, the identity of G. The morphisms of $\mathscr{F}$ are equivariant maps $f: X\to Y$, i.e., $f(gx)=gf(x)$. Given an object $X\in\mathscr{F}$, set $\tilde{X}=X/G$, the corresponding orbit space with the identification topology and let $p: X\to\tilde{X}$ denote the associated identification map.

This category $\mathscr{F}$ may be identified with the category Prin G of locally trivial principal G-bundles with paracompact base by means of the functor

$$\mathscr{P}: X\to(X,p,\tilde{X},G).$$

To see this requires a few remarks. First, we recall the ingredients of a locally trivial principal G-bundle with paracompact base, i.e., an object in Prin G:

(2.1) *Definition.* A locally trivial principal G-bundle $\xi=(X,p,B,G)$ with paracompact base is:

(i) A triple (X,p,B) where $p: X\to B$ is a surjective map of topological spaces and B is paracompact.

(ii) A free right action $\psi: X\times G\to X$ (with $\psi(x,g)$ written xg and $xg=x$ only when $g=1$) such that:

(iii) Let $\Delta=\{(x_1,x_2)\in X\times X: p(x_1)=p(x_2)\}$. Then $(x_1,x_2)\in\Delta$ if, and only if, there is a (unique) $g=\alpha(x_1,x_2)$ in G such that $x_1g=x_2$ and the function $\alpha:\Delta\to G$ is continuous.

(iv) p admits local sections, i.e., there is an open cover $\{U_j\}$ of B and maps $\sigma_j: U_j\to X$ such that $p\sigma_j(b)=b$, $b\in U_j$.

Recall also that for $\xi=(X,p,B,G)\in \operatorname{Prin} G$, all the fibers $p^{-1}(x)$, $x\in X$, are homeomorphic to G and we have a local product structure

$$\begin{array}{ccc} p^{-1}(U_j) & \underset{\psi_j}{\overset{\varphi_j}{\rightleftarrows}} & U_j\times G \\ & \searrow \quad \swarrow & \\ & U_j & \end{array}$$

given by $\varphi_j(x)=(p(x), \alpha(\sigma_j(p(x)),x))$, $\psi_j(b,g)=\sigma_j(b)\,g$.

Returning to the functor $\mathscr{P}\colon \mathscr{F}\to \operatorname{Prin} G$ we transform the left G-space X to a right space in the conventional manner by setting $xg=g^{-1}x$. Thus, $\xi=(X,p,\tilde{X},G)$ satisfies (i) and (ii), leaving the paracompactness of $\tilde{X}$ aside for the moment. Since X is a fortiori completely regular, the now classical cross section theorem of Gleason [13] applies to give both (iii) and (iv). Now that ξ is locally trivial, it is a simple exercise to show that $\tilde{X}$ is paracompact using the paracompactness of X and the compactness of G. Thus, $\mathscr{P}$ provides a bijective correspondence on the category $\mathscr{F}$ as asserted. We might note that in both $\mathscr{F}$ and $\operatorname{Prin} G$ we are identifying equivalent objects where the morphisms in $\mathscr{F}$ are equivariant maps and those in $\operatorname{Prin} G$ are bundle maps.

In the presence of paracompactness of the base B, every $\xi=(X,p,B,G)\in \operatorname{Prin} G$ is a *numerable* principal G-bundle in the sense of Dold [14] and hence there is a universal numerable principal G-bundle $\eta=(E_G,q,B_G,G)$ giving rise to a natural equivalence

$$T\colon [B,B_G]\approx \operatorname{Prin}_G B$$

where $[B,B_G]$ is the set of homotopy classes of maps from B to B_G and $\operatorname{Prin}_G B$ is the set of (equivalence classes of) principal G-bundles with base B (see [14]). The transformation T assigns to $f\colon B\to B_G$ the induced bundle $f^*(\eta)$ over B. Thus, given a principal G-bundle $\xi=(E,p,B,G)$, there is a map $f\colon B\to B_G$, called the *classifying map* which induces ξ (up to equivalence) and f is unique up to homotopy.

In our case G is a compact Lie group and a universal G-bundle $\eta=(E_G,q,B_G,G)$ may be constructed as follows. First realize G as a subgroup of some orthogonal group $\mathscr{O}(k)$ for k sufficiently large. Let $V_{n,k}$ denote the space of orthonormal k-frames in $\mathbb{R}^{n+k}$ so that

$$V_{0,k}\subseteq V_{1,k}\subseteq\cdots\subseteq V_{n,k}\subseteq\cdots.$$

Then, $V_{\infty,k}=\bigcup_{n\geq 0} V_{n,k}$ is the total space of the universal $\mathscr{O}(k)$-bundle $\zeta=(V_{\infty,k},p_\infty,G_{\infty,k},\mathscr{O}(k))$, where $G_{\infty,k}$ is the union of the Grassmannians $G_{n,k}=V_{n,k}/\mathscr{O}(k)$. $V_{\infty,k}$ is paracompact and contractible and $G_{\infty,k}$ is a CW-complex [15, 16]. $G\subset\mathscr{O}(k)$ acts freely on the total space $V_{\infty,k}$ and hence if we set $E_G=V_{\infty,k}$ and $B_G=V_{\infty,k}/G$ with identification map $q\colon E_G\to B_G$, we have a principal G-bundle $\eta=(E_G,q,B_G,G)$ which is numerable because B_G is paracompact; and universal for arbitrary numerable principal G-bundles because E_G is contractible [14]. In particular, because $V_{\infty,k}$ and G are locally contractible so is B_G and hence the singular cohomology of B_G and the Čech-Alexander-Spanier cohomology of B_G are isomorphic [17].

The cohomology employed, unless otherwise stated, will be Čech-Alexander-Spanier [17].

3. The α-Index

Let G denote a compact Lie group and choose an element (characteristic class) α in the cohomology group $H^q(B_G, \Lambda)$, where $B_G = V_{\infty,k}/G$ is the base space of the universal bundle η described at the end of the previous section; and Λ is a (non-trivial) principal ideal domain serving as (simple) coefficients.

(3.1) *Definition.* Let $X \in \mathscr{F}$ denote a paracompact free G-space. Let $\mathscr{P}(X) = (X, p, \tilde{X}, G)$ denote the corresponding principal G-bundle and let $f: \tilde{X} \to B_G$ denote a classifying map for $\mathscr{P}(X)$. Set

$$\text{index}_\alpha X = \max(k : f^*(\alpha^k) \neq 0,\ k \geqq 0).$$

(3.2) *Remarks.* $H^*(\tilde{X}, \Lambda)$ and $H^*(B_G, \Lambda)$ are rings with the usual cup product structure [17] and f^* above is a ring homomorphism. We set $\alpha^0 = 1$, the unit element, so that when X is non-empty $f^*(1) = 1 \neq 0$ and hence $\text{index}_\alpha X \geqq 0$ for $X \neq \emptyset$. If $X = \emptyset$ we set $\text{index}_\alpha X = -1$. If $f^*(\alpha^k) \neq 0$ for infinitely many k, we set $\text{index}_\alpha X = \infty$. Notice, also that $\text{index}_\alpha X$ is independent of f since classifying maps for equivalent bundles are homotopic.

We now proceed to verify the basic properties of index_α.

(3.3) **Proposition** (Monotonicity). *Let $\varphi: X \to Y$ denote a morphism of $\mathscr{F}$,* i.e., *φ is an equivariant map of paracompact free G-spaces. Then,*

$$\text{index}_\alpha X \leqq \text{index}_\alpha Y.$$

Proof. Let f denote a classifying map for $\mathscr{P}(Y) = (Y, p_Y, \tilde{Y}, G)$. Then φ induces a bundle map and a diagram

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & Y & \longrightarrow & E_G \\ {\scriptstyle p_X}\downarrow & & {\scriptstyle p_Y}\downarrow & & \downarrow \\ \tilde{X} & \xrightarrow{\tilde{\varphi}} & \tilde{Y} & \xrightarrow{f} & B_G \end{array}$$

with $f\tilde{\varphi}$ serving as a classifying map for $\mathscr{P}X = (X, p_X, \tilde{X}, G)$. Thus, $\tilde{\varphi}^* f^*(\alpha^k) \neq 0$ implies $f^*(\alpha^k) \neq 0$ so that

$$\text{index}_\alpha X \leqq \text{index}_\alpha Y.$$

(3.4) **Corollary.** *Let $\varphi: X \to Y$ denote an equivalence in $\mathscr{F}$,* i.e. *an equivariant homeomorphism. Then,* $\text{index}_\alpha X = \text{index}_\alpha Y$.

(3.5) **Proposition** (Continuity). *Let X denote an object in $\mathscr{F}$ and A a closed invariant subset of X,* i.e., *$ga \in A$, $a \in A$, $g \in G$. Then, there exists a closed invariant neighborhood N of A such that* $\text{index}_\alpha N = \text{index}_\alpha A$.

Proof. Let $\mathcal{N}$ denote the family of invariant neighborhoods of A directed by inclusion and let $\mathcal{P}$ denote the subfamily of paracompact invariant neighborhoods of A. Given any $N \in \mathcal{N}$, let C denote a closed neighborhood of A such that $A \subset C \subset N$. Since G is compact GC is again a closed neighborhood of A and being closed GC is paracompact with $A \subset GC \subset N$. This shows that $\mathcal{P}$ is cofinal in $\mathcal{N}$, and hence [17, p. 316]

$$(1) \quad \varinjlim_{\mathcal{P}} H^q(\tilde{N}) \approx H^q(\tilde{A}).$$

Since for $N \in \mathcal{P}$, $A \subset N$, we have $\operatorname{index}_\alpha N \geqq \operatorname{index}_\alpha A$. If $\operatorname{index}_\alpha A = \infty$, then for every $N \in \mathcal{P}$, $\operatorname{index}_\alpha N = \infty$ so that we may assume $\operatorname{index}_\alpha A = k < \infty$. Let f denote a classifying map for $(N, p_N, \tilde{N}, G)$, $N \in \mathcal{P}$, and consider the maps

$$\tilde{A} \xrightarrow{\ i\ } \tilde{N} \xrightarrow{\ f_N\ } B_G$$

with $i^* f_N^*(\alpha^{k+1}) = 0$. Using the isomorphism (1) there must exist an $N_0 \in \mathcal{P}$ such that $f_{N_0}^*(\alpha^{k+1}) = 0$ so that $\operatorname{index}_\alpha N_0 \leqq k$. Thus, $\operatorname{index}_\alpha N_0 = k$ and the proposition follows.

(3.6) **Proposition** (Subadditivity). *Let X denote an object in $\mathcal{F}$ and A and B closed invariant subsets of X such that $X = A \cup B$. Then*

$$\operatorname{index}_\alpha (A \cup B) \leqq \operatorname{index}_\alpha A + \operatorname{index}_\alpha B + 1.$$

Proof. The proof will make use of the cup product in Alexander-Spanier cohomology ([17], p. 315)

$$H^r(X, A) \otimes H^s(X, B) \to H^{r+s}(X, A \cup B)$$

which requires that the interiors of A and B cover X. However, in view of Proposition (3.5) we may assume without loss of generality that this is the case and proceed. Observe also that we need only concern ourselves with the case when $\operatorname{index}_\alpha A = a$ and $\operatorname{index}_\alpha B = b$ are finite. Consider the diagram

$$\begin{array}{ccccc} \tilde{A} & & & & \\ & \searrow^{i_1} & & & \\ & & \tilde{X} & \xrightarrow{\ f\ } & B_G \\ & \nearrow_{i_2} & & & \\ \tilde{B} & & & & \end{array}$$

where i_1, i_2 are inclusions and f is a classifying map for $(X, p_X, \tilde{X}, G)$. Then, $i_1^* f^*(\alpha^{a+1}) = 0 = i_2^* f^*(\alpha^{b+1})$ so that if $j_1: \tilde{X} \to (\tilde{X}, \tilde{A})$, $j_2: \tilde{X} \to (\tilde{X}, \tilde{B})$ are also inclusions $f^*(\alpha^{a+1})$ and $f^*(\alpha^{b+1})$ pull back under j_1^* and j_2^*, respectively and then the diagram

$$\begin{array}{ccc} H^r(\tilde{X}, \tilde{A}) \otimes H^s(\tilde{X}, \tilde{B}) & \longrightarrow & H^{r+s}(\tilde{X}, \tilde{A} \cup \tilde{B}) \\ \downarrow & & \downarrow \\ H^r(\tilde{X}) \otimes H^s(\tilde{X}) & \longrightarrow & H^{r+s}(\tilde{X}) \end{array}$$

with $r=(a+1)\dim\alpha$, $s=(b+1)\dim\alpha$ shows that $f^*(\alpha^{a+b+2})=0$ so that $\text{index}_\alpha X \leqq a+b+1$

(3.7) **Proposition** (Normalization). $\text{index}_\alpha G=0$.

Proof. This is immediate because $\tilde{G}$ is a point.

(3.8) **Proposition** (Dimension). *If* $X\in\mathscr{F}$, $\tilde{X}=X/G$, *then*

$$(\text{index}_\alpha X)(\dim\alpha)\leqq\dim\tilde{X}.$$

Proof. This is immediate also because the cohomology of $\tilde{X}$ vanishes in dimensions bigger than $\dim X$ ([17], p. 359).

