Inventiones math. 25, 159–198 (1974)

Regular Elements of Finite Reflection Groups

T.A. Springer (Utrecht)

Introduction

If G is a finite reflection group in a finite dimensional vector space V then $v \in V$ is called regular if no nonidentity element of G fixes v. An element $g \in G$ is regular if it has a regular eigenvector (a familiar example of such an element is a Coxeter element in a Weyl group). The main theme of this paper is the study of the properties and applications of the regular elements. A review of the contents follows.

In § 1 we recall some known facts about the invariant theory of finite linear groups. Then we discuss in § 2 some, more or less familiar, facts about finite reflection groups and their invariant theory. § 3 deals with the problem of finding the eigenvalues of the elements of a given finite linear group G. It turns out that the explicit knowledge of the algebra R of invariants of G implies a solution to this problem. If G is a reflection group, R has a very simple structure, which enables one to obtain precise results about the eigenvalues and their multiplicities (see 3.4). These results are established by using some standard facts from algebraic geometry. They can also be used to give a proof of the formula for the order of finite Chevalley groups. We shall not go into this here. In the case of eigenvalues of regular elements one can go further, this is discussed in § 4. One obtains, for example, a formula for the order of the centralizer of a regular element (see 4.2) and a formula for the eigenvalues of a regular element in an irreducible representation (see 4.5).

In § 5 we give, using a case by case analysis, the regular elements of the various irreducible Coxeter groups. The results of §§ 3, 4 are extended in § 6 to the "twisted" case, where certain outer automorphisms of reflection groups come into play. This is used in § 7 to establish properties of "twisted Coxeter elements" of Weyl groups, partly via a case by case discussion.

The main result of § 8 is a reduction theorem (8.4) which can be used to obtain properties of arbitrary elements of Weyl groups from those of regular elements. We use it to give proofs of two results (8.5 and 8.7) which so far were established only by using elaborate case by case checkings. In [10] Kostant has given a method to set up a connection between the class of regular nilpotent elements of a complex semisimple Lie algebra and the Coxeter class of the corresponding Weyl group. His method is extended in §9 to obtain in a few other cases a connection between a nilpotent class of a simple Lie algebra and a regular Weyl group class. One can verify that among the nilpotent classes which occur are the subregular ones of the exceptional Lie algebras (these subregular classes were studied by Brieskorn, Steinberg and Tits, a brief report can be found in Brieskorn's paper in the proceedings of the Nice congress). Moreover two other cases occur in type E_8. A closer study of these two extra cases might be interesting.

In this paper we have not made a thorough study of regular elements of complex reflection groups which are not Coxeter groups. That such elements may be useful is shown by a result of A.M. Cohen, who in a systematic study of the classification of irreducible complex reflection groups has made use of theorems of §4 to obtain results of [15] about degrees of the generators of the algebra of invariants.

The author is grateful to R.W.Carter, I.G.Macdonald and R.Steinberg for various discussions about the topics of this paper.

1. Invariants of Finite Linear Groups

1.1. Let k be a field, let V be a finite dimensional vector space over k. We put $n = \dim V$. Denote by S the symmetric algebra of the dual V' of V. From the definition of S it follows that there exists an algebra homomorphism α of S into the k-algebra of k-valued functions on V. We call the elements of αS *polynomial functions* on V. If k is infinite then α is injective. Below we shall mainly be interested in the case that k is the field of complex numbers. Then, and also in the case of an infinite k, we shall identify S with αS, and simply speak of S as the algebra of polynomial functions on V.

The algebra S is (non-canonically) isomorphic to the polynomial algebra $k[T_1, \ldots, T_n]$ in n indeterminates. There is a canonical grading

$$S = \coprod_{i \geqq 0} S_i,$$

corresponding to the usual grading of the polynomial algebra.

1.2. Let G be a group of nonsingular linear transformations of V. The action of G on V carries over to an action on V', S and αS. If f is a polynomial function on V, then the action of $g \in G$ on f is given by

$$(g \cdot f)(v) = f(g^{-1} \cdot v) \quad (v \in V).$$

We denote by R the subalgebra of S formed by the G-invariant elements, i.e.

$$R = \{s \in S \mid g \cdot s = s \text{ for all } g \in G\}.$$

We next recall without proof some known results about the case of a finite group G.

1.3. **Proposition.** *Let G be finite.*

(i) *S is an R-module of finite type.*

(ii) *R is a k-algebra of finite type.*

This is classical. (ii) goes back (at least) to E. Noether [13].

In the situation of 1.2, G acts on the set of prime ideals of S.

1.4. **Lemma.** *Let G be finite, let P and Q be two prime ideals of S. The following properties are equivalent:*

(i) *There exists $g \in G$ with $g \cdot P = Q$.*

(ii) *$P \cap R = Q \cap R$.*

The following result is a consequence of 1.4.

1.5. **Proposition.** *Let G be finite, let $v, w \in V$. The following properties are equivalent:*

(i) *There exists $g \in G$ with $g \cdot v = w$.*

(ii) *For all $f \in R$ we have $(\alpha f)(v) = (\alpha f)(w)$.*

If we take k to be an algebraically closed field, then V can be viewed as an affine algebraic variety over k and 1.5 then means that we can put a structure of affine algebraic variety on the orbit space $G \diagdown V$, whose algebra is R.

2. Finite Reflection Groups, Recollections and Auxiliary Results

We keep the notations of 1.1 and 1.2.

2.1. We say that a linear transformation of V is a *reflection*, if it is diagonalizable and if all but one of its eigenvalues are equal to 1 (what we call reflection is called pseudo-reflection in [4, p. 66]). G is called a *reflection group* (in V) if G is generated by reflections. A *reflection subgroup* of G is a subgroup of G which, viewed as a group of linear transformations of V, is a reflection group.

The reflection group G is called a *Coxeter group* if the following holds: (a) k is the field $\mathbb{C}$ of complex numbers and there is a structure on V over the field $\mathbb{R}$ of real numbers which is G-stable (this means that there is a G-stable $\mathbb{R}$-subspace V_0 of V, such that the canonical map $V_0 \otimes_{\mathbb{R}} \mathbb{C} \to V$ is bijective), (b) G is generated by reflections of order 2. Observe that (b) is a consequence of (a) if G is finite.

A finite Coxeter group G is called a *Weyl group* if, with the above notations, there is a lattice L in V_0 which is stabilized by G. Then there is a structure on V over the field $\mathbb{Q}$ of rational numbers which is G-stable.

From now on we shall assume the base field k to be *infinite*. As announced in 1.1, we then identify S with the algebra of polynomial functions on V, so that we shall drop the homomorphism α. In the sequel the letter G will stand for a finite group, which most of the times is a reflection group.

We next recall some basic results about the invariant theory of finite reflection groups.

2.2. **Theorem.** *Let G be a finite reflection group. If* $\mathrm{char}(k)=0$ *then R is generated as a k-algebra by $n=\dim V$ algebraically independent homogeneous polynomial functions $f_1, \ldots, f_n$.*

This is due to Chevalley [7], see also [4, p. 107].

The f_i being as in 2.2, let $d_i=\deg(f_i)$ be the degree. The family $\{d_1, \ldots, d_n\}$ is then uniquely determined, i.e. is independent of the particular choice of the f_i (subject to the conditions of 2.2). This is also proved in [4]. It also follows that the order $|G|$ of G equals $\prod_{i=1}^{n} d_i$. We shall give a different proof of the invariance statement.

2.3. **Proposition.** *Assume that R is generated over k by n algebraically independent homogeneous polynomial functions $f_1, \ldots, f_n$. Let $d_i=\deg(f_i)$ and assume the ordering to be such that $d_1 \leqq d_2 \leqq \cdots \leqq d_n$.*

(i) *d_i is the minimal degree of a homogeneous element of R which is algebraically independent of $f_1, \ldots, f_{i-1}$.*

(ii) *Let $g_1, \ldots, g_n$ be n algebraically independent homogeneous elements of R, let $e_i=\deg(g_i)$. Assume that $e_1 \leqq e_2 \leqq \cdots \leqq e_n$. Then $d_i \leqq e_i$ $(1 \leqq i \leqq n)$.*

(iii) *If in the situation of* (ii) *we have $d_i=e_i$ $(1 \leqq i \leqq n)$ then the g_i generate the k-algebra R.*

Let $f \in R$ be homogeneous of degree d. There exists $F \in k[T_1, \ldots, T_n]$ such that $f=F(f_1, \ldots, f_n)$. The algebraic independence of the f_i then implies that for any monomial $T_1^{e_1} \ldots T_n^{e_n}$ which occurs in F with a nonzero coefficient we must have $e_1 d_1+e_2 d_2+\cdots+e_n d_n=d$. Hence if $e_j \neq 0$ for some $j \geqq i$, we have $d \geqq d_i$. This implies (i). Observe that for $i=1$ the assertion of (i) means that d_1 is the smallest degree of a nonconstant homogeneous element of R.

To prove (ii) we use induction on i. That $d_1 \leqq e_1$ follows from what we just observed. Now assume that $d_j \leqq e_j$ $(1 \leqq j \leqq i-1)$. There exist polynomials $F_j \in k[T_1, \ldots, T_n]$ such that $g_j=F_j(f_1, \ldots, f_n)$ $(1 \leqq j \leqq n)$. Because of algebraic independence, the F_j with $j \leqq i$ cannot all involve the indeterminates $T_1, \ldots, T_{i-1}$ only. Hence there is $j \leqq i$ such that F_j involves an indeterminate T_h with $h \geqq i$. It follows that $e_j \geqq d_h$. Hence $e_i \geqq e_j \geqq d_h \geqq d_i$, which implies (ii).

Assume that we have $d_i = e_i$ $(1 \leq i \leq n)$. We then prove that there exist polynomials $G_i \in k[T_1, \ldots, T_n]$ such that $f_i = G_i(g_1, \ldots, g_n)$, which will prove (iii). Assume that the existence of the G_i has already been established for all i with $d_i < d$. We shall then prove the same for the i with $d_i = d$ (the induction starts with $d = 1$). The F_j being as before we have, for any monomial $T_1^{e_1} \ldots T_n^{e_n}$ occurring in F_i with nonzero coefficient, that $e_1 d_1 + \cdots + e_n d_n = d_i$. If $d_i = d$, then such monomials of degree > 1 involve only T_j with $d_j < d$. By our inductive assumption it follows that if $d_i = d$ there exist $a_{ij} \in k$ such that

$$g_i = \sum_{d_j = d} a_{ij} f_j + h_i,$$

where $h_i \in k[g_1, \ldots, g_n]$ involves only g_j with $d_j < d$. The algebraic independence of the g_j implies that the matrix (a_{ij}) is a non-singular square matrix. This implies the existence of the G_i for all i with $d_i = d$, finishing the proof of 2.3.

Application of 2.3 in the situation of 2.2 shows that then the family $\{d_1, \ldots, d_n\}$ is independent of the choice of the generators f_i. We call the integers d_i the *degrees* of the reflection group G. They are only defined if $\mathrm{char}(k) = 0$.

The next theorem is a slight refinement of a result of Shephard and Todd [15, p. 288].

2.4. **Theorem.** *Assume that $k = \mathbb{C}$, let G be a finite group. Assume that $f_1, \ldots, f_n$ are $n = \dim V$ algebraically independent homogeneous elements of R, let $d_i = \deg(f_i)$.*

(i) $|G| \leq d_1 d_2 \ldots d_n$.

(ii) *If equality holds in* (i) *then G is a reflection group, R is generated by the f_i and the d_i are the degrees of G.*

(iii) *If the f_i generate the k-algebra R, then equality holds in* (i) *and G is a reflection group.*

Let $R_i = R \cap S_i$ be the space of homogeneous invariants of degree i and put $a_i = \dim R_i$. We then have the following identity of formal power series:

$$|G|^{-1} \sum_{g \in G} (\det(1 - g\,T))^{-1} = \sum_{i=0}^{\infty} a_i\, T^i \tag{1}$$

(see [4, p. 110–111]). It is clear that the power series

$$\sum_{i=0}^{\infty} a_i\, t^i$$

converges for all real t with $0 \leq t < 1$.

Put $b_i = \dim(k[f_1, \ldots, f_n] \cap S_i)$, then $b_i \leq a_i$. Moreover we have (see [loc. cit., p. 103])

$$\sum_{i=0}^{\infty} b_i\, T^i = \prod_{i=1}^{n} (1 - T^{d_i})^{-1}.$$

It follows that

$$\sum_{i=0}^{\infty} a_i t^i \geqq \prod_{i=1}^{n} (1-t^{d_i})^{-1},$$

if $0 \leqq t < 1$. Multiplying by $(1-t)^n$ and letting t tend to 1, we infer from (1) that (i) holds.

