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Triple Collision in the Collinear Three-Body Problem*

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1. Introduction

Consider n point masses moving in k-dimensional space according to the laws of classical mechanics. If particle i has mass $m_i > 0$ and position $q_i \in R^k$, then the negative potential energy is given by

$$U = \sum_{i<j} \frac{m_i m_j}{\|q_i - q_j\|}, \tag{1.1}$$

where $\| \ \|$ denotes the Euclidean norm in R^k. The motion of the particles is described by the system of differential equations

$$m_i \ddot{q}_i = \nabla_{q_i} U, \quad i = 1, 2, \ldots, n, \tag{1.2}$$

where $\nabla_{q_i} U$ is the gradient of U with respect to q_i.

A position $(q_1, \ldots, q_n)$ of the particles will be called a collision if $q_i = q_j$ for some $i \neq j$. The above system of equations is defined everywhere except at collisions. Suppose we are given the position and momentum of the particles at time $t=0$. If we do not start at a collision, then the standard theorems of differential equations assure the existence and uniqueness of a solution of Eqs. (1.2) on some maximal interval $[0, t^*)$. If $t^* < \infty$, then the solution is said to experience a singularity at t^*.

The behavior of a solution as it approaches a singularity is not fully understood, but some of the possibilities are known. If all of the particles approach a limiting position as $t \to t^*$, it is not difficult to show that the limiting position must be a collision [12, 17]. The singularity is then said to be due to collision and the solution is said to end in collision. If m of the particles coincide while the rest have distinct positions, then the collision is called an m-tuple collision. It is unknown whether there are singularities not due to collision.

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For the two-body problem one can change variables so that a double collision transforms to a regular point of the equations [6]. Such a transformation is called a regularization of the double collision. The solution can then be extended through the singularity. The extension corresponds physically to an elastic bounce.

Sundman [14] showed that double collisions can also be regularized in the three-body problem. That is, one can transform the variables in such a way that the solution can be continued through the double collision as an analytic function of a new time variable. Again the extension corresponds to an elastic bounce.

Collisions involving more than two particles are of course more complicated, but some aspects of their behavior are known. We define the configuration of the particles to be the position divided by a norm which corresponds physically to the moment of inertia. Sundman [15] has shown that, for triple collision in the three-body problem, the configuration approaches one of the five so-called central configurations (cf. [12, 17], and Sections 6 and 7 below). Wintner [17] has observed that Sundman's techniques can be used to show that solutions ending in n-tuple collision in the n-body problem also approach central configurations. However, little is known about central configurations for $n \geqq 4$.

Since double collisions can be regularized, one is led to ask whether the same can be done to other singularities. Siegel [11] has addressed this question for triple collision in the three-body problem. He found that most solutions cannot be extended through triple collision as analytic functions of some transformed time variable. He also showed that the set of orbits ending in triple collision forms a smooth submanifold of the phase space.

In this paper we consider the singularity due to triple collision. We wish to describe not only the solutions with such a singularity but also the solutions close to the singular ones. The simplest case of triple collision occurs in the collinear three-body problem. In this case $n=3$ and $k=1$, so a position (q_1, q_2, q_3) of the particles is a point in R^3. Fixing the center of mass at the origin confines the position to the plane $\mathbf{Q}$ determined by $m_1 q_1 + m_2 q_2 + m_3 q_3 = 0$. The momentum is also confined to a plane and Eqs. (1.2) can be made into a four-dimensional first order system. Conservation of energy further reduces the system to a vectorfield on a three-dimensional constant energy surface.

Triple collision corresponds to the origin in $\mathbf{Q}$. We shall make a transformation which blows up the origin to a circle. With an appropriate change of coordinates in the momentum space this transformation has the effect of pasting a two-dimensional boundary, which we shall call the "triple collision manifold", onto the constant energy surface. A time transformation scales the vectorfield so that it can be extended to the boundary. The boundary is invariant for the extended vectorfield. All points on this boundary correspond to triple collision in the original coordinates. The flow on the triple collision manifold is entirely ficticious since orbits on it do not correspond to any orbits in the original coordinates. However, the flow on the entire constant energy surface, including the ficticious boundary, is continuous. Hence the flow close to the boundary follows the flow on the boundary for an arbitrarily long time. Therefore the behavior of orbits on the triple collision manifold can be used to determine the behavior of orbits close to the manifold, *i.e.* close to triple collision.

The triple collision manifold is shown in Fig. 2. The flow on it is gradient-like and has two rest points, both saddles. All orbits ending in triple collision are asymptotic to the triple collision manifold.

As examples of how the properties of the triple collision manifold can be exploited, we shall give new proofs of two known results. The first is the previously noted result of Sundman that triple collision orbits approach a central configuration. We shall show that a triple collision orbit must be asymptotic to one of the two rest points. This property will be shown to imply Sundman's result in the collinear case. The second result, due to Siegel, is that the set of orbits ending in triple collision forms a smooth submanifold of the constant energy surface. We shall show for the collinear case that this result follows from the stable manifold theorem applied to the two rest points.

We shall also use the properties of the triple collision manifold to examine the question of whether orbits can be extended through triple collision. We shall adopt the viewpoint of Easton [4, 5], rather than that of Sundman and Siegel. Whereas Sundman and Siegel ask if a single solution can be extended as an analytic function of time, Easton asks if it can be extended so as to be continuous with respect to nearby solutions. By examining the flow on the triple collision manifold, we shall show that triple collision cannot be "regularized" in the sense of Easton, at least for some values of the masses.

Finally, the flow on the triple collision manifold will be used to show that the following property holds for some values of the masses: After passing close to triple collision the system emerges with arbitrarily high kinetic energy. That is, one of the particles emerges with an arbitrarily large velocity in one direction, while the other two particles are close together and moving in the opposite direction with large velocity. This behavior is rather surprising and may have some bearing on the question of whether there exist singularities not due to collision in the n-body problem. Painlevé [8] showed that such singularities cannot occur in the three-body problem, but the question is open for $n \geqq 4$. We shall speculate on this question for $n=5$.

2. Preliminaries

We begin by writing the equations of motion in Hamiltonian form. Let $\mathbf{q}=(q_1, q_2, q_3)\in R^3$ and let

$$M=\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}.$$

For the collinear three-body problem, Eqs. (1.1) and (1.2) become

$$\begin{aligned} U(q) &= \frac{m_1 m_2}{|q_1-q_2|}+\frac{m_1 m_3}{|q_1-q_3|}+\frac{m_2 m_3}{|q_2-q_3|} \\ M\ddot{\mathbf{q}} &= \nabla U(\mathbf{q}) \end{aligned} \tag{2.1}$$

Let $p_i=m_i \dot{q}_i$ be the momentum of particle i and let $\mathbf{p}=(p_1, p_2, p_3)\in R^3$. We define the kinetic energy of the system as

$$T(\mathbf{p})=\tfrac{1}{2}\left(\frac{p_1^2}{m_1}+\frac{p_2^2}{m_2}+\frac{p_3^2}{m_3}\right)=\tfrac{1}{2}\mathbf{p}^T M^{-1}\mathbf{p}.$$

If we now define the Hamiltonian

$$H(\mathbf{q},\mathbf{p})=T(\mathbf{p})-U(\mathbf{q}), \tag{2.2}$$

Eq. (2.1) can be written:

$$\begin{aligned} \dot{\mathbf{q}} &= H_p(\mathbf{q},\mathbf{p})=\nabla T(\mathbf{p})=M^{-1}\mathbf{p} \\ \dot{\mathbf{p}} &= -H_q(\mathbf{q},\mathbf{p})=\nabla U(\mathbf{q}). \end{aligned} \tag{2.3}$$

The function T is defined everywhere on R^3. The function U is defined everywhere except at collision points. Let

$$\Delta=\{\mathbf{q}\in R^3\colon q_1=q_2,\ q_2=q_3,\ \text{or}\ q_3=q_1\}$$

denote the set of collision points. Eqs. (2.3) define a vectorfield on $(R^3-\Delta)\times R^3$. We shall use the phrase "vectorfield with singularities" to describe a vectorfield which is undefined at some points. Thus we shall say that Eqs. (2.3) define a vectorfield with singularities on $R^3\times R^3$.

We can reduce the dimension of system (2.3) by removing the center of mass and the linear momentum. Fixing the center of mass at the origin restricts the position coordinates to the linear subspace

$$\mathbf{Q}=\{\mathbf{q}\in R^3\colon m_1 q_1+m_2 q_2+m_3 q_3=0\}$$

(see Fig. 1) and the momentum coordinates to the subspace

$$\mathbf{P}=\{\mathbf{p}\in R^3\colon p_1+p_2+p_3=0\}.$$

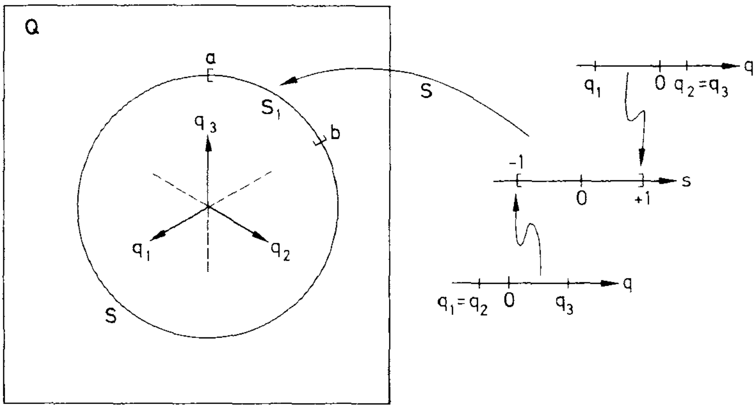

Fig. 1. Position coordinates

Eqs. (2.3) determine a vectorfield with singularities on $\mathbf{Q}\times\mathbf{P}$, a four-dimensional linear space. They determine a vectorfield on $(\mathbf{Q}-\Delta)\times\mathbf{P}$.

Since the Hamiltonian (2.2) is constant along solutions of (2.3), it defines an invariant set

$$\mathbf{M}(h)=\{(\mathbf{q},\mathbf{p})\in(\mathbf{Q}-\Delta)\times\mathbf{P}\colon H(\mathbf{q},\mathbf{p})=h\} \tag{2.4}$$

for each real constant h. This set is a three-dimensional manifold, which we shall call a constant energy surface. Thus Eqs. (2.3) define a vectorfield on $\mathbf{M}(h)$.

However, this vectorfield is not complete, i.e. solutions do not exist for all time. In finite time some solutions tend to collision and hence leave $\mathbf{M}(h)$. Solutions which end in double collision will be extended by a technique similar to Sundman's [14]. Solutions which end in triple collision will be slowed down so that they approach collision in infinite time.

3. The Singularities Due to Triple Collision

We first examine the singularities at $\mathbf{q}=0$. Define

$$r=(m_1 q_1^2+m_2 q_2^2+m_3 q_3^2)^{\frac{1}{2}}=(\mathbf{q}^T M\mathbf{q})^{\frac{1}{2}}.$$

Note that r^2 is the moment of inertia of the system of particles and that triple collision corresponds to $r=0$. Let

$$\mathbf{S}=\{\mathbf{q}\in\mathbf{Q}\colon r^2=\mathbf{q}^T M\mathbf{q}=1\}$$

be the unit circle in $\mathbf{Q}$ in the norm given by the moment of inertia. A point on $\mathbf{S}$ is called a configuration for the system of particles. We think of $(r,\mathbf{s})\in(0,\infty)\times\mathbf{S}$ as polar coordinates on $\mathbf{Q}-\{0\}$ by the map: $(r,\mathbf{s})\mapsto r\,\mathbf{s}$.