(3.9) **Proposition** (Piercing Property). *Let* $X\in\mathscr{F}$ *and suppose* $X=X_0\cup X_1$ *where* X_0 *and* X_1 *are closed invariant subsets. Suppose further that* $A\in\mathscr{F}$ *and* $\varphi: A\times I\to X$ *is equivariant imbedding, i.e.,* $\varphi(ga,t)=g\varphi(a,t)$, $g\in G$, $a\in A$, $t\in I=[0,1]$. *We assume also that* $\varphi(A\times I)$ *is closed in* X. *If* $A_0=\varphi(A\times\{0\})\subset X_0$ *and* $A_1=\varphi(A\times\{1\})\subset X_1$, *then*

$$\text{index}_\alpha\,\varphi(A\times I)\cap X_0\cap X_1=\text{index}_\alpha A.$$

Proof. First, there is no loss of generality in assuming that $A_0=A$ and $\varphi(a,0)=a$, $a\in A$. Let $\gamma=\text{proj}_1\circ\varphi^{-1}:\varphi(A\times I)\to A\times I\to A$, $C=\varphi(A\times I)\cap X_0\cap X_1$ and $\gamma_C=\gamma\,|\,C: C\to A$ and observe that the maps γ and γ_C are equivariant. Thus, $\text{index}_\alpha C\leqq\text{index}_\alpha A$. Now, to prove equality it suffices to show that γ_C induces injections $\tilde{\gamma}_C^*: H^q(\tilde{A})\to H^q(\tilde{C})$ for all q, where $\tilde{A}=A/G$ and $\tilde{C}=C/G$. This is done as follows. Introduce the notation $B_0=X_0\cap\varphi(A\times I)$, $B_1=X_1\cap\varphi(A\times I)$ and the inclusion maps

$$\begin{array}{ccc} A_0 & \xrightarrow{k_0} & B_0 \\ & {\scriptstyle j_0}\searrow \quad \swarrow{\scriptstyle i_0} & \\ & B_0\cup B_1 & \\ & l_0: C\subset B_0 & \end{array} \qquad \begin{array}{ccc} A_1 & \xrightarrow{k_1} & B_1 \\ & {\scriptstyle j_1}\searrow \quad \swarrow{\scriptstyle i_1} & \\ & B_0\cup B_1 & \\ & l_1: C\subset B_1 & \end{array}$$

where $B_0\cap B_1=C$. All these sets and maps are equivariant and working in the corresponding orbit spaces $\tilde{A}_0=A_0/G$, etc. we have a Mayer-Vietoris Sequence

$$\cdots\to H^q(\tilde{B}_0\cup\tilde{B}_1)\xrightarrow{\zeta}H^q(\tilde{B}_0)\oplus H^q(\tilde{B}_1)\xrightarrow{\eta}H^q(\tilde{B}_0\cap\tilde{B}_1)\to\cdots$$

where $\zeta=(\tilde{i}_0^*,-\tilde{i}_1^*)$, $\eta=\tilde{l}_0^*+\tilde{l}_1^*$. We assert first that $\tilde{l}_0^*$ is an injection for suppose $\tilde{l}_0^*(x)=0$. This implies that $\eta(x,0)=0$ and hence $\zeta(y)=(x,0)$ for some $y\in H^q(\tilde{B}_0\cup\tilde{B}_1)$. This forces $\tilde{i}_1^*(y)=0$ and hence $\tilde{j}_1^*(y)=0$. But j_1 and hence $\tilde{j}_1: \tilde{A}_1\to\tilde{B}_0\cup\tilde{B}_1$ are homotopy equivalences which forces $y=0$ and hence $i_0^*(y)=x=0$. Thus $\tilde{l}_0^*$ is an injection. Finally, let $\gamma_0=\gamma\,|\,B_0: B_0\to A$ which is a retraction of B_0 to A. The diagram

$$\begin{array}{ccc} H^q(\tilde{B}_0) & \xrightarrow{\tilde{l}_0^*} & H^q(\tilde{C}) \\ & {\scriptstyle \tilde{\gamma}_0^*}\nwarrow \quad \nearrow{\scriptstyle \tilde{\gamma}_C^*} & \\ & H^q(\tilde{A}) & \end{array}$$

then exhibits $\tilde{\gamma}_C^*$ as the composition of injections and the proof is complete.

(3.10) **Corollary.** *If in Proposition* (3.9), *we assume only that* φ *is an equivariant map* (*not necessarily an imbedding*), *then*

$$\mathrm{index}_\alpha\, \varphi(A \times I) \cap X_0 \cap X_1 \geqq \mathrm{index}_\alpha A.$$

Proof. Let $Y = A \times I$, $Y_0 = \varphi^{-1}(X_0)$, $Y_1 = \varphi^{-1}(X_1)$. Then,

$$\varphi\colon Y_0 \cap Y_1 \to \varphi(A \times I) \cap X_0 \cap X_1$$

is equivariant and

$$\mathrm{index}_\alpha\, Y_0 \cap Y_1 \leqq \mathrm{index}_\alpha\, \varphi(A \times I) \cap X_0 \cap X_1.$$

By applying Proposition (3.9) to (Y, Y_0, Y_1) we obtain

$$\mathrm{index}_\alpha\, Y_0 \cap Y_1 = \mathrm{index}_\alpha A$$

and the required inequality.

(3.11) *Remark.* As pointed to us by L. Sonneborn, the result in Corollary (3.10) for the special case of a free $\mathbb{Z}_2$-action, is proved and employed by Yang in the proof of his Generalized Kakutani-Yamabe-Yujobô Theorem [18].

4. Specializations

We now consider three special cases of the α-index where the compact Lie Group G is S^0, S^1 or S^3, i.e. G is the unit sphere in $\mathbb{F}$ where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, i.e. $\mathbb{F}$ is the reals, complex numbers or quaternions.

(4.1) *Definition.* Define $\mathrm{index}_{\mathbb{F}}$ in the three cases $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ as follows:

(a) $\mathbb{F} = \mathbb{R}$. Here $G = S^0 = \mathbb{Z}_2$ and $\mathscr{F}$ is the category of paracompact spaces on which $\mathbb{Z}_2$ acts freely. The coefficient ring Λ is $\mathbb{Z}_2$ and the universal classifying space $B_{\mathbb{Z}_2}$ is $\mathbb{R}P^\infty$ with $H^*(\mathbb{R}P^\infty; \mathbb{Z}_2)$ the polynomial ring on a single generator $w \in H^1(\mathbb{R}P^\infty, \mathbb{Z}_2)$. We set

$$\mathrm{index}_{\mathbb{R}} X = \mathrm{index}_w X, \qquad X \in \mathscr{F}.$$

(b) $\mathbb{F} = \mathbb{C}$. Here $G = S^1$, the complex numbers of norm 1, and $\mathscr{F}$ is the category of paracompact spaces on which S^1 acts freely. The coefficient ring Λ is $\mathbb{Z}$ and the universal classifying space B_{S^1} is $\mathbb{C}P^\infty$ with $H^*(\mathbb{C}P^\infty; \mathbb{Z})$ the polynomial ring on a single generator $c \in H^2(\mathbb{C}P^\infty, \mathbb{Z})$. We set

$$\mathrm{index}_{\mathbb{C}} X = \mathrm{index}_c X.$$

(c) $\mathbb{F}=\mathbb{H}$. Here $G=S^3$ the group of quaternions of norm 1, and $\mathscr{F}$ is the category of free paracompact S^3-spaces. The coefficient ring Λ is $\mathbb{Z}$ and the universal classifying space B_{S^3} is $\mathbb{H}P^\infty$ with $H^*(\mathbb{H}P^\infty, \mathbb{Z})$ the polynomial ring on a single generator $\sigma \in H^4(\mathbb{H}P^\infty, \mathbb{Z})$. We set

$$\operatorname{index}_{\mathbb{H}} X = \operatorname{index}_\sigma X, \qquad X \in \mathscr{F}.$$

(4.2) *Remarks.* The first case (a) is equivalent, in a restricted category to the index of Yang [2, 3]. It appears also in Conner-Floyd [4] where it is denoted by co-index$_{\mathbb{Z}_2}$ X and also in Holm-Spanier [5]. An alternative development which includes a variant form of the "piercing property" (Proposition (3.9)) along with an application to a bifurcation theorem is contained in [1].

Furthermore, the class w in (a) is the first universal Stiefel-Whitney class, while c in (b) is the first universal Chern class [19].

We now proceed to prove some special properties of index$_{\mathbb{F}}$ where $\mathbb{F}=\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$.

(4.3) **Proposition** (Stability). *Let G denote the unit sphere in $\mathbb{F}$. Then, if $X \in \mathscr{F}$, let $X \circ G$ denote the join of X and G with G acting by $g(x, t, y) = (gx, t, gy)$, $x \in X$, $y \in G$, $t \in I$. Then, if X is locally contractible*

$$\operatorname{index}_{\mathbb{F}} X \circ G = \operatorname{index}_{\mathbb{F}} X + 1.$$

Proof. First we remark that $X \circ G$ is paracompact, and following a suggestion of K. Kunen, a proof of this result may be effected using a result of Michael [20]. Furthermore, $X \circ G$ is easily seen to be locally contractible. This forces the orbit spaces $\tilde{X} = X/G$ and $\tilde{B} = X \circ G/G$ to be paracompact and locally contractible. Then index$_{\mathbb{F}}$ $X \circ G$ is defined and we may equivalently employ singular cohomology ([17]) in dealing with the notion of index$_{\mathbb{F}}$. Now, we have the following inequalities

$$\operatorname{index}_{\mathbb{F}} X \leqq \operatorname{index}_{\mathbb{F}} X \circ G \leqq \operatorname{index}_{\mathbb{F}} X + 1.$$

The first holds because X equivariantly imbeds in $X \circ G$ and the second because $X \circ G$ can be written as the union of two closed invariant subsets A_0 and A_1 with $\operatorname{index}_{\mathbb{F}} A_0 = \operatorname{index}_{\mathbb{F}} X$ and $\operatorname{index}_{\mathbb{F}} A_1 = 0$ and Proposition (3.6) applies. To complete the proof we make use of a standard argument using the Gysin sequence ([17], p. 260). Recall that our G-bundles are now orientable sphere bundles since G is 0-connected when $\mathbb{F}=\mathbb{C}$ or $\mathbb{H}$ and we are using $\mathbb{Z}_2$-coefficients when $\mathbb{F}=\mathbb{R}$. Let $B = X \circ G$ and consider the following diagram of Gysin sequences for the bundles $(X, p_X, \tilde{X}, G)$, $(B, p_B, \tilde{B}, G)$, where $i: X \to B$ is the inclusion map $i(x) = [x, 0, G]$, and $d = \dim G$.

$$\begin{array}{ccccccccc}
\longrightarrow & H^{k+d}(\tilde{X}) & \xrightarrow{p_X^*} & H^{k+d}(X) & \longrightarrow & H^k(\tilde{X}) & \xrightarrow{\psi_X^*} & H^{k+d+1}(\tilde{X}) & \longrightarrow \cdots \\
 & \uparrow{\scriptstyle \tilde{\imath}^*} & & \uparrow{\scriptstyle i^*} & & \uparrow{\scriptstyle \tilde{\imath}^*} & & \uparrow{\scriptstyle i^*} & \\
\longrightarrow & H^{k+d}(\tilde{B}) & \xrightarrow{p_B^*} & H^{k+d}(B) & \longrightarrow & H^k(\tilde{B}) & \xrightarrow{\psi_B^*} & H^{k+d+1}(\tilde{B}) & \longrightarrow \cdots
\end{array}$$

Suppose, as we may, that $\operatorname{index}_{\mathbb{F}} X = \operatorname{index}_{\mathbb{F}} B = k < \infty$. Then, we would have a non-zero element $u \in H^k(\tilde{B})$ such that $\psi_B^*(u) = 0$ and hence which pulls back to $H^{k+d}(B)$ and which also has the property that $\tilde{i}^*(u) \neq 0$. This forces i^* to be non-trivial. But when $k+d>0$, i^* is trivial which is a contradiction unless $k=0$, $d=0$. This special case is disposed by observing that B contains $S^0 \circ S^0 = S^1$ as an invariant subset, and $\operatorname{index}_{\mathbb{R}} S^1 = 1$.

(4.4) *Remark.* Proposition (4.3) in the case $\mathbb{F} = \mathbb{R}$ and X is compact is due to Conner-Floyd [4]. The proof is a simple adaptation of theirs.

(4.5) **Corollary.** $\operatorname{index}_{\mathbb{F}} S^{(d+1)n+d} = n$, *where* $S^{(d+1)n+d}$ *is the unit sphere in* $\mathbb{F}^{n+1}$ *and* $d = \dim G$.

Corollary (4.5) has the following extension. The special case $\mathbb{F} = \mathbb{R}$ is similar to a result of Holm and Spanier [5]. Our proof is different making use of the transfer map [21]. The action on $\mathbb{F}^{n+1}$ is scalar multiplication.

(4.6) **Proposition** (Boundary of Invariant Neighborhoods). *Let M denote a topological G-manifold of dimension $(d+1)(n+1) = \dim \mathbb{F}^{n+1}$, and U an open invariant set in M with compact closure. Let $\mathbb{F}^{n+1}$ denote Euclidean $(n+1)$-space over $\mathbb{F}$ and φ: $(\bar{U}, \partial U) \to (\mathbb{F}^{n+1}, \mathbb{F}^{n+1} - 0)$ an equivariant map, where ∂U represents the boundary of U and G acts freely on ∂U. Then, if the degree of φ is $\neq 0$ (using $\mathbb{Z}_2$ in case $\mathbb{F} = \mathbb{R}$),*

$$\operatorname{index}_{\mathbb{F}} \partial U = n.$$

Proof. We may assume without loss of generality that φ is defined on an equivariant neighborhood V of $\bar{U}$ and $\varphi^{-1}(S) = \partial U$ where S is the unit sphere in $\mathbb{F}^{n+1}$. Thus, we have $\varphi: (V, V - \partial U) \to (\mathbb{C}^{n+1}, \mathbb{C}^{n+1} - S)$ and φ has non-zero degree δ by assumption, i.e. if $B = \partial U$ and o_B and o_S are respectively fundamental classes, $\varphi_*(o_B) = \delta o_S$. Thus, we are in a position to apply the transfer map $t: H^*(B) \to H^*(S)$ with $t\varphi^* = \delta(\mathrm{id})$. Thus $\varphi^*: H^*(S) \to H^*(B)$ injects. Now, look at the bundle map

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & S \\ \downarrow & & \downarrow \\ \tilde{B} & \xrightarrow{\tilde{\varphi}} & \mathbb{F}P^n \end{array}$$

where $\tilde{B} = B/G$ and, of course, $\mathbb{F}P^n = S/G$. Now a simple Gysin sequence argument (over $\mathbb{Z}_2$ in case $\mathbb{F} = \mathbb{R}$) tells us that in the top dimension $r = (d+1)n$, $\tilde{\varphi}^*: H^r(\mathbb{F}P^n) \to H^r(\tilde{B})$ injects which forces $\operatorname{index}_{\mathbb{F}} \partial U = n$.

(4.7) *Remark.* C. Conley pointed out to us that the special case of Proposition (4.6) for $M = \mathbb{F}^{n+1}$ and U a bounded open set containing the origin follows immediately from the Piercing Property (Proposition (3.9)).