If equality holds in (i), it follows from (1) that

$$|G|^{-1} \sum_{g \neq 1} (\det(1-tg))^{-1} \geqq \prod_{i=1}^{n} (1-t^{d_i})^{-1} - |G|^{-1}(1-t)^{-n},$$

if $0 \leqq t < 1$ (observe the left-hand side is a real number). Multiplying by $(1-t)^{n-1}$ and letting t tend to 1, it follows from an argument of Shephard-Todd (see [4, p. 111–112]) that the number of reflections in G is at least $\sum_{i=1}^{n} (d_i - 1)$. Let G' be the subgroup of G generated by the reflections contained in G. Let $e_1, \dots, e_n$ be the degrees of the reflection group G'. By 2.3(ii) we may assume that $e_i \leqq d_i$ $(1 \leqq i \leqq n)$. Since the number of reflections in G' is $\sum_{i=1}^{n} (e_i - 1)$ [loc. cit., p. 111], we find that

$$\sum_{i=1}^{n} (e_i - 1) \geqq \sum_{i=1}^{n} (d_i - 1).$$

But then we must have $e_i = d_i$. Since G' is a reflection group, we have $|G'| = \prod_{i=1}^{n} d_i$ [loc. cit., p. 110], hence $G' = G$. This establishes (i).

If the f_i generate R, then the right-hand side of (1) equals $\sum_{i=1}^{n} (1-T^{d_i})^{-1}$. Multiplying both sides of (1) by $(1-T)^n$ and putting $T=1$ we obtain equality in (i). Then (iii) follows from (ii).

2.5. In the remainder of this section we assume that G is a finite reflection group over $k = \mathbb{C}$, with degrees d_i. The integers $p_i = d_i - 1$ $(1 \leqq i \leqq n)$ are called the *exponents* of G. The result 2.6 to be discussed now is a known one (although not stated in the literature). It was first pointed out to me by Macdonald. Let I be the ideal in the algebra S of polynomial functions generated by the invariants of strictly positive degree, it is a G-stable graded ideal. Hence the quotient algebra S/I is a graded G-module. By a theorem of Chevalley one knows that the representation of G in S/I is equivalent to the regular representation (see [7, p. 779]). Let χ be an irreducible character of G, i.e. the character of an irreducible complex representation ρ of G. Then ρ occurs $\chi(1)$ times in the representation of G in S/I.

The grading gives a decomposition

$$S/I = \coprod_{i \geqq 0} (S/I)_i$$

into a finite number of G-stable subspaces. Let $(p_j(\chi))_{1 \leqq j \leqq \chi(1)}$ be the set of degrees i such that ρ occurs in $(S/I)_i$, each i occurring in this set with a multiplicity equal to that of ρ in $(S/I)_i$. The $p_j(\chi)$ are called the χ-*exponents* of G. If χ is the trivial character there is only one, viz. 0.

Define the polynomial $f_\chi \in \mathbb{Z}[T]$ by

$$f_\chi(T) = \sum_{j=1}^{\chi(1)} T^{p_j(\chi)}.$$

We then have the following result.

2.6. **Proposition.** *Let G be a complex reflection group with degrees d_i $(1 \leqq i \leqq n)$. Let χ be an irreducible character of G. Then*

$$|G|^{-1} \sum_{g \in G} \chi(g) (\det(1 - gT))^{-1} = f_\chi(T) \prod_{i=1}^{n} (1 - T^{d_i})^{-1}. \tag{2}$$

If χ is the trivial character, (2) reduces to a well-known formula which follows from (1) (see [4, p. 111]). We use the notations of 2.5. Let A be a graded G-stable subspace of S such that S is the direct sum of A and I (the existence of A follows by the complete reducibility of the action of G on the homogeneous components S_i). One then proves, by a straightforward induction on degrees, that the product map of S induces a bijection of G-modules

$$A \otimes_{\mathbb{C}} R \tilde{\rightarrow} S. \tag{3}$$

Now observe that by [loc. cit.], the left-hand side of (2) can be written as a formal power series

$$\sum_{i=0}^{\infty} a_i(\chi) T^i,$$

where $a_i(\chi)$ is the multiplicity of ρ in the representation of G in S_i. The isomorphism (3) allows one to determine these multiplicities, leading to (2). The argument is much the same as that used to deal with the special case $\chi = 1$.

2.7. If one assumes that G acts irreducibly in V then one may take for χ the character of the corresponding representation of G. The left-hand side of (2) can then be computed, by using a formula of Solomon [16], which shows that we then have

$$f_\chi(T) = \sum_{i=1}^{n} T^{d_i - 1},$$

so that we now recover the original exponents of G.

Another special case is that of a character χ of G of degree 1. Then there is only one exponent $p_1(\chi)$. The following result, which is an immediate consequence of the isomorphism (3), shows how this exponent can be found.

2.8. **Corollary.** *Let χ be a character of degree* 1 *of G. Its exponent $p_1(\chi)$ is the minimal degree of a nonzero homogeneous $f \in S$ such that*

$$g \cdot f = \chi(g) f \quad (g \in G).$$

Let δ be the character of degree 1 of G defined by

$$\delta(g) = (\det g).$$

If χ is an irreducible character of G, then $\delta\chi$ is also one, of the same degree. Denote by $\bar{\chi}$ the complex conjugate of χ, so that $\bar{\chi}(g) = \chi(g^{-1})$. We then have a connection between the exponents of χ and those of $\delta\bar{\chi}$.

2.9. **Lemma.** *Let $N = \sum_{i=1}^{n} p_i$. The family of $\delta\bar{\chi}$-exponents of G is $(N - p_i(\chi))_{1 \leq i \leq \chi(1)}$.*

This follows by replacing T by T^{-1} in (2).

2.9 implies that, for any χ, the χ-exponents are $\leq N$. It also follows that a χ-exponent equals N if and only if $\chi = \delta$. In that case there is a well-known polynomial f with the property of 2.8 (see [4, p. 113], our notations are slightly different).

3. Eigenvalues of Elements of Linear Groups, in Particular of Reflection Groups

3.1. We keep the notations of 1.1 and 1.2. We assume the field k to be algebraically closed. We shall use in this section some facts from algebraic geometry. In particular, we shall view V as an affine algebraic variety over k.

Fix a finite set of homogeneous invariants $f_1, \ldots, f_m$ which generate the k-algebra R (such a set exists by 1.3(ii)). We put $d_i = \deg f_i$ and we assume, as we may, that all d_i are > 0. If $k = \mathbb{C}$ and G is a reflection group we take f_i as in 2.2. In particular, we then have $m = n$.

Conversely, if $k = \mathbb{C}$ and if we have a set of generators f_i with $m = n$ then G is a reflection group. For if $m = n$ the f_i must be algebraically independent because of 1.3 (i), so that 2.4 (iii) applies.

We denote by H_i the algebraic hypersurface in V defined by f_i, i.e.

$$H_i = \{v \in V \mid f_i(v) = 0\}.$$

Fix a root of unity $\zeta \in k^*$, of order d. If $g \in G$ denote by $V(g, \zeta)$ the eigenspace of V corresponding to the eigenvalue ζ, i.e.

$$V(g,\zeta)=\{v\in V \mid g\cdot v=\zeta v\}.$$

We say that $V(g, \zeta)$ is *maximal* if it is not properly contained in another $V(h, \zeta)$.

3.2. **Proposition.** (i) *We have*

$$\bigcup_{g\in G} V(g,\zeta)=\bigcap_{d\nmid d_i} H_i.$$

(ii) *The irreducible components of the algebraic set* $\bigcap_{d\nmid d_i} H_i$ *are the maximal* $V(g, \zeta)$.

It follows from 1.5 that, given $v \in V$, there exists $g \in G$ with $g\cdot v=\zeta v$ if and only if $f(v)=0$ for all $f\in R$ which are homogeneous of a degree not divisible by d. It then suffices to require that $f_i(v)=0$ for all i such that d does not divide d_i. This proves (i). It follows that $\bigcap_{d\nmid d_i} H_i$ is the union of the distinct maximal $V(g,\zeta)$. Since each $V(g,\zeta)$ is an irreducible algebraic variety, (ii) now follows from a standard elementary result in algebraic geometry [12, p. 15].

3.3. **Corollary.** *Assume moreover that* G *acts irreducibly in* V. *Let* $p=\operatorname{char}(k)$ *and denote by* e *the part prime to* p *of the greatest common divisor of* $d_1, \ldots, d_m$ *(*e *equals this* $g\cdot c\cdot d$, *if* $p=0$*). Then the center of* G *is cyclic of order* e.

The center of G consists of scalar multiplications by Schur's lemma. If scalar multiplication by ζ lies in this center, then (using the previous notations) the set $\bigcap_{d\nmid d_i} H_i$ must be all of V by 3.2(i). This means that all d_i are divisible by d. Conversely, if this is so then $\zeta\cdot\mathrm{id}$ lies in the center of G, by 3.2(i), (ii). Then 3.3 follows by observing that, if $p>0$, the order d has to be prime to p.

Remark. If G is a Weyl group then 3.3 gives a well-known result [4, p.112].

From now on we assume that $k=\mathbb{C}$ and that G is a finite reflection group. So $m=n$ and the d_i are the degrees of G. For each integer d, we denote by $a(d)$ the number of d_i divisible by d. We can now sharpen 3.2 considerably.

3.4. **Theorem.** *Let* G *be a complex reflection group.*

(i) $\max_{g\in G} \dim V(g,\zeta)=a(d)$. *In particular, there exists* $g\in G$ *with eigenvalue* ζ *if and only if* d *divides a degree* d_i.

(ii) *For any* $g \in G$ *there exists* $h \in G$ *such that* $V(h, \zeta)$ *has maximal dimension* $a(d)$ *and that* $V(g, \zeta) \subset V(h, \zeta)$.

(iii) *If* $\dim V(g, \zeta) = \dim V(g', \zeta) = a(d)$ *then there exists* $h \in G$ *with* $h \cdot V(g, \zeta) = V(g', \zeta)$.

It follows from 1.5, with $w=0$, that $\bigcap_{i=1}^{n} H_i = \{0\}$. Hence, if C_i is an irreducible component of H_i (an irreducible hypersurface in V) we also have $\bigcap_{i=1}^{n} C_i = \{0\}$. It follows that the n irreducible hypersurfaces C_i intersect *properly* in the sense of [22, p. 146]. But then for any subset A of $\{1, \ldots, n\}$ the C_i with $i \in A$ also intersect properly, so that each irreducible component of $\bigcap_{i \in A} C_i$ has dimension $|A|(n-1)-(|A|-1)n = n-|A|$.

Consequently, each irreducible component of a partial intersection $\bigcap_{i \in A} H_i$ has dimension $n-|A|$. Applying this to the case that $A = \{i \mid d \nmid d_i\}$ and using 3.2(ii) we obtain (i). (ii) is also a consequence of 3.2(ii).

We come now to the proof of (iii). Let $\dim V(g, \zeta) = a(d)$. Number the f_i such that $f_1, \ldots, f_a$ (where $a = a(d)$) are those with degree divisible by d. We claim that the restrictions of the $(f_i)_{1 \leq i \leq a}$ to $V(g, \zeta)$ are *algebraically independent* polynomial functions on $V(g, \zeta)$. To show this, consider the morphism of algebraic varieties ϕ of $V(g, \zeta)$ to $\mathbb{C}^a$ with

$$\phi(v) = (f_1(v), \ldots, f_a(v)).$$

From $\bigcap_{i=1}^{n} H_i = \{0\}$ one concludes that the fibre $\phi^{-1}(0)$ consists of $\{0\}$ only (observe that the f_i with $i > a$ all vanish on $V(g, \zeta)$).

But by a known result [12, p. 92] this fibre has dimension at least $a - \dim \phi(V(g, \zeta))$. It follows that $\dim \phi(V(g, \zeta)) = a$, which can only be if, as we claimed, the restrictions of the $(f_i)_{1 \leq i \leq a}$ to $V(g, \zeta)$ are algebraically independent.

Let $U \subset V(g, \zeta)$ be the set of those elements which are not contained in a maximal space $V(h, \zeta)$ distinct from $V(g, \zeta)$. It is a Zariski-open subset of $V(g, \zeta)$.

Doing the same things for $V(g', \zeta)$ we obtain a morphism $\phi' : V(g', \zeta) \to \mathbb{C}^a$ and a Zariski-open subset U' of $V(g', \zeta)$. Now $\phi(U)$ and $\phi'(U')$ both contain a nonempty open subset of $\mathbb{C}^a$, hence have a nonempty intersection. This means that there exist $v \in V(g, \zeta)$ and $v' \in V(g', \zeta)$ such that (a) v' is not contained in a maximal $V(h, \zeta)$ distinct from $V(g', \zeta)$, (b) $f_i(v) = f_i(v')$ $(1 \leq i \leq n)$. By 1.5, it follows from (b) that there is $h \in G$ with $h \cdot v = v'$. But then $v' \in h \cdot V(g, \zeta) = V(h g h^{-1}, \zeta)$. By (a) this can only be if $h \cdot V(g, \zeta) = V(g', \zeta)$, which proves (iii).

Now fix $g \in G$ such that $\dim V(g, \zeta) = a(d)$. Let H be the stabilizer of $V(g, \zeta)$ in G and let H' be the normal subgroup of H whose elements fix $V(g, \zeta)$ elementwise. Let H_0 be the group of restrictions of elements of H to $V(g, \zeta)$. Clearly H_0 is isomorphic to H/H'.

3.5. **Proposition.** (i) *We have* $|H_0| \leqq \prod_{d|d_i} d_i$.

(ii) *If equality holds in* (i) *then* H_0 *is a complex reflection group in* $V(g, \zeta)$ *whose algebra of invariants is generated by the restrictions to* $V(g, \zeta)$ *of the* f_i *with* $d|d_i$.