We now define the variables:

$$r=(\mathbf{q}^T M\mathbf{q})^{\frac{1}{2}},$$

$$\mathbf{s}=r^{-1}\mathbf{q},$$

$$y=\mathbf{p}^T\mathbf{s},$$

$$\mathbf{x}=\mathbf{p}-yM\mathbf{s}.$$

Note that $\mathbf{s}\in\mathbf{S}$ and that $\mathbf{x}^T\mathbf{s}=0$. Thus we have broken the momentum $\mathbf{p}$ into a radial component y and a tangential component $\mathbf{x}$. We now let

$$\mathbf{T}=\{(\mathbf{q},\mathbf{p})\in\mathbf{Q}\times\mathbf{P}\colon \mathbf{q}\in\mathbf{S}, \mathbf{p}^T\mathbf{q}=0\},$$

which can be thought of as the tangent bundle of $\mathbf{S}$. We then have $r\in(0,\infty)$, $y\in R^1$, and $(\mathbf{s},\mathbf{x})\in\mathbf{T}$. Note that the old variables can be written in terms of the new variables:

$$\mathbf{q}=r\mathbf{s}$$

$$\mathbf{p}=\mathbf{x}+yM\mathbf{s}.$$

Thus we have defined a real analytic diffeomorphism:

$$\begin{gathered}(0,\infty)\times R^1\times\mathbf{T}\to(\mathbf{Q}-\{0\})\times\mathbf{P}\colon\\ (r,y,(\mathbf{s},\mathbf{x}))\mapsto(r\mathbf{s},\mathbf{x}+yM\mathbf{s}).\end{gathered}\tag{3.1}$$

In these new coordinates the kinetic energy can be written:

$$T(\mathbf{p})=\tfrac{1}{2}(\mathbf{x}^T M^{-1}\mathbf{x}+y^2),$$

while the potential energy becomes:

$$U(\mathbf{q})=r^{-1}U(\mathbf{s}).$$

Thus the energy relation $H(\mathbf{q},\mathbf{p})=h$ can be written

$$\tfrac{1}{2}(\mathbf{x}^T M^{-1}\mathbf{x}+y^2)-r^{-1}U(\mathbf{s})=h.\tag{3.2}$$

The equations of motion (2.3) become:

$$\begin{aligned}\dot r&=y\\ \dot y&=r^{-1}\mathbf{x}^T M^{-1}\mathbf{x}-r^{-2}U(\mathbf{s})\\ \dot{\mathbf{s}}&=r^{-1}M^{-1}\mathbf{x}\\ \dot{\mathbf{x}}&=-r^{-1}y\mathbf{x}-r^{-1}(\mathbf{x}^T M^{-1}\mathbf{x})M\mathbf{s}+r^{-2}U(\mathbf{s})M\mathbf{s}+r^{-2}\nabla U(\mathbf{s}).\end{aligned}\tag{3.3}$$

The computation to derive these equations is straightforward if one notes that U is homogeneous of degree -1. Hence ∇U is homogeneous of degree -2 and Euler's formula implies

$$\mathbf{q}^T\nabla U(\mathbf{q})=-U(\mathbf{q}).$$

Eqs. (3.3) define a vectorfield with singularities on $[0, \infty) \times R^1 \times \mathbf{T}$. We have now expanded the singularities due to triple collision. Whereas for Eqs. (2.3) the set of triple collision points was $\{0\} \times \mathbf{P}$, the set of triple collision points for Eqs. (3.3) is now $\{0\} \times R^1 \times \mathbf{T}$, i.e. the set where $r=0$. The set of double collision points is $(0, \infty) \times R^1 \times (\mathbf{T} \cap (\Delta \times \mathbf{P}))$, i.e. the set where $\mathbf{s}=(s_1, s_2, s_3)$ satisfies $s_1=s_2$, $s_2=s_3$, or $s_3=s_1$. We shall deal with double collisions in Section 5, but first we wish to remove the singularities at $r=0$.

We again introduce new variables:

$$\mathbf{u}=r^{\frac{1}{2}}\mathbf{x},$$
$$v=r^{\frac{1}{2}}y.$$

That is, we define a real analytic diffeomorphism:

$$\begin{gathered}(0, \infty) \times R^1 \times \mathbf{T} \to (0, \infty) \times R^1 \times \mathbf{T}: \\ (r, v, (\mathbf{s}, \mathbf{u})) \mapsto (r, r^{-\frac{1}{2}} v, (\mathbf{s}, r^{-\frac{1}{2}}\mathbf{u})).\end{gathered} \tag{3.4}$$

Then the energy relation (3.2) can be written:

$$\tfrac{1}{2}(\mathbf{u}^T M^{-1}\mathbf{u}+v^2)-U(\mathbf{s})=r h, \tag{3.5}$$

and the equations of motion (3.3) become

$$\begin{aligned}\dot{r}&=r^{-\frac{1}{2}} v\\ \dot{v}&=r^{-\frac{3}{2}}[\tfrac{1}{2}v^2+\mathbf{u}^T M^{-1}\mathbf{u}-U(\mathbf{s})]\\ \dot{\mathbf{s}}&=r^{-\frac{3}{2}} M^{-1}\mathbf{u}\\ \dot{\mathbf{u}}&=r^{-\frac{3}{2}}[-\tfrac{1}{2}v\mathbf{u}-(\mathbf{u}^T M^{-1}\mathbf{u}) M\mathbf{s}+U(\mathbf{s}) M\mathbf{s}+\nabla U(\mathbf{s})].\end{aligned} \tag{3.6}$$

As before, these equations define a vectorfield with singularities on $[0, \infty) \times R^1 \times \mathbf{T}$. The double and triple collision points are exactly the same as those for Eqs. (3.3). However, now we can remove the singularities at $r=0$ by scaling the vectorfield with the time transformation:

$$dt=r^{\frac{3}{2}}\, dt'. \tag{3.7}$$

Eqs. (3.6) then become:

$$\begin{aligned}\frac{dr}{dt'}&=r v\\ \frac{dv}{dt'}&=\tfrac{1}{2}v^2+\mathbf{u}^T M^{-1}\mathbf{u}-U(\mathbf{s})\\ \frac{d\mathbf{s}}{dt'}&=M^{-1}\mathbf{u}\\ \frac{d\mathbf{u}}{dt'}&=-\tfrac{1}{2}v\mathbf{u}-(\mathbf{u}^T M^{-1}\mathbf{u}) M\mathbf{s}+U(s) M\mathbf{s}+\nabla U(\mathbf{s}).\end{aligned} \tag{3.8}$$

Since the above equations do not have singularities at $r=0$ we have extended the equations of motion to include triple collision. Note that $\{r=0\}$ is invariant for Eqs. (3.8). Time transformation (3.7) acts to slow down the orbits for small r so that a solution ending in a triple collision now takes an infinite amount of time to reach it. The set of orbits ending in triple collision is now the set of orbits asymptotic to the invariant set $\{r=0\}$. Also, orbits on the invariant set $\{r=0\}$ can be used to describe orbits of (3.8) for small r, i.e. orbits passing close to triple collision.

4. Coordinates

Before discussing the invariant set introduced at triple collision, we wish to eliminate the singularities due to double collisions. In the next section we shall extend orbits through double collision by an elastic bounce. To facilitate the computations we first construct a new coordinate system, in which Eqs. (3.8) transform to a vectorfield on R^4.

If we begin with the particles ordered on the line, $q_1<q_2<q_3$, they will retain that ordering after any double collision. Hence the ordering is preserved along any orbit. Let

$$\mathbf{S}_0=\{\mathbf{s}\in\mathbf{S}:\ s_1<s_2<s_3\},$$

$$\mathbf{S}_1=\{\mathbf{s}\in\mathbf{S}:\ s_1\leqq s_2\leqq s_3\},$$

$$\mathbf{T}_0=\{(\mathbf{s},\mathbf{u})\in\mathbf{T}:\ \mathbf{s}\in\mathbf{S}_0\},$$

$$\mathbf{T}_1=\{(\mathbf{s},\mathbf{u})\in\mathbf{T}:\ \mathbf{s}\in\mathbf{S}_1\}.$$

Also let $\mathbf{a}=(a_1,a_2,a_3)$ and $\mathbf{b}=(b_1,b_2,b_3)$ be the unique points on $\mathbf{S}$ with $a_1=a_2<a_3$ and $b_1<b_2=b_3$. We see that $\mathbf{S}_1$ is a closed interval on $\mathbf{S}$ with endpoints $\mathbf{a}$ and $\mathbf{b}$, while $\mathbf{S}_0$ is the corresponding open interval. The endpoints correspond to double collisions. (See Fig. 1.)

Note that $\mathbf{T}_1$ is homeomorphic to a compact interval crossed with the real line. Since it is more convenient to work with a subset of R^2 than with a subset of R^6, we next define a diffeomorphism between $[-1,1]\times R^1$ and $\mathbf{T}_1$. Let

$$A_1=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix},\qquad A_2=\begin{bmatrix}0&1&-1\\-1&0&1\\1&-1&0\end{bmatrix},$$

and define

$$A=\frac{1}{m_1+m_2+m_3}A_1M+\left(\frac{m_1m_2m_3}{m_1+m_2+m_3}\right)^{\frac{1}{2}}M^{-1}A_2.$$

Some elementary calculations show that A has the following properties:

$$A^TMA=M, \tag{4.1}$$

$$A:\ \mathbf{Q}\to\mathbf{Q}, \tag{4.2}$$

$$\mathbf{q}^T M A \mathbf{q} = 0 \quad \text{if } \mathbf{q} \in \mathbf{Q}, \tag{4.3}$$

$$A^2 \mathbf{q} = -\mathbf{q} \quad \text{if } \mathbf{q} \in \mathbf{Q}, \tag{4.4}$$

$$\mathbf{a}^T A^T M \mathbf{b} > 0. \tag{4.5}$$

If we consider $\mathbf{Q}$ as having an inner product induced by M, then A is a rotation by 90° in $\mathbf{Q}$, $\mathbf{a}$ and $\mathbf{b}$ have unit length, and $\{\mathbf{a}, A\mathbf{a}\}$ is an orthonormal basis for $\mathbf{Q}$. Therefore we have that

$$\mathbf{b} = (\mathbf{a}^T M \mathbf{b})\, \mathbf{a} + (\mathbf{a}^T A^T M \mathbf{b})\, A \mathbf{a}. \tag{4.6}$$

Note that $\mathbf{a}^T M \mathbf{b}$ is a constant depending only on the masses and that

$$0 < \mathbf{a}^T M \mathbf{b} < 1.$$

Choose λ to be the smallest positive number such that

$$\cos 2\lambda = \mathbf{a}^T M \mathbf{b}. \tag{4.7}$$

Note that λ is a constant depending only on the masses m_1, m_2, and m_3. Eqs. (4.6) and property (4.5) then give that

$$\mathbf{b} = (\cos 2\lambda)\, \mathbf{a} + (\sin 2\lambda)\, A \mathbf{a}. \tag{4.8}$$

Now for any real s define

$$S(s) = (\sin 2\lambda)^{-1} \left[(\sin \lambda(1-s))\, \mathbf{a} + (\sin \lambda(1+s))\, \mathbf{b}\right]. \tag{4.9}$$

Proposition 4.1. S: $[-1, 1] \to \mathbf{S}_1$ *is a real analytic diffeomorphism such that*

$$S'(s) = \lambda A\, S(s). \tag{4.10}$$

Proof. Clearly S is real analytic, $S(-1) = \mathbf{a}$, and $S(1) = \mathbf{b}$. Since $\mathbf{a}, \mathbf{b} \in \mathbf{Q}$, $S(s) \in \mathbf{Q}$ for all real s. Eq. (4.8) and some trigonometric manipulation give

$$S(s) = (\cos \lambda(1+s))\, \mathbf{a} + (\sin \lambda(1+s))\, A \mathbf{a}. \tag{4.11}$$

Properties (4.1) and (4.3) then yield $S(s)^T M S(s) = 1$ and hence $S(s) \in \mathbf{S}$ for all s. Now using property (4.4) and formula (4.11) we can derive Eq. (4.10). We have only left to show that S: $[-1, 1] \to \mathbf{S}_1$ is one-to-one, since Eq. (4.10) and the inverse function theorem then imply that S is a real analytic diffeomorphism. Let

$$S(s) = (S_1(s), S_2(s), S_3(s)).$$

We then have

$$\begin{aligned} S_2(s) - S_1(s) &= (\sin 2\lambda)^{-1} (b_2 - b_1) \sin \lambda(1+s) \\ S_3(s) - S_2(s) &= (\sin 2\lambda)^{-1} (a_3 - a_2) \sin \lambda(1-s) \\ S_3(s) - S_1(s) &= (\sin 2\lambda)^{-1} [(b_2 - b_1) \sin \lambda(1+s) + (a_3 - a_2) \sin \lambda(1-s)]. \end{aligned} \tag{4.12}$$

Since $0<\lambda<\frac{\pi}{4}$ and $s\in[-1,1]$, $\sin\lambda(1+s)$ and $\sin\lambda(1-s)$ have inverses and all three of the above numbers are positive. Therefore $S\colon[-1,1]\to\mathbf{S}$ is one-to-one and $S(s)\in\mathbf{S}_1$ for $s\in[-1,1]$. The proof of Proposition 4.1 is complete.