5. A Reformulation

Let $\mathscr{B}$ denote a normal linear space over $\mathbb{F}=\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Furthermore, if $\mathscr{B}_* = \mathscr{B}-\{0\}$ and G is the unit sphere in $\mathbb{F}$, G acts freely on $\mathscr{B}_*$. Let $\mathscr{E}$ denote the family of invariant subsets of $\mathscr{B}_*$. Then, each $X \in \mathscr{E}$ is a paracompact free G-space and we define

$$\operatorname{Index}_{\mathbb{F}} X = \operatorname{index}_{\mathbb{F}} X + 1.$$

Letting $\mathbb{N}$ denote the non-negative integers we may summarize the contents of the previous sections in this setting as follows:

(5.1) **Theorem.** *The function* $\operatorname{Index}_{\mathbb{F}}: \mathscr{E} \to \mathbb{N}$ *possesses the following properties; where* $X, Y, \dots \in \mathscr{E}$*:*

1° If $X=\emptyset$, $\operatorname{Index}_{\mathbb{F}} X=0$; if $X \neq \emptyset$, $\operatorname{Index}_{\mathbb{F}} X \geqq 1$

2° (Normalization) $\operatorname{Index}_{\mathbb{F}} G=1$

3° (Dimension) $\operatorname{Index}_{\mathbb{F}} X \cdot \dim \mathbb{F} \leqq \dim X$

4° (Monotonicity) $\psi: X \to Y$ equivariant implies that $\operatorname{Index}_{\mathbb{F}} X \leqq \operatorname{Index}_{\mathbb{F}} Y$. In particular, equality holds if ψ is also a homeomorphism.

5° (Continuity.) If X is closed, there exists a closed invariant neighborhood N of X such that

$$\operatorname{Index}_{\mathbb{F}} X = \operatorname{Index}_{\mathbb{F}} Y$$

for any invariant set Y, $X \subset Y \subset N$. If X is compact N may be chosen as a uniform neighborhood

$$N_\delta(X)=\{b \in \mathscr{B}: \|b-X\| \leqq \delta\}.$$

6° (Subadditivity.) $\operatorname{Index}_{\mathbb{F}}(X \cup Y) \leqq \operatorname{Index}_{\mathbb{F}} X + \operatorname{Index}_{\mathbb{F}} Y$

7° (Neighborhood of Zero.) If $\mathscr{B}=\mathbb{F}^{n+1}$ and U is a bounded open invariant neighborhood of 0, then

$$\operatorname{Index}_{\mathbb{F}} \partial U = n+1.$$

8° (Stability.) If X is closed, and $X \circ G$ is the join of X with G, realized in $\mathscr{B} \oplus \mathbb{F}$, then

$$\operatorname{Index}_{\mathbb{F}} X \circ G = \operatorname{Index}_{\mathbb{F}} X + 1.$$

9° (Piercing Property.) Let X_0, X_1, A, denote closed subsets in $\mathscr{E}$ and φ: $A \times I \to X_0 \cup X_1$ an equivariant imbedding, i.e. $\varphi(ga,t)=g\varphi(a,t)$. Suppose further that $\varphi(A \times I)$ is a closed subset and $\varphi(A \times \{0\}) \subset X_0$ and $\varphi(A \times \{1\}) \subset X_1$. Then

$$\operatorname{Index}_{\mathbb{F}} \varphi(A \times I) \cap X_0 \cap X_1 = \operatorname{Index}_{\mathbb{F}} A.$$

10° (Infinity.) If $\mathscr{B}$ has infinite dimension and S is the unit sphere in $\mathscr{B}$; then $\operatorname{Index}_{\mathbb{F}} S = \infty$.

(5.2) We need to make a remark about 8° since we did not assume that X was locally contractible. This is because 5° allows us to replace X by a locally convex neighborhood. Also 10° follows because $\mathrm{Index}_{\mathbb{F}} S$ is certainly defined and S contains invariant spheres of arbitrarily high (finite) dimensions.

6. α-Index for Non-Free Actions

In this section we develop the general theory for actions which are not necessarily free. Our compact Lie Group will be fixed throughout this section and the notation for our principal G-bundle $\eta=(E_G, q, B_G, G)$ (Section 2) will be shortened to $\eta=(E, q, B)$. We note the important fact that our universal total space E is $V_{\infty,k}$ which is the union of countably many compact sets (σ-compact). Hence, ([20]) $E\times X$ is paracompact, whenever X is paracompact. Accordingly, we let $\mathscr{F}_*$ denote the category of all paracompact (Hausdorff) G-spaces X, making no asumptions that the action be free or even non-trivial. We also fix once and for all an element $\alpha\in H^q(B, \Lambda)$, where Λ is a (simple) coefficient ring.

(6.1) *Remark.* If one wanted to extend these ideas to include more general topological groups G, the G-space X would have to be restricted to have the property that if E_G is paracompact, then $E_G\times X$ is also paracompact. This is the case, e.g., when X is locally compact or σ-compact (see Dugundji [22] and Michael [20, 23]).

Now, take a G-space $X\in\mathscr{F}_*$. Then, G acts *freely* on $E\times G$ by the usual action

$$g(e,x)=(eg^{-1}, gx), \qquad g\in G, \qquad e\in E, \qquad x\in X.$$

The resulting orbit space $(E\times X)/G$ which is usually designated by $E\times_G X$ is the total space of the associated bundle $(E\times_G X, p_X, B)$, where p_X is induced by $q\,j_1$ where $j_1: E\times X\to E$ is projection on the first factor. Notice then that $E\times X$, with this *free* G-action, belongs to our category $\mathscr{F}$ of Section 3 and we may introduce the following definition.

(6.2) *Definition.* For $X\in\mathscr{F}_*$, set

$$\mathrm{index}^*_\alpha X=\mathrm{index}_\alpha E\times X$$

where $\mathrm{index}_\alpha E\times X$ is as defined in §3. Alternatively, consider the diagram

$$\begin{array}{ccc} E\times X & \xrightarrow{j_1} & E \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q} \\ E\times_G X & \xrightarrow{p_X} & B \end{array}$$

where j_1 is projection on the first factor and set

$$\mathrm{index}^*_\alpha X=\max\{k: p_X^*(\alpha^k)\neq 0,\ k\geqq 0\}.$$

Before we investigate the properties of $\mathrm{index}^*_\alpha: \mathscr{F}_*\to\mathbb{Z}$ we first check consistency.

(6.3) **Lemma.** *If* $X \in \mathscr{F}$

$$\operatorname{index}_\alpha X = \operatorname{index}^*_\alpha X.$$

Proof. Consider the diagram

$$\begin{array}{ccc} X & \xleftarrow{j_2} & E \times X \\ \downarrow & & \downarrow \\ X/G & \xleftarrow{\tilde{j}_2} & E \times_G X \end{array}$$

where j_2 is (equivariant) projection on X. Then, one shows easily that $\tilde{j}_2$ is a locally trivial map with contractible fiber E. This forces $\tilde{j}_2$ to be a homotopy equivalence [14] and hence $\operatorname{index}_\alpha X = \operatorname{index}_\alpha E \times_G X = \operatorname{index}^*_\alpha X$.

We now proceed to verify the properties of this index on $\mathscr{F}_*$.

(6.4) **Proposition** (Monotonicity). *Let* $\varphi: X \to Y$ *denote a morphism of* $\mathscr{F}_*$, *i.e.,* φ *is an equivariant map of paracompact G-spaces. Then,*

$$\operatorname{index}^*_\alpha X \leqq \operatorname{index}^*_\alpha Y.$$

Proof. Immediate, since $1 \times \varphi: E \times X \to E \times Y$ is equivariant.

Before we establish the Continuity Theorem in this setting we recall [24, 25] that our universal space $E = V_{\infty, k}$ has the property that E is the ascending union of compact manifolds

$$E^1 \subset E^2 \subset \cdots \subset E^m \subset E^{m+1} \subset \cdots$$

with the following properties, where $X \in \mathscr{F}_*$,

a) The homotopy groups $\pi_i(E^m) = 0$, $i < m$.

b) The inclusion map $E^m \subset E^{m+1}$ induces isomorphisms (any coefficients)

$$H^q(E^{m+1} \times_G X) \to H^q(E^m \times_G X), \qquad q < m.$$

c) Since $E \times_G X$ is paracompact

$$\varprojlim H^q(E^m \times_G X) \approx H^q(E \times_G X)$$

so that the inclusion map induces

d) $H^q(E^m \times_G X) \approx H^q(E \times_G X)$, $q < m$.

(6.5) **Proposition** (Continuity). *Let* X *denote an object in* $\mathscr{F}_*$ *and* A *a closed invariant subset of* X, *i.e.,* $ga \in A$ *when* $a \in A$ *and* $g \in G$. *Then, there exists a closed invariant neighborhood* N *on* A *such that* $\operatorname{index}^*_\alpha N = \operatorname{index}^*_\alpha A$.

Proof. The proposition is obvious for $\operatorname{index}^*_\alpha A = \infty$, so we may assume that $\operatorname{index}^*_\alpha A < \infty$.

First choose a closed invariant neighborhood V of A in X. Then $E \times A$ is a closed invariant subset of $E \times V \in \mathscr{F}$. By the Continuity Theorem for free actions

(Proposition (3.5)), there is a closed invariant neighborhood W of $E\times A$ in $E\times V$ such that

$$\mathrm{index}_\alpha E\times A=\mathrm{index}_\alpha W.$$

In particular, if $\alpha\in H^d(B, A)$ and $\mathrm{index}^*_\alpha A=k$, for a classifying map

$$f\colon W/G\to B$$

we have $f^*(\alpha^{k+1})=0$. Choose $m>d(k+1)$. Using the fact that E^m is compact, we can find a closed invariant neighborhood N of A in X $(N\subset V)$ such that $E^m\times N\subset W$. Now, using the diagram

$$\begin{array}{ccccccc} E\times A & \longrightarrow & W & \longrightarrow & E\times V & \longrightarrow & E\\ & & \uparrow & & \uparrow & & \\ & & E^m\times N & \longrightarrow & E\times N & & \end{array}$$

and the fact that $H^q(E^m\times_G N)\approx H^q(E\times_G N)$ for $q<m$ we see that the classifying map

$$f\colon E\times_G N\to B$$

has the property that $f^*(\alpha^{k+1})=0$ and hence

$$\mathrm{index}^*_\alpha N=\mathrm{index}_\alpha E\times N=\mathrm{index}_\alpha E\times A=\mathrm{index}^*_\alpha A.$$

(6.6) **Proposition** (Subadditivity). *Let X denote an object in $\mathscr{F}_*$ and A and B closed invariant subsets of X such that $X=A\cup B$. Then,*

$$\mathrm{index}^*_\alpha(A\cup B)\leqq\mathrm{index}^*_\alpha A+\mathrm{index}^*_\alpha B+1.$$

Proof.

$$\begin{aligned}\mathrm{index}^*_\alpha(A\cup B)&=\mathrm{index}_\alpha E\times(A\cup B)=\mathrm{index}_\alpha(E\times A)\cup(E\times B)\\ &\leqq\mathrm{index}_\alpha E\times A+\mathrm{index}_\alpha E\times B+1\\ &\leqq\mathrm{index}^*_\alpha A+\mathrm{index}^*_\alpha B+1.\end{aligned}$$

(6.7) **Proposition** (Normalization). $\mathrm{index}^*_\alpha G=0$.

Proof. By Lemma (6.3), $\mathrm{index}^*_\alpha G=\mathrm{index}_\alpha G=0$, using Proposition (3.7).

(6.8) *Remark.* The fact that X/G is finite dimensional will not guarantee that $\mathrm{index}^*_\alpha X$ is finite. For this reason we don't have a direct analogue of the Dimension Property (3.8). We will explore this question further, however, at the end of this section.

(6.9) **Proposition** (Piercing Property). *Let $X\in\mathscr{F}_*$ and suppose $X=X_0\cup X_1$, where X_0 and X_1 are closed invariant subsets. Suppose further that $A\in\mathscr{F}_*$ and $\varphi\colon A\times I\to X$ is an equivariant imbedding, i.e., $\varphi(ga, t)=g\varphi(a, t)$, $g\in G$, $a\in A$, $t\in I$ $=[0,1]$. We assume also that $\varphi(A\times I)$ is closed in X. If $A_0=\varphi(A\times\{0\})\subset X_0$ and*

$A_1 = \varphi(A \times \{1\}) \subset X_1$, *then*

$$\text{index}_\alpha^* \, \varphi(A \times I) \cap X_0 \cap X_1 = \text{index}_\alpha^* \, A.$$

Proof. If $\varphi: A \times I \to X$ is an equivariant imbedding, $1 \times \varphi: E \times A \times I \to E \times X$ is also and we apply the Piercing Property Proposition (3.9) to this situation to obtain

$$\text{index}_\alpha E \times [\varphi(A \times I) \cap X_0 \cap X_1] = \text{index}_\alpha (E \times A)$$

which gives the desired result.

(6.10) **Corollary.** *If in Proposition* (6.9), *we assume only that* φ *is an equivariant map (not necessarily an imbedding) then*

$$\text{index}_\alpha^* \, \varphi(A \times I) \cap X_0 \cap X_1 \geqq \text{index}_\alpha^* \, A.$$

Contrary to the free situation, where $\text{index}_\alpha X$ is finite when the dimension of X is finite, $\text{index}_\alpha^* X$ may be infinite even when X is compact and finite dimensional. In fact, consider the case where the Lie group $G = S^1$, the circle group, and we take as coefficients $\Lambda = \mathbb{Q}$, the additive group of rationals. Furthermore, let $\alpha \in H^2(\mathbb{C}P^\infty; \mathbb{Q})$, denote a generator. Suppose $X \in \mathscr{F}_*$ has a non-empty fixed point set $F \subseteq X$, i.e. $x \in F$, if and only if, $gx = x$ for every $g \in S^1$. Then, on one hand

$$\text{index}_\alpha^* F \leqq \text{index}_\alpha^* X$$

and furthermore $E \times_G F = B \times F$, where $E = S^\infty$, $B = \mathbb{C}P^\infty$, and the diagram

$$\begin{array}{ccc} E \times F & \longrightarrow & E \\ \downarrow & & \downarrow \\ B \times F & \xrightarrow{p} & B \end{array}$$

where p = projection, tells us that $p^*: H^*(B, \mathbb{Q}) \to H^*(B \times F, \mathbb{Q})$ is an injection so that $p^*(\alpha^k) \neq 0$ for all $k \geqq 1$, forcing

$$\text{index}_\alpha^* F = \infty = \text{index}_\alpha^* X.$$

Thus, $\text{index}_\alpha^* X$ may not prove useful in the presence of fixed points belonging to X. However, $\text{index}_\alpha X$ is finite quite often, in particular when the isotropy groups are finite. Recall that for $x \in X$, the isotropy group G_x is defined by

$$G_x = \{g \in G : gx = x\}.$$

Thus, $G_x = G$ implies that $x \in F$, the fixed point set of the action.

(6.11) **Lemma.** *Suppose* $X \in \mathscr{F}_*$ *and all the isotropy groups* G_x, $x \in X$, *are finite. Then, the map* $\tilde{j}: E \times_G X \to X/G$, *induced by projection* $j: E \times X \to X$, *induces isomorphisms*

$$\tilde{j}^*: H^q(X/G, \mathbb{Q}) \to H^q(E \times_G X, \mathbb{Q})$$

in all dimensions q, over the field of rationals $\mathbb{Q}$.

