Inventiones math. 25, 143–157 (1974)

The Intersection Matrix of Brieskorn Singularities

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Abstract. To each isolated singularity of a hypersurface of dimension n, one associates the local fundamental group G of the moduli space minus the discriminant locus, and a representation $\sigma: G \to \mathrm{Aut}(H)$, where H is the n-homology group, with integer coefficients, of the non singular fibre. Although, in general it is very difficult to determine even a presentation of G, we show that the image of σ can be computed rather easily, by exploiting some relations in a first aproximate presentation of G, in the case of Brieskorn polynomials namely, polynomials of the type $x_0^{a_0} + \cdots + x_n^{a_n}$.

In this way we solve an open problem stated by Brieskorn [1] and Pham [8].

1. Introduction

The object of this paper is to solve an open problem, stated by Brieskorn [1] and Pham [8], about the intersection matrix between the vanishing cycles for a particular class of isolated hypersurface singularities, namely the Brieskorn singularities. In doing so, we also exhibit an explicit expression for the intersection matrix over a geometrical basis.

Finally, we give a method to derive explicitely the matrix of linking numbers from an intersection matrix over a geometrical basis, for an isolated hypersurface singularity.

Let $\varphi: (\mathbb{C}^{n+h}, 0) \to (\mathbb{C}^h, 0)$ be a flat morphism whose fibre $\varphi^{-1}(0)$ has an isolated singularity at the origin. Let $(\Delta, 0) \hookrightarrow (\mathbb{C}^h, 0)$ designate the discriminant of φ (in the sense of [6])[1] and define μ to be the multiplicity of Δ at the origin.

Define $B_\varepsilon(r) = \{x \in \mathbb{C}^r / \|x\| < \varepsilon\}$ and $S_\varepsilon(r) =$ boundary of $B_\varepsilon(r)$.

Hamm in [4] has proved the following:
There exists a pair of positive numbers ε_0, η_0 such that φ is defined in a neighborhood of $B^*_{\varepsilon_0}(n+h)$ ($B^* =$ closure of B) in such a way that its restriction

$$\varphi_{\varepsilon_0, \eta_0}: M_{\varepsilon_0, \eta_0} \to B_{\eta_0}(h)$$

* This work was conducted while both authors belonged to the G.N.S.A.G.A of the C.N.R.

[1] $(\Delta, 0)$ is in general a not reduced space and it can be defined in the following way: let $\sigma: (X, x) \to (T, t)$ be the semiuniversal deformation of $(\varphi^{-1}(0), 0)$; then φ is induced by a map $\tau: (\mathbb{C}^h, 0) \to (T, t)$. If (D, t) denotes the (reduced) discriminant locus of σ, then $(\Delta, 0)$ is $(\tau^{-1}(D), 0)$ with the fibre structure.

(written also for the sake of brevity as $\varphi\colon M\to B$, where $M\equiv M_{\varepsilon_0,\eta_0}=B^*_{\varepsilon_0}(n+h)\cap\varphi^{-1}(B)$, $B\equiv B_{\eta_0}(h)$), has the following properties:

i) M is a differentiable submanifold of $\mathbb{C}^{n+h}$ with a boundary and φ is a proper differentiable surjection which is of maximal rank on the boundary of M and outside $\varphi^{-1}(\Delta)$.

ii) For each $\lambda\in B\cap\Delta$, $\varphi^{-1}(\lambda)$ has only isolated singularities while for each $\lambda\in B-\Delta$, $\varphi^{-1}(\lambda)$ is a compact differentiable parallelizable manifold which is canonically oriented and