We now use the function S to define new variables for $(\mathbf{s},\mathbf{u})\in T_1$. Let

$$s=S^{-1}(\mathbf{s}),$$
$$u=\mathbf{s}^T A^T \mathbf{u}.$$

Then $s\in[-1,1]$ and $u\in R^1$. Letting

$$\mathfrak{R}=[0,\infty)\times R^1\times[-1,1]\times R^1$$

we have defined a real analytic diffeomorphism

$$\mathfrak{R}\to[0,\infty)\times R^1\times\mathbf{T}_1\colon\ (r,v,s,u)\mapsto(r,v,(S(s),uMAS(s))). \tag{4.13}$$

Note that this map restricted to $[0,\infty)\times R^1\times(-1,1)\times R^1$ is a diffeomorphism onto $[0,\infty)\times R^1\times\mathbf{T}_0$.

Next we transform the vectorfield given by Eqs. (3.8). Let

$$V\colon(-1,1)\to R^1\colon s\mapsto U(S(s)).$$

For future reference we use Eqs. (4.12) to explicitly write:

$$\begin{aligned}V(s)=\sin 2\lambda\Bigg[&\frac{m_1 m_2}{(b_2-b_1)\sin\lambda(1+s)}+\frac{m_2 m_3}{(a_3-a_2)\sin\lambda(1-s)}\\&+\frac{m_1 m_3}{(b_2-b_1)\sin\lambda(1+s)+(a_3-a_2)\sin\lambda(1-s)}\Bigg].\end{aligned} \tag{4.14}$$

Eq. (4.10) enables us to compute the derivative of this map:

$$V'(s)=\lambda DU(S(s))\,AS(s). \tag{4.15}$$

In the new variables defined by transformation (4.13) the energy relation (3.5) becomes

$$\tfrac{1}{2}(u^2+v^2)-V(s)=rh, \tag{4.16}$$

while the equations of motion (3.8) become

$$\begin{aligned}\frac{dr}{dt'}&=rv\\ \frac{dv}{dt'}&=\tfrac{1}{2}v^2+u^2-V(s)\end{aligned} \tag{4.17}$$

$$\frac{ds}{dt'}=\lambda^{-1}u$$
$$\frac{du}{dt'}=-\tfrac{1}{2}vu+\lambda^{-1}V'(s). \tag{4.17}$$

The above equations define a vectorfield with singularities on $\mathfrak{R}$. The singularities occur when $s=\pm 1$. The vectorfield is diffeomorphic by transformation (4.13) to the vectorfield on $[0,\infty)\times R^1\times \mathbf{T}_1$ given by Eqs. (3.8). Thus $\{r=0\}$ corresponds to triple collision while $\{s=\pm 1\}$ corresponds to double collision. Note that we have restricted attention to the case $q_1 \leqq q_2 \leqq q_3$. Therefore the only double collisions which can occur are between particles 1 and 2 ($s=-1$) or between particles 2 and 3 ($s=+1$). In the next section we extend orbits through double collision by transforming Eqs. (4.17) to a vectorfield without singularities on $\mathfrak{R}$.

5. Regularization of Double Collisions

It is well-known that orbits can be extended through double collisions even for the three-body problem in three dimensions. Sundman [14] gives an analytic technique for such an extension. Easton [4, 5] gives a topological technique which he uses to describe the regularized energy manifolds in the planar three-body problem. Here we use a transformation similar to Sundman's to globally regularize all double collisions on an energy manifold. The regularization corresponds physically to an elastic bounce.

For $s\in(-1,1)$, define

$$W(s)=2(1-s^2)\,V(s). \tag{5.1}$$

Using Eq. (4.14) we can rewrite this function:

$$W(s)=2\lambda^{-1}\sin 2\lambda\,[W_1(s)+W_2(s)+W_3(s)], \tag{5.2}$$

where

$$W_1(s)=\frac{m_1 m_2(1-s)}{(b_2-b_1)\,Sn(\lambda(1+s))},$$

$$W_2(s)=\frac{m_2 m_3(1+s)}{(a_3-a_2)\,Sn(\lambda(1-s))},$$

$$W_3(s)=\frac{\lambda m_1 m_3(1-s^2)}{(b_2-b_1)\sin\lambda(1+s)+(a_3-a_2)\sin\lambda(1-s)},$$

and where

$$Sn(x)=\frac{\sin x}{x}.$$

Since $Sn(x)$ can be extended to a positive real analytic function on $\left[0, \frac{\pi}{2}\right]$ and since $0<\lambda<\frac{\pi}{4}$, W_1 and W_2 become real analytic functions on $[-1, 1]$. Thus W can be extended to a positive real analytic function on $[-1, 1]$, which we denote again by W.

Now define a new variable

$$w=(1-s^2)\, W(s)^{-\frac{1}{2}} u.$$

That is, define the real analytic diffeomorphism

$$\begin{aligned}[0, \infty) \times R^1 \times (-1, 1) \times R^1 \to [0, \infty) \times R^1 \times (-1, 1) \times R^1 &: \\ (r, v, s, w) \mapsto \left(r, v, s, (1-s^2)^{-1}\, W(s)^{\frac{1}{2}} w\right). &\end{aligned} \tag{5.3}$$

The energy relation (4.16) becomes

$$w^2+s^2-1+(1-s^2)^2\, W(s)^{-1} (v^2-2rh)=0, \tag{5.4}$$

and the equations of motion (4.17) become

$$\begin{aligned}
\frac{dr}{dt'} &= r v, \\
\frac{dv}{dt'} &= \tfrac{1}{2} v^2 - \frac{W(s)}{2(1-s^2)} \left(1-\frac{2w^2}{1-s^2}\right), \\
\frac{ds}{dt'} &= \frac{W(s)^{\frac{1}{2}}}{\lambda(1-s^2)} w, \\
\frac{dw}{dt'} &= -\tfrac{1}{2} v w + \frac{W(s)^{\frac{1}{2}}}{\lambda(1-s^2)} \left[s\left(1-\frac{2w^2}{1-s^2}\right)+\tfrac{1}{2}\frac{W'(s)}{W(s)}(1-s^2-w^2)\right].
\end{aligned}$$

The singularities due to double collision occur at $s=\pm 1$. We can now remove them by a time transformation and by making use of the energy relation. First we make the time transformation

$$dt'=\lambda(1-s^2)\, W(s)^{-\frac{1}{2}} d\tau. \tag{5.5}$$

The above equations then become:

$$\begin{aligned}
\frac{dr}{d\tau} &= \frac{\lambda(1-s^2)}{W(s)^{\frac{1}{2}}} r v \\
\frac{dv}{d\tau} &= \frac{\lambda}{2}\left[\frac{(1-s^2)}{W(s)^{\frac{1}{2}}} v^2 - W(s)^{\frac{1}{2}}\left(1-\frac{2w^2}{1-s^2}\right)\right] \\
\frac{ds}{d\tau} &= w \\
\frac{dw}{d\tau} &= s\left(1-\frac{2w^2}{1-s^2}\right)+\tfrac{1}{2}\frac{W'(s)}{W(s)}(1-s^2-w^2)-\frac{\lambda(1-s^2)}{2W(s)^{\frac{1}{2}}} v w.
\end{aligned} \tag{5.6}$$

The energy relation (5.4) gives us

$$1-\frac{2w^2}{1-s^2}=\frac{2(1-s^2)}{W(s)}(v^2-2rh)-1.$$

Substituting the above expression into Eqs. (5.6), we have

$$\begin{aligned}
\frac{dr}{d\tau}&=\frac{\lambda(1-s^2)}{W(s)^{\frac{1}{2}}}rv\\
\frac{dv}{d\tau}&=\frac{\lambda}{2}W(s)^{\frac{1}{2}}\left[1-\frac{(1-s^2)}{W(s)}(v^2-4rh)\right]\\
\frac{ds}{d\tau}&=w\\
\frac{dw}{d\tau}&=-s+\frac{2s(1-s^2)}{W(s)}(v^2-2rh)+\tfrac{1}{2}\frac{W'(s)}{W(s)}(1-s^2-w^2)-\frac{\lambda(1-s^2)}{2W(s)^{\frac{1}{2}}}vw.
\end{aligned}\tag{5.7}$$

These equations define a real analytic vectorfield on $\mathfrak{R}$.

For each real h let

$$N(h)=\{(r,v,s,w)\in\mathfrak{R}:\ (5.4)\ \text{holds}\}.$$

Since the gradient of expression (5.4) does not vanish on $N(h)$, $N(h)$ is a three-dimensional real analytic submanifold of $\mathfrak{R}$. Since (5.4) is the transformed Hamiltonian, $N(h)$ is invariant under vectorfield (5.6). Therefore (5.6) is a vectorfield on the part of $N(h)$ where it is defined and (5.7) is the extension of (5.6) to all of $N(h)$.

At this point we wish to review what we have accomplished so far. Let

$$\begin{aligned}
N_3(h)&=\{(r,v,s,w)\in N(h):\ r=0\},\\
N_2(h)&=\{(r,v,s,w)\in N(h):\ s=\pm 1\},\\
N_1(h)&=N(h)-(N_3(h)\cup N_2(h)).
\end{aligned}$$

We began with a vectorfield (2.3) defined on a manifold $\mathbf{M}(h)$ for each fixed real h. We successively made transformations (3.1), (3.4), (4.13), and (5.13). The composition of these transformations defines an embedding: $\mathbf{M}(h)\to N(h)$. In fact, this embedding is a real analytic diffeomorphism onto $N_1(h)$. After a scaling defined by the time transformations (3.7) and (5.5), the vectorfield (2.3) on $\mathbf{M}(h)$ is carried to the vectorfield (5.7) on $N_1(h)$. However, this new vectorfield can be extended to a real analytic vectorfield on $N(h)$. The extension to points in $N_2(h)$ corresponds to the regularization of double collisions. The extension to points in $N_3(h)$ corresponds to "pasting on" an invariant boundary of triple collision points. Any orbit on $\mathbf{M}(h)$ is carried to an orbit on $N_1(h)$. However an orbit which ended in a double collision on $M(h)$ is now extended through a point in $N_2(h)$. An orbit which ended in triple collision on $\mathbf{M}(h)$ is now slowed down so that it asymptotically approaches $N_3(h)$.

Solutions which end in triple collision are now defined for all time. The vectorfield (5.7) would be complete except that the time transformation (3.7) increases the speed of orbits for large r so that some solutions now get to infinity in finite time. This problem could have been avoided by replacing (3.7) with

$$dt = \frac{r^{\frac{3}{2}}}{1+r^{\frac{3}{2}}}\, dt'.$$

Vectorfield (5.7) would then be divided by the scalar $1+r^{\frac{3}{2}}$ and would be complete. However, since in this paper we are concerned only with orbits near $r=0$, we have chosen the simpler transformation (3.7).

For any point $x=(r, v, s, w) \in N(h)$ we shall denote by $\varphi(x, t)$ the solution of Eqs. (5.7) starting at x when $t=0$. We shall somewhat loosely refer to φ as the "flow determined by the vectorfield (5.7)".

6. The Triple Collision Manifold

We now turn our attention to the invariant set of triple collision points. We have a three-dimensional manifold $N(h)$, a vectorfield (5.7) on $N(h)$, and a flow φ given by the vectorfield. The invariant boundary $N_3(h)$ corresponds to triple collision. If we set $r=0$ in the energy relation (5.4), we have

$$w^2 + s^2 - 1 + (1-s^2)^2\, W(s)^{-1}\, v^2 = 0. \qquad (6.1)$$

Thus

$$N_3(h) = \{(r, v, s, w) \in \mathfrak{R} \colon r=0 \text{ and } (6.1) \text{ is satisfied}\}$$

which we shall henceforth refer to as "the triple collision manifold." Note that the triple collision manifold is independent of the energy h. For ease of notation we write

$$C = N_3(h).$$

The triple collision manifold C is a two-dimensional manifold homeomorphic to a two-sphere minus four points. (See Fig. 2.) It is ficticious in the sense that it is introduced by the transformations and that orbits on C are not actual orbits of the original equations. However, the continuity of the flow φ allows us to use the flow on C to describe the flow near C and hence to describe the solutions of the original equations near triple collision.