Proof. The proof we give is standard and is included for the reader's convenience. We again make use of the filtration

$$E^1 \subset E^2 \subset \cdots \subset E^m \subset E^{m+1} \subset \cdots$$

of our universal total space E as in the proof of Proposition (6.5). We consider the diagram, for each m,

$$\begin{array}{ccc} E^m \times X & \xrightarrow{j^m} & X \\ \downarrow & & \downarrow \\ E^m \times_G X & \xrightarrow{\tilde{j}^m} & X/G \end{array}$$

Note that $\tilde{j}^m$ (induced by the projection j^m) is a closed map because E^m is compact and furthermore the preimage sets (fibers) of $\tilde{j}^m$ are all of the form E^m/G_x, where G_x is a finite isotropy group. Applying the Vietoris-Begle mapping theorem [17] and noting that $H^q(E^m/G_x, \mathbb{Q}) = 0$ for $q < m$, we have isomorphisms induced by $\tilde{j}^m$

$$H^q(X/G, \mathbb{Q}) \to H^q(E^m \times_G X, \mathbb{Q}), \qquad q < m.$$

Then, $\tilde{j}^*$ is just the composition of this isomorphism and the isomorphism $H^q(E^m \times_G X, \mathbb{Q}) \approx H^q(E \times_G X, \mathbb{Q})$, $q < m$.

We are now in a position to state the analogue of the "dimension property", Proposition (3.8).

(6.12) **Proposition** (Dimension). *Suppose* $X \in \mathscr{F}_*$ *and all the isotropy groups* G_x *are finite. Let* $\dim X/G$ *denote the covering dimension of the orbit space* X/G. *Then, over the rational field* $\mathbb{Q}$,

$$(\text{index}_\alpha^* X)(\dim \alpha) \leqq \dim X/G.$$

Proof. We may assume $\dim X/G < \infty$. Then, by the above lemma, $H^q(E \times_G X, \mathbb{Q}) = 0$ for $q > \dim X/G$. Thus, if $\alpha \in H^d(B, \mathbb{Q})$, and $f \colon E \times_G X \to B$ is a classifying map, we have $f^*(\alpha^k) = 0$ for $kd > \dim X/G$. Thus,

$$(\text{index}_\alpha^* X)(\dim \alpha) \leqq \dim X/G.$$

(6.13) *Remark.* Note that under the hypotheses of Proposition (6.12), we have for $m > \dim X/G$,

$$\text{index}_\alpha^* X = \text{index}_\alpha E^m \times X.$$

In fact, this equality holds for m sufficiently large whenever $\text{index}_\alpha^* X$ is finite.

(6.14) Perhaps the simplest criterion for X/G to be finite dimensional is obtained under the hypothesis that X is a separable metric space. Then, X/G is again a separable metric space and $\dim X/G \leqq \dim X$ (see [26]), so that X/G is finite dimensional whenever X is.

(6.15) Just as in the free case it is sometimes convenient to increase the index by one and set

$$\mathrm{Index}^*_\alpha X = \mathrm{index}^*_\alpha X + 1.$$

It is a simple matter to restate the properties of index^*_α in terms of Index^*_α. The monotone, continuity and piercing properties are verbatim the same, just capitalize the *i*. Just as in the free case, we have the following alterations in the others.

(Subadditivity)	$\mathrm{Index}^*_\alpha (A \cup B) \leqq \mathrm{Index}^*_\alpha A + \mathrm{Index}^*_\alpha B$
(Normalization)	$\mathrm{Index}^*_\alpha G = 1$
(Dimension)	$(\mathrm{Index}^*_\alpha X) \dim \alpha \leqq \dim X/G + \dim \alpha$

whenever X has only finite isotropy groups.

(6.16) *Remark.* We close this section with a simple observation to be used later. When $\mathrm{Index}^*_\alpha X > 1$ and all the isotropy groups G_x are finite, then X/G must be an infinite set.

7. Some Special Cases

We consider now three examples which will be employed in our applications. Throughout this section our Lie group G is the circle group S^1 and thus our category $\mathscr{F}_*$ is paracompact spaces with an S^1-action. Furthermore our universal S^1-bundle (E, p, B) is the inductive limit of the classical Hopf-fibrations

$$\begin{array}{ccccccccc} S^3 & \subset & S^5 & \subset \cdots \subset & S^{2n+1} & \subset \cdots \\ \downarrow & & \downarrow & & \downarrow & \\ S^2 & \subset & \mathbb{C}P^2 & \subset \cdots \subset & \mathbb{C}P^n & \subset \cdots \end{array}$$

i.e. $E = S^\infty$ and $B = \mathbb{C}P^\infty$. Notice also that if X is an S^1-space and $x \in X$, either the isotropy group G_x is finite or $G_x = S^1$. We employ rational coefficients $\mathbb{Q}$ for cohomology and $\mathbb{Q}$ will not be displayed when rational coefficients are understood. Finally, our index theory will be based on the universal Chern class $c_1 \in H^2(\mathbb{C}P^2, \mathbb{Z})$ and so we choose $c \in H^2(\mathbb{C}P^\infty)$ corresponding to this class. Following the notation in section 4 set

$$\mathrm{Index}^*_{\mathbb{C}} X = \mathrm{Index}^*_c X, \qquad X \in \mathscr{F}_*.$$

(7.1) *Notation.* Given a G-space X, set $\mathrm{Fix}(X) = \{x : gx = x, g \in G\}$. $\mathrm{Fix}(X)$ is thus the set of points fixed under the action. (It is also denoted by X^G in the literature.)

(7.2) **Proposition.** *If $X \in \mathscr{F}_*$ and the orbit space X/S^1 is finite dimensional* (e.g. *X is separable metric and finite dimensional*), *then $\mathrm{Index}^*_{\mathbb{C}} X$ is finite if and only if* $\mathrm{Fix}\, X = \emptyset$.

Proof. This is immediate from Proposition (6.12) and remarks made preceding this proposition.

Just as in the free case (§ 4), $\text{Index}_{\mathbb{C}}^*$ satisfies a stability condition which we formulate as follows. Let $X \circ S^1$ denote the join of $X \in \mathscr{F}_*$ with S^1 and let k denote a non-zero integer. Define an S^1 action on $X \circ S^1$ by

$$g(x, t, z) = (gx, t, g^k z), \qquad x \in X, \ z \in S^1, \ t \in I$$

where $g^k z$ is ordinary multiplication.

(7.3) **Proposition.** *If X is locally contractible and $X \in \mathscr{F}_*$, then*

$$\text{Index}_{\mathbb{C}}^* X \circ S^1 = \text{Index}_{\mathbb{C}}^* X + 1.$$

Proof. The proof is almost identical with the proof of Proposition (4.3) so we content ourselves with a brief sketch. First of all $X \circ S^1 \in \mathscr{F}_*$ and $X \circ S^1$ is locally contractible. Hence $E \times X$, $E \times (X \circ S^1)$ are both locally contractible and singular cohomology may be employed in our argument. We also note that we may assume that $\text{Index}_{\mathbb{C}}^* X$ is finite so that all the isotropy subgroups S_x^1, $x \in X$ are finite.

Just as in (4.3),

$$\text{Index}_{\mathbb{C}}^* X \leqq \text{Index}_{\mathbb{C}}^* X \circ S^1 \leqq \text{Index}_{\mathbb{C}}^* X + 1$$

where a simple computation shows that $\text{Index}_{\mathbb{C}}^* S^1 = 1$ and where S^1 is given the action $gz = g^k z$, $k \neq 0$.

Now, use the diagram of Gysin sequences as in (4.3) with the following replacements

replace X by $E \times X$,

replace $\tilde{X}$ by $E \times_{S^1} X$,

replace B by $E \times (X \circ S^1)$,

replace $\tilde{B}$ by $E \times_{S^1} (X \circ S^1)$

to show that the inequality $\text{Index}_{\mathbb{C}}^* X = \text{Index}_{\mathbb{C}}^* X \circ S^1$ is impossible.

Example 1

Let $\mathbb{C}^N$ denote the space of k-tuples $(c_1, \ldots, c_k)$ with entries $c_i \in \mathbb{C}^n$, where c_i may be thought of as an n-vector over the complex field $\mathbb{C}$. Thus $N = nk$. For a given k-tuple of *non-zero* integers $(n_1, \ldots, n_k)$, the circle group S^1 acts on $\mathbb{C}^N$ by

$$g(c_1, \ldots, c_k) = (g^{n_1} c_1, \ldots, g^{n_k} c_k)$$

Then, for every invariant set $X \subset \mathbb{C}_*^N = \mathbb{C}^N - 0$, $\text{Index}_{\mathbb{C}}^* X$ is defined and on this category of invariant subsets of $\mathbb{C}_*^N$, $\text{Index}_{\mathbb{C}}^*$ satisfies all the properties of Index_α^* discussed in Section 6 as well as the stability property (Proposition (7.3)). In particular, for $X \subset \mathbb{C}^N_*$,

$$2\,\mathrm{Index}^*_{\mathbb{C}} X \leqq \dim X + 2$$

so that our index is finite over invariant subsets of $\mathbb{C}^N_*$.

We compute first the index of the unit sphere $S = S^{2N-1}$ in $\mathbb{C}^N$. By definition,

$$\mathrm{Index}^*_{\mathbb{C}} S = \mathrm{Index}_c E \times S$$

where c is the first rational Chern class. To compute the R.H.S., we use standard techniques as follows. Consider the bundle map

$$\begin{array}{ccc} E \times S & \xrightarrow{\hat{p}} & E \\ \downarrow & & \downarrow \\ E \times_{S^1} S & \xrightarrow{p} & B \end{array}$$

where p is induced by equivariant projection $\hat{p}$. Then, notice that the fiber of the fiber map p is a $(2N-1)$-sphere. Using the Gysin sequence [17], we conclude that (for any coefficients)

$$H^i(B) \xrightarrow{p^*} H^i(E \times_{S^1} S)$$

is an isomorphism for $i \leqq 2N-1$. Thus, $\mathrm{Index}^*_{\mathbb{C}} S \geqq N$. On the other hand, using Lemma (6.11), we have isomorphisms

$$\tilde{j}^*: H^q(S/S^1) \to H^q(E \times_{S^1} S)$$

and since $H^{2N}(S/S^1) = 0$, we have $H^{2N}(E \times_{S^1} S) = 0$. Thus, $\mathrm{Index}_{\mathbb{C}} S \leqq N$ and we have verified

(7.4) **Proposition.** $\mathrm{Index}^*_{\mathbb{C}} S^{2N-1} = N$.

(7.5) **Corollary.** $\mathrm{Index}\ \mathbb{C}^N_* = N$.

(7.6) **Corollary.** *Let K denote an invariant linear subspace of $\mathbb{C}^N$ of (complex) dimension k, then* $\mathrm{Index}^*_{\mathbb{C}} K_* = k$, *where* $K_* = K - 0$.

(7.7) **Corollary.** *Let K denote an invariant linear subspace of $\mathbb{C}^N$ of dimension k and let $X \subset \mathbb{C}^N_*$ denote a closed invariant subset such that* $k + \mathrm{Index}^*_{\mathbb{C}} X > N$. *Then* $X \cap K \neq \emptyset$. *More precisely*

$$\mathrm{Index}^*_{\mathbb{C}} X \cap K \geqq \mathrm{Index}^*_{\mathbb{C}} X - (N-k) > 0.$$

Proof. Let $K^\perp$ denote the orthogonal complement of K. $K^\perp$ is invariant and the orthogonal projection $\pi: \mathbb{C}^N \to K^\perp$ is equivariant. By continuity (Proposition (6.5)), there is a closed invariant neighborhood A of $X \cap K$ in X such that $\mathrm{Index}^*_{\mathbb{C}} X \cap K = \mathrm{Index}^*_{\mathbb{C}} A$. Let B denote X minus the interior of A (in X). Then,

$\pi|B: B \to K_*^{\perp}$ tells us that $\mathrm{Index}_{\mathbb{C}}^* B \leqq N-k$ and hence using subadditivity

$$\mathrm{Index}_{\mathbb{C}}^* X \leqq \mathrm{Index}_{\mathbb{C}}^* X \cap K + (N-k)$$

which is the desired result.

Before we consider the index of the boundary of an invariant neighborhood, we make one more comment. Let $\operatorname{cat} X$ denote the Ljusternik-Schnirelman category of X. Recall that $\operatorname{cat} X = \lambda$, if X can be covered by λ open sets each of which is contractible in X and λ is minimal with this property.

(7.8) **Corollary.** $\operatorname{cat} S/S^1 = N$.

Proof. The remarks above show that S/S^1 has non-trivial cup products of length $N-1$, so that $\operatorname{cat} S \geqq N$. To see that $\operatorname{cat} S \leqq N$ we proceed by induction on N, representing S as N-tuples $(x_1, \ldots, x_N)$, $x_i \in \mathbb{C}$, $x_i \bar{x}_i = 1$. Let A denote the orbit containing $(0, 0, \ldots, 1)$. Then, by induction $\operatorname{cat}(S-A)/S^1 = N-1$. On the other hand, S/S^1 is an ANR ([27]), so that S/S^1 is locally contractible at the point corresponding to the orbit A. Thus, $\operatorname{cat} S/S^1 \leqq N$ so that our proof is complete.

(7.9) **Proposition** (Boundary of Invariant Neighborhoods). *Let $M \in \mathscr{F}_*$ denote an orientable $2N$-manifold and U an open invariant set in M with compact closure. Let $\varphi: (\bar{U}, \partial U) \to (\mathbb{C}^N, \mathbb{C}_*^N)$ be an equivariant map of non-zero degree. Then,* $\mathrm{Index}_{\mathbb{C}}^* \partial U = N$.