(iii) *If no nontrivial element of* G *fixes* $V(g, \zeta)$ *elementwise then equality holds in* (i) *and* H *is isomorphic to* H_0.

We have seen in the proof of 3.4(iii) that the restrictions to $V(g, \zeta)$ of the f_i with $d|d_i$ are algebraically independent. (i) and (ii) now follow from 2.4(i) and 2.4(ii), respectively.

In the proof of (iii), we denote by V' the projective space defined by V and by H_i' the projective hypersurface defined by H_i ($1 \leqq i \leqq n$). It follows from 3.4(iii) that G acts transitively on the irreducible components of the intersection $\bigcap_{d \nmid d_i} H_i'$. By the definition of H, the number of these components is $|G|\,|H|^{-1}$.

Since the H_i with $d \nmid d_i$ intersect properly in V, the corresponding projective hypersurfaces H_i' intersect properly in V' and hence an intersection multiplicity is defined for each irreducible component of the latter intersection (see [22, p. 200]). Because of the transitivity of G, the multiplicity is the same for all components. Let μ be this common multiplicity.

By Bezout's theorem we have that the total number of irreducible components, each counted with its multiplicity, equals the product of the degrees of the intersecting projective hypersurfaces (for Bezout's theorem see [14, p. 107]).

It follows that

$$\prod_{d \nmid d_i} d_i = \mu |G|\,|H|^{-1}.$$

Since $|G| = \prod_{i=1}^{n} d_i$, we obtain

$$|H| = \mu \prod_{d|d_i} d_i.$$

Using (i) we find that $\mu \leqq |H'|$. Under the hypothesis of (iii) we have $|H'| = 1$, whence $\mu = 1$ and $|H| = |H_0| = \prod_{d|d_i} d_i$. This proves (iii).

4. Regular Elements in Reflection Groups

In this section, G is a complex reflection group. The notations are as before.

A vector $v \in V$ is called *regular* if v is not contained in any of the reflecting hyperplanes of G. The following known result, proved in [18], provides a characterization of the regular vectors.

4.1. **Proposition.** *Let $v \in V$, let G_v be the stabilizer of v in G.*

(i) *v is regular if and only if $G_v = \{1\}$.*

(ii) *G_v is generated by the reflections which it contains.*

As in no. 3 we denote by ζ a primitive d-th root of unity.

4.2. **Theorem.** *Let $g \in G$ be such that the eigenspace $V(g, \zeta)$ contains a regular vector. Then we have the following:*

(i) $g^d = 1$.

(ii) $\dim V(g, \zeta) = a(d)$.

(iii) *The centralizer of g in G is isomorphic to a reflection group in $V(g, \zeta)$, whose degrees are the d_i divisible by d and whose order is $\prod_{d \mid d_i} d_i$.*

(iv) *The elements of G with the property* (ii) *form a single conjugacy class.*

(v) *The eigenvalues of g are ζ^{-p_i}, where the p_i are the exponents of G.*

Let $v \in V(g, \zeta)$ be regular. Then g^d fixes v, hence is the identity by 4.1(i), which proves (i). By 3.4(ii) there is $h \in G$ with $\dim V(h, \zeta) = a(d)$, $V(g, \zeta) \subset V(h, \zeta)$. Then $h^{-1}g$ fixes v, whence, again by 4.1(i), $h = g$. This proves (ii). A similar argument, using 3.4(iii), proves (iv). (iii) follows from 3.5(iii). The proof of (v) is similar to the one given in [4, p. 122] for the case of a Coxeter element in a Weyl group. We briefly sketch the argument. Let $(e_i)_{1 \leqq i \leqq n}$ be a basis of V consisting of eigenvectors of g, with $e_1 \in V(g, \zeta)$ regular. By (i) the eigenvalues of g are d-th roots of unity, let ζ^{h_i} be the one corresponding to e_i. Let D_i be the derivation of S defined by e_i, and let

$$J = \det(D_i f_j)_{1 \leqq i, j \leqq n}.$$

By [4, p. 113] we have $J(e_1) \neq 0$. Consequently, there is a permutation s of $\{1, \dots, n\}$ such that

$$D_{s(i)} f_i(e_1) \neq 0 \qquad (1 \leqq i \leqq n).$$

It follows that the polynomial $f_i\left(\sum_{j=1}^{n} x_j e_j\right)$ must involve a monomial $x_1^{d_i - 1} x_{s(i)}$. The invariance of f_i then gives that $\zeta^{d_i - 1 + h_{s(i)}} = 1$, which implies (v).

We shall say that $g \in G$ is *regular* if g has a regular eigenvector. By 4.2(i) the order of the corresponding eigenvalue equals the order of g, hence is uniquely determined by g. By 4.2(ii), (iv) the regular elements of a given order are conjugate.

4.3. **Corollary.** *Let $g \in G$ be a regular element of order d, such that there is only one d_i divisible by d. Then the centralizer of g is cyclic of order d.*

This follows from 4.2(iii).

Now let G be a Weyl group, with an irreducible root system. Let $c \in G$ be Coxeter element. For this notion and for the properties of Coxeter elements we refer to [4, p. 116–123].

4.4. **Corollary.** *The centralizer of c in G is the cyclic group generated by c.*

Let the degrees d_i be ordered in increasing magnitude. One knows that the smallest degree d_1 equals 2 and that $d_2 > 2$ (because of the irreducibility assumption). Also we have that $d_i + d_{n+1-i}$ is constant, say $h+2$. Then $d_n = h$ and $d_{n-1} < h$. Now a Coxeter element has a regular eigenvector, whose eigenvalue is a h-th root of unity. 4.4 then follows from 4.3. For another proof of 4.4 see [5, p. 35–37].

We next discuss miscellaneous results about regular elements.

4.5. **Proposition.** *Let g be as in 4.2. Let ρ be an irreducible complex representation of G, with character χ. Then the eigenvalues of g in the representation ρ are the $\zeta^{-p_i(\chi)}$, where the $p_i(\chi)$ are the χ-exponents of G defined in 2.5.*

From formula (2) of no. 2 we obtain

$$|G|^{-1} \sum_{h \in G} \chi(h)\left(\det(1 - \zeta^{-1} h T)\right)^{-1} = f_\chi(\zeta^{-1} T) \prod_{i=1}^{n} (1 - \zeta^{-d_i} T^{d_i})^{-1}. \quad (1)$$

Multiply both sides of (1) by $(1-T)^{a(d)}$ and then put $T=1$. In the left-hand side of (1) we get nonzero contributions only from the h with $\dim V(h, \zeta) = a(d)$, all of which are conjugate to g by 4.2(iv). Using 4.2(iii), (v) a simple computation then gives that

$$\chi(g) = f_\chi(\zeta^{-1}).$$

Since any power g^i also satisfies the hypothesis of 4.2, with ζ^i instead of ζ, we have

$$\chi(g^i) = f_\chi(\zeta^{-i}).$$

This determines the restriction of χ to the cyclic group generated by g. The assertion then follows from the connection between f_χ and the χ-exponents (see 2.5).

4.5 is somewhat similar to a result proved for Coxeter elements of Weyl groups by Kostant in [11, p. 399].

4.6. **Proposition.** *Assume that G is a finite Coxeter group. Let $g \neq 1$ be a regular element of G which is not an involution.*

(i) *There exist two involutions* $s, t\in G$ *such that* $g=st$.

(ii) *If* s_1, t_1 *is a second pair of such involutions then there exists* $h\in G$ *centralizing* g *such that* $s_1=sh^{-1}$, $t_1=ht$.

We may assume that there is a real vector space V_0 such that $V=V_0\otimes_{\mathbb{R}}\mathbb{C}$ and that G stabilizes $V_0\otimes 1$. Let $v\mapsto\bar{v}$ be the semilinear map of V with $\overline{v_0\otimes c}=v_0\otimes\bar{c}$ (the last bar denoting complex conjugation). We have $\overline{g\cdot v}=g\cdot\bar{v}$ if $g\in G$. Moreover we may assume that the polynomials f_i are real, i.e. satisfy $\overline{f_i(v)}=f_i(\bar{v})$ $(1\leqq i\leqq n)$.

Let $v\in V(g,\zeta)$ be regular. Since g is not an involution, the order d of ζ is >2, so that ζ is not a real number.

From $g\cdot v=\zeta v$ it follows that $g^{-1}\cdot\bar{v}=\zeta\bar{v}$, whence $\overline{V(g,\zeta)}=V(g^{-1},\zeta)$. Now an argument like that used in the proof of 3.4 (ii) shows that there exists a regular $v\in V(g,\zeta)$ with $f_i(v)\in\mathbb{R}$ for all i. This implies that $f_i(v)=f_i(\bar{v})$ $(1\leqq i\leqq n)$, so that by 1.5 there is $t\in G$ with $t\cdot v=\bar{v}$. Then $t^2\cdot v=v$, whence $t^2=1$. It also follows that

$$(t^{-1}gt)\cdot v=\zeta^{-1}v=g^{-1}\cdot v,$$

which implies that $s=gt$ is also an involution. Hence $g=st$, as we claimed in (i).

We come now to the proof of (ii). Let v as before (with all $f_i(v)\in\mathbb{R}$) and put $v_1=t_1\cdot v$. Then $g^{-1}\cdot v_1=\zeta v_1$, so $v_1\in V(g^{-1},\zeta)$. Also $f_i(v_1)=f_i(v)=f_i(\bar{v})$. Let H be the centralizer of g in G, let H_0 be its restriction to $V(g^{-1},\zeta)$.

By 3.5 (ii), (iii) H_0 is a reflection group, whose algebra of invariants is generated by the restrictions to $V(g^{-1},\zeta)$ of the f_i with $d|d_i$. 1.5 now implies that there exists $h\in H$ with $v_1=h\cdot\bar{v}$. Hence $t_1\cdot v=(ht)\cdot v$, whence $t_1=ht$. Then $s_1=h^{-1}$. This finishes the proof of (ii).

Remark. 4.6 (i) was proved by Carter [5, p. 45] for all elements of Weyl groups, using a case by case discussion. We shall give in no. 8 another proof of Carter's result, based on 4.6.

4.7. **Proposition.** *Assume that* G *is a Weyl group. Let* g *be a regular element of order* d. *Then for each* i *prime to* d *we have that* g^i *is conjugate to* g *in* G.

Since G is a Weyl group, G stabilizes a structure over the field of rationals in V. It follows that if g has a regular eigenvector with eigenvalue ζ, a primitive d-th root of unity, then so does g^i. The assertion now follows from 4.2 (iv).

4.8. **Corollary.** *Let* G *and* g *be as in* 4.7. *Let* χ *be the character of a complex representation of* G. *Then* $\chi(g)$ *is a rational integer.*

This is clear. Again, this is true for any element of a Weyl group. We shall return in no. 8 to the more general result.

4.9. From now on we assume that G is a Weyl group. Let R be its root system. The roots are elements in V, they span a real vector space $V_0 \subset V$, such that $V \simeq V_0 \otimes_{\mathbb{R}} \mathbb{C}$.

We fix a positive definite symmetric bilinear form on $V_0 \times V_0$ which is G-invariant. We extend it to a positive definite hermitian form $(,)$ on $V \times V$.

An ordering on V_0 defines an ordering of R, which in turn fixes a basis B of R (in the sense of [4, p. 153]) and a set of generators S of G. The elements of S are reflections of order 2. One can speak of the length $l_S(g)$ of an element of G with respect to S.

If $R_1 \subset R$ is a closed subsystem of R [4, p. 160], the reflections defined by the roots in R_1 generate a subgroup G_1 of G, called a *Weyl subgroup*. The *rank* of G_1 is the dimension of the subspace of V generated by the roots of R_1.

4.10. **Proposition.** *Let $g \in G$ be a regular element of order d.*

(i) *The elements of R are permuted by g in orbits of length d.*

(ii) *There exists an ordering of R such that, with respect to the set S of generators of G defined by it, g has length $d^{-1}|R|$.*

(iii) *If g has no eigenvalue 1, the length of g with respect to any set S of generators, defined by an ordering of R, is at least $d^{-1}|R|$.*

Let v be a regular vector in $V(g,\zeta)$, ζ being a primitive d-th root of unity. Suppose that $\alpha \in R$, $g^s \cdot \alpha = \alpha$. Then

$$(v,\alpha) = \zeta^s (v,\alpha).$$

Since v is regular, we have $(v,\alpha) \neq 0$ for all $\alpha \in R$ (otherwise v would lie in the reflecting hyperplane defined by α). Hence $\zeta^s = 1$ and s is divisible by d, which implies (i) (this argument is due to Kostant, see [10, p. 1021]).

Since G is a Weyl group, together with ζ all its conjugates over the rationals occur as eigenvalues of g. Hence we may assume that $\zeta = e^{2\pi i d^{-1}}$. Multiplying v by a suitable nonzero scalar, we may and shall assume that the real parts $\mathrm{Re}(v,\alpha)$ are nonzero for all $\alpha \in R$. We define an ordering on R by declaring $\alpha \in R$ to be positive if $\mathrm{Re}(v,\alpha) > 0$ and negative otherwise.