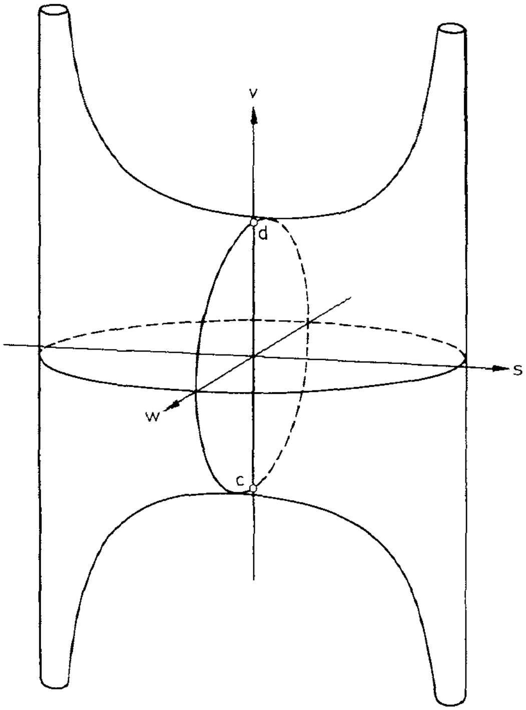

Fig. 2. The triple collision manifold

The vectorfield on C is given by letting $r=0$ in Eqs. (5.7).

$$\frac{dv}{d\tau}=\frac{\lambda}{2}W(s)^{\frac{1}{2}}\left[1-v^2\frac{(1-s^2)}{W(s)}\right], \tag{6.2a}$$

$$\frac{ds}{d\tau}=w, \tag{6.2b}$$

$$\frac{dw}{d\tau}=-s+\frac{2s(1-s^2)}{W(s)}v^2+\frac{1}{2}\frac{W'(s)}{W(s)}(1-s^2-w^2)-\frac{\lambda}{2}\frac{(1-s^2)}{W(s)^{\frac{1}{2}}}vw. \tag{6.2c}$$

Before examining this vectorfield it is convenient to recall the notion of central configuration (*cf.* Wintner [17]).

Definition. A point $\mathbf{s}_0 \in \mathbf{S}$ is called a *central configuration* if

$$\nabla U(\mathbf{s}_0)=\mu M \mathbf{s}_0$$

for some real number μ.

Part of the importance of central configurations derives from the so-called homographic solutions. Recall the original second order equations of motion (2.1). Suppose $\rho(t)$ is a positive function satisfying

$$\ddot{\rho}(t)=\mu\rho(t)^{-2}. \tag{6.3}$$

One then sees that $\mathbf{q}(t)=\rho(t)\,\mathbf{s}_0$ is a solution of (2.1). This solution is called homographic since the configuration of the particles does not change with time. Since all solutions of (6.3) tend to zero in some finite time, homographic solutions either begin or end in triple collision. Thus triple collision orbits exist.

In the coordinates we have introduced, central configurations correspond to critical points of V.

Proposition 6.1. $\mathbf{s}_0=S(s_0)$ *is a central configuration if and only if* $V'(s_0)=0$.

Proof. Suppose $\mathbf{s}_0$ is a central configuration. Eq. (4.15) then implies

$$V'(s_0)=\lambda\, DU(\mathbf{s}_0)\, A\mathbf{s}_0=\lambda\mu\,\mathbf{s}_0^T MA\mathbf{s}_0.$$

So, by property (4.3), $V'(s_0)=0$.

Now suppose $V'(s_0)=0$. Then, again by Eq. (4.15), we have

$$\mathbf{s}_0^T A^T \nabla U(\mathbf{s}_0)=DU(\mathbf{s}_0)\, A\mathbf{s}_0=0.$$

Since $\nabla U(\mathbf{s}_0)\in P$, we can write

$$\nabla U(\mathbf{s}_0)=\alpha M\mathbf{s}_0+\beta MA\mathbf{s}_0,$$

for some real α and β. Therefore, by properties (4.1) and (4.3), we have

$$0=\alpha\,\mathbf{s}_0^T A^T M\mathbf{s}_0+\beta\,\mathbf{s}_0^T A^T MA\mathbf{s}_0=\beta,$$

and hence $\nabla U(\mathbf{s}_0)=\alpha M\mathbf{s}_0$. Thus $\mathbf{s}_0$ is a central configuration and the proof is complete.

The central configurations of the three-body problem are well-known (*cf.* Winter [17]). Of the so-called collinear central configurations there is only one so that $s_1<s_2<s_3$.

Proposition 6.2. *There is exactly one central configuration* $\mathbf{s}_c\in\mathbf{S}_1$.

Proof. By the previous proposition we must prove that V has exactly one critical point on $(-1,1)$. Since $V(s)\to\infty$ as $s\to\pm1$, V has a critical point. Using Eqs. (4.15), (4.10), (4.4), and Euler's formula we compute

$$V''(s)=\lambda^2\left[D^2U(S(s))\,(AS(s),AS(s))+V(s)\right]. \tag{6.4}$$

From the definition of U we compute

$$D^2U(\mathbf{q})(\xi,\eta)=\sum_{i<j}\frac{2m_i m_j(\xi_i-\xi_j)(\eta_i-\eta_j)}{|q_i-q_j|^3}.$$

Hence $D^2 U(\mathbf{q})$ is positive definite. Since $V(s)>0$, Eq. (6.4) implies that $V''(s)>0$ for all $s \in (-1, 1)$. Thus V has a unique critical point (its minimum) and the proof is complete.

We are now able to compute the rest points for the flow on C. Let $\mathbf{s}_c = S(s_c)$ be the central configuration on $\mathbf{S}_1$ and let

$$v_c = (2V(s_c))^{\frac{1}{2}} = \left(\frac{W(s_c)}{1-s_c^2}\right)^{\frac{1}{2}}. \tag{6.5}$$

Now define two points on C (see Fig. 2):

$$c = (0, -v_c, s_c, 0),$$
$$d = (0, v_c, s_c, 0).$$

Proposition 6.3. *The flow φ restricted to C has exactly two rest points, c and d.*

Proof. The point $x=(0, v, s, w) \in C$ is a rest point of φ if and only if it is a zero for the vectorfield (6.2). From Eqs. (6.2b) and (6.2a) we see that x is a zero only if $w=0$ and

$$v^2 = \frac{W(s)}{(1-s^2)}. \tag{6.6}$$

Thus Eq. (6.2c) gives us

$$s + \tfrac{1}{2}\frac{W'(s)}{W(s)}(1-s^2) = 0.$$

But from the definition of W we have

$$\frac{W'(s)}{W(s)} = -\frac{2s}{1-s^2} + \frac{V'(s)}{V(s)}$$

and hence

$$\tfrac{1}{2}\frac{V'(s)}{V(s)} = 0.$$

Therefore $s = s_c$. From (6.6) we then have that $v = \pm v_c$, i.e. that $x = c$ or d. The proof is complete.

We show below that the coordinate v increases along solutions of Eqs. (6.2). Thus the flow φ on C exhibits a property which we shall call "gradient-like".

Definition. Let ψ be a flow on a complete metric space X. Suppose there is a continuous function $g: X \to R'$ such that

$$g(\psi(x, t)) < g(x) \quad \text{if } t>0$$

unless x is a rest point. Suppose further that the rest points of ψ are isolated. Then ψ is called *gradient-like* (with respect to g).

Proposition 6.4. *The flow φ restricted to C is gradient-like with respect to*

$$g: \ C \to R^1: \ (0, v, s, w) \mapsto -v.$$

Proof. Eq. (6.2a) implies that $\frac{dv}{d\tau} > 0$ if $s = \pm 1$. Combining (6.2a) with (6.1) we have

$$\frac{dv}{d\tau} = \frac{\lambda}{2} W(s)^{\frac{1}{2}} \frac{w^2}{1-s^2}$$

if $s \neq \pm 1$. Therefore

$$\frac{dv}{d\tau} > 0 \quad \text{if } w \neq 0 \text{ or } s = \pm 1.$$

When $w = 0$ and $s \neq \pm 1$ we can compute

$$\frac{dv}{d\tau} = \frac{d^2 v}{d\tau^2} = 0,$$

$$\frac{d^3 v}{d\tau^3} = \frac{\lambda(1-s^2)^3}{W(s)^{\frac{3}{2}}} V'(s)^2.$$

By Proposition 6.1 this last expression is positive except when $s = s_c$. Thus v is increasing everywhere except at the two rest points and the proof is complete.

Propositions 6.3 and 6.4 give a description of the flow on the triple collision manifold C. In Section 10 we shall develop a more complete description for certain values of the masses, but first we discuss two known theorems about the set of orbits ending in triple collision.

7. Asymptotic Behavior of Triple Collision Orbits

Sundman [15] proved for the three-body problem in three dimensions that an orbit ending in triple collision asymptotically approaches a central configuration. In this section we offer a different proof for the collinear problem.

A solution $(\mathbf{q}(t), \mathbf{p}(t)) \in \mathbf{Q} \times \mathbf{P}$ of Eqs. (2.3) will be called a triple collision orbit if, for some real number t_1, $\mathbf{q}(t) \to 0$ as $t \to t_1$. The orbit will be said to end in triple collision if $\mathbf{q}(t) \to 0$ as $t \to t_1 -$; it will be said to begin in triple collision if $\mathbf{q}(t) \to 0$ as $t \to t_1 +$.

Theorem 7.1 (Sundman). *Let $(\mathbf{q}(t), \mathbf{p}(t))$ be a triple collision orbit. Then as $t \to t_1$*

(a) $$\frac{\mathbf{q}(t)}{r(t)} \to \mathbf{s}_c \quad \textit{and}$$

(b) $$r(t) \sim \kappa (t_1 - t)^{\frac{2}{3}}.$$

Recall that $r(t)^2$ is the moment of inertia. Thus Sundman not only proved that triple collision orbits approach a central configuration but that the moment of inertia goes to zero like $(t_1 - t)^{\frac{4}{3}}$. Statements (a) and (b) are sometimes combined to read:

$$\mathbf{q}(t) \sim \kappa (t_1 - t)^{\frac{2}{3}} \mathbf{s}_c .$$

The value of the constant κ is given in the proof below.

Our proof of Sundman's Theorem in the collinear case uses the transformations introduced in the previous sections and is based on general considerations of flows on metric spaces. For a flow ψ on a complete metric space X denote the ω-limit set of a point $x_0 \in X$ by

$$\omega(x_0) = \bigcap_{t>0} \overline{\psi(x, [t, \infty))}.$$

Here the bar represents topological closure. Theorem 7.1 is a consequence of the following lemma, the proof of which is given in the appendix.

Lemma 7.2. *Let ψ be a flow on a locally compact metric space X. Let $x_0 \in X$ be such that $\omega(x_0)$ is a non-empty compact set. Suppose ψ restricted to $\omega(x_0)$ is gradient-like. Then $\omega(x_0)$ is a single point.*

Theorem 7.1 follows almost immediately from the above lemma if we can show that the ω-limit set of a triple collision orbit is non-empty and compact. We need the following notation, which we shall use again in Section 9.