Proof. Just as in Proposition (4.6), we assume that φ is defined on an equivariant neighborhood V of $\bar{U}$ and $\varphi^{-1}(S) = \partial U$, where S is the unit sphere in $\mathbb{C}^N$. Thus, we have $\varphi: (V, V - \partial U) \to (\mathbb{C}^N, \mathbb{C}^N - S)$ and, by assumption, φ has non-zero degree δ, i.e. if $o_1 \in H_{2N}(V, V-\partial U)$, $o_2 \in H_{2N}(\mathbb{C}^N, \mathbb{C}^N - S)$ are fundamental classes, $\varphi_*(o_1) = \delta\, o_2$. The map φ also induces a map

$$\varphi_0 = 1 \times \varphi: E \times V \to E \times \mathbb{C}^N, \quad E = S^\infty$$

and it suffices to show that for m sufficiently large

$$\tilde{\varphi}_0: S^m \times_{S^1} \partial U \to S^m \times_{S^1} S$$

induces an injection in rational cohomology in dimension $2N-2$. Now, if μ is a fundamental class of the sphere S^m, let $\bar{o}_1 \in H_{m+2N}(S^m \times (V, V-\partial U))$, $\bar{o}_2 \in H_{m+2N}(S^m \times (\mathbb{C}^N, \mathbb{C}^N - S))$ denote fundamental classes corresponding to $\mu \times o_1, \mu \times o_2$, respectively. Then, $\varphi_0^*(\bar{o}_1) = \delta \bar{o}_2$ and the transfer map $t: H^*(S^m \times \partial U, \mathbb{Q}) \to H^*(S^m \times S, \mathbb{Q})$ applies to force $\varphi_0^*: H^*(S^m \times S, \mathbb{Q}) \to H^*(S^m \times \partial U, \mathbb{Q})$ to inject in all dimensions. Now, we look at the bundle map

$$\begin{array}{ccc} S^m \times \partial U & \xrightarrow{\varphi_0} & S^m \times S \\ \downarrow & & \downarrow \\ S^m \times_{S^1} \partial U & \xrightarrow{\tilde{\varphi}_0} & S^m \times_{S^1} S \end{array}$$

and proceed, just as in the proof of Proposition (4.6), via a Gysin sequence argument over the rationals. Keep in mind that the action of S^1 on ∂U has all isotropy groups finite so that $H^*(S^m \times_{S^1} \partial U, \mathbb{Q}) = H^*(\partial U/S^1, \mathbb{Q})$. In particular, $H^q(S^m \times_{S^1} \partial U, \mathbb{Q}) = 0$ for $q \geqq 2N-1$, since $\dim \partial U/S^1 \leqq 2N-2$.

Example 2

This example is similar to Example 1 except that we allow fixed points. Let $\mathbb{C}^M$ denote the space of $(k+1)$-tuples $(c_0, c_1, \ldots, c_k)$ with $c_i \in \mathbb{C}^n$ so that $M = (k+1)n = N+n$, where $N = nk$ and $\mathbb{C}^N$, as in Example 1, is naturally imbedded in $\mathbb{C}^M$. For a given k-tuple of non-zero integers $(n_1, \ldots, n_k)$, we define an action of the circle group S^1 on $\mathbb{C}^M$ by

$$g(c_0, c_1, \ldots, c_k) = (c_0, g^{n_1} c_1, \ldots, g^{n_k} c_k)$$

so that the 0-th coordinate remains fixed. Then, the fixed point set $F = \mathrm{Fix}(\mathbb{C}^M)$ of this action is the subspace given by $c_i = 0$, $i \geqq 1$. Furthermore, the invariant subspace $\mathbb{C}^N$, defined by $c_0 = 0$, is precisely Example 1. For any invariant subset $X \subset \mathbb{C}^M$, *just as in Example* 1,

$$\mathrm{Index}^*_{\mathbb{C}} X = \mathrm{Index}^*_c X$$

where c is the first (rational) Chern class. Thus, we have a index theory on the invariant subsets of $\mathbb{C}^M$ satisfying the properties in Section 6, but which is not finite on sets which intersect the fixed point set F. However, in the complement of F things still behave nicely. The projection.

$$\eta: (c_0, c_1, \ldots, c_k) \mapsto (c_1, \ldots, c_k)$$

takes $C^M - F$ equivariantly onto $\mathbb{C}^N_*$ and this, together with the inclusion map in the other direction tells us that

(7.10) **Proposition.**

$$\mathrm{Index}^*_{\mathbb{C}} \mathbb{C}^M - F = \mathrm{Index}^*_{\mathbb{C}} \mathbb{C}^N_* = N.$$

Now, let S^{2M-1} denote the unit sphere in $\mathbb{C}^M$, S the unit sphere in $\mathbb{C}^N$, as in Example 1, and $F_0 = F \cap S^{2M-1}$. F_0 is then the $(2n-1)$-sphere given by $c_0 \bar{c}_0 = 1$, $c_i = 0$, $i \geqq 1$. Clearly $S \subset S^{2M-1}$ and η above induces an equivariant map

$$\bar{\eta}: S^{2M-1} - F_0 \to S$$

by

$$\bar{\eta}(c_0, c_1, \ldots, c_k) = \left(\sum_{i=1}^{k} c_i \bar{c}_i \right)^{-\frac{1}{2}} (c_i, \ldots, c_k).$$

(7.11) **Proposition.** *Let A denote any invariant subset of S^{2M-1} such that $F_0 \subset A \subset S^{2M-1} - S$. Then,*

$$\mathrm{Index}^*_{\mathbb{C}} S^{2M-1} - A = \mathrm{Index}^*_{\mathbb{C}} S = N.$$

Proof. Use $\bar{\eta}$ and Proposition (7.4).

(7.12) **Proposition** (Boundary of Invariant Neighborhood of 0.). *Let U denote a bounded open invariant set in $\mathbb{C}^M$ containing the origin $0 \in \mathbb{C}^M$. Then if ∂U denotes the boundary of U, we have*

$$\mathrm{Index}^*_{\mathbb{C}}(\partial U - F) = N.$$

Proof. Let $V = U \cap \mathbb{C}^N$. Then $\mathrm{Index}^*_{\mathbb{C}} \partial V = N$. Since $\partial V \subset \partial U - F$, we have $\mathrm{Index}^*_{\mathbb{C}}(\partial U - F) \geqq N$. On the other hand $\eta(\partial U - F) \subset S$ so that $\mathrm{Index}^*_{\mathbb{C}}(\partial U - F) \leqq N$.

(7.13) We close this section with a few remarks concerning Ljusternik-Schirelmann category. First of all, $\bar{\eta}$ above is an equivariant homotopy equivalence and hence

a) $\mathrm{cat}(S^{2M-1} - F_0)/S^1 = \mathrm{cat}(S/S') = N.$

For $\varepsilon > 0$, let V_ε denote the ε-neighborhood of F_0 in $\mathbb{C}^M$, i.e.

$$V_\varepsilon = \{c \in \mathbb{C}^M \colon \|c - p\| < \varepsilon \text{ for some } p \in F_0\}.$$

Then, $\bar{\eta} \colon S^{2M-1} - V_\varepsilon \to S$ remains an equivariant homotopy equivalence and

b) $\mathrm{cat}(S^{2M-1} - V_\varepsilon)/S^1 = \mathrm{cat}(S/S^1) = N.$

More generally if $\mathrm{cat}_X A$ denotes the category of A in X (open sets covering A are in X and contractions are in X), then for any invariant set $A \subset S$ we have

c) $\mathrm{cat}_X \tilde{A} = \mathrm{cat}_Y \tilde{A} = \mathrm{cat}_Z \tilde{A}$

where $\tilde{A} = A/S^1$, $X = S/S^1$, $Y = (S^{2M-1} - F_0)/S^1$ and $Z = (S^{2M-1} - V_\varepsilon)/S^1$.

Now, the function $\gamma(A) = \mathrm{cat}_Y \tilde{A}$, defined on invariant subsets of $S^{2M-1} - F_0$, where $Y = (S^{2M-1} - F_0)/S^1$, satisfies many of the properties of $\mathrm{Index}^*_{\mathbb{C}}$ e.g. monotonicity, continuity and subadditivity. However, we are not sure how γ behaves in relation to the piercing property (Prob. (6.9)) (we conjecture against it) and this is one of the reasons why $\mathrm{Index}^*_{\mathbb{C}}$ is better suited to our techniques.

Example 3

Examples 1 and 2 are finite dimensional versions of the following infinite dimensional example. First, we identify as usual the reals mod 2π with $S^1 (t \leftrightarrow e^{it})$ and we denote by $W^{1,2}(S^1)$ the Hilbert space of real valued functions $z \colon S^1 \to \mathbb{R}$ such that z and $\dot{z} = \dfrac{dz}{dt}$ are square integrable with inner product

$$(z_1, z_2) = \frac{1}{2\pi} \int_0^{2\pi} [\dot{z}_1 \dot{z}_2 + z_1 z_2] \, dt$$

S^1 acts on this space $W^{1,2}(S^1)$ as follows. For $g \in S^1$ set

$$(gz)(e^{it}) = z(g e^{it})$$

where ge^{it} is ordinary complex multiplication. Alternatively, the action may be written

$$(L_\theta z)(t) = z(t+\theta)$$

where θ corresponds to $g = e^{i\theta} \in S^1$ and L_θ is the linear transformation corresponding to the action of the element g on $W^{1,2}(S^1)$. This space $W^{1,2}(S^1)$ can be identified with the space of Fourier series $\sum_{-\infty}^{\infty} c_n e^{int}$ subject to the conditions

$$\bar{c}_{-n} = c_n, \qquad \sum_{-\infty}^{\infty} (1+n^2)|c_n|^2 < \infty, \qquad c_n \in \mathbb{C}.$$

Consequently, $W^{1,2}(S^1)$ can be identified with the space of infinite sequences $(c_0, c_1, \ldots, c_k, \ldots)$ subject to the conditions

$$c_0 \in \mathbb{R}, \quad \sum_{0}^{\infty} (1+2n^2)|c_n|^2 < \infty, \quad c_n \in \mathbb{C}.$$

The S^1 action translates into

$$g(c_0, c_1, \ldots, c_k, \ldots) = (c_0, gc_1, \ldots, g^k c_k, \ldots)$$

and it is clear that each g corresponds to a unitary transformation of $W^{1,2}(S^1)$.

This action is not free. In fact isotropy groups of all orders appear. Nevertheless our index theory $\text{Index}^*_{\mathbb{C}}$ applies to all invariant subsets of $W^{1,2}(S^1)$.

(7.14) We close this section with a few comments concerning the analogue of Section 5, in the non-free case. Let $\mathscr{B}$ denote any normed linear space over $\mathbb{C}$. Then, any S^1 action on $\mathscr{B}$ induces an index theory $\text{Index}^*_{\mathbb{C}}$ on the family $\mathscr{E}$ of invariant subsets of $\mathscr{B}$. Furthermore, the function

$$\text{Index}^{\mathbb{C}}_*: \mathscr{E} \to \mathbb{N}$$

possesses properties analogous to those in Theorem (5.1), with some obvious changes. We leave the formalities to the reader.

8. An application

In this section we shall show how the index theory of Sections 6 and 7 can be applied to study the bifurcation of time periodic solutions from an equilibrium solution for Hamiltonian systems of ordinary differential equations.

Let $p, q \in \mathbb{R}^n$ and $H = H(p,q) \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ with $H(0,0) = 0$, $H_p(0,0) = 0 = H_q(0,0)$. Consider the Hamiltonian system of ordinary differential equations:

(8.1) $$\frac{dp}{dt} = -H_q, \qquad \frac{dq}{dt} = H_p.$$

Letting $z=(p,q)$ and $\mathscr{J}=\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}$, (8.1) can be rewritten as

$$(8.2)\quad \frac{dz}{dt}=\mathscr{J}H_z.$$

Our assumptions on $H_z(0)$ imply that (8.2) possesses the trivial equilibrium solution $z\equiv 0$ which is periodic with any period. Of interest is the existence of small nontrivial periodic solutions of (8.2). The Lyapunov Center Theorem is an old result of this nature [28]. To state it, observe that if (8.2) is linearized about $z=0$, the resulting equation is

$$(8.3)\quad \frac{dw}{dt}=\mathscr{J}H_{zz}(0)\,w.$$

The Lyapunov result then says that if $\mathscr{J}H_{zz}(0)$ possesses purely imaginary eigenvalues: $\pm\zeta_1, \pm\zeta_2, \dots, \pm\zeta_n$ and if ζ_j/ζ_1 is not an integer for $j\neq 1$, a family of periodic solutions with periods near $2\pi\zeta_1^{-1}$ bifurcates from $z=0$.

Lyapunov's irrationality condition on the eigenvalues of $\mathscr{J}H_{zz}(0)$ was eliminated by A. Weinstein [10, 29] who assumed instead that $H_{zz}(0)$ is a positive definite matrix. He then showed that for all small $\varepsilon>0$, the manifold $H=\varepsilon$ contains at least n distinct periodic orbits whose periods are near those of the linearized problem (8.3). He also discussed the case of indefinite $H_{zz}(0)$.

Recently J. Moser [11] generalized and simplified Weinstein's result. Moser showed that if $\mathbb{R}^{2n}=E_1\oplus E_2$ where E_1 and E_2 are invariant subspaces for (8.3), if all solutions of (8.3) with initial data in E_1 have a common period $T>0$ while no solutions of (8.3) in $E_2-\{0\}$ have period T, and if $H_{zz}(0)$ is positive definite on E_1, then for all small $\varepsilon>0$, (8.1) possesses at least $\frac{1}{2}\dim E_1$ distinct periodic orbits on $H=\varepsilon$ whose periods are near T.

Observe that both the Weinstein and Moser results provide lower bounds for the number of distinct periodic solutions of (8.1) on $H=\varepsilon$. In contrast in this section we will use the index theory of sections 6 and 7 to obtain lower bounds for the number of distinct small nontrivial periodic orbits of (8.1) as a function of the period. This procedure will be carried out under more general hypotheses than those considered by Moser. Given the index theory of Section 6 and 7, the techniques we use to find the periodic solutions and the results we obtain are closely related to our earlier paper [1]. However we will give a self contained development here.

Our main result is:

(8.4) **Theorem.** *Let $H\in C^2(\mathbb{R}^{2n},\mathbb{R})$ with $H(0)=0$, $H_z(0)=0$. Let $\mathbb{R}^{2n}=E_1\oplus E_2$ where E_1 and E_2 are invariant subspaces for the flow given by* (8.3). *Suppose all solutions of* (8.3) *with initial data in E_1 are T periodic, no solutions of* (8.3) *with initial data in $E_2-\{0\}$ are T periodic, and there are no equilibrium solutions of* (8.3) *in $E_1-\{0\}$. If the signature 2ν of the quadratic form $(H_{zz}(0)\,\zeta,\zeta)$, $\zeta\in E_1$, is nonzero, then either:* (i) *0 is a nonisolated T-periodic solution of* (8.1); *or* (ii) *there exist a pair of integers k, $m\geqq 0$ with $k+m\geqq|\nu|$, and a left neighborhood, $\mathscr{I}_l$, and a* *right neighborhood, $\mathscr{I}_r$, of T in $\mathbb{R}$ such that for all $\lambda \in \mathscr{I}_l$ (resp. $\mathscr{I}_r$),* (8.1) *possesses at least k (resp. m) distinct non-trivial λ-periodic solutions.*

(8.5) **Remark.** That the signature is even follows from the hypotheses on E_1. A more precise count of the number of distinct nontrivial solutions for fixed λ will be given in the course of the proof of Theorem (8.4). See Theorem (8.48) and Corollary (8.51). Observe that under Moser's hypotheses, since $H_{zz}(0)$ is positive definite on E_1, (8.3) possesses no equilibrium solutions in $E_1 - \{0\}$ and $v = \frac{1}{2}\dim E_1 \neq 0$. Thus our result applies to his case.