Observe that $\mathrm{Re}(v, g\cdot\alpha) = \mathrm{Re}(\zeta^{-1}(v,\alpha))$. Let $\{\alpha, g\alpha, \dots, g^{d-1}\cdot\alpha\}$ be a g-orbit in R. The d complex numbers $(v, g^i\alpha)_{0 \leq i < d}$ are located at the vertices of a regular d-gon in the complex plane.

The action of g on the orbit determines a rotation over an angle $-2\pi d^{-1}$ of the d-gon. It follows that there is exactly one i with $0 \leqq i < d$ such that $\mathrm{Re}(v, g^i\cdot\alpha) > 0$, $\mathrm{Re}(v, g^{i+1}\cdot\alpha) < 0$. From (i) we conclude that the number of roots $\alpha<0$ with $g\cdot\alpha<0$ equals $d^{-1}|R|$. One knows that this number equals the length of g, see [21, p. 7], whence (ii). The argument is similar to the one given by Steinberg in [19, p. 57–58]. If g has no eigenvalue 1, the sum of the roots in a g-orbit must be 0, so that each g-orbit contains at least one $\alpha>0$ with $g\cdot\alpha<0$. So (iii) is a consequence of (i).

We terminate this section by two results which can be used to decide whether a given element of a Weyl group is regular.

4.11. **Lemma.** *An element g of a Weyl group whose eigenvalues are distinct primitive d-th roots of unity is regular.*

Let ζ be one of the eigenvalues. If $v\in V(g,\zeta)$ is nonregular, there exists $\alpha\in R$ such that v lies in the hyperplane H defined by α. There is a G-stable $\mathbb{Q}$-structure on V. We may assume that v is defined over some Galois extension $k/\mathbb{Q}$ containing ζ. It then follows that for all s in the Galois-group of $k/\mathbb{Q}$ we have that $sv\in H$. But our assumptions imply that these sv generate V, which leads to a contradiction.

4.12. **Lemma.** *Let G be a Weyl group. If $v\in V(g,\zeta)$ is nonzero and nonregular, there exists a Weyl subgroup G_1 of G whose rank is strictly smaller than* $\dim V$, *such that the degree d of ζ divides one of the degrees of G_1. The same is true for Coxeter groups.*

We use the notations of 4.9. If g has an eigenvalue 1, g fixes an element of $V_0\otimes 1$, and the assertion follows from 4.1 (ii).

Assume now that g has no eigenvalue 1. By 4.1 (ii) there is $\alpha\in R$ such that the reflection r_α defined by α fixes v. Let $w=(g-1)^{-1}\cdot\alpha$, then $g\cdot w=w+\alpha$. It follows that $2(w,\alpha)+(\alpha,\alpha)=0$, which implies that $r_\alpha\cdot w=w+\alpha$. But then $r_\alpha\cdot g$ has an eigenvalue 1, and $v\in V(r_\alpha\cdot g,\zeta)$, so that we are in the case considered first. The proof for Coxeter groups is quite similar, it is left to the reader.

5. Regular Elements in Coxeter Groups

As an illustration and application of the results of the preceding section we shall discuss now the regular elements of an irreducible Coxeter group G. We denote by $g\in G$ a regular element of order $d\geqq 2$. If G is a Weyl group, the conjugacy class of g is uniquely determined by d (see 4.2). The $d\geqq 2$ which occur as orders of regular elements are called the *regular numbers* of G.

We first deal with the case of a Weyl group G. Let R be its root system. We discuss now the various types of irreducible root systems R. We refer to [4] for the properties of root systems to be used below.

5.1. *Type* A_n ($n \geq 1$). We take

$$V = \left\{ x = (x_1, \ldots, x_{n+1}) \in \mathbb{C}^{n+1} \,\middle|\, \sum_{i=1}^{n+1} x_i = 0 \right\}.$$

Let $(e_i)_{1 \leq i \leq n+1}$ be the canonical basis of $\mathbb{C}^{n+1}$. The root system R consists of the vectors $e_i - e_j$ ($i \neq j$). The Weyl group G is the symmetric group $\mathfrak{S}_{n+1}$, its action on V being the restriction of the action on $\mathbb{C}^{n+1}$ defined by

$$s \cdot e_i = e_{s(i)} \qquad (s \in \mathfrak{S}_{n+1}).$$

The degrees of G are given by $d_i = i+1$ ($1 \leq i \leq n$). We have $a(d) = [d^{-1}(n+1)]$.

Now if $s \in \mathfrak{S}_{n+1}$ has $e(j)$ orbits of length j in $\{1, \ldots, n+1\}$ then its characteristic polynomial $P_s(T)$ in V is

$$\prod_{j \geq 1} (T^j - 1)^{e(j)} (T-1)^{-1}$$

as one easily sees. By 4.2(v) we find that $P_g(T)$ is divisible by $(T^d - 1)^{a(d)}$, so that g has $[d^{-1}(n+1)]$ orbits of length d. If $[d^{-1}(n+1)]\, d < n$, we would find from 4.2(v) that g had an eigenvalue ζ^{-1} with multiplicity $> a(d)$, which is impossible. It follows that there are two possibilities:

(a) $d \mid n+1$ and g has $d^{-1}(n+1)$ orbits of length d,

(b) $d \mid n$, g fixes one element and permutes the others in $d^{-1} n$ orbits of length d.

In the first case g is a power of a Coxeter element of G, in the second case it is a power of a Coxeter element of a Weyl subgroup of type A_{n-1}. Since such Coxeter elements have a regular eigenvector in V (as one easily sees) and since obviously a power of a regular element is also regular, it follows that the elements of types (a) and (b) are indeed regular.

Thus the regular numbers are the divisors ≥ 2 of n and $n+1$.

5.2. *Types* B_n, C_n ($n \geq 2$). We have $V = \mathbb{C}^n$. Now G consists of the group of all linear transformations t of V with

$$t \cdot e_i = \varepsilon_i e_{s(i)}, \tag{1}$$

$\varepsilon_i = \pm 1$, $s \in \mathfrak{S}_n$. As before, (e_i) is the canonical basis. G is the Weyl group of the two root systems B_n and C_n, the first one consisting of the vectors $\pm e_i \pm e_j$ ($i \neq j$) and $\pm e_i$, the second one of the vectors $\pm e_i \pm e_j$ ($i \neq j$) and $\pm 2 e_i$.

The degrees of G are the integers $2, 4, \ldots, 2n$. It follows that $a(d) = [d^{-1} n]$ if d is odd and $a(d) = [2 d^{-1} n]$ if d is even. Let t be as above. An orbit $\{i, s \cdot i, \ldots, s^{l-1} \cdot i\}$ of s in $\{1, \ldots, n\}$ is called a *positive* (*negative*) orbit of t if

$$\prod_{j=0}^{l-1} \varepsilon_{s^j \cdot i} = +1 \, (-1),$$

see [5, p. 25]. If t has $e_+(j)$ positive orbits of length j and $e_-(j)$ negative orbits of that length, then its characteristic polynomial in V is

$$P_t(V)=\prod_{j\geq 1}(X^j-1)^{e_+(j)}(X^j+1)^{e_-(j)}.$$

Using 4.2(v) one finds, in a manner similar to that of 5.1, that there are the following possibilities:

(a) d is an odd divisor of n, g has $d^{-1}n$ positive orbits of length d,

(b) d is an even divisor of $2n$, g has $2d^{-1}n$ negative orbits of length $\frac{1}{2}d$.

The Coxeter elements of G are those with a negative orbit of length n. One easily sees that each element g with the properties of (a) or (b) is a power of a Coxeter element.

Since a Coxeter is regular (in the present case this can also easily be checked directly) it follows that (a) and (b) describe the regular elements of G.

The regular numbers are the divisors ≥ 2 of $2n$.

5.3. *Type* D_n $(n\geq 4)$. Again, $V=\mathbb{C}^n$. The Weyl group G is the group of all linear transformations of the form (1), with $\prod_{i=1}^{n}\varepsilon_i=1$. Hence G is the subgroup of the group considered in 5.2 consisting of the elements with an even number of negative orbits.

R consists of the vectors $\pm e_i \pm e_j$ $(i\neq j)$. The degrees of G are the integers $2, 4, \ldots, 2n-2, n$. Put $\delta(x)=1$ if x is an integer and $\delta(x)=0$ otherwise. Then

$$a(d)=[d^{-1}(n-1)]+\delta(d^{-1}n) \qquad \text{if } d \text{ is odd},$$
$$a(d)=[d^{-1}(2n-2)]+\delta(d^{-1}n) \qquad \text{if } d \text{ is even}.$$

We find the following possibilities:

(a) d is an odd divisor of n, all orbits of g are positive of length d,

(b) d is an odd divisor of $n-1$, g has one positive orbit of length 1 and $d^{-1}(n-1)$ positive orbits of length d,

(c) n is even, d is an even divisor of n, all orbits of g are negative of length $\frac{1}{2}d$.

(d) d is an even divisor of $2n-2$, g has $d^{-1}(2n-2)$ negative orbits of length $\frac{1}{2}d$ and moreover another orbit of length 1, which is positive or negative according as $d^{-1}(2n-2)$ is even or odd.

The Coxeter elements are those of type (d), with $d=2n-2$. The powers of the Coxeter elements are those of the form (b) and (d). An element of the form (a) or (c) is a power of an element with one positive orbit of length n or of one with 2 negative orbits of lengths $\frac{1}{2}n$. The regular numbers are the divisors $\geqq 2$ of $2n-2$ and n.

5.4. *The Exceptional Types.* Here we shall give the results in the form of tables. We have listed in the tables below for the exceptional types the possible regular numbers d and for each d the degrees of the centralizer of g (which, by 4.2 (iii), is isomorphic to a reflection group) and the characteristic polynomial of g. We denote by Φ_s the cyclotomic polynomial whose roots are the primitive s-th roots of unity.

Table 1. Type E_6 (degrees 2, 5, 6, 8, 9, 12)

d	Degrees centralizer	Characteristic polynomial
2	2, 6, 8, 12	$\Phi_2^4\,\Phi_1^2$
3	6, 9, 12	Φ_3^3
4	8, 12	$\Phi_4^2\,\Phi_1^2$
6	6, 12	$\Phi_6^2\,\Phi_3$
8	8	$\Phi_8\,\Phi_2\,\Phi_1$
9	9	Φ_9
12	12	$\Phi_{12}\,\Phi_3$

Table2. Type E_7 (degrees 2, 6, 8, 10, 12, 14, 18)

d	Degrees Centralizer	Characteristic polynomial
2	2, 6, 8, 10, 12, 14, 18	Φ_2^7
3	6, 12, 18	$\Phi_3^3\,\Phi_2$
6	6, 12, 18	$\Phi_6^3\,\Phi_2$
7	14	$\Phi_{14}\,\Phi_1$
9	18	$\Phi_9\,\Phi_1$
14	14	$\Phi_{14}\,\Phi_2$
18	18	$\Phi_{18}\,\Phi_2$

Table 3. Type E_8
(degrees 2, 8, 12, 14, 18, 20, 24, 30)

d	Degrees centralizer	Characteristic polynomial
2	2, 8, 12, 14, 18, 20, 24, 30	Φ_2^8
3	12, 18, 24, 30	Φ_3^4
4	8, 12, 20, 24	Φ_4^4
5	20, 30	Φ_5^2
6	12, 18, 24	Φ_6^4
8	8, 24	Φ_8^2
10	20, 30	Φ_{10}^2
12	12, 24	Φ_{12}^2
15	30	Φ_{15}
20	20	Φ_{20}
24	24	Φ_{24}
30	30	Φ_{30}

Table 4. Type F_4
(degrees 2, 6, 8, 12)

d	Degrees centralizer	Characteristic polynomial
2	2, 6, 8, 12	Φ_2^4
3	6, 12	Φ_3^2
4	8, 12	Φ_4^2
6	6, 12	Φ_6^2
8	8	Φ_8
12	12	Φ_{12}

Finally, in type G_2 there are three regular classes, viz. those of the Coxeter elements and their powers ± 1, so that the regular numbers are 2, 3, 6.

To establish the results contained in these tables, one first observes that the regular numbers d must be divisors of degrees d_i, by 3.4 (i). The maximal value of d is the order of a Coxeter element, which is regular. Using that a power of a regular element is regular one then finds already quite a few of the regular numbers.

To illustrate the arguments we shall discuss the case of type E_8. The Coxeter element then gives that 2, 3, 5, 6, 10, 15, 30 are regular numbers. The remaining ones occur among the numbers 4, 7, 8, 9, 12, 14, 18, 20, 24. Using 4.11 one sees that 20 and 24 are regular numbers. It follows that 4, 8, 12 are also regular. There remain the numbers 7, 9, 14, 18 which must be ruled out. If $d=18$ were regular, then 4.2 (v) would give the eigenvalues of the corresponding element. However one sees that the eigenvalues given by 4.2 (v) cannot be those of a linear transformation over the field of rationals, so $d=18$ is nonregular.

For 7, 9, 14 the argument is similar.

Most of the results of the tables are established by this kind of argument. In a few cases an additional reasoning is needed:

(a) To prove that 8 is a regular number for E_6 one observes that 4 is one (since the order of a Coxeter element is 12). If an element g having a primitive 8th root of unity as eigenvalue were nonregular then an argument like that of the proof of 4.11 gives that there is a root orthogonal to the space spanned by the eigenvectors for such eigenvalues. But since g^2 is regular, this space contains a regular vector, which is a contradiction.