For $\nu > \alpha > 0$, define the following subsets of the transformed constant energy manifold $N(h)$ (see Fig. 3):

$$\begin{aligned}
\mathbf{B}(h, \alpha) &= \{(r, v, s, w) \in N(h) \colon r \leqq \alpha\}, \\
B_0(h, \alpha, \nu) &= \{(r, v, s, w) \in \mathbf{B}(h, \alpha) \colon |v| \leqq \nu - r\}, \\
B^-(h, \alpha, \nu) &= \{(r, v, s, w) \in \mathbf{B}(h, \alpha) \colon v \geqq \nu - r\}, \\
B^+(h, \alpha, \nu) &= \{(r, v, s, w) \in \mathbf{B}(h, \alpha) \colon -v \geqq \nu - r\}, \\
b^\pm(h, \alpha, \nu) &= \{(r, v, s, w) \in B^\pm(h, \alpha, \nu) \colon r = \alpha\}, \\
\sigma^\pm(h, \alpha, \nu) &= B^\pm(h, \alpha, \nu) \cap B_0(h, \alpha, \nu).
\end{aligned}$$

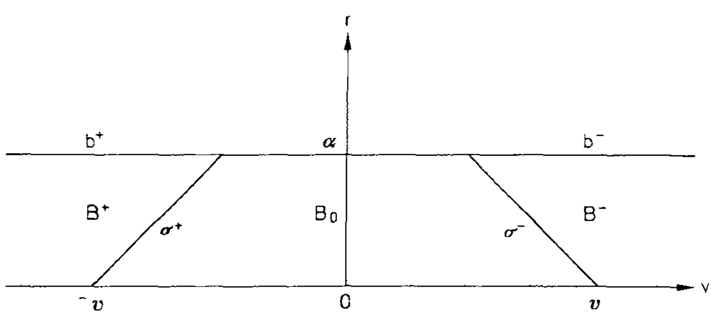

Fig. 3. The isolating block $\mathbf{B}(h, a)$. Note that the coordinates s and w are not shown

We shall say an orbit segment $\varphi(x_0, [\tau, \tau'])$ is maximal in a closed set K if it is a subset of K but $\varphi(x_0, I) \not\subset K$ for any larger interval $I \supset [\tau, \tau']$.

Proposition 7.3. *Let* $\nu > \alpha - h$. *If* $\varphi(x_0, [\tau, \tau'])$ *is a maximal orbit segment in* $B^-(h, \alpha, \nu)$, *then* $\varphi(x_0, \tau) \in \sigma^-(h, \alpha, \nu)$ *and* $\varphi(x_0, \tau') \in b^-(h, \alpha, \nu)$.

Proof. Adding the first two of Eqs. (5.7) and inserting (5.4) into the result, we have

$$\frac{d}{d\tau}(v+r) = \lambda \left[\frac{W(s)^{\frac{1}{2}}}{2(1-s^2)} w^2 + r \frac{(1-s^2)}{W(s)^{\frac{1}{2}}} (h+v) \right],$$

for $s \neq \pm 1$, while, for $s = \pm 1$,

$$\frac{d}{d\tau}(v+r) = \tfrac{1}{2} \lambda W(s)^{\frac{1}{2}}.$$

If $(r, v, s, w) \in \sigma^-$, then $v = \nu - r$ and $r \leqq \alpha$, so $v \geqq \nu - \alpha$. By hypothesis, $\nu > \alpha - h$, so $v + h > 0$. Therefore, $\frac{d}{d\tau}(v+r) > 0$ for $(r, v, s, w) \in \sigma^-$. Thus points on σ^- are entering B^-, so $\varphi(x_0, \tau') \in b^-$. Since $(r, v, s, w) \in b^-$ implies $v > 0$, the first of Eqs. (5.7) implies that points on b^- are leaving B^-, so $\varphi(x_0, \tau) \in \sigma^-$. The proof is complete.

A similar argument proves the following:

Proposition 7.4. *Let* $\nu > \alpha - h$. *If* $\varphi(x_0, [\tau, \tau'])$ *is a maximal orbit segment in* $B^+(h, \alpha, \nu)$, *then* $\varphi(x_0, \tau) \in b^+(h, \alpha, \nu)$ *and* $\varphi(x_0, \tau') \in \sigma^+(h, \alpha, \nu)$.

Proof of Theorem 7.1. We consider only the case of orbits ending in triple collision. The proof for orbits beginning in triple collision is essentially the same.

Let $x_0 = (r_0, v_0, s_0, w_0) \in N(h)$ be the image of $(q(0), p(0))$ under the transformations (3.1), (3.4), (4.13), and (5.3). Thus the orbit

$$\{(q(t), p(t)) \colon t \in [0, t_1)\}$$

is mapped to the orbit $\varphi(x_0, [0, \tau_1))$. Since by definition of a triple collision orbit $r(\tau) \to 0$ as $\tau \to \tau_1$, and since $\{r = 0\}$ is an invariant set for the flow, we must have that $\tau_1 = \infty$.

Fix $\alpha > 0$. Since $r(\tau) \to 0$ there is a $\tau_2 > 0$ so that $r(\tau) < \alpha$, and hence $\varphi(x_0, \tau) \in \mathbf{B}(h, \alpha)$, for all $\tau \geqq \tau_2$. Choose ν so that

$$\nu > \alpha + \max(-h, |v(\tau_2)|).$$

Then $\varphi(x_0, \tau_2) \in B_0(h, \alpha, \nu)$. We shall show by contradiction that

$$\varphi(x_0, \tau) \in B_0(h, \alpha, \nu) \qquad \text{for all } \tau \geqq \tau_2. \tag{7.1}$$

Suppose $\varphi(x_0, \tau_3) \in B^-(h, \alpha, \nu)$ for some $\tau_3 \geqq \tau_2$. Then Proposition 7.3 and $r(\tau) < \alpha$ imply that $\varphi(x_0, \tau) \in B^-$ for all $\tau \geqq \tau_3$. But $v > 0$ for points in B^-, so the first of Eqs. (5.7) implies that $\frac{dr}{d\tau} \geqq 0$ for $\tau \geqq \tau_3$. But this contradicts $r(\tau) \to 0$. Therefore $\varphi(x_0, \tau) \notin B^-$ for $\tau \geqq \tau_2$. Proposition 7.4 implies that $\varphi(x_0, \tau) \notin B^+$ for $\tau \geqq \tau_2$. Thus we have established (7.1).

Since B_0 is compact, $\omega(x_0)$ is a non-empty compact set. Since $r(\tau) \to 0$, $\omega(x_0) \in C$. By Proposition 6.4, the flow restricted to $\omega(x_0)$ is gradient-like. Therefore, by Lemma 7.2, $\omega(x_0)$ is exactly one point, necessarily a rest point. By Proposition 6.3, the only two rest points are c and d, so

$$(r, v, s, w) \to (0, \pm v_c, s_c, 0) \quad \text{as } \tau \to \infty.$$

Thus $r^{-1}\mathbf{q} = \mathbf{s} \to \mathbf{s}_c$ as $\tau \to \infty$, and part (a) of Theorem 7.1 is established.

The first of Eqs. (5.7) implies that $v(\tau) \not\to v_c$, since $\frac{dr}{d\tau} > 0$. Therefore

$$\varphi(x_0, \tau) \to c \quad \text{as } \tau \to \infty,$$

and we have

$$\frac{dr}{d\tau} \sim -\frac{\lambda(1-s_c^2)}{W(s_c)^{\frac{1}{2}}} v_c r \quad \text{as } \tau \to \infty.$$

Time transformations (3.7) and (5.5) imply that

$$\frac{dr}{dt} \sim -v_c r^{-\frac{1}{2}}$$

and hence

$$r(t) \sim (\tfrac{3}{2} v_c)^{\frac{2}{3}} (t_1 - t)^{\frac{2}{3}} \quad \text{as } t \to t_1 - .$$

Thus we have established part (b) and completed the proof of Theorem 7.1.

One should note that orbits beginning in triple collision have the property that

$$\varphi(x_0, \tau) \to d \quad \text{as } \tau \to \infty.$$

8. The Set of Triple Collision Orbits

Siegel [11, 12] has described the set of all triple collision orbits for the three-body problem in three dimensions. He was mainly concerned with the question of whether triple collision can be "regularized". We shall discuss the regularization question in the next section but for now we are concerned with a corollary of Siegel's work wherein he showed that the set of orbits ending in triple collision forms a smooth submanifold of the energy surface. In this section we give a proof of Siegel's corollary in the collinear case.

The proof follows from the stable manifold theorem applied to the critical point c on the triple collision manifold. By Theorem 7.1 the set of orbits ending in triple collision is exactly the stable manifold of the point c. We shall compute that the Jacobian at c has one positive and two negative eigenvalues and hence that the stable manifold of c is two-dimensional. (See Fig. 4.)

Theorem 8.1 (Siegel). *The set of orbits ending in triple collision forms a real-analytic two-dimensional immersed submanifold of the three-dimensional constant energy surface.*

Proof. By the above remarks it is only necessary to compute the eigenvalues of the Jacobian

$$J:\ T_c N(h) \to T_c N(h),$$

where $T_c N$ denotes the tangent space to N at c.

Let $F = (F_1, F_2, F_3, F_4)$: $\mathfrak{R} \to R^4$ be the vectorfield defined by Eqs. (5.7). Then $DF(c)$: $R^4 \to R^4$ leaves $T_c N(h)$ invariant, and J is the restriction of $DF(c)$ to $T_c N(h)$.

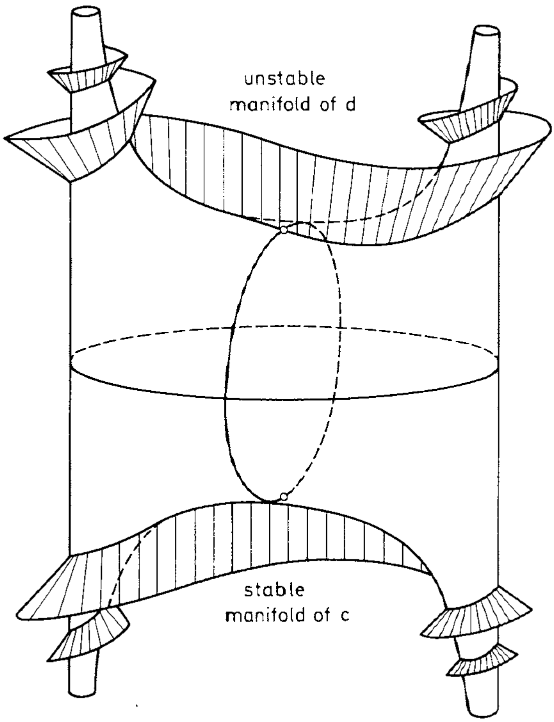

Fig. 4. The upper (lower) shaded surface is the set of orbits beginning (ending) in triple collision

Recall that $c = (0, -v_c, s_c, 0)$. Using Eqs. (5.1) and (6.5) we compute:

$DF_1(c)(\rho, \gamma, \sigma, \chi) = -\lambda(1 - s_c^2)^{\frac{1}{2}} \rho,$

$$DF_2(c)(\rho,\gamma,\sigma,\chi)=\frac{\lambda(1-s_c^2)^{\frac{1}{2}}}{v_c}(2h\rho+v_c\gamma),$$

$$DF_3(c)(\rho,\gamma,\sigma,\chi)=\chi,$$

$$DF_4(c)(\rho,\gamma,\sigma,\chi)=-\frac{4s_c}{v_c^2}(h\rho+v_c\gamma)+\frac{V''(s_c)}{v_c^2}(1-s_c^2)\sigma+\frac{\lambda}{2}(1-s_c^2)^{\frac{1}{2}}\chi.$$

From the definition (5.4) of $N(h)$ we see that

$$T_c N(h)=\{(\rho,\gamma,\sigma,\chi)\in R^4\colon h\rho+v_c\gamma=0\}.$$

Thus, for $(\rho,\gamma,\sigma,\chi)\in T_c N(H)$, we have

$$J(\rho,\gamma,\sigma,\chi)=DF(c)(\rho,\gamma,\sigma,\chi)$$
$$=\left(-\lambda(1-s_c^2)^{\frac{1}{2}}\rho,\ \frac{\lambda(1-s_c^2)^{\frac{1}{2}}}{v_c}h\rho,\ \chi,\ \frac{V''(s_c)}{v_c^2}(1-s_c^2)\sigma+\frac{\lambda}{2}(1-s_c^2)^{\frac{1}{2}}\chi\right).$$

We now choose a basis $\{\xi_1,\xi_2,\xi_3\}$ for $T_c N(h)$ as follows:

$$\xi_1=(-v_c,h,0,0),$$
$$\xi_2=(0,0,1,0),$$
$$\xi_3=(0,0,0,1).$$

Note that $\{\xi_2,\xi_3\}$ is a basis for $T_c C$. The matrix for J in this basis is

$$\begin{bmatrix} -\lambda(1-s_c^2)^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & (1-s_c^2)\frac{V''(s_c)}{v_c^2} & \frac{\lambda}{2}(1-s_c^2)^{\frac{1}{2}} \end{bmatrix}.$$

Thus ξ_1 is an eigenvector with eigenvalue $-\lambda(1-s_c^2)^{\frac{1}{2}}$. The characteristic equation for J restricted to $T_c C$ is

$$x^2-\frac{\lambda}{2}(1-s_c^2)^{\frac{1}{2}}x-(1-s_c^2)\frac{V''(s_c)}{v_c^2}=0.$$

In the proof of Proposition 6.2 we showed that $V''(s_c)>0$. Therefore the above equation has one negative and one positive root. Thus c has a one-dimensional stable manifold and a one-dimensional unstable manifold in C. On $N(h)$ we add a negative eigenvalue and thus have a two-dimensional stable manifold. The proof is complete.