While completing the final draft of this paper we learned of the work of Chow & Mallet-Paret [12] who have obtained a special case of Theorem (8.4) for (8.1) where $E_1 = \{(z_1, \dots, z_r, 0, \dots, 0, z_{n+1}, \dots, z_{n+r}, 0, \dots, 0)\}$ and H restricted to E_1 has the form

$$H(z) = \tfrac{1}{2} \sum_{j=1}^{l} z_j^2 + z_{n+j}^2 - \tfrac{1}{2} \sum_{j=l+1}^{r} z_j^2 + z_{n+j}^2 + o(|z|^2).$$

This form for H on E_1 implies the hypotheses required of E_1 are automatically satisfied with $T = 2\pi$ and 2π is the minimal period for solutions of (8.3) in E_1. This has the effect of inducing a free S^1 action on our problem making it tractable by a simple extension of the index theory of [1]. Chow and Mallet-Paret also have some more refined results when H is analytic.

The proof of Theorem (8.4) will be carried out in several steps. The basic idea is to convert the problem to that of finding critical points of a real valued function g defined near 0 in a finite dimensional space of periodic functions. Critical points of g then will be obtained using minimax arguments.

To begin, we normalize the problem by fixing the period at 2π. Thus let $\tau = \lambda^{-1} t$. Then (8.2) becomes

$$\dot{z} = \lambda \mathscr{J} H_z \tag{8.6}$$

where $\dot{z} = dz/d\tau$. Any 2π periodic solution of (8.6) is a $2\pi\lambda$ periodic solution of (8.2). Observe that $\mathscr{J}^2 = -I$. For our later purposes it is convenient to replace (8.6) by the equivalent equation

$$\mathscr{J} \dot{z} = -\lambda H_z \tag{8.7}$$

Finally set $\mathscr{F}(\lambda, z) = \mathscr{J}\dot{z} + \lambda H_z$. The solutions of (8.7) will be obtained as the zeroes of $\mathscr{F}$. To introduce the class of functions in which (8.7) is studied, we identify $\mathbb{R}/[0, 2\pi]$ with S^1. Let $W^{1,2}(S^1)$ denote the real Hilbert space of 2π periodic functions which have square integrable first derivatives and let $E = (W^{1,2}(S^1))^{2n}$. Then E is a real Hilbert space under the norm

$$\|z\|_E^2 = \frac{1}{2\pi} \int_0^{2\pi} (|\dot{z}(\tau)|^2 + |z(\tau)|^2)\, d\tau$$

Let $Y = (L^2(S^1))^{2n}$. The smoothness assumptions on H imply $\mathscr{F} \in C^1(\mathbb{R} \times E, Y)$. Let $\mu = 2\pi T^{-1}$. The Frechet derivative of $\mathscr{F}$ with respect to z at $(\mu, 0)$ is

$$\mathscr{F}_z(\mu, 0)\, w = \mathscr{J} \dot{w} + \mu H_{zz}(0)\, w. \tag{8.8}$$

Comparing (8.8) to (8.3), we see that $\mathscr{F}_z(\mu, 0)$ has a null space $\mathscr{N}$ of dimension $2N \equiv \dim E_1$ of vectors of the form

$$(8.9)\quad z(t) = \sum_{j=-N}^{N} e^{ik_j\tau} e_j$$

where $k_j \in \mathbb{Z}$, $k_{-j} = -k_j$, $e_j \in \mathbb{C}^{2n}$, $e_{-j} = \bar{e}_j$, and e_j is an eigenvector of $\mathscr{J} H_{zz}(0)$. In fact $\mathscr{N}$ is isomorphic to E_1, the isomorphism being given by $z(t) = S(t)z(0)$ where $z(0) \in E_1$ and $S(t)$ is the semigroup for the initial value problem for (8.3). It is straightforward to check that $\mathscr{F}_z(\mu, 0)$ is a Fredholm map of index zero.

We seek zeroes of $\mathscr{F}$ in $\mathbb{R} \times E$ for λ near μ and z near 0. We already have the trivial family of zeroes $\{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$. Using the method of Lyapunov-Schmidt, (8.7) can be reduced to a finite dimensional problem. (We do not use the same finite dimensional reduction carried out by Moser but the analogue of [5].) Let $\mathscr{N}^\perp$ denote the L^2 orthogonal complement of $\mathscr{N}$ in E, i.e.

$$\mathscr{N}^\perp = \left\{ z \in E \mid \int_0^{2\pi} (z(t), w(t))_{\mathbb{R}^{2n}} dt = 0 \text{ for all } w \in \mathscr{N} \right\}.$$

Let P and $P^\perp$ denote the (L^2 orthogonal) projectors of E onto $\mathscr{N}$ and $\mathscr{N}^\perp$ respectively. Then (8.7) is equivalent to the pair of equations:

$$(8.10)\quad P\mathscr{F}(\lambda, z) = 0, \qquad P^\perp \mathscr{F}(\lambda, z) = 0.$$

Any $z \in E$ can be written uniquely as $z = x + y$ where $x \in \mathscr{N}$ and $y \in \mathscr{N}^\perp$. Define

$$(8.11)\quad F(\lambda, x, y) = P^\perp \mathscr{F}(\lambda, z).$$

Then $F(\mu, 0, 0) = 0$ and by construction $\mathscr{F}_y(\mu, 0, 0)$ is an isomorphism from $\mathscr{N}^\perp$ to $\mathscr{N}^\perp \cap Y$. Therefore by the implicit function theorem, there exists a neighborhood Ω of $(\mu, 0)$ in $\mathbb{R} \times \mathscr{N}$ and a mapping $\varphi \in C^1(\Omega, \mathscr{N}^\perp)$ such that $F(\lambda, x, y) = 0$ for λ near μ and z near 0 is equivalent to $y = \varphi(\lambda, x)$. Moreover since

$$(8.12)\quad 0 = F(\lambda, x, \varphi(\lambda, x)) = F_y(\lambda, 0, 0)\,\varphi + o(\|x + \varphi\|_E)$$

and $F_y(\lambda, 0, 0)$ is an isomorphism from $\mathscr{N}^\perp$ to $\mathscr{N}^\perp \cap Y$ for all λ near μ, it follows that

$$(8.13)\quad \varphi(\lambda, x) = o(\|x\|_E)$$

at $x = 0$ uniformly for λ near μ.

Thus to solve (8.7), it suffices to solve the finite dimensional problem

$$(8.14)\quad P\mathscr{F}(\lambda, x + \varphi(\lambda, x)) = 0.$$

Before discussing this question, we observe some invariance properties of our operators. For $z \in E$ and $\theta \in [0, 2\pi]$, set $L_\theta z = z(t + \theta)$. This defines an S^1 action on E. (See Example 3 of § 7.) It is easy to see that $\mathscr{F}$ commutes with L_θ, i.e. $\mathscr{F}(\lambda, L_\theta z) = L_\theta \mathscr{F}(\lambda, z)$. Note further that both $\mathscr{N}$ and $\mathscr{N}^\perp$ are invariant under L_θ. It then follows from (8.11) that $F(\lambda, \cdot)$ commutes with L_θ. The same is true of

$\varphi(\lambda,\cdot)$. Indeed

$$F(\lambda, x(t), y(t))=0,$$

where $y=\varphi(\lambda, x)$, implies that

$$0=F(\lambda, x(t+\theta), y(t+\theta))=F(\lambda, L_\theta x, L_\theta y).$$

Hence by the implicit function theorem $L_\theta \varphi(\lambda, x)=\varphi(\lambda, L_\theta x)$. Following standard usage, as in earlier sections we will refer to functions with values in E, $\mathcal{N}$, or $\mathcal{N}^\perp$ that commute with L_θ as being equivariant. The same term will be applied to real valued functions d for which $d(L_\theta(z))=d(z)$. Sets A such that $L_\theta A=A$ for all $\theta\in[0,2\pi]$ will be called invariant.

The next step in the proof of Theorem (3.4) is to show that the solutions of (8.14) can be determined as the critical points of an appropriate function. Some additional notation is required. If $z\in E$, $z=(z_1(t),\dots,z_{2n}(t))$. Let $p(t)\equiv P_1 z\equiv(z_1(t),\dots,z_n(t))$ and let $q(t)\equiv P_2 z\equiv(z_{n+1}(t),\dots,z_{2n}(t))$.

Define

$$g(\lambda, x)=\int_0^{2\pi}[(p(t),\dot q(t))_{\mathbb{R}^n}-\lambda H(p(t),q(t))]\,dt \tag{8.15}$$

where $z=x+\varphi(\lambda,x)$ and $(\cdot,\cdot)_{\mathbb{R}^n}$ denotes the $\mathbb{R}^n$ inner product. Thus $g\in C^1(\Omega,\mathbb{R})$ and it is easily checked that $g(\lambda,\cdot)$ is equivariant. Moreover for fixed λ, the critical points of $g(\lambda,\cdot)$ satisfy (8.14). Before showing this, it is technically convenient to renorm $\mathcal{N}$ by taking the L^2 norm on $\mathcal{N}$ which is equivalent to the E norm on $\mathcal{N}$. Henceforth we denote the new norm by $\|\cdot\|_{\mathcal{N}}$.

Now suppose x is a critical point of $g(\lambda,\cdot)$. Then for all $\xi\in\mathcal{N}$,

$$\begin{aligned}(g_x(\lambda,x),\xi)_{\mathcal{N}}=0=\int_0^{2\pi}\Big[&\Big(p, P_2\frac{d}{d\tau}(\xi+\varphi_x(\lambda,x)\xi)\Big)_{\mathbb{R}^n}\\ &+(P_1(\xi+\varphi_x(\lambda,x)\xi),q)_{\mathbb{R}^n}-\lambda(H_p(p,q),P_1(\xi+\varphi_x(\lambda,x)\xi))_{\mathbb{R}^n}\\ &-\lambda(H_q(p,q),P_2(\xi+\varphi_x(\lambda,x)\xi))_{\mathbb{R}^n}\Big]\,d\tau\end{aligned} \tag{8.16}$$

where $p=P_1(x+\varphi(\lambda,x))$ and $q=P_2(x+\varphi(\lambda,x))$. An integration by parts yields:

$$\begin{aligned}0=&\int_0^{2\pi}[(\dot q-\lambda H_p(p,q),P_1(\xi+\varphi_x(\lambda,x)\xi))_{\mathbb{R}^n}\\ &-(\dot p+\lambda H_p(p,q),P_2(\xi+\varphi_x(\lambda,x)\xi))_{\mathbb{R}^n}]\,d\tau\\ =&-\int_0^{2\pi}(\mathcal{F}(\lambda,z),\xi+\varphi_x(\lambda,x)\xi)_{\mathbb{R}^{2n}}\,d\tau.\end{aligned} \tag{8.17}$$

Since $P^\perp\mathcal{F}(\lambda,z)=0$ and $\varphi_x(\lambda,x)\xi\in\mathcal{N}^\perp$, (8.17) implies that

$$\int_0^{2\pi}(\mathcal{F}(\lambda,z),\xi)_{\mathbb{R}^{2n}}\,d\tau=0 \tag{8.18}$$

for all $\xi\in\mathcal{N}$ which is equivalent to (8.14).

Thus to solve (8.7), it suffices to find small nontrivial critical points of $g(\lambda,\cdot)$ in $\mathcal{N}$. If 0 is not an isolated critical point of $g(\mu,\cdot)$, the first alternative of Theorem (8.4) obtains. Hence for the remainder of this section, *we assume* 0 *is an isolated critical point of* $g(\mu,\cdot)$.

To continue several preliminaries are required. Consider the ordinary differential equation

$$(8.19)\quad \frac{d\psi}{ds}=-g_x(\mu,\psi),\qquad \psi(0,x)=x$$

for x near 0 in $\mathcal{N}$. By (8.16)–(8.18),

$$(8.20)\quad g_x(\lambda,x)=P\mathscr{F}(\lambda,x+\varphi(\lambda,x))$$

so g_x is continuously differentiable in x near $(\mu,0)$. Hence (8.19) possesses a unique solution for all x near 0 in $\mathcal{N}$. We will show ψ is equivariant.

(8.21) **Lemma.** *If $V(x)$ is a locally Lipschitz continuous map of $\mathcal{N}$ to $\mathcal{N}$ and is equivariant, the solution $\eta(s,x)$ of*

$$(8.22)\quad \frac{d\eta}{ds}=V(\eta),\qquad \eta(0,x)=x$$

is equivariant.

Proof. Let $w=L_\theta\eta(s,x)$. Then

$$(8.23)\quad \frac{dw}{ds}=L_\theta\frac{d\eta}{ds}=L_\theta V(\eta)=V(L_\theta\eta)=V(w)$$

and $w(0)=L_\theta x$. Therefore $w(s)=\eta(s,L_\theta x)=L_\theta\eta(s,x)$.

(8.24) **Corollary.** *$\psi(s,x)$ is equivariant.*

Proof. By Lemma (8.21), all we need show is that $g_x(\mu,x)$ is equivariant. Since $g(\mu,x)$ is equivariant,

$$(8.25)\quad (g_x(\mu,x),\xi)_{\mathcal{N}}=(g_x(\mu,L_\theta x),L_\theta\xi)_{\mathcal{N}}$$

for all $\xi\in\mathcal{N}$. It is easy to verify from (8.9) that L_θ is a unitary transformation so $L_\theta^{-1}=L_\theta^*$. Thus choosing $\xi=L_\theta^{-1}\alpha$ in (8.25) yields

$$(8.26)\quad (g_x(\mu,x),L_\theta^*\alpha)_{\mathcal{N}}=(L_\theta g_x(\mu,x),\alpha)_{\mathcal{N}}=(g_x(\mu,L_\theta x),\alpha)_{\mathcal{N}}$$

for all $\alpha\in\mathcal{N}$ which implies the equivariance of $g_x(\mu,x)$.

(8.27) *Remark.* The above argument also shows that $g_x(\lambda,x)$ is equivariant for all λ near μ.

With the aid of $\psi(s,x)$, the neighborhood of 0 in $\mathcal{N}$ will be constructed in which we will find critical points of $g(\lambda,\cdot)$.

(8.28) **Lemma.** *There is a constant $c>0$ and an open invariant neighborhood Q of 0 in $\mathcal{N}$ such that*

1° *If $x \in Q$, $|g(\mu, x)|<c$ and $\psi(s,x) \in Q$ for all s such that* $|g(\mu, \psi(s,x))|<c$.