(b) To prove that 14 is a regular number for E_7 one uses 4.12, noticing that all degrees of Weyl groups of rank $\leqq 6$ are less than 14.

5.5. *The Other Coxeter Groups.* The remaining finite irreducible Coxeter groups are either dihedral or are of types H_3, H_4 (see [4, p. 194]). The dihedral case is easily dealt with and is left to the reader. The Tables 5 and 6 give the regular numbers for H_3 and H_4. For the degrees of H_3 and H_4 see [8, p. 772].

Table 5. Type H_3 (degrees 2, 6, 10)

d	Degrees centralizer	Number of classes
2	2, 6, 10	1
3	6	1
5	10	2
6	6	1
10	10	2

Table 6. Type H_4 (degrees 2, 12, 20, 30)

d	Degrees centralizer	Number of classes
2	2, 12, 20, 30	1
3	12, 30	1
4	12, 30	1
5	20, 30	2
6	12, 30	1
10	20, 30	2
12	12	1
15	30	2
20	20	2
30	30	2

The results about the regular numbers are established in a way similar to that followed before. One first uses the existence of Coxeter elements and also an appropriate extension of 4.11 (taking into account that H_3 and H_4 are defined over $\mathbb{Q}(\sqrt{5})$). Some trouble is given by $d=6$ for type H_3. There one can establish regularity by using the geometric description of H_3 as a group of orthogonal transformations in $\mathbb{R}^3$ leaving invariant an icosahedron.

For any of the above d there exists an element in the group G of type H_3 or H_4 having an eigenvalue ζ which is a primitive d-th root of unity. For different ζ, these elements need not be conjugate. Using 4.2 (v) one can see how many classes there are which such eigenvalues. The result is given in the tables.

5.6. The results of this section lead to certain complex reflection groups. Consider, for example, a regular element g of order 10 in a Weyl group of type E_8. From Table 3 one sees that the centralizer of g is isomorphic to a 2-dimensional complex reflection group H in a suitable space $V(g, \zeta)$, with degrees 20 and 30. One easily sees that H acts irreducibly. By 3.3 the centre Z of H is cyclic of order 10. Then H/Z is isomorphic to the icosahedral group (see [15, p. 286]). The discussion in [loc.cit.] of H and of similar groups is bases on the properties of Klein's polyhedral groups. One can proceed the other way round and derive the main results about polyhedral groups from the theory of Weyl groups, making use of the results of this paper. We shall not go into this matter here.

6. An Extension

Let G be a reflection group in the complex vector space V, let $f_1, \ldots, f_n$ be, as before, algebraically independent generators of the algebra of invariants of G. We put $d_i = \deg(f_i)$. We denote by σ a linear transformation of V of finite order such that $\sigma G \sigma^{-1} = G$. In this section we shall extend a number of results, obtained in the previous sections about eigenvalues of elements of G, to eigenvalues of elements of $G\sigma$.

6.1. **Lemma.** (i) *The f_i may be chosen such that $\sigma \cdot f_i = \varepsilon_i f_i$ $(1 \leqq i \leqq n)$, with suitable roots of unity ε_i.*

(ii) *The set of pairs $(d_i, \varepsilon_i)_{1 \leqq i \leqq n}$ is independent of the choice of the f_i.*

Denote by R_j the space of homogeneous invariants of degree j. Then σ acts on R_j so that R_j may be decomposed into eigenspaces for σ. Define inductively f_i to be an eigenvector of σ of degree d_i which is algebraically independent of $f_1, \ldots, f_{i-1}$. By 2.3 (iii) these f_i will be as required. (ii) follows by the same kind of argument as was used to prove 2.3 (iii).

We call the ε_i the *factors* of σ. Fix a primitive d-th root of unity ζ. Let $a(d, \sigma)$ be the number of i such that

$$\varepsilon_i \zeta^{d_i} = 1 \quad (1 \leqq i \leqq n).$$

We denote by $V(g\sigma, \zeta)$ the eigenspace of $g\sigma$ for the eigenvalue ζ.

6.2. **Theorem.** (i) $\max_{g\in G} \dim V(g\sigma,\zeta)=a(d,\sigma)$. *In particular, there exists* $g\in G$ *such that* $g\sigma$ *has an eigenvalue* ζ *if and only if* $\varepsilon_i\zeta^{d_i}=1$ *for some i.*

(ii) *For any* $g\in G$ *there exists* $h\in G$ *such that* $V(h\sigma,\zeta)$ *has maximal dimension* $a(d,\sigma)$ *and that* $V(g\sigma,\zeta)\subset V(h\sigma,\zeta)$.

(iii) *If* $\dim V(g\sigma,\zeta)=\dim V(g'\sigma,\zeta)=a(d,\sigma)$ *then there exists* $h\in G$ *such that* $h\cdot V(g\sigma,\zeta)=V(g'\sigma,\zeta)$.

The proof runs parallel to that of 3.4 and will be left to the reader.

Fix $g\in G$ such that $V(g\sigma,\zeta)$ has maximal dimension $a(d,\sigma)$. Let H be the stabilizer of $V(g\sigma,\zeta)$ in G and let H' be the normal subgroup of H whose elements fix $V(g\sigma,\zeta)$ elementwise. Let H_0 be the group of restrictions of elements of H to $V(g\sigma,\zeta)$.

6.3. **Proposition.** (i) *We have* $|H_0|\leqq \prod_{\varepsilon_i\zeta^{d_i}=1} d_i$.

(ii) *If equality holds in* (i) *then* H_0 *is a reflection group in* $V(g\sigma,\zeta)$, *whose algebra of invariants is generated by the restrictions to* $V(g\sigma,\zeta)$ *of the* f_i *with* $\varepsilon_i\zeta^{d_i}=1$.

(iii) *If no nontrivial element of G fixes* $V(g\sigma,\zeta)$ *elementwise, then equality holds in* (i) *and H is isomorphic to* H_0.

This is an extension of 3.5. The proof is as before. We deduce from it the following extension of 4.2.

6.4. **Theorem.** *Let* $g\in G$ *be such that* $V(g\sigma,\zeta)$ *contains a regular vector. Then we have the following:*

(i) *If* $\sigma^d=1$ *then* $g\sigma$ *has order d.*

(ii) $\dim V(g\sigma,\zeta)=a(d,\sigma)$.

(iii) *The centralizer of* $g\sigma$ *in G is isomorphic to a reflection group in* $V(g\sigma,\zeta)$, *whose degrees are the* d_i *with* $\varepsilon_i\zeta^{d_i}=1$, *its order being the product of these* d_i.

(iv) *If* $\dim V(g\sigma,\zeta)=\dim V(g'\sigma,\zeta)=a(d,\sigma)$ *then* $g\sigma$ *and* $g'\sigma$ *are conjugate by an element of G.*

(v) *The eigenvalues of* $g\sigma$ *are the* $\varepsilon_i^{-1}\zeta^{-p_i}$ $(1\leqq i\leqq n)$, *where the* p_i *are the exponents of G.*

The proof is like that of 4.2.

If $g\sigma$ has a regular eigenvector we say that $g\sigma$ is a *regular element* of $G\sigma$. There is an interesting particular case (which becomes trivial in the situation of no. 4).

6.5. **Corollary.** *Assume that* σ *has a regular eigenvector with eigenvalue* 1;

(i) *The eigenvalues of* σ *are the* ε_i^{-1} $(1\leqq i\leqq n)$.

(ii) *The centralizer* G_σ *of* σ *in G is a reflection group in* $V(\sigma,1)$, *whose degrees are the* d_i *such that* $\varepsilon_i=1$.

Remark. If, in the situation of 6.5, G is a Coxeter group, so that V has a structure over $\mathbb{R}$ and if σ is defined over $\mathbb{R}$, then G_σ is also a Coxeter group. Similarly for a Weyl group.

6.6. Now let G be a Weyl group, with root system R. Assume that there is an ordering of R such that σ stabilizes the set of all positive roots. Put

$$2\rho = \sum_{\alpha>0} \alpha.$$

Then ρ is a regular vector, as is well-known, which is fixed by σ. So 6.5 applies.

This situation was discussed by Steinberg in [21]. 6.5 (i) and the first part of 6.5 (ii) are proved in [loc. cit., p. 22 and p. 15]. The last assertion of 6.5 (ii) is stated in [loc. cit., p. 22] as an observation.

We next give extensions of 4.6 and 4.10 (i). The proofs are as before.

6.7. **Proposition.** *Assume that G is a Coxeter group, that $\sigma^2 \in G$ and that $g\sigma$ is a regular element $\neq 1$ which is not an involution.*

(i) *There exists an involution $s \in G\sigma$ and an involution $t \in G$ such that $g\sigma = st$.*

(ii) *If s_1, t_1 is a second pair of such involutions then there exists $h \in G$ centralizing $g\sigma$ such that $s_1 = sh^{-1}$, $t_1 = ht$.*

6.8. **Lemma.** *Let G be a Weyl group with root system R. Assume that σ fixes R. Let $g\sigma$ have a regular eigenvector whose eigenvalue is a primitive d-th root of unity.*

(i) *$g\sigma$ permutes the roots of R in orbits of length d.*

(ii) *There exists an ordering of R such that each orbit of $g\sigma$ contains exactly one $\alpha > 0$ with $g\sigma \cdot \alpha < 0$.*

Let G be an irreducible Weyl group with root system R. The non-trivial possibilities for σ are discussed in [21]. We shall review them below, and find the regular elements of $G\sigma$ in these cases. The discussion will be quite similar to that of no. 5. $g\sigma$ denotes a regular element, the corresponding eigenvalue being a primitive d-th root of unity ζ, with $d > 1$.

6.9. *Type A_n ($n \geq 2$).* Let the notations be as in 5.1. There is only one possible coset $G\sigma$, viz. $-\mathfrak{S}_{n+1}$. So we may take $\sigma = -1$. We have $d_j = j+1$, $\varepsilon_j = (-1)^{j+1}$ ($1 \leq j \leq n$). It follows that

$$\begin{aligned} a(d,\sigma) &= [\tfrac{1}{2} d^{-1}(n+1)] && \text{if } d \text{ is odd},\\ &= [d^{-1}(n+1)] && \text{if } d \equiv 0 \pmod 4,\\ &= [2d^{-1}(n+1)] && \text{if } d \equiv 2 \pmod 4,\ d > 2,\\ &= n && \text{if } d = 2. \end{aligned}$$

Proceeding as in 5.1 we find the following possibilities for $g\sigma$:

(a) n and d are odd, g has $\frac{1}{2}d^{-1}(n+1)$ orbits of length $2d$,

(b) n is even, d is odd, g has $\frac{1}{2}d^{-1}n$ orbits of length $2d$ and one of length 1;

(c) $n \equiv 3 \pmod 4$, $d \equiv 0 \pmod 4$, g has $d^{-1}(n+1)$ orbits of length d;

(d) $n \equiv d \equiv 0 \pmod 4$, g has d^{-1} orbits of length d and one of length 1,

(e) $d \equiv 2 \pmod 4$, g has $2d^{-1}(n+1)$ orbits of length $\frac{1}{2}d$,

(f) $d \equiv 2 \pmod 4$, g has $2d^{-1}n$ orbits of length $\frac{d}{2}$ and one of length 1.

The maximal value of d is $2n+2$ if n is even and $2n$ if n is odd (given by the element of type (e) and (f), respectively).

6.10. *Types* B_2, C_2. Notations being as in 5.2, with $n=2$, we may take σ to be given by

$$\sigma \cdot e_1 = 2^{-\frac{1}{2}}(-e_1+e_2),$$
$$\sigma \cdot e_2 = 2^{-\frac{1}{2}}(e_1+e_2).$$

We have $d_1=2$, $d_2=4$, $\varepsilon_1=1$, $\varepsilon_2=-1$. One checks that the possibilities for a regular $g\sigma$ are:

(a) $d=2$, $g=1$,

(b) $d=4$, g a Coxeter element of G.

6.11. *Type* D_n $(n \geqq 4)$. In this case there is a σ of order 2, which (with the notations of 5.3) we can take to be given by $\sigma \cdot e_i = e_i$ $(1 \leqq i \leqq n-1)$, $\sigma \cdot e_n = -e_n$. We have $d_i = 2i$, $\varepsilon_i = 1$ $(1 \leqq i \leqq n-1)$, $d_n = n$, $\varepsilon_n = -1$. It follows that

$$a(d,\sigma) = [d^{-1}(n-1)] \qquad \text{if } d \text{ is odd},$$
$$= [d^{-1}(2n-2)] + \delta(d^{-1}n + \tfrac{1}{2}) \qquad \text{if } d \text{ is even},$$

δ being as in 5.3.

$g\sigma$ can now be considered to be an element of the Weyl group of type B_n and must then have an odd number of negative orbits. One finds the following possibilities:

(a) d is an odd divisor of $n-1$, $g\sigma$ has $d^{-1}(n-1)$ positive orbits of length d and one negative one of length 1,

(b) d is an even divisor of $2n-2$, $g\sigma$ has $2d^{-1}(n-1)$ negative orbits of length $\frac{1}{2}d$ and one orbit of length d,

(c) $d>2$ is an even divisor of $2n$ such that $2d^{-1}n$ is odd, $g\sigma$ has $2d^{-1}n$ negative orbits of length $\frac{1}{2}d$.