The same theorem holds for the set of orbits beginning in triple collision. In this case one can prove that the unstable manifold of d is two-dimensional. The argument is exactly the same if one replaces $-v_c$ with $+v_c$.

9. The Isolating Block about Triple Collision

One can ask if orbits can be extended through triple collision. Of course, an orbit ending in triple collision can be connected arbitrarily to one beginning in triple collision. The problem is whether the connection can be made in a meaningful way.

Siegel has investigated this problem from an analytic viewpoint [11, 12]. He concentrated on a single orbit ending in triple collision and asked if that orbit, as a function of time, has a continuation. He found that in general it does not.

However, in the context of flows on manifolds, another viewpoint may be more natural. One can ask whether every orbit ending in triple collision can be connected to an orbit beginning in triple collision in such a way that a flow results. To make this connection it is necessary that each two orbits starting close together and close to a triple collision orbit remain close together for long periods of time.

Easton has given a definition of regularization which makes the above notion precise [4]. He has used his definition to describe the integral surfaces of the three-body problem after double collisions have been extended [5]. In this section we briefly describe Easton's definition and use it to show that orbits cannot be extended through triple collision for some values of the masses. For a thorough discussion of the motivation behind these definitions we refer to the papers of Easton [4, 5], Conley and Easton [3], and Churchill [1].

We must first introduce some notation. Let ψ be a flow on a manifold M and let $\mathbf{B}$ be a submanifold of the same dimension. Let $\mathbf{b}$ be the boundary of $\mathbf{B}$ and define

$$\mathbf{b}^+ = \{x \in \mathbf{b} : \psi(x, (-\varepsilon, 0)) \cap \mathbf{B} = \emptyset \text{ for some } \varepsilon > 0\},$$

$$\mathbf{b}^- = \{x \in \mathbf{b} : \psi(x, (0, \varepsilon)) \cap \mathbf{B} = \emptyset \text{ for some } \varepsilon > 0\}.$$

Definition. We say $\mathbf{B}$ is an isolating block if $\mathbf{b}^+ \cup \mathbf{b}^- = \mathbf{b}$.

Now let

$$\mathbf{a}^+ = \{x \in \mathbf{b}^+ : \psi(x, t) \in \mathbf{B} \text{ for all } t \geqq 0\},$$

$$\mathbf{a}^- = \{x \in \mathbf{b}^- : \psi(x, t) \in \mathbf{B} \text{ for all } t \geqq 0\}$$

and define $\Psi: \mathbf{b}^+ - \mathbf{a}^+ \to \mathbf{b}^- - \mathbf{a}^-$ by following the flow until it first hits $\mathbf{b}^- - \mathbf{a}^-$. We shall refer to Ψ as the "map across the block $\mathbf{B}$". An important property of isolating blocks is that the map Ψ is a homeomorphism [3, 1].

Definition. We say $\mathbf{B}$ is *regularizable* if Ψ can be extended to a homeomorphism: $\mathbf{b}^+ \to \mathbf{b}^-$. Otherwise we say $\mathbf{B}$ is *non-regularizable.*

Now consider the flow φ given by Eqs. (5.7). Recall the definition of $\mathbf{B}(h, \alpha)$ given in Section 7. We see that, for any given h, $\mathbf{B}(h, \alpha)$ is an isolating block for small enough α. In fact,

$$\mathbf{b}^+(h,\alpha)=\{(r,v,s,w)\in\mathbf{B}(h,\alpha)\colon r=\alpha, v\leqq 0\},$$

$$\mathbf{b}^-(h,\alpha)=\{(r,v,s,w)\in\mathbf{B}(h,\alpha)\colon r=\alpha, v\geqq 0\}.$$

Furthermore, $\mathbf{a}^+(h,\alpha)$ is the intersection of $\mathbf{b}^+(h,\alpha)$ with the stable manifold of c, while $\mathbf{a}^-(h,\alpha)$ is the intersection of $\mathbf{b}^-$ with the unstable manifold of d. If $\mathbf{B}(h,\alpha)$ is non-regularizable, then $\mathbf{B}(h,\gamma)$ is non-regularizable for all $\gamma\leqq\alpha$. Thus we shall say that triple collision is non-regularizable for energy h if $\mathbf{B}(h,\alpha)$ is non-regularizable for some α. We wish to prove the following theorem.

Theorem 9.1. *There exist masses m_1, m_2 and m_3 such that triple collision is non-regularizable for all energies.*

The proof of this theorem depends upon a close examination of the flow on the triple collision manifold C. In particular, we must determine the location of the two orbits comprising the unstable manifold of c.

Definition. We shall say the flow φ on C is *totally degenerate* if the unstable manifold of c coincides exactly with the stable manifold of d. (See Fig. 5.)

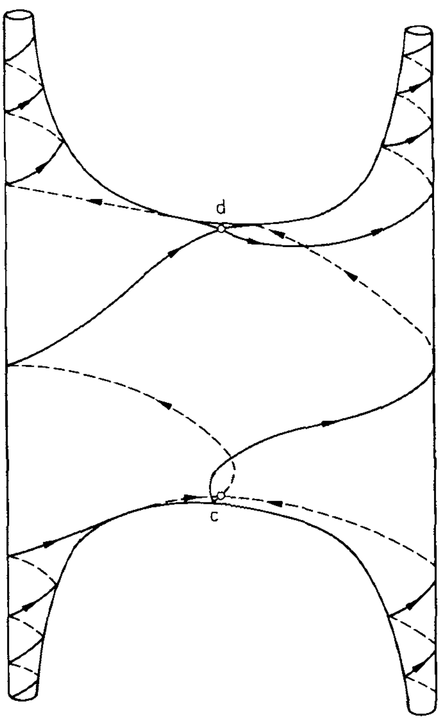

Fig. 5. Flow on the triple collision manifold in the totally degenerate case

In the next section we show that φ is not totally degenerate for some values of the masses. In the remainder of this section we shall prove that triple collision is non-regularizable if φ is not totally degenerate.

Recall the definitions of B^-, b^-, and σ^- given in Section 7. Henceforth we shall fix an energy h and choose an $\alpha > 0$ so that $\mathbf{B}(h, \alpha)$ is an isolating block. To streamline notation, we shall write $\mathbf{B} = \mathbf{B}(h, \alpha)$, $B^-(v) = B^-(h, \alpha, v)$, etc. Let Φ be the map across the block $\mathbf{B}$.

Proposition 9.2. *Suppose φ on C is not totally degenerate. Let $a_0 \in \mathbf{a}^+$, let U be an open subset of $\mathbf{b}^+$ containing a_0, and let $v > v_c$. Then $\exists\, x_v \in U$ such that $\Phi(x_v) \in b^-(v)$.* (See Fig. 6.)

Proof. From the gradient-like property of φ on C and the non-degeneracy condition we know that at least one branch of the unstable manifold of c intersects $\sigma^-(v)$. We know from the proof of Proposition 7.3 that $\sigma^-(v)$ is a section for the flow. We also know from the definition that $\mathbf{b}^+$ is a section. Since φ can be approximated by its linear part in a neighborhood of c, there exist a point $x_v \in U$ and a point $y_v \in \sigma^-(v)$ such that $\varphi(x_v, \tau) = y_v$ for some $\tau > 0$. Thus the orbit through x_v crosses $\sigma^-(v)$ and enters $B^-(v)$. By Proposition 7.3 the orbit can leave $B^-(v)$ only on $b^-(v)$. Hence $\Phi(x_v) \in b^-(v)$ and the proof is complete.

Proposition 9.3. *If φ on C is not totally degenerate, then triple collision is non-regularizable.*

Proof. Let $a_0 \in \mathbf{a}^+$, $v > v_c$. We can find points x arbitrarily close to a_0 such that $\Phi(x) \in b^-(v)$. Since $\bigcap \{b^-(v) \colon v > v_c\} = \emptyset$, Φ cannot be extended to a_0 and the proof is complete.

One should note here a difference between the regularization defined by Easton and the regularization explored by Siegel. Proposition 9.3 shows that the homographic orbit cannot be extended through triple collision. However, if one examines Eq. (6.3) describing the homographic solution, one sees that this solution behaves in the same way as a double collision. Thus the homographic orbit can be extended as a function of time, but the extension does not result in continuity with respect to nearby orbits.

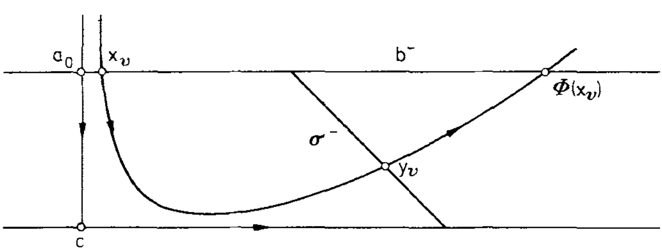

Fig. 6. An orbit passing close to triple collision

At first glance the above results seem highly technical and only vaguely related to the problem of three particles moving along a line. However, when one examines the relation of the coordinates r, v, s, and w to the original coordinates $\mathbf{q}$ and $\mathbf{p}$, one finds a somewhat surprising implication of Proposition 9.2.

First recall that r^2 is the moment of inertia of the system of particles. The isolating block $\mathbf{B}$ about triple collision is the set of points with $r \leqq \alpha$. The boundary $\mathbf{b}$ of $\mathbf{B}$ is the set of points with $r=\alpha$. On $\mathbf{b}^+$ we have the one-dimensional set $\mathbf{a}^+$ of points whose orbits end in triple collision. Proposition 9.2 tells us that orbits starting close to $\mathbf{a}^+$ emerge from the isolating block with arbitrarily large values of v.

Now consider the energy relation (5.4). Since points in the isolating block have $r \leqq \alpha$, we see that $|w| \leqq 1$ for large v. We also see that a large value of v forces a small value of $1-s^2$. Hence, for large values of ν, the set $B^-(\nu)$ has two components, one containing points with s close to $+1$, the other containing points with s close to -1. Therefore orbits passing close to triple collision emerge from the isolating block with $r=\alpha$, $|w| \leqq 1$, v large, and s close to ± 1. Recall that s near $+1$ corresponds to a configuration with particles 2 and 3 close together while s near -1 corresponds to particles 1 and 2 close together.

Working through transformations (3.1), (3.4), (4.13), and (5.3), we can write the momentum vector

$$\mathbf{p}=\frac{1}{r^{\frac{1}{2}}}\left[v\,MS(s)+w\,\frac{W(s)^{\frac{1}{2}}}{1-s^2}\,MAS(s)\right].$$

Using Eq. (4.11) we compute the momentum of particle 3:

$$p_3=\frac{m_3\,a_3}{r^{\frac{1}{2}}}\left[v\cos\lambda(1+s)-w\,\frac{W(s)^{\frac{1}{2}}}{1-s^2}\sin\lambda(1+s)\right].$$

Now consider an orbit passing close to triple collision and emerging from the isolating block with s near -1. Since $r=\alpha$, $|w| \leqq 1$, and v is large, p_3 must be large. Since s is near -1, particles 1 and 2 must be close together. A similar argument shows that, for an orbit passing close to triple collision and emerging with s near $+1$, the momentum of particle 1 must be large. Thus we have established the statement in the introduction: After passing close to triple collision one of the particles emerges with an arbitrarily high velocity in one direction while the other two particles emerge close together and moving rapidly in the opposite direction.

Now consider the set $\{(r, v, s, w) \in C\colon v > \nu\}$. As noted above, this set for large ν has two components: $C_+(\nu)$ containing points with s close to $+1$, and $C_-(\nu)$ containing points with s close to -1. An interesting case of a non-degenerate flow on C occurs when one branch of the unstable manifold of c intersects $C_+(\nu)$ while the other branch intersects $C_-(\nu)$.