2° *If $x \in \partial Q$, $|g(\mu, x)|=c$ or $\psi(s,x) \in \partial Q$ for all s satisfying* $|g(\mu, \psi(s,x))| \leqq c$.

Proof. Since 0 is an isolated critical point of $g_x(\mu, 0)$, there is a neighborhood X of 0 in $\mathcal{N}$ in which 0 is the only critical point of $g_x(\mu, 0)$. We restrict ourselves to X. Let $S^+=\{x \in X \mid \psi(s,x) \in X$ for all $s>0\}$ and $S^-=\{x \in X \mid \psi(s,x) \in X$ for all $s<0\}$. It is easy to see that at least one of these sets is nonempty. In particular if there are points near 0 where $g(\mu, \cdot)$ is positive, $S^+ \neq \emptyset$ for then we can find a sequence $x_m \to 0$ such that $g(\mu, x_m)>0$. If $B_r=\{x \in \mathcal{N} \mid \|x\|_{\mathcal{N}}<r\}$, then for some small r, and all large m, the orbit $\psi(-s, x_m)$ will intersect ∂B_r at $s=s_m>0$. Since $x_m \to 0$, $s_m \to \infty$. A subsequence of $\psi(-s_m, x_m)$ converges to $\tilde{x} \in \partial B_r$ and our construction implies $\psi(s, \tilde{x}) \in X$ for all $s>0$. A similar argument shows that $S^- \neq \emptyset$ if there are points near 0 where $g(\mu, \cdot)$ is negative. Let x be near S^+, say $\|x-S^+\|_{\mathcal{N}} \leqq \rho$ and $x \notin S^+$. Then for $\rho \leqq \rho^+$ there is a $b^+(\rho)>0$ such that $\psi(s,x)$ will cross all level sets $g(\mu, \cdot)=b$ as s increases provided that $b \leqq g(\mu, x)$ and $|b| \leqq b^+(\rho)$. Similarly if $\|x-S^-\|_{\mathcal{N}} \leqq \rho \leqq \rho^-$ and $x \in S^-$, there is a $b^-(\rho)>0$ such that $\psi(s,x)$ will cross all level sets $g(\mu, \cdot)=b$ as s decreases provided that $b \geqq g(\mu, x)$ and $|b| \leqq b^-(\rho)$. Thus choosing $\rho=\min(\rho^+, \rho^-)$ and $c \in (0, \min b^+(\rho), b^-(\rho))$, we can take Q to be the union of all orbit segments $\psi(s,x)$ starting in B_ρ and lying between $g(\mu, \cdot)=c$ and $g(\mu, \cdot)=-c$. Then Q satisfies 1° and 2°. Moreover since L_θ is unitary, $\|L_\theta x\|_{\mathcal{N}}=\|x\|_{\mathcal{N}}$ so if $x \in B_\rho$, $L_\theta x \in B_\rho$. Hence Q is invariant.

(8.29) *Remark.* The index theory of § 7 is applicable to invariant subsets of $\mathcal{N}$. For such $A \in \mathcal{N}$ we set $i(A)=\mathrm{Index}_{\mathbb{C}}^* A$. Since $\bar{Q}$ is equivariant and is a neighborhood of 0 in $\mathcal{N}$ with $\dim \mathcal{N}=2N$, it follows from Proposition (7.7.) that $i(\partial Q)=N$. Set $T^{\pm}=S^{\pm} \cap \partial Q$. The indices of these sets play an important role in determining the number of critical points of $g(\lambda, \cdot)$ in Q. The next result gives an estimate for these numbers.

(8.30) **Theorem.** $i(T^-)+i(T^+) \geqq N$.

Proof. Let X be as in Lemma (8.28) and $r>0$ such that $B_r \subset X$. By the construction of Lemma (8.28) with X replaced by B_r, there is a neighborhood Q_b of 0 in $\mathcal{N}$ satisfying 1°–3° of Lemma (8.28) with c replaced by b. Let $Q_b^+ =\{x \in \partial Q_b \mid g(\mu, x)=b\}$ and $Q_b^-=\{x \in \partial Q_b \mid g(\mu, x)=-b\}$. If $x \in Q_b^-$, there is a unique $\kappa(x)>0$ such that $g(\mu, \psi(\kappa(x), x))=-c$. Since

$$\begin{aligned} g(\mu, \psi(\kappa(x), x)) &= g(\mu, L_\theta \psi(\kappa(x), x))=g(\mu, \psi(\kappa(x), L_\theta x)) \\ &= -c=g(\mu, \psi(\kappa(L_\theta x), L_\theta x)) \end{aligned}$$

by the equivariance of $g(\mu, \cdot)$ and $\psi(s, \cdot)$, it follows that κ is equivariant. Therefore so is the map $v(x)=\psi(\kappa(x), x)$ and $v \in C(Q_b^-, Q_c^-)$ (where $Q_c^{\pm}$ has the obvious meaning). In particular $v(S^- \cap Q_b^-) = T^-$. Hence by (6.4) and (6.5), there is a neighborhood U of T^- such that $i(U \cap Q_c^-) = i(T^-)$.

If r is sufficiently small, $v(Q_b^-) \subset U \cap Q_c^-$ for if not, for all $r_m \to 0$, there exists $b_m \to 0$ and $x_m \in B_{r_m}$ such that $g(\mu, x_m) = b_m > 0$ and $v(x_m) \in Q_c^- - U$. Along some subsequence we have $v(x_m) \to w \in Q_c^-$ and $w \notin T^-$. However since $x_m \to 0$, $\kappa(x_m) \to \infty$ which shows $w \in T^-$, a contradiction. Thus we can assume $v(Q_b^-) \subset U \cap Q_c^-$.

By (6.4),

(8.31) $i(S^- \cap Q_b^-) = i(T^-) \leqq i(Q_b^-) \leqq i(U \cap Q_c^-) = i(T^-)$.

Hence

(8.32) $i(Q_b^-) = i(T^-)$.

Similarly

(8.33) $i(Q_b^+) = i(T^+)$.

Next let $x \in \partial Q_b - Q_b^-$. Then there exists a unique $\omega(x) \leqq 0$ such that $g(\mu, \psi(\omega(x), x)) = b$. An above argument implies $\rho(x) = \psi(\omega(x), x)$ is equivariant and $\rho \in C(\overline{\partial Q_b - Q_b^-}, Q_b^+)$. Hence by (6.4)

(8.34) $i(\overline{\partial Q_b - Q_b^-}) \leqq i(Q_b^+) \leqq i(\overline{\partial Q_b - Q_b^-})$

Combining (8.32)–(8.34) and using (6.6) yields

(8.35) $N = i(\partial Q_b) \leqq i(Q_b^-) + i(\overline{\partial Q_b - Q_b^-}) = i(T^-) + i(T^+)$

(8.36) *Remark.* The number of critical points we obtain for $g(\lambda, \cdot)$ in Q depends on the interplay between $g(\lambda, \cdot)$ near ∂Q and $g(\lambda, \cdot)$ near 0. The estimates just obtained for $i(T^{\pm})$ are a quantiative measure of the behavior of $g(\mu, \cdot)$ near ∂Q and therefore of $g(\lambda, \cdot)$ near ∂Q for λ near μ since such a perturbation does not change the behavior of g near ∂Q. On the other hand, the quadratic part of $g(\mu, \cdot)$ vanishes identically while for $\lambda \neq \mu$, the quadratic terms in $g(\lambda, \cdot)$ are dominant near 0. These terms are governed by the quadratic part of H restricted to E_1. We will make these statements more precise in what follows. To help determine the behavior of $g(\lambda, \cdot)$ near 0, we have the following lemma. We are indebted to Mark Adler who assisted in the proof.

(8.37) **Lemma.** *Under the hypotheses of Theorem* (8.4), *the quadratic form* $(H_{zz}(0)\zeta, \zeta)$, $\zeta \in E_1$, *is nondegenerate.*

Proof. From (8.9), we see ζ has the form

$$\zeta = \sum_{j=-N}^{N} \alpha_j e_j$$

where $\alpha_{-j} = \bar{\alpha}_j$. Therefore

(8.38) $$(H_{zz}(0)\zeta, \zeta) = \sum_{|i|, |j| \leqq N} \alpha_i \bar{\alpha}_j (H_{zz}(0) e_i, e_j) \equiv (\hat{H}\alpha, \alpha)$$

where $\hat{H}_{ij}=(H_{zz}(0)e_i, e_j)$. Thus $(H_{zz}(0)\zeta,\zeta)$ is nondegenerate on E_1 if and only if $\hat{H}$ has no nontrivial null vectors. If there is an $\alpha\in\mathbb{C}^{2N}$ such that $\hat{H}\alpha=0$, then

$$(8.39)\quad \sum_{|j|\leq N}\hat{H}_{ij}\alpha_j=0=(H_{zz}(0)e_i, \sum_{|j|\leq N}\alpha_j e_j) \\ =(e_i, H_{zz}(0)\hat{e}), \quad |i|\leq N$$

where $\hat{e}=\sum_{|j|\leq N}\alpha_j e_j$. Thus $H_{zz}(0)\hat{e}$ is orthogonal to E_1. Since E_1 is invariant under $\mathscr{J}H_{zz}(0)$, $\mathscr{J}H_{zz}(0)E_1=E_1$ and $H_{zz}(0)E_1=\mathscr{J}^{-1}E_1=-\mathscr{J}E_1=\mathscr{J}E_1$. Thus $H_{zz}(0)\hat{e}=\mathscr{J}\tilde{e}$ with $\tilde{e}\in E_1$ and (8.39) implies

$$(8.40)\quad (E_1,\mathscr{J}\tilde{e})\equiv[E_1,\tilde{e}]=0.$$

Since E_2 is also invariant under $\mathscr{J}H_{zz}(0)$, $[E_1,E_2]=0$ (see Moser [11]). Hence $\tilde{e}=0$ and the lemma is proved.

(8.41) **Lemma.** *If* $z\in\mathscr{N}$,

$$(8.42)\quad \int_0^{2\pi}(H_{zz}(0)z(t), z(t))\,dt=2\pi(H_{zz}(0)z(\theta), z(\theta))$$

for any $\theta\in[0,2\pi]$.

Proof. Let $H_2(z)$ denote the quadratic part of $H(z)$ at $z=0$, i.e.

$$H_2(z)=\tfrac{1}{2}(H_{zz}(0)z,z).$$

The elements of $\mathscr{N}$ are just the solutions of the Hamiltonian system corresponding to H_2:

$$(8.43)\quad \dot{z}=\mu\mathscr{J}H_{2z}$$

having initial data in E_1. Hence $H_2(z(t))$ is constant along such solutions of (8.43) from which (8.42) follows.

(8.44) *Remark.* Let E_1^+,E_1^- denote the subspaces of E_1 on which $H_{zz}(0)$ is respectively positive and negative definite. Since if $z\in\mathscr{N}$, $z(t)\in E_1$ for each $t\in\mathbb{R}$, we see from Lemma (8.41) that $(H_{zz}(0)z(t), z(t))$ is independent of t. It then follows from (8.9) that E_1^+, E_1^- are even dimensional with dimensions $2\beta, 2\gamma$ respectively. Moreover Lemma (8.37) implies $\beta+\gamma=N$. Let $\mathscr{N}^+,\mathscr{N}^-$ denote the subspaces of $\mathscr{N}$ of dimension 2β, 2γ corresponding to E_1^+, E_1^-. Note that they are equivariant.

With the observations, we can determine the behavior of $g(\lambda,\cdot)$ near 0. Let

$$(8.45)\quad H(z)\equiv H_2(z)+\hat{H}(z)$$

so $\hat{H}(z)=o(|z|^2)$ at $z=0$. From (8.15), (8.13), and (8.38) we have

$$(8.46)\quad g(\lambda,x)=\int_0^{2\pi}[(P_1x,P_2\dot{x})_{\mathbb{R}^n}-\lambda H_2(P_1x,P_2x)]\,d\tau+o(\|x\|_{\mathscr{N}}^2)$$

at $x=0$. Since x satisfies (8.43), on integrating by parts in (8.46) and using the homogeneity of H_2 we find:

$$(8.47)\quad g(\lambda,x)=\int_0^{2\pi}[\tfrac{1}{2}(P_1x,P_2\dot{x})_{\mathbb{R}^n}-\tfrac{1}{2}(P_2x,P_1\dot{x})_{\mathbb{R}^n}$$
$$-\lambda H_2(P_1x,P_2x)]\,d\tau+o(\|x\|^2_{\mathcal{N}})$$
$$=\int_0^{2\pi}\left\{\frac{\mu}{2}[(P_1x,H_{2p}(P_1x,P_2x))_{\mathbb{R}^n}+(P_2x,H_{2q}(P_1x,P_2x))_{\mathbb{R}^n}]\right.$$
$$\left.-\lambda H_2(P_1x,P_2x)\right\}d\tau+o(\|x\|^2_{\mathcal{N}})$$
$$=(\mu-\lambda)\int_0^{2\pi}H_2(P_1x,P_2x)\,d\tau+o(\|x\|^2_{\mathcal{N}})$$

at $x=0$. Thus by Lemma (8.41) and Remark (8.44), for $\lambda<\mu$, $g(\lambda,\cdot)>0$ on $\mathcal{N}^+$ and <0 on $\mathcal{N}^-$ in a deleted neighborhood of 0; if $\lambda>\mu$, these inequalities are reversed.

Theorem (8.4) is now a consequence of the following two results:

(8.48) **Theorem.** *Suppose that*

(8.49) $i(T^-)>\gamma$

(resp. (8.50) $i(T^-)>\beta$).

Then there is a $\delta>0$ such that if $\lambda\in(\mu-\delta,\mu)$ (resp. $\lambda\in(\mu,\mu+\delta)$), $g(\lambda,\cdot)$ has at least $i(T^-)-\gamma$ (resp. $i(T^-)-\beta$) positive critical values with a corresponding number of distinct critical points, $x(\lambda)$ such that $x(\lambda)\to 0$ as $\lambda\to\mu$.

(8.51) **Corollary.** *Suppose that*

(8.52) $i(T^+)>\gamma$

(resp. (8.53) $i(T^+)>\beta$).

Then there is a $\delta>0$ such that if $\lambda\in(\mu,\mu+\delta)$ (resp. $\lambda\in(\mu-\delta,\mu)$, $g(\lambda,\cdot)$ has at least $i(T^+)-\gamma$ (resp. $i(T^+)-\beta$) negative critical values with a corresponding number of distinct critical points, $x(\lambda)$, *such that* $x(\lambda)\to 0$ as $\lambda\to\mu$.