The maximal possible value of d is $2n$, given by a $g\sigma$ which has one negative orbit of length n.

6.12. *Exceptional Types.* In the remaining cases we say that (G, σ) is of *exceptional type.* They are the following ones: G is of type D_4 and σ has order 3, G is of type E_6, F_4, G_2 and σ has order 2. The factors are given in [21, p. 81–82].

The results are given in the tables below, which are similar to the ones in no. 5.

Table 7. Type D_4 (degrees 2, 4, 4, 6; factors 1, $e^{2\pi i/3}$, $e^{4\pi i/3}$, 1)

d	Degrees centralizer	Characteristic polynomial
2	2, 6	$\Phi_2^2 \Phi_1^2$
3	4, 6	Φ_3^2
6	4, 6	Φ_6^2
12	4	Φ_{12}

Table 8. Type E_6 (degrees 2, 5, 6, 8, 9, 12; factors 1, −1, 1, 1, −1, 1)

d	Degrees centralizer	Characteristic polynomial
2	2, 5, 6, 8, 9, 12	Φ_2^6
3	6, 12	$\Phi_6 \Phi_3^2$
4	8, 12	$\Phi_4^2 \Phi_2^2$
6	6, 9, 12	Φ_6^3
8	8	$\Phi_8 \Phi_2 \Phi_1$
12	12	$\Phi_{12} \Phi_6$
18	9	Φ_{18}

Table 9. Type F_4 (degrees 2, 6, 8, 12; factors 1, −1, 1, −1)

d	Degrees centralizer	Characteristic polynomial
2	2, 8	$\Phi_2^2 \Phi_1^2$
4	6, 8	Φ_4^2
8	8, 12	$(T^2 - 2^{\frac{1}{2}} T + 1)^2$
12	6	Φ_{12}
24	12	$\left(T^2 - 2\cos\frac{5\pi}{12} T + 1\right)\left(T^2 - 2\cos\frac{7\pi}{12} T + 1\right)$

In type G_2 the degrees are 2 and 6, with factors $1, -1$. There is a regular element with $d=2$, centralizer of order 2 and characteristic polynomial $\Phi_2 \Phi_1$ and one with $d=12$, centralizer of order 6 and characteristic polynomial $\left(T^2 - 2\cos\frac{\pi}{6} T + 1\right)$.

These results are derived in much the same manner as in no. 5. We also use another lemma. To state it, we recall that σ induces a permutation ρ of the root system R, see [21, p. 73].

6.13. **Lemma.** *Assume that there is a basis B of R which is stabilized by ρ and such that the ρ-orbits in B consist of mutually orthogonal roots. Then there exists a Coxeter element of G which commutes with σ.*

Let $\alpha_1, \ldots, \alpha_n$ be an ordering of the roots of B such that those in a ρ-orbit are consecutive. Let $r_\alpha \in G$ be the reflection defined by $\alpha \in R$. Then $c = r_{\alpha_1} \ldots r_{\alpha_n}$ is a Coxeter element with the required property.

That in the case of D_4 we find $d=12$ is proved by an argument like that of 4.11. Taking powers we can deal with $d=3, 6$. The case $d=2$ is given by the element $-\sigma$.

Table 8 comes from Table 1, using that we may take $\sigma=-1$ if G is of type E_6.

The case $d=24$ in Table 9 is taken care of by the argument of 4.11. It then also follows that $d=8$ occurs in Table 9. The case $d=12$ comes from 6.13, and gives $d=4$. Finally, $d=2$ is given again by $-\sigma$.

We restrict ourselves to these indications, leaving the details to the reader.

7. Twisted Coxeter Elements

7.1. Let G be a Weyl group in V, with root system R. Let σ be a linear transformation of V of finite order, with $\sigma\notin G$, $\sigma G\sigma^{-1}=G$. Assume that there is a basis B of R which is fixed by σ (so that σ fixes R). In that situation one can introduce a "twisted" version of Coxeter elements, as was observed by Steinberg (unpublished). We shall show how the results of no. 6 can be used to establish the properties of the twisted Coxeter elements.

As a preliminary we give below, in the cases where R is irreducible the maximal possible order h_σ of the eigenvalue of an element $g\sigma$. We also have listed the dimension n_σ of the space of vectors of V fixed by σ. It is readily seen that n_σ equals the number of σ-orbits in B.

Table 10

Type	h_σ	n_σ
A_n (n even)	$2n+2$	$\frac{1}{2}n$
A_n (n odd and >1)	$2n$	$\frac{1}{2}(n+1)$
D_n ($\sigma^2=1$)	$2n$	$n-1$
D_4 ($\sigma^3=1$)	12	2
E_6	18	4

7.2. **Lemma.** (i) *We have* $h_\sigma n_\sigma=|R|$.

(ii) *An element* $g\sigma$ *with an eigenvalue of order* h_σ *is regular.*

(i) follows from Table 10 by inspection. (ii) follows from the results of no. 6.

7.3. Let $r_\alpha\in G$ be the reflection defined by $\alpha\in R$. For each orbit of σ in B pick out a root α in that orbit. Let g be the product of the corresponding r_α, in any order, and put $c=g\sigma$. We call c a *twisted Coxeter element* of (G,σ). We shall show presently that they possess properties similar to those of the usual Coxeter elements of Weyl groups. We first establish some auxiliary results. We denote by (,) a positive definite hermitian form on $V \times V$, which is invariant for G and σ.

7.4. **Lemma.** *Assume that G fixes no nonzero vector of V. Let $c = g\sigma$ be a twisted Coxeter element.*

(i) *c fixes no nonzero vector of V.*

(ii) *If g' is a product of n_σ distinct reflections r_α with $\alpha \in B$, then $g'\sigma$ has no eigenvalue 1 if and only if the α lie in different σ-orbits.*

Let $v \in V$, $c \cdot v = v$. We then find that

$$\sigma \cdot v = v + \sum_{\alpha \in S} c_\alpha \alpha, \tag{1}$$

where α runs through a set S of representatives of the σ-orbits in B. Assume that $\sigma^e = 1$. Applying $\sigma, \sigma^2, \ldots, \sigma^{e-1}$ to (1) and adding the results we find from (1) that

$$\sum_{\alpha \in B} d_\alpha \alpha = 0,$$

where we have $d_\alpha = e_\alpha c_\alpha$ with $e_\alpha > 0$ if $\alpha \in S$. It follows that $c_\alpha = 0$ for all $\sigma \in S$, hence $\sigma v = v$ and $g v = v$. Now g is a Coxeter element of a Weyl subgroup of G. Using that a Coxeter element of an irreducible Weyl group has no eigenvalue 1 [4, p. 118], we see that $(v, \alpha) = 0$ for all $\alpha \in S$. From $\sigma \cdot v = v$ it then follows that $(v, \alpha) = 0$ for all $\alpha \in B$. This proves (i).

Let g' be, as in (ii), the product of n_σ reflections r_α, α running through a subset S of B, with $|S| = n_\sigma$. If not all σ-orbits are represented in S, there is a proper σ-invariant subspace W of V, containing S. Let W' be its orthogonal complement. Clearly g' acts trivially on W'. Since the space of fixed points of σ is not contained in W, there must be a nonzero vector in W' fixed by both σ and g'. This establishes (ii).

The following lemma is a generalization of [4, Lemma 1, p. 117]. It is due to Steinberg.

7.5. **Lemma.** *Let X be a finite forest, let ϕ be an automorphism of X. Let $x \mapsto g_x$ be a map of X into a group Γ, such that g_x and g_y commute whenever x and y are not joined in X. Let ψ be an automorphism of Γ with $g_{\phi(x)} = \psi(g_x)$ $(x \in X)$. Let Y be the set of ϕ-orbits in X, let $\mathcal{T}$ be the set of total orderings of Y. For each section s: $Y \to X$ and each $\xi \in \mathcal{T}$ denote by $p_{s,\xi}$ the product in Γ of the sequence $(g_{s(y)})_{y \in Y}$, defined by ξ. Then if $p_{s,\xi}$ and $p_{t,\eta}$ are two such elements there exists $g \in \Gamma$ with $p_{t,\eta} = g\, p_{s,\xi} (\psi g)^{-1}$.*

The proof is by induction on $n = |X|$. The case $n = 1$ is trivial, assume $n \geqq 2$. Let a be an endpoint of X, let $b \in X - \{a\}$ be joined to a if such a point exists; if not let b be an arbitrary point of $X - \{a\}$.

We first show: for each $p_{s,\xi}$ there exists $g \in \Gamma$ such that $g\, p_{s,\xi}(\psi g)^{-1} = p_{t,\eta}$ where tY contains a and b.

That we may achieve $a \in tY$ follows from the trivial formula

$$p_{s,\xi}^{-1}\, p_{s,\xi}\, \psi(p_{s,\xi}) = p_{\phi\circ s,\xi}. \tag{2}$$

Now if s is such that $a \in sY$, $b \notin sY$, we have that $p_{s,\xi}$ is a product of g_a and some g_x ($x \in X - \{a\}$) which commute with g_a. We can then establish the result by using once more (2), but now for the forest $X - \{a\}$.

To prove 7.5, it now suffices to consider only the $p_{s,\xi}$ with $a, b \in sY$. The proof then goes along the lines of that of the lemma in [4] quoted before. We leave the details to the reader.

The next theorem gives the properties of a twisted Coxeter element.

7.6. **Theorem.** *Assume that R is irreducible. Let c be a twisted Coxeter element.*

(i) *The G-conjugacy class of c in $G\sigma$ is uniquely determined.*

(ii) *c has a regular eigenvector whose eigenvalue is a primitive h_σ-th root of unity, with multiplicity* 1.

(iii) *The eigenvalues of c are $\varepsilon_j^{-1} e^{-2\pi i p_j/h_\sigma}$ $(1 \leq j \leq n)$, no eigenvalue equals* 1.

(iv) *c permutes the roots in n_σ orbits of length h_σ.*

(v) *The centralizer of c in G is cyclic and consists of the powers of c which lie in G.*

(i) follows from 7.5 and the fact (easily checked) that two bases of R which are fixed by σ are conjugate by an element of G which commutes with σ.

Now let $c' = g\sigma$ be a regular element with an eigenvalue of order h_σ, such elements exist by 7.2(ii). By 6.8(ii) we may take an ordering on R such that each orbit of c' in R contains exactly one $\alpha > 0$ with $c' \cdot \alpha < 0$. We may (replacing σ by a conjugate) assume that σ keeps positive roots positive.

It then follows that there are n_σ roots α with $\alpha > 0$, $g \cdot \alpha < 0$, hence g has length n_σ. Using 6.4(v) a case by case inspection shows that c has no eigenvalue 1. But then it follows from 7.4(ii) that c' is a Coxeter element, hence conjugate to c by (i). (ii) now also follows. (iii), (iv), (v) are consequences of 6.4(v), 6.8 and 6.4(iii), respectively.

7.7. *Remarks.* We have deduced here the properties of twisted Coxeter elements from the results of no. 6. This involves a case by case discussion. If $\sigma^2 = 1$ one may, however, also prove the regularity of twisted Coxeter elements (which is the essential point) along the lines of Steinberg's proof for the usual Coxeter elements (see [4, p. 118–121]).

8. A Reduction Theorem for Elements of Weyl Groups

In this section we prove a result about elements of Weyl groups (8.4) which will allow us to extend 4.6 and 4.8 to nonregular elements.

8.1. Let G be a Weyl group in V. We use the notations of 4.9. Denote by Q the lattice in V spanned by the roots $\alpha \in R$ and by P the weight lattice of R, which is the set of $v \in V$ with

$$2(v, \alpha)(\alpha, \alpha)^{-1} \in \mathbb{Z},$$

for all $\alpha \in R$. Then P/Q is a finite abelian group. We denote its order by e.

8.2. **Proposition.** *Let $g \in G$.*

(i) *We have $(g-1) P \subset Q$.*

(ii) *Let l be a prime number. If $v \in P$ is such that $(g-1) v \in l Q$, then g is a product of reflections r_α $(\alpha \in R)$ of G such that $(r_\alpha - 1) v \in l Q$.*

Since the reflection $r_\alpha \in G$ defined by $\alpha \in R$ is given by

$$r_\alpha(x) = x - 2(x, \alpha)(\alpha, \alpha)^{-1} \alpha,$$

(i) is true if g is a reflection. If g is arbitrary, write it as a product of reflections $g = r_{\alpha_1} \dots r_{\alpha_h}$ and put $g' = r_{\alpha_2} \dots r_{\alpha_h}$. Then

$$(g-1) x = r_{\alpha_1}(g'-1) x + (r_{\alpha_1} - 1) x,$$

from which (i) follows by induction on h.

To prove (ii), we have to use the affine Weyl group G_a of the dual root system. This is the group of affine transformations of V generated by the elements of G and the translations defined by the elements of Q, it is isomorphic to the semi-direct product of G and Q. For a discussion of affine Weyl groups we refer to [4, p. 173] (we are working here in a complex vector space, whereas in [loc. cit.] one is over the reals, but this is an immaterial difference).