(See Fig. 7.) In the next section we shall show that such a non-degenerate flow on C occurs for some values of the masses. In such a case, Proposition 9.2 implies that orbits passing close to triple collision emerge from the isolating block with s close to $+1$ or -1 depending upon on which side of the stable manifold of c they begin. Thus orbits starting close to each other and close to a triple collision orbit can emerge with totally different configurations, as well as arbitrarily high velocities.

Theorem 9.1 follows from Proposition 9.3 if we can show the existence of masses m_1, m_2, and m_3 for which the flow φ on C is not totally degenerate. The flow φ is a continuous function of the masses. If φ is not totally degenerate, then it remains so under small perturbations. Therefore the the set of masses for which φ is not totally degenerate is open. In the following section we show that this set is non-empty, thus proving Theorem 9.1.

10. A Special Case

In this section we consider the special case when the two outside masses are equal and the inside mass is small. Let

$$m_1 = m_3 = m, \qquad m_2 = \varepsilon m.$$

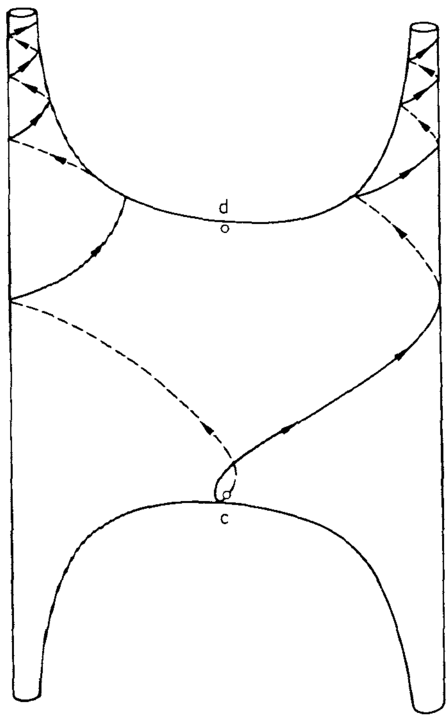

Fig. 7. Flow on the triple collision manifold in a non-degenerate case

Recall the definitions of **a** and **b** given at the beginning of Section 4 and the definition of λ given by Eq. (4.7). In this case we can explicitly write

$$\mathbf{a}=\left(m(1+\varepsilon)(2+\varepsilon)\right)^{-\frac{1}{2}}(-1,\,-1,\,1+\varepsilon)$$
$$\mathbf{b}=\left(m(1+\varepsilon)(2+\varepsilon)\right)^{-\frac{1}{2}}(-1-\varepsilon,\,1,\,1).$$

We compute that $\mathbf{a}^T M \mathbf{b}=(1+\varepsilon)^{-1}$ and hence that λ is the smallest positive number satisfying

$$\cos 2\lambda=\frac{1}{1+\varepsilon}.$$

Thus $\lambda\to 0$ as $\varepsilon\to 0$.

For this choice of masses we can rewrite Eq. (4.14) as

$$V(s)=m^{\frac{5}{2}}\left(\frac{1+\varepsilon}{2+\varepsilon}\right)^{\frac{1}{2}} \cdot \sin 2\lambda\left\{\frac{\varepsilon}{\sin\lambda(1+s)}+\frac{\varepsilon}{\sin\lambda(1-s)}+\frac{1}{\sin\lambda(1+s)+\sin\lambda(1-s)}\right\}. \tag{10.1}$$

From the above expression and the definition of $W(s)$ given by Eq. (5.1) we see that V and W are both even functions.

Proposition 10.1. *For any fixed m there are values of ε for which the flow φ on C is not totally degenerate.*

Proof. A branch of the unstable manifold of c is a single orbit $\varphi(x_0,(-\infty,\infty))$ such that $\varphi(x_0,\tau)\to c$ as $\tau\to-\infty$. Let $\tau_1\in(-\infty,\infty]$ be such that the value of v at $\varphi(x_0,\tau_1)$ is $+v_c$. We include $\tau_1=+\infty$ in case $\varphi(x_0,\tau)\to d$ as $\tau\to+\infty$. From the gradient-like property of φ, τ_1 is unique. By Eq. (6.2c) the line $\{s=+1\}$ is a section of the flow on C. Let $Z(\varepsilon)$ be the number of times $\varphi(x_0,(-\infty,\tau_1))$ intersects the line $\{s=+1\}$. Again from the gradient-like property of φ, $Z(\varepsilon)$ is a finite positive integer.

Now assume φ is totally degenerate for all ε. Then $Z(\varepsilon)$ is constant. We shall show that $Z(\varepsilon)\to\infty$ as $\varepsilon\to 0$, thus contradicting the assumption and proving the proposition.

Recall that in the proof of Proposition 6.2 we showed that V has a unique minimum on $[-1,1]$. In the present case V is even, so the minimum must occur at $s=0$. From Eq. (10.1) we see that

$$V(0)\to 2^{-\frac{1}{2}}m^{\frac{5}{2}} \quad \text{as } \varepsilon\to 0.$$

Now let

$$\mu=v_c^{-2}=(2V(0))^{-1}=W(0)^{-1}.$$

Then μ is a constant depending on the masses and approaches a positive limit as $\varepsilon\to 0$.

Now consider

$$g(v,s)=\frac{1-(1-s^2)v^2W(s)^{-1}}{1-\mu v^2}. \tag{10.2}$$

Then g is a positive continuous function on $(-v_c, v_c)\times[-1,1]$. We define a new variable

$$\eta = g(v,s)^{-\frac{1}{2}} w.$$

That is, we define a transformation

$$(-v_c, v_c)\times[-1,1]\times R^1 \to (-v_c, v_c)\times[-1,1]\times R^1:$$
$$(v,s,w)\mapsto\left(v,s,g(v,s)^{-\frac{1}{2}} w\right).$$

Now let

$$C_0 = \{(v,s,w)\in C:\ -v_c < v < v_c\}.$$

From Eq. (6.1) we see that the above transformation is a diffeomorphism from C_0 to

$$C' = \{(v,s,\eta)\in(-\mu,\mu)\times[-1,1]\times R:\ \eta^2-(1-s^2)(1-\mu v^2)=0\}.$$

The vectorfield (6.2) when transformed to C' becomes

$$\begin{aligned}
\frac{dv}{d\tau} &= \frac{\lambda}{2} W(s)^{\frac{1}{2}} g(v,s)(1-\mu v^2)\\
\frac{ds}{d\tau} &= g(v,s)^{\frac{1}{2}}\eta\\
\frac{d\eta}{d\tau} &= -g(v,s)^{\frac{1}{2}} s(1-\mu v^2) - \frac{\lambda}{2}\mu W(s)^{\frac{1}{2}} g(v,s)\, v\eta.
\end{aligned} \tag{10.3}$$

If we now make the time transformation

$$d\tau = g(v,s)^{\frac{1}{2}} d\tau'$$

the vectorfield on C' becomes

$$\begin{aligned}
\frac{dv}{d\tau'} &= \frac{\lambda}{2} W(s)^{\frac{1}{2}} g(v,s)^{\frac{1}{2}}(1-\mu v^2)\\
\frac{ds}{d\tau'} &= \eta\\
\frac{d\eta}{d\tau'} &= -s(1-\mu v^2) - \frac{\lambda}{2}\mu W(s)^{\frac{1}{2}} g(v,s)^{\frac{1}{2}}\, v\eta.
\end{aligned} \tag{10.4}$$

Now from Eq. (5.2) we see that

$$W(s)\to\sqrt{2}\, m^{\frac{5}{2}}(1-s^2) \qquad \text{as } \lambda\to 0$$

uniformly on $[-1,1]$. Thus Eq. (10.2) gives us

$$W(s)\, g(v,s)\to\sqrt{2}\, m^{\frac{5}{2}}(1-s^2) \qquad \text{as } \lambda\to 0$$

uniformly on compact subsets of $(-v_c, v_c) \times [-1, 1]$. Therefore

$$\lambda\, W(s)^{\frac{1}{2}}\, g(v, s)^{\frac{1}{2}} \to 0 \qquad \text{as } \lambda \to 0$$

uniformly on compact subsets. Hence the orbits of the vectorfield (10.4) become close to the orbits of the vectorfield

$$\begin{aligned} \frac{dv}{d\tau'} &= 0 \\ \frac{ds}{d\tau'} &= \eta \\ \frac{d\eta}{d\tau'} &= -s(1-\mu v^2) \end{aligned} \tag{10.5}$$

as $\lambda \to 0$. But the orbits of the above vectorfield are just the circles on C' given by constant v. Thus orbits of the vectorfield (10.4) must wrap around the dylinder C' an arbitrarily large number of times as $\lambda \to 0$. Therefore the same statement holds for the vectorfield (10.3). Since the flow on C' given by (10.3) is homeomorphic to the flow on C_0 given by (6.2), we have the required statement that $Z(\varepsilon) \to \infty$ as $\varepsilon \to 0$. The proof of Proposition 10.1 is complete.

One can give the following interpretation of the vectorfield (10.4). The masses of the particles are $m_1 = m_3 = m$ and $m_2 = \varepsilon m$. As $\lambda \to 0$, $\varepsilon \to 0$ and $m_2 \to 0$. Thus Eqs. (10.5) represent triple collision when the central particle has mass zero. It is difficult to physically interpret $\varepsilon = 0$ since the various transformations become singular. However, for small ε, the rapid spiralling around C' represents the central particle bouncing back and forth between the two outer particles many times while passing close to triple collision. The number of bounces goes to infinity as $\varepsilon \to 0$.

In the present case, where $m_1 = m_3$, we have a symmetry for the flow on C. Since W is even, Eqs. (6.2) are invariant under the transformation

$$(v, s, w) \to (v, -s, -w).$$

Therefore the two branches of the unstable manifold of c are reflections of each other. Recall the definitions of $C_+(v)$ and $C_-(v)$ given at the end of Section 9. Because of the symmetry, if one branch of the unstable manifold of c intersects $C_+(v)$, the other branch must intersect $C_-(v)$. Thus Proposition 10.1 establishes the existence of flows as shown in Fig. 7.

11. Remarks and Speculations

Many questions remain unanswered by this work. For example, can one describe the set of values of the three masses for which triple collision is non-regularizable? The problem is made somewhat simpler by noting that the original equations of motion (2.1) remain invariant if one makes the transformation

$$M = \mu M',$$

$$\mathbf{q} = \mu^{\frac{1}{3}} \mathbf{q}'$$

for $\mu > 0$. Thus triple collision is non-regularizable for masses (m_1, m_2, m_3) if and only if triple collision is non-regularizable for masses $(\mu m_1, \mu m_2, \mu m_3)$ for any $\mu > 0$. Therefore one need only consider values of the masses on the simplex

$$\mathfrak{M} = \{(m_1, m_2, m_3): m_1 + m_2 + m_3 = 1,\ m_k > 0\ \forall k\}.$$

In Section 9 we showed that triple collision is non-regularizable if the flow on the triple collision manifold is not totally degenerate. We also saw that the set of masses for which the flow is not totally degenerate is open. One expects that this set is large.

Conjecture. The set of masses for which the flow on the triple collision manifold is not totally degenerate is dense in $\mathfrak{M}$.

If this conjecture were true, then the set of masses for which triple collision is non-regularizable would be dense in $\mathfrak{M}$. There still remains the question of whether triple collision is regularizable when the flow on the triple collision manifold is totally degenerate. The eigenvalues at the two rest points may play a role here, but the author does not have a conjecture.

One can ask whether the results and methods of this paper apply to situations other than triple collision in the collinear three-body problem. The flow for the collinear problem is contained in the flow for the problem in higher dimensions as an invariant set. Therefore if triple collision cannot be regularized on the line, it is automatically non-regularizable in the plane or 3-space. However, certain mass ratios may be regularizable in the collinear problem, but non-regularizable in the planar problem.

The transformations used in Section 3 can be generalized to apply to n-tuple collision in the n-body problem in any dimension. That is, n-tuple collision can be made into an invariant manifold by blowing up the origin, and the time variable can be transformed so that no orbit arrives at n-tuple collision in finite time. The author has not yet studied the transformed equations in enough detatil to see if any new results can be obtained from this method.