Proof of Corollary (8.51). Replace $g(\lambda,\cdot)$ by $-g(\lambda,\cdot)$. This has the effect of reversing the roles of T^+ and T^- and changing the sign of the factor $(\mu-\lambda)$ in (8.47). Hence the result obtains via Theorem (8.48).

Assuming Theorem (8.48) for now, we can finally give the:

Proof of Theorem (8.4). We assume 0 is an isolated T periodic solution of (8.1). Thus we must produce k, m, $\mathcal{I}_l$, and $\mathcal{I}_r$ as in the statement of the theorem. Since $v=\beta-\gamma\neq 0$, $\beta\neq\gamma$. Without loss of generality, we can take $\beta>\gamma$ and $i(T^-)\geqq i(T^+)$. If $i(T^-)\geqq\beta$, then (8.49) is satisfied so by Theorem (8.48) we can take $\mathcal{I}_l=(\mu-\delta,\mu)$, $\mathcal{I}_r=\emptyset$, $k=i(T^-)-\gamma\geqq v$, and $m=0$. Thus suppose $i(T^-)<\beta$. Then by Theorem (8.30), $N-i(T^+)<\beta$ or $i(T^+)>\gamma$. We claim

(8.54) $i(T^-)-\gamma+i(T^+)-\gamma\geqq v.$

Indeed by Theorem (8.30) again,

$$i(T^-)-\gamma+i(T^+)-\gamma \geqq N-2\gamma=\nu.$$

Hence by Theorem (8.48) and Corollary (8.51), we can take $\mathscr{I}_l=(\mu-\delta,\delta)$, $\mathscr{I}_r=(\mu,\mu+\delta)$, $k=i(T^-)-\gamma$, and $m=i(T^+)-\gamma$.

It remains to prove Theorem (8.48). The idea is to obtain the critical points of $g(\lambda,\cdot)$ by taking the minimax of $g(\lambda,\cdot)$ over appropriate subsets of Q. This requires several additional preliminaries. First we construct the desired subsets of Q.

Let $\mathscr{E}=\{A\in\bar{Q}\mid A$ is closed and invariant$\}$. Let $K\subset T^-$ and define $\Phi(K)=\{\psi(s,x)\mid(s,x)\in(-\infty,0)\times K\}$. Thus $\Phi(K)$ is a cone over K. Set $\mathscr{M}=\{\chi\in C(\bar{Q},\bar{Q})\mid\chi$ is $1-1$, equivariant, and $\chi(x)=x$ if $x\in T^-\}$. For $1\leqq j\leqq i(T^-)$, define $G_j=\{\chi(\Phi(K))\mid\chi\in\mathscr{M},\ K\subset T^-,\ i(K)\geqq j\}$. By Corollary (8.24), $\Phi(K)\in\mathscr{E}$. Hence $\chi(\Phi(K))\in\mathscr{E}$. Lastly define

$$\Gamma_j=\{\overline{A-W}\mid A\in G_k \text{ for some } k, j\leqq k\leqq i(T^-),\ W\in\mathscr{E}, \text{ and } i(W)\leqq k-j\}.$$

(8.55) **Lemma.** *The sets Γ_j possess the following properties:*

1° $\Gamma_{j+1}\subset\Gamma_j$, $1\leqq j<i(T^-)$.

2° If $\chi\in\mathscr{M}$ and $B\in\Gamma_j$, then $\chi(B)\in\Gamma_j$.

3° If $B\in\Gamma_j$ and $Z\in\mathscr{E}$ with $i(Z)\leqq m<j$, then $\overline{B-Z}\in\Gamma_{j-m}$.

Proof. 1° is trivial. Let $B=\overline{A-W}$ as in the definition of Γ_j. Then $\chi(\overline{A-W})=\overline{\chi(A-W)}=\overline{\chi(A)-\chi(W)}$. Since $A\in G_k$ implies $\chi(A)\in G_k$ and $i(\chi(W))=i(W)$ by (6.4), $\chi(B)\in\Gamma_j$ and 2° is verified. To check 3°, again let $B=\overline{A-W}$. Then $\overline{B-Z}=\overline{\overline{A-W}-Z}=\overline{A-(W\cup Z)}$. Since $A\in G_k$ and $i(W\cup Z)\leqq k-j+m=k-(j-m)$ by (6.6), $\overline{B-Z}\in\Gamma_{j-m}$.

With the aid of these sets, we define

$$\text{(8.56)}\quad c_j(\lambda)=\inf_{B\in\Gamma_j}\max_{x\in B} g(\lambda,x),\qquad 1\leqq j\leqq i(T^-).$$

We will show that an appropriate subset of these numbers provides us with the critical values whose existence was asserted in Theorem (8.48).

(8.57) **Lemma.** *If $i(T^-)>\gamma$ and $0<\mu-\lambda$ is small, then $c_j(\lambda)>0$ for $\gamma<j\leqq i(T^-)$.*

Proof. By 1° of Lemma (8.55), $c_j\leqq c_{j+1}$. Thus it suffices to show $c_{\gamma+1}(\lambda)>0$. For ρ sufficiently small and $x\in\partial B_\rho\cap\mathscr{N}^+$, it follows from (8.47) that

$$\text{(8.58)}\quad g(\lambda,x)\geqq a(\mu-\lambda)\rho^2$$

where a is a constant independent of ρ. (In fact a is a multiple of the smallest positive eigenvalue of $(H_{zz}(0)\zeta,\zeta)$ for $\zeta\in E_1$.) Since $\dim\mathscr{N}^+=2\beta$, by (7.7), $i(\partial B_\rho\cap\mathscr{N}^+)=\beta$. Let $B\in\Gamma_{\gamma+1}$ so $B=\overline{\chi(\Phi(K))-W}$ with $K\in T^-$, $i(K)=m\geqq\gamma+1$, and $i(W)\leqq m-(\gamma+1)$. For $s\geqq\omega$ depending on χ and K, $\chi(\psi(-s,K))\subset B_\rho$. By the

Piercing Property (6.9),

(8.59) $i(\chi(\psi([-\omega, 0] \times K)) \cap \partial B_\rho) = i(K) = m.$

Therefore by (6.4) and (6.5)

(8.60) $$i(B \cap \partial B_\rho) = i(\overline{(\chi(\Phi(K)) \cap \partial B_\rho) - W}) \geqq i(\chi(\overline{\Phi(K)}) \cap \partial B_\rho) - i(W) \geqq m - m + \gamma + 1 = \gamma + 1.$$

Corollary (7.7) now implies

(8.61) $B \cap \partial B_\rho \cap \mathcal{N}^+ \neq \emptyset.$

Let $\xi \in B \cap \partial B_\rho \cap \mathcal{N}^+$. By (8.58),

(8.62) $$\max_{x \in B} g(\lambda, x) \geqq g(\lambda, \xi) \geqq \min_{x \in \partial B_\rho \cap \mathcal{N}^+} g(\lambda, x) \geqq a(\mu - \lambda)\, \rho^2.$$

Since (8.61)–(8.62) are valid for all $B \in \Gamma_{\gamma+1}$, it follows that

(8.63) $c_{\gamma+1} \geqq a(\mu - \lambda)\, \rho^2.$

(8.64) **Corollary.** *If* $i(T^-) > \beta$ *and* $0 < \lambda - \mu$ *is small, then* $c_j(\lambda) > 0$ *for* $\beta < j \leqq i(T^-)$.

Proof. Same as that of Lemma (8.57) with $\mathcal{N}^+$ replaced by $\mathcal{N}^-$.

To show that the c_j's of Lemma (8.57) are critical values of $g(\lambda, \cdot)$, requires a variant of a standard result from the calculus of variations. Let $A_{\lambda b} = \{x \in \bar{Q} \mid g(\lambda, x) \leqq b\}$ and $K_{\lambda b} = \{x \in \bar{Q} \mid g(\lambda, x) = b,\ g_x(\lambda, x) = 0\}$.

(8.65) **Lemma.** *If* λ *is near* μ, $b \in (0, c)$, $\bar{\varepsilon} > 0$, *and* U *is any neighborhood of* $K_{\lambda b}$, *then there exists an* $\varepsilon \in (0, \bar{\varepsilon})$ *and* $\eta \in C([0, 1] \times \bar{Q}, \bar{Q})$ *such that*

1° $\eta(s, \cdot)\colon \mathscr{E} \to \mathscr{E}$, $s \in [0, 1]$,

2° $\eta(s, x) = x$ if $g(\lambda, x) \in [b - \bar{\varepsilon}, b + \bar{\varepsilon}]$,

3° $\eta(s, x)$ is a homeomorphism from $\bar{Q}$ to $\bar{Q}$ for all $s \in [0, 1]$,

4° $\eta(1, A_{\lambda, b+\varepsilon} - U) \subset A_{\lambda, b-\varepsilon}$,

5° If $K_{\lambda, b} = \emptyset$, $\eta(1, A_{\lambda, b+\varepsilon}) \subset A_{\lambda, b-\varepsilon}$.

Proof. A proof of Lemma (8.65) (without 1°) for the case in which Q is a real Banach space can be found in [30] or [31]. Thus we merely indicate the modifications required here to employ the earlier proofs. To satisfy 1°, it suffices to obtain η as the solution of an ordinary differential equation of the form (8.22) where V is a locally Lipschitz continuous map of $\mathcal{N}$ to $\mathcal{N}$ and is equivariant. Let $w\colon \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $w(r) = 1$ if $0 \leqq r \leqq \rho$, $w(r) = 0$ if $r \geqq 2\rho$, and $w(r)$ is linear between ρ and 2ρ. For the moment, ρ is free. Define $d(x) = \|x - \partial Q\|_{\mathcal{N}}$. Then $d(L_\theta x) = d(x)$ since $\hat{x} \in \partial Q$ implies $L_\theta \hat{x} \in \partial Q$. Set $\varphi(x) = w(d(x))$. Thus φ is equivariant and Lipschitz continuous in $\bar{Q}$ as is $\hat{V}(x) = -\varphi(x)\, g_x(\mu, x) - (1 - \varphi(x))\, g_x(\lambda, x)$ via Remark (8.27).

A vector field Ψ on $\tilde{Q}=\bar{Q}-\{x\in\bar{Q}|v(x)=0\}$ is called a pseudogradient vector field for $v(x)$ if Ψ is locally Lipschitz continuous in $\tilde{Q}$ and

$$\begin{aligned}&\|\Psi(x)\|\leqq 2\|v(x)\|\\&(\Psi(x), v(x))\geqq\|v(x)\|^2\end{aligned} \tag{8.66}$$

for all $x\in\tilde{Q}$. Since $g_x(\mu, x)$ has no critical points near ∂Q, neither does $g_x(\lambda, x)$ for λ near μ. Hence for ρ sufficiently small, if $\hat{V}$ is appropriately scaled in $\{x\in\bar{Q}\,|\,d(x)\leq 2\rho\}$, $\hat{V}$ is a pseudogradient vector field for $-g_x(\lambda, x)$. Multiplication of $\hat{V}$ by another scalar Lipschitz continuous equivariant function as in [30] or [31] produces a V for which the corresponding flow satisfies 1°–5°.

(8.67) **Lemma.** *Under the hypotheses of Lemma* (8.57), $c_j(\lambda)$ *is a critical value of* $g(\lambda,\cdot)$, $\gamma<j\leqq i(T^-)$. *Moreover if* $c_j=\cdots=c_{j+r-1}\equiv b$, $i(K_{\lambda b})\geqq r$.

Proof. It suffices to prove the second assertion. Clearly $K_{\lambda b}\in\mathscr{E}$. If $i(K_{\lambda b})<r$, by (6.5), there is a neighborhood U of $K_{\lambda b}$ such that $i(U)<r$. Choose $\bar{\varepsilon}=\frac{1}{2}b$ in Lemma (8.65). By that lemma with the above choice of U, there is an $\varepsilon\in(0,\bar{\varepsilon})$ and an $\eta\in C([0,1]\times\bar{Q},\bar{Q})$ such that

$$\eta(1, A_{\lambda, b+\varepsilon}-U)\subset A_{\lambda, b-\varepsilon}. \tag{8.68}$$

Choose $B\in\Gamma_{j+r-1}$ so that

$$\max_{x\in B} g(\lambda, x)\leqq b+\varepsilon=c_{j+r-1}+\varepsilon. \tag{8.69}$$

By 3° of Lemma (8.55), $\overline{B-U}\in\Gamma_j$. If λ is close enough to μ so that $g(\lambda, x)<0$ for $x\in T^-$, by 1°–3° of Lemma (8.57), $\eta(1,\cdot)\in\mathscr{M}$. Hence by 2° of Lemma (8.55), $\eta(1,\overline{B-U})\in\Gamma_j$. Therefore

$$\max_{x\in\eta(1,\overline{B-U})} g(\lambda, x)\geqq b=c_j \tag{8.70}$$

which contradicts (8.68)–(8.69).

(8.71) *Remark.* A similar argument shows the c_j's of Corollary (8.64) are also critical values of $g(\lambda,\cdot)$ with a corresponding multiplicity statement. Observe also that if $i(K_{\lambda b})>1$, by Remark (6.16), $K_{\lambda b}$ contains infinitely many distinct critical points.

(8.72) **Lemma.** *Under the hypotheses of Lemma* (8.57) *for* $\gamma<j\leqq i(T^-)$, *let* $x_j(\lambda)\in Q$ *be a critical point of* $g(\lambda,\cdot)$ *corresponding to* $c_j(\lambda)$. *Then* $x_j(\lambda)\to 0$ *as* $\lambda\to\mu^-$.

Proof. Observe that $\overline{\Phi(T^-)}\in\Gamma_j$, $1\leqq j\leqq i(T^-)$ and if $x\in\Phi(T^-)$, $g(\mu, x)<0$. Since $0\in\overline{\Phi(T^-)}$,

$$\max_{x\in\overline{\Phi(T^-)}} g(\mu, x)=0$$

Moreover since $g(\lambda, x) \to g(\mu, x)$ uniformly for $x \in \bar{Q}$ as $\lambda \to \mu$,

$$(8.73) \quad 0 < c_j(\lambda) \leqq \max_{x \in \overline{\Phi(T^-)}} g(\lambda, x) \to 0$$

as $\lambda \to \mu^-$. Therefore along a subsequences of λ's converging to μ^-, we have $x_j(\lambda) \to x \in \bar{Q}$ with $g(\mu, x) = 0$ and $g_x(\mu, x) = 0$. Since 0 is the unique critical point of $g(\mu, \cdot)$ in $\bar{Q}$, the result follows.

Proof of Theorem (8.48). Immediate from Lemma (8.67), Remark (8.71), and Lemma (8.72).

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