Now let v be as in (ii) and put $w = l^{-1} v$. The assumption made in (ii) then implies that w is fixed by an element t of G_a. According to a theorem of Steinberg [21, p. 10–11] we can conclude that t is a product of reflections contained in G_a and fixing w. These affine reflections are the affine transformations $r_{\alpha,k}$ $(\alpha \in R, k \in \mathbb{Z})$ with

$$r_{\alpha,k}(x) = x - 2\big((\alpha, x) - k\big)(\alpha, \alpha)^{-1} \alpha.$$

Now $r_{\alpha,k}(w) = w$ means that $(r_\alpha - 1) v \in l Q$. The fact that h is a product of such $r_{\alpha,k}$ implies that g is a stated in (ii).

8.3. **Corollary.** *Let* $g \in G$.

(i) $\det(g-1)$ *is an integer, which is divisible by* e.

(ii) *If* $\det(g-1) \neq \pm e$ *there is a proper Weyl subgroup of* G *which contains* g.

(i) is a direct consequence of 10.2 (i). If $\det(g-1) \neq \pm e$, then either $\det(g-1)=0$, in which case g fixes some nonzero vector $v \in V$ and (ii) follows from 4.1 or there is a prime number l such that $\det(g-1)$ is divisible by le. In the latter case there exists $v \in P - lP$ such that $(g-1)\,v \in lQ$.

Denote by R_1 the subset of R consisting of the roots α with

$$2(v,\alpha)(\alpha,\alpha)^{-1} \equiv 0 \pmod{l}.$$

One checks that R_1 is a root system, let G_1 be its Weyl group. By 10.2 (ii) we have $g \in G_1$. Since $v \notin lP$, we have $R_1 \neq R$ and $G_1 \neq G$. This proves (ii).

8.4. **Theorem.** *Let* G *be the Weyl group of an irreducible root system* R, *let* $g \in G$. *There are the following (non-exclusive) possibilities:*

(a) g *is contained in a proper Weyl subgroup of* G,

(b) g *is regular,*

(c) $-1 \in G$ *and* $-g$ *is in case* (a) *or* (b),

(d) R *is of type* D_n *and* g *is an element with only* 2 *orbits, which are both negative.*

The notations in case (d) are those of 5.3.

By 9.3 (ii), it suffices to prove this for elements g with $\det(g-1) = \pm e$, where e is as before. We shall check the various irreducible root systems, using the results of no. 5.

Type A_n. The description of characteristic polynomials made in 5.1 shows that $\det(g-1) \neq 0$ if and only if g is a Coxeter element, hence regular.

Type B_n, C_n. We have $e=2$. From 5.2 it follows that $\det(g-1) = \pm 2$ if and only if g has only one negative orbit, of length n. Then g is a Coxeter element.

Type D_n. Now $e=4$. A similar argument shows that $\det(g-1) = \pm 4$ if and only if g is as in case (d).

In dealing with the exceptional types we use the fact that the characteristic polynomial f_g of g is of the form

$$f_g = \prod_j \Phi_{e_j}$$

(where Φ_h is the cyclotomic polynomial of primitive h-th roots of unity), the e_j being such that the $e^{2\pi i/e_j}$ are eigenvalues of g, each Φ_{e_j} occurring with the appropriate multiplicity. Hence

$$\det(g-1) = \pm \prod_j \Phi_{e_j}(1). \tag{1}$$

It is well-known that

$$\begin{aligned} \Phi_h(1) &= 1 \quad \text{if } h \text{ is not a prime power},\\ &= l \quad \text{if } h = l^k,\ k \geqq 1. \end{aligned}$$

Type E_6. We have $e=3$. By 3.4(i), the e_j occur among the integers 2, 3, 4, 5, 6, 8, 9, 12. If $\det(g-1) = \pm 3$, the only possibilities for the characteristic polynomial of g are $\Phi_9, \Phi_6^2\Phi_3, \Phi_{12}\Phi_3$. All of these possibilities lead to regular elements, by Table 1 in 5.4.

Type E_7. Now $e=2$. The e_j occur among the integers 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 18. If $\det(g-1) = \pm 2$, f_g must have a factor Φ_2, Φ_4 or Φ_8. But since the degree of Φ_h is even for all $h \neq 2$ in this list of integers, a factor Φ_2 must occur. So, if $\det(g-1) = \pm 2$, we have that g is in case (c), since $-1 \in G$.

In the remaining cases E_8, F_4, G_2 we have $e=1$. We shall only discuss the most complicated case E_8, the discussion of the other two is left to the reader.

Type E_8. The e_j occur among the integers 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 18, 20, 24, 30. If $\det(g-1) = \pm 1$, we need only consider those e_i for which $\Phi_{e_i}(1) = 1$. Also, if some e_i is twice a power of an odd prime, then $|\det(1+g)| > 1$ from which it follows (since $-1 \in G$) that then g is in case (c). Hence it suffices to consider the case that the e_j occur among the integers 12, 15, 20, 24, 30. If one of the last four occurs, then g is regular by Table 3 in 5.4. There only remains the case that g has characteristic polynomial Φ_{12}^2, in which case this table shows that g is again regular.

We now discuss some applications of 8.4.

8.5. **Theorem.** *Let G be a Weyl group, let χ be the character of a complex representation of G. Then χ takes rational integral values.*

We use induction on the order $|G|$. The starting case $|G| = 2$ is trivial. Assume that 8.5 is true for all Weyl groups of order less than $|G|$. We may assume G to be irreducible.

Let $g \in G$. The induction assumption and 4.8 imply that $\chi(g) \in \mathbb{Z}$ unless we are in the cases (c) and (d) of 8.4. If $-1 \in G$ and if χ is an irreducible character, then $\chi(-g) = \pm\chi(g)$. This observation implies that $\chi(g) \in \mathbb{Z}$ also in case (c).

Finally, if we are in case (d), an easy verification shows that all powers g^j, with j prime to the order g, are conjugate to g, from which it again follows that $\chi(g)\in\mathbb{Z}$. This proves 8.5.

8.6. *Remark.* 8.5 is a known result. It can be deduced by inspection from the known character tables of the irreducible Weyl groups. For references to the literature we refer to [1]. The present proof seems somewhat more satisfying. It would be interesting to prove by an argument similar to the above, the sharper result that all representations of a Weyl group are rational (see [loc.cit.]).

The next result was proved in the case of Weyl groups by Carter [5, p. 45].

8.7. **Theorem.** *An element $\neq 1$ of a finite Coxeter group is a product of at most two involutions.*

Let G be a finite Coxeter group, let $g\in G$. For the proof of 8.7 it suffices to assume that G is irreducible and that $g^2\neq 1$. First let G be a Weyl group. Using 8.4 and 4.6, an argument like that given in the proof of 8.5 establishes 8.7.

In the case (d) of 8.4 one uses that then there is an involution i in G such that $igi=g^{-1}$ (which is easily checked).

To complete the proof, we have to deal with the cases that G is dihedral or is of type H_3, H_4. The dihedral case is easy (and the assertion is, of course, well-known in that case).

Type H_3. From 3.3, using that the degrees are 2, 6, 10, it follows that $-1\in G$. Since G is a group of orthogonal transformations in an odd-dimensional vector space V, it is well-known that any $g\in G$ has an eigenvalue ± 1. Hence $\pm g$ fixes a vector in V, and an application of 4.1 (ii) shows that we can reduce the situation to that of a 2-dimensional Coxeter group, which is a dihedral or a Weyl group.

Type H_4. We have $-1\in G$. From 3.4, using that the degrees are now 2, 12, 20, 30, we obtain that the eigenvalues of $g\in G$ are primitive d-th roots of unity, where d occurs among the following integers: 2, 3, 4, 5, 6, 10, 12, 15, 20, 30. If one of the last four integers occurs then one reads off from Table 6 that g is regular, and 4.6 shows that g has the required property. If g has an eigenvalue -1, then $-g$ has an eigenvalue 1 and 4.1 (ii) shows that we are reduced to a lower-dimensional situation, which has already been discussed. So we may assume that g has no eigenvalue -1.

Now from Table 6 and 4.6 (v) it follows that if g is not regular, we may assume that its characteristic polynomial is a product of two distinct real quadratic polynomials, whose roots are d-th roots of unity, where d occurs the integers 3, 4, 5, 6, 10. If g has two eigenvalues with relatively prime orders, we can write g as a product of two commuting elements, each fixing elementwise a 2-dimensional subspace. We then are again reduced to a lower-dimensional situation. Since we are free to replace g by $-g$, it follows, finally, that we have only to consider the case that the eigenvalues of g are primitive roots of unity of orders 3, 6 and 5, 10 respectively. In both cases, g^2 is regular, as follows from Table 6 and 4.2 (ii). Also, 4.2 (iii) implies that the centralizer H of g^2 is isomorphic to a 2-dimensional complex reflection group with degrees 12, 30 and 20, 30 respectively, acting in a suitable space $V(g^2, \zeta)$.

Let $\Pi \subset SL_2(\mathbb{C})$ be the binary icosahedral group. Denoting by μ_d the group of scalar multiplications in $\mathbb{C}^2$ by d-th roots of unity it follows from the results of [15] that H is isomorphic to $\Pi \cdot \mu_3$ or $\Pi \cdot \mu_5$. But then g would lead to an element of $\Pi \cdot \mu_d$ whose eigenvalues are a d-th and a $2d$-th primitive root of unity ($d=3, 5$). Taking determinants one sees that this is impossible. This establishes 8.7 for type H_4 and finishes the proof of 8.7.

Question (posed in [5, p. 45]). Can one prove 8.7 directly from the abstract definition of finite Coxeter groups?

9. Certain Nilpotent Elements of Semisimple Lie Algebras

In a well-known paper [10], Kostant has exhibited an explicit connexion between the regular nilpotent elements of a complex semisimple Lie algebra (called "principal nilpotent elements" in [loc. cit.]) and the Coxeter elements of the corresponding Weyl group. In this section that result will be generalized, so as to obtain a similar connexion for certain other nilpotent elements. We first fix notations and recall some known facts.

9.1. Let G be a complex connected semisimple Lie group, let $\mathfrak{g}$ be its Lie algebra. If $X \in \mathfrak{g}$ we denote by $Z(X)$ its centralizer in G, i.e.

$$Z(X) = \{g \in G \mid \mathrm{Ad}(g)\, X = X\}$$

(Ad denoting the adjoint representation). We denote by $Z(X)^{\circ}$ the connected centralizer of X in G (i.e. the identity component of $Z(X)$) and by $\mathfrak{z}(X)$ the centralizer of X in $\mathfrak{g}$, which coincides with the Lie algebra of $Z(X)$.

For results to be recalled see [3, E III, § 4]. Let A be a nonzero nilpotent element of $\mathfrak{g}$. By the theorem of Jacobson-Morozov there exist $B, H \in \mathfrak{g}$ such that

$$[H, A] = 2A, \quad [H, B] = -2B, \quad [A, B] = H.$$

If $i \in \mathbb{Z}$, put

$$\mathfrak{g}(i) = \{X \in \mathfrak{g} \mid [H, X] = iX\}.$$

Then $\mathfrak{g}$ is the direct sum of the $\mathfrak{g}(i)$, moreover we have

$$[\mathfrak{g}(i), \mathfrak{g}(j)] \subset \mathfrak{g}(i+j),$$

so that the $\mathfrak{g}(i)$ define a structure of graded Lie algebra on $\mathfrak{g}$. All this is unique up to an automorphism of $\mathfrak{g}$ induced by an element of $Z(A)^\circ$.

From now on we make the following assumptions on A:

(a) $\mathfrak{z}(A)$ *consists of nilpotent elements,*

(b) A *is even, i.e.* $\mathfrak{g}(i) = 0$ *if* i *is odd.*

Actually, (b) is a consequence of (a), as can be checked from the classification of nilpotent conjugacy classes (given in [3, E IV] for the simple Lie algebras of classical type and in [9] for those of exceptional type). This matter will be discussed in a forthcoming paper by Bala and Carter.

9.2. **Lemma.** *Assumption* (a) *holds if and only if* $\mathfrak{z}(A) \cap \mathfrak{g}(i) = \{0\}$ *for* $i \leqq 0$.

This follows from [3, p. 238].

Let a be the smallest integer such that $\mathfrak{g}(i) = 0$ for $i > 2a$. Since $\dim \mathfrak{g}(i) = \dim \mathfrak{g}(-i)$ we then have $\mathfrak{g}(-2a) \neq \{0\}$.

9.3. **Lemma.** *Let* $M \in \mathfrak{g}(-2a)$ *and put* $C = A + M$.

(i) C *is nilpotent if and only if* $M = 0$.

(ii) $\mathfrak{z}(C) \cap \mathfrak{g}(0) = \{0\}$.

If C is nilpotent, the Jacobson-Morozov theorem shows that there exists $X \in \mathfrak{g}$ with $[X, C] = 2C$. Write $X = \Sigma X_i$, with $X_i \in \mathfrak{g}(2i)$. Looking at homogeneous components, we obtain

$$[X_0, A] = 2A, \qquad [X_0, M] = 2M.$$

From the first formula we obtain, using 9.2 that $X_0 = H$. The second one then gives $(2a+2)M = 0$, whence $M = 0$. This proves (i).