The question of the existence of singularities other than collision has long been outstanding. Painlevé [8] proved for the three-body problem that all singularities are due to collisions. The work of Painlevé, Von Zeipel [16], and Sperling [13] shows that a collision is the only singularity that can occur in a solution of the n-body problem, if the positions of all the particles remain bounded. Therefore finding a singularity not due to collision is equivalent to finding an orbit for which at least one of the positions becomes unbounded in finite time.

The results of Section 9 indicate that such an orbit may exist. Consider the collinear 5-body problem with all double collisions regularized. Let $m_1, \dots, m_5$ be the masses of the five particles in the order in which they appear on the line. Choose m and ε so that the flow on the triple collision manifold for the three-body problem with masses $(m, \varepsilon m, m)$ is not totally degenerate. Let $m_1 = m_3 = m_5 = m$ and $m_2 = m_4 = \varepsilon m$. If triple collision in the five-body problem can be shown to behave like triple collision in the three-body problem, then, as we noted in Section 9, the system can pick up arbitrarily high kinetic energy whenever particles 1, 2, and 3 or particles 3, 4, and 5 pass close to triple collision.

We shall now describe an orbit which becomes unbounded in less than unit time. Particles 1 and 2 always remain close together and we shall call them binary system A. Particles 4 and 5 also remain close together and we shall call them binary system B. Particle 3 bounces back and forth between the two binary systems. At time $t=0$, particle 3 is moving so that it overtakes binary system A in time $t_1 < \frac{1}{2}$. After particles 1, 2, and 3 pass close to triple collision, particle 3 emerges with a velocity high enough to overtake binary system B in time $t_2 < \frac{1}{4}$. After particles 3, 4, and 5 pass close to triple collision, particle 3 emerges with enough velocity to overtake binary system A in time $t_3 < \frac{1}{8}$, etc. In less than unit time binary system A goes to $-\infty$, binary system B goes to $+\infty$, and particle 3 bounces back and forth between them an infinite number of times.

Conjecture. The orbit described above exists.

Clearly it will not be a simple matter to prove this conjecture. However, the author hopes that the methods used in this paper will provide a first step towards such a proof.

One should note here an apparent contradiction between the above conjecture and a theorem of Saari [9]. Saari proved that all singularities are due to collision in the collinear n-body problem. However, Saari does not extend orbits through double collisions. One sees that the orbit described above must contain an infinite number of binary collisions.

Appendix

We wish to prove Lemma 7.2. Using ideas of Sacker and Sell [10], a fairly direct proof can be given. However, we choose to use methods introduced by Conley [2], since they provide a characterization of the

ω-limit set of a point. This characterization is given by Theorem A.2 below and seems to be interesting in its own right.

Our method is to introduce the concept of a "chain-minimal" flow and to prove that the ω-limit set of a point is chain-minimal. We then show that a gradient-like chain-minimal flow is a single point, thus proving the lemma.

The following two definitions were developed by Conley [2].

Definition. Let φ be a flow on a complete metric space X with metric d. Let $x, y \in X$ and let ε and T be positive numbers. We say a collection $(x_1, \ldots, x_{n+1}, t_1, \ldots, t_n)$ is an $(\varepsilon, T, \varphi)$-chain from x to y provided the following conditions hold for $i = 1, \ldots, n$:

(a) $x_i \in X,\ x = x_1,\ y = x_{n+1}$,

(b) $t_i \geqq T$,

(c) $d(\varphi(x_i, t_i), x_{i+1}) \leqq \varepsilon$.

Definition. Let $x, y \in X$. We shall write $x > y$ provided there is an $(\varepsilon, T, \varphi)$-chain from x to y for all positive ε and T.

Conley shows that " $>$ " is a transitive relation on X but that it is neither reflexive nor anti-symmetric. We shall somewhat loosely refer to " $>$ " as an ordering. For a gradient-like flow this ordering is consistent with the gradient function in the following sense.

Lemma A.1. *Let φ be a flow on a compact metric space X. Suppose φ is gradient-like with respect to g, and let $x, y \in X$. Then*

$$x > y \Rightarrow g(x) \geqq g(y).$$

Proof. Suppose $x > y$ and $g(x) < g(y)$. Choose real numbers a_1 and a_2 so that

$$g(x) < a_1 < a_2 < g(y)$$

and $K = g^{-1}([a_1, a_2])$ contains no rest points. Let

$$K_1 = g^{-1}((-\infty, a_1]), \qquad K_2 = g^{-1}([a_2, \infty)).$$

Then K_1 and K_2 are disjoint compact sets. Let $\delta = d(K_1, K_2)$. Since K is compact and contains no rest points, there is a positive T so that

$$x \in K, \qquad t \geqq T \Rightarrow \varphi(x, t) \in K_1.$$

Thus, for $\varepsilon < \delta$ and $T' > T$, there is no $(\varepsilon, T', \varphi)$-chain from x to y. This contradicts $x > y$ and proves the lemma.

We now introduce the following generalization of a minimal flow.

Definition. Let φ be a flow on a complete metric space X. We shall say φ is *chain-minimal* if $x > y$ for all $x, y \in X$.

Conley [2] calls a flow "chain-recurrent" if $x > x$ for all $x \in X$. One can show that chain-recurrent and chain-minimal are equivalent on compact connected spaces.

We now have the following property of ω-limit sets.

Theorem A.2. *Let φ be a flow on a locally compact metric space X. Let $x_0 \in X$ be such that $\omega(x_0)$ is a non-empty compact set. Then φ restricted to $\omega(x_0)$ is chain-minimal.*

It is important to note that we are considering the flow *restricted* to the ω-limit set. It is trivial to prove that $x, y \in \omega(x_0)$ implies $x > y$ for the ordering given by φ on X. However, the ordering on $\omega(x_0)$ given by the restricted flow is different from the restricted ordering. This distinction is easily seen if we consider non-wandering points. All points in $\omega(x_0)$ are non-wandering for the flow φ. However, they may wander in the flow restricted to $\omega(x_0)$.

One should also note that, if X is compact, $\omega(x_0)$ is automatically non-empty and compact. Thus $\omega(x_0)$ is always chain-minimal for flows on compact metric spaces.

To prove Theorem A.2 we need the following proposition.

Proposition A.3. *Let φ be a flow on a locally compact metric space X. Let $x_0 \in X$ be such that $\omega(x_0)$ is a non-empty compact set. Then $\omega(x_0)$ is connected. Furthermore, for each open U containing $\omega(x_0)$, there is a positive t' so that $\varphi(x_0, t) \in U$ for $t \geqq t'$.*

When X is compact, Proposition A.3 is a standard result [7]. The assumption that $\omega(x_0)$ is non-empty and compact is enough to insure that the standard proof is valid when X is locally compact.

Proof of Theorem A.2. Let $\Omega = \omega(x_0)$ and let ψ denote the flow φ restricted to Ω. Let $y, y' \in \Omega$, and let ε and T be given. We must construct an (ε, T, ψ)-chain from y to y'.

Let U be an open subset of X containing Ω with compact closure. Choose $\delta < \varepsilon/2$ so that

$$u_1, u_2 \in \bar{U}, \qquad d(u_1, u_2) \Rightarrow d(\varphi(u_1, t), \varphi(u_2, t)) < \frac{\varepsilon}{2} \qquad \forall\, t \in [0, 2T]. \tag{A.1}$$

Let V be an open subset of U so that $\Omega \subset V$ and $d(x, \Omega) < \delta$ for all $x \in V$. By Proposition A.3 there is a t' so that $\varphi(x_0, [t', \infty)) \subset V$.

Choose $x_1 \in \varphi(x_0, [t', \infty))$ so that $d(x_1, y) < \delta$. Choose $T' > T$ so that $d(\varphi(x_1, T'), y') < \varepsilon/2$. Let n be the greatest integer in T'/T and let $x_i = \varphi(x_{i-1}, T)$ for $i = 2, \ldots, n$. Let $x_{n+1} = \varphi(x_n, T' - (n-1)T) = \varphi(x_1, T')$. For $i = 2, \ldots, n$, choose $y_i \in \Omega$ so that $d(y_i, x_i) < \delta$. Let $y_1 = y$ and $y_{n+1} = y'$. Let $t_i = T$ for $i = 1, \ldots, n-1$, and let $t_n = T' - (n-1)T$.

We claim that $(y_1, \ldots, y_{n+1}, t_1, \ldots, t_n)$ is an (ε, T, ψ)-chain from y to y'. Since (a) and (b) of the definition are satisfied by construction we need only show (c), i.e.

$$d(\psi(y_i, t_i), y_{i+1}) \leqq \varepsilon \quad \text{for } i=1, \ldots, n. \tag{A.2}$$

But $d(y_i, x_i) < \delta$ and $t_i < 2T$, so by (A.1),

$$d(\psi(y_i, t_i), x_{i+1}) = d(\varphi(y_i, t_i), \varphi(x_i, t_i)) < \varepsilon/2.$$

Since $d(y_{i+1}, x_{i+1}) < \delta < \frac{\varepsilon}{2}$, (A.2) follows from the triangle inequality. Thus $(y_1, \ldots, y_{n+1}, t_1, \ldots, t_n)$ is an (ε, T, ψ)-chain from y to y' and the proof of Theorem A.2 is complete.

Lemma 6.2 now follows as a corollary of Theorem A.2, Proposition A.3, and the following proposition.

Proposition A.4. *Let φ be a chain-minimal gradient-like flow on a compact connected non-empty metric space X. Then X consists of a single point.*

Proof. Let $x, y \in X$. Since $x > y$, $g(x) \geqq g(y)$. Since $y > x$, $g(y) \geqq g(x)$. Thus g is constant on X, so X contains only rest points. Since the rest points are isolated and X is compact, X contains only finitely many points. Since X is connected and non-empty it consists of a single point.

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References

1. Churchill, R.: Isolated invariant sets in compact metric spaces. J. Differential Equations **12**, 330–352 (1972)
2. Conley, C.: The gradient structure of a flow: I. IBM Research Report, Yorktown Heights, N.Y. 1972
3. Conley, C., Easton, R.: Isolated invariant sets and isolating blocks. Trans. A.M.S. **158**, 35–61 (1971)
4. Easton, R.: Regularization of vector fields by surgery. J. Differential Equations **10**, 92–99 (1971)
5. Easton, R.: The topology of the regularized integral surfaces of the 3-body problem. J. Differential Equations **12**, 361–384 (1972)
6. Moser, J.: Regularization of Kepler's problem and the averaging method on a manifold. Comm. Pure Appl. Math. **XXIII**, 609–636 (1970)
7. Nemytskii, V., Stepanov, V.: Qualitative Theory of Differential Equations. Princeton Univ. Press 1960
8. Painlevé, P.: Lecons sur la théorie analytique des équations différentielles (Stockholm, 1895). Hermann: Paris 1897
9. Saari, D.: Singularities and collisions of Newtonian gravitational systems. Arch. Rational Mech. Anal. **49**, 311–320 (1973)

10. Sacker, R., Sell, G.: On the existence of nontrivial recurrent motions. Duke Math. J., to appear
11. Siegel, C.: Der Dreierstoß. Ann. Math. **42**, 127–168 (1941)
12. Siegel, C., Moser, J.: Lectures on Celestial Mechanics. Berlin-Heidelberg-New York: Springer 1971
13. Sperling, H.: On the real singularities of the *n*-body problem. J. Reine Angew. Math. **245**, 14–50 (1970)
14. Sundman, K.: Mémoire sur le problème des trois corps. Acta Math. **36**, 105–179 (1912)
15. Sundman, K.: Nouvelles recherches sur le problème des trois corps. Acta Soc. Sci. Fenn. **35**, No. 9 (1909)
16. Zeipel, H. von: Sur les singularités du problème des *n* corps. Ark. Mat. Astr. Fys. **4**, No. 32 (1908)
17. Wintner, A.: The Analytical Foundations of Celestial Mechanics. Princeton Univ. Press 1941

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Note Added in Proof. Orbits similar to the one described at the end of Section 11 have been proved to exist. (Mather, J., McGehee, R.: Orbits for the Collinear Four-Body Problem Which Become Unbounded in Finite Time. *Battelle Rencontres 1974 Proceedings*, to appear.)