Now let $X \in \mathfrak{g}(0) \cap \mathfrak{z}(C)$. We find, similarly, that $[X, A] = 0$, whence $X = 0$ by 9.2.

Let S be the 1-dimensional torus in G whose Lie algebra is $\mathbb{C}H$. Let γ be the character of S with

$$\mathrm{Ad}(s)\, A = s^\gamma A \qquad (s \in S).$$

We then have, if $X \in \mathfrak{g}(2i)$,

$$\mathrm{Ad}(s)\, X = s^{i\gamma} X. \tag{1}$$

Fix a primitive $(a+1)$-th root of unity ζ and choose $c \in S$ such that $c^{\gamma}=\zeta$. The eigenspaces of $\mathrm{Ad}(c)$ are

$$V_i=\mathfrak{g}(2i)+\mathfrak{g}(2i-2a-2) \quad (0 \leqq i \leqq a),$$

the eigenvalue for V_i being ζ^i. In particular, the fixed point set of $\mathrm{Ad}(c)$ is $\mathfrak{g}(0)$.

Let $M \neq 0$ be in $\mathfrak{g}(-2a)$ and put $C=A+M$, as before.

9.4. **Lemma.** (i) $\mathrm{Ad}(c)$ *stabilizes* $\mathfrak{z}(C)$ *and has no nonzero fixed point in* $\mathfrak{z}(C)$.

(ii) c *normalizes* $Z(C)^{\circ}$ *and centralizes no element of* $Z(C)^{\circ}$ *outside the center of* G.

(iii) $Z(C)^{\circ}$ *is a connected solvable linear algebraic group.*

Since $\mathrm{Ad}(c)\, C=\zeta C$, it follows that $\mathrm{Ad}(c)$ stabilizes $\mathfrak{z}(C)$ and that c normalizes $Z(C)^{\circ}$. (i) now follows from 9.3 (ii). Then (ii) is a consequence of known results [2, p. 229] and the same is true for (iii) [21, p. 71].

9.5. **Lemma.** *The following properties are equivalent:*

(i) $Z(C)^{\circ}$ *is a maximal torus of* G.

(ii) C *is regular semisimple.*

(iii) C *is semisimple.*

M can be chosen such that C has these properties if and only if the following assumption is verified:

(c) V_1 *contains a regular semisimple element.*

The implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) being known, it suffices for the proof of the first part to show that (iii) implies (i). If (iii) holds, the connected centralizer of C is on the one hand reductive (by a general result) and on the other hand solvable (by 9.4 (iii)).

It must then be a maximal torus, which proves (i).

The "only if" part of the last statement follows from the first one, since $C \in V_1$. Now assume that (c) holds. It is known that there is an $\mathrm{Ad}(G)$-invariant polynomial function F on $\mathfrak{g}$ such that $X \in \mathfrak{g}$ is regular semisimple if and only if $F(X) \neq 0$.

Let $L=Z(H)^{\circ}$. One knows that the orbit $\mathrm{Ad}(L)\, A$ is a Zariski-open subset of $\mathfrak{g}(2)$. It then follows from (c) that there are $Y \in \mathrm{Ad}(L)\, A$ and $Z \in \mathfrak{g}(-2a)$ with $Z \neq 0$ such that $F(Y+Z) \neq 0$. Choose $l \in L$ such that $Y=\mathrm{Ad}(l)\, A$ and put $M=\mathrm{Ad}(l)^{-1} Z$. Then $C=A+M$ is as required.

9.6. **Lemma.** *Assumption* (c) *of* 9.5 *is a consequence of the following one:*

(d) $$\dim \mathfrak{g}(4)=\dim \mathfrak{g}(2)-1.$$

We shall prove, more precisely, that if (d) holds any $C=A+M$ with nonzero $M\in\mathfrak{g}(-2a)$ is semisimple. This will imply 9.6 (by 9.5). Consider such a C and let C_n be its nilpotent part [2, p. 151]. The uniqueness of the Jordan decomposition in $\mathfrak{g}$ [loc. cit.] shows that $\mathrm{Ad}(c)\,C_n=\zeta\,C_n$, whence $C_n\in V_1$. Let $C_n=X_2+X_{-2a}$ be the decomposition in its homogeneous components. Since C_n commutes with C, we have

$$[X_2+X_{-2a}, A+M]=0,$$

from which it follows that $[A, X_2]=0$. Now $X\mapsto[A, X]$ is a surjective linear map of $\mathfrak{g}(2)$ onto $\mathfrak{g}(4)$ (a similar result is in [3, p. 240]). If (d) holds, its kernel must be $\mathbb{C}A$. Then X_2 is a multiple of A. Using 9.3 (i) it follows that C_n is a multiple of either A or M. Using that $[C_n, C]=0$ it follows in both cases from 9.2 that $C_n=0$. Hence C is semisimple, as we claimed.

Assume that (c) holds. We write $T=Z(C)^\circ$, this is a maximal torus of G. Its Lie algebra $\mathfrak{t}=\mathfrak{z}(C)$ is a Cartan subalgebra of $\mathfrak{g}$. We denote by W the Weyl group of G with respect to T, which is a reflection group in $\mathfrak{t}$. Since c normalizes T by 9.4 (ii), it defines an element $w\in W$.

9.7. **Proposition.** (i) *w is a regular element of W, it has C as a regular eigenvector, with eigenvalue ζ. No eigenvalue of w equals* 1.

(ii) *If* (d) *holds, the eigenvalue ζ of w has multiplicity* 1.

A regular semisimple element of $\mathfrak{t}$ is regular in the sense of no. 4 for W, as is well-known. This proves the first part of (i). The second one follows from 9.3 (ii).

An argument like that used in the proof of 9.6 shows that if $X_2+X_{-2a}\in\mathfrak{t}\cap V_1$, the assumption (d) implies that X_2 is a multiple of A. On the other hand we have $\mathfrak{t}\cap\mathfrak{g}(-2a)=\{0\}$, since the elements of $\mathfrak{g}(-2a)$ are nilpotent (which follows from (1)). These observations imply (ii).

Let $n=\dim\mathfrak{t}$ and denote by $\{d_1,\ldots,d_n\}$ the degrees of the reflection group W. Let $2N$ be the number of roots of the root system R of G with respect to T (or of W). Then $N=\sum_{i=1}^{n}(d_i-1)$.

9.8. **Corollary.** (i) *W contains a regular element of order $a+1$.*

(ii) *If* (d) *holds there is exactly one d_i which is divisible by $a+1$.*

(i) follows from 9.7 (i) and 4.2 (i), (ii) from 9.7 (ii) and 3.4 (i).

We still assume (c).

9.9. **Lemma.** $\dim\mathfrak{g}(0)=\dim\mathfrak{g}(2)=2(a+1)^{-1}N$.

If $\alpha\in R$, denote $X_\alpha\in\mathfrak{g}$ a corresponding root vector. By 9.7 and 4.10, all orbits of w in R have length $a+1$. Now if $\alpha\in R$ is fixed we can normalize the root vectors for the roots in its orbit such that

$$\mathrm{Ad}(c)^i \cdot X_\alpha = X_{w^i \cdot \alpha} \quad (0 \leqq i < a+1).$$

We then have

$$\mathrm{Ad}(c)^{a+1} \cdot X_\alpha = \varepsilon X_\alpha, \tag{2}$$

where $\varepsilon = \pm 1$ (this follows by taking the X_α to belong to a Chevalley basis, see e.g. [20]).

If $\varepsilon = 1$, then

$$X_\alpha + X_{w \cdot \alpha} + \cdots + X_{w^a \cdot \alpha}$$

is an eigenvector of $\mathrm{Ad}(c)$ for the eigenvalue 1, and all such eigenvectors are linear combinations of those of this form, for suitable α. If $\varepsilon = -1$ in (2), put $\theta = e^{\pi i (a+1)^{-1}}$. Then

$$X_\alpha + \theta X_{w \cdot \alpha} + \cdots + \theta^a X_{w^a \cdot \alpha}$$

is an eigenvector of $\mathrm{Ad}(c)$ whose eigenvalue is not an $(a+1)$-th root of unity, which is impossible. Hence $\varepsilon = 1$ always, from which it follows that $\dim \mathfrak{g}(0)$ equals the number $2(a+1)^{-1} N$ of w-orbits in R. That $\dim \mathfrak{g}(0) = \dim \mathfrak{g}(2)$ is a known fact which follows from [3, p. 240] and 9.2.

9.10. *Regular Nilpotent Elements.* We shall now indicate how the preceding results may be used to establish Kostant's connection between regular nilpotent elements of $\mathfrak{g}$ and Coxeter elements in the corresponding Weyl group (the method used is Kostant's original one). We assume G to be simple. Let T_0 be a fixed maximal torus of G, let R_0 be the root system of G with respect to T_0. Let B_0 be a basis of R_0. If $\alpha \in R_0$, we denote by $X_\alpha \in \mathfrak{g}$ a corresponding root vector. We put

$$A = \sum_{\alpha \in B_0} X_\alpha.$$

9.11. **Proposition.** (i) *A is a nilpotent element which satisfies* (a), (b) *and* (d).

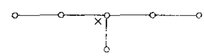
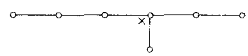
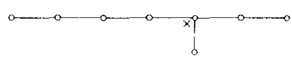
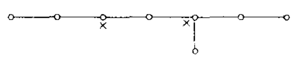
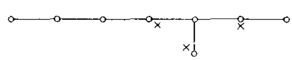
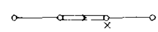
(ii) *The corresponding element w is a Coxeter element of W.*

It is known that $Z(A)^\circ$ is unipotent, which implies (a). It is also known that (b) holds and that (with the notations of 9.1) we have that for $i > 0$ the space $\mathfrak{g}(2i)$ consists of the linear combinations of the X_α where α has height i, with respect to the basis B_0. For all this see for example [17]. It readily follows that (d) holds. Let n be the rank of G. By 9.9 we have that the element w has order $2n^{-1} N$. By [4, p. 119] it is then a Coxeter element.

The conjugacy class of A is uniquely determined, its elements are those nilpotents whose centralizer has dimension n. See [17].

9.12. We now consider the more interesting case that A is not regular and satisfies (a), (b), (d). By 9.7 we must then look for non-Coxeter elements w in Weyl groups, without eigenvalue 1 which have an eigenvalue whose eigenvectors are regular with multiplicity 1. Assume that the root system R is irreducible. The results of no. 5 show which elements w and which orders $a+1$ can occur. In the classical types A_n, B_n, C_n, D_n there are none. The results for the other types are contained in the first two columns of Table 11.

Table 11

Type	$a+1$	$\dim \mathfrak{g}(0)$	Dynkin diagram
E_6	9	8	
E_7	14	9	
E_8	24	10	
E_8	20	12	
E_8	15	16	
F_4	8	6	
G_2	3	4	

We do not yet know that these possibilities can all be realized as coming from a nilpotent A satisfying (a), (b), (d). First observe that if this is the case, $\dim \mathfrak{g}(0)$ must be as stated in the third column of Table 11 (by 9.9).

Put $L = Z(H)^\circ$, as in the proof of 9.9. This is a reductive group and $\dim L = \dim \mathfrak{g}(0)$. There is a parabolic subgroup P of G, whose Lie algebra is $\coprod_{i \geq 0} \mathfrak{g}(i)$, which has L as a Levi subgroup, see [3, p. 240].

Consider the Dynkin diagram of A, defined in [3, p. 243]. This is the Dynkin graph D of W, together with a distribution of the numbers 0 and 2 on the vertices of D (in the case of a nilpotent satisfying (b)). The Dynkin graph of the semisimple part of L can be found from D by omitting the vertices to which the number 2 is attached and the segments ending in one of these vertices [loc. cit.].

In [14, Table 23, p. 190] one finds for the exceptional types a list of all possible Dynkin diagrams of nilpotents satisfying (a). It is easy to check which ones lead to the values of $a+1$ and $\mathfrak{g}(0)$ of Table 11. They are listed in the last column of that table. We have put a cross at the nodes to which the number 0 is attached. In all these cases we have $\dim \mathfrak{g}(4) = \dim \mathfrak{g}(2) - 1$, and they also come from nilpotents A satisfying (a), as follows from [loc. cit.]. Hence all Weyl group classes containing elements with the properties enunciated in the first paragraph of 9.12 arise from nilpotents A satisfying (a), (b), (d).

9.13. *Remarks.* (1) In [6] a bijective map is described from a certain set of nilpotent conjugacy classes of $\mathfrak{g}$ to a set of conjugacy classes of the Weyl group of G. From the results discussed above one can deduce a similar map. It can be checked that these two maps give the same image of a class on which both are defined.

(2) In this no. we have assumed that we worked over $\mathbb{C}$. This is not really necessary. The same results are obtained if one works over an algebraically closed field K of characteristic p, where p is sufficiently large. We refer to [3, E III, § 4] for a discussion of the restrictions to be imposed on p. The results of no. 3 and no. 4 remain valid in this more general situation, as follows by using [4, p. 107, Th. 3]. It should also be remarked that the proof of 9.11 indicated above works in characteristic p, if p is bigger than the order of a Coxeter element (Kostant's original proof only works in characteristic 0).

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(Received December 7, 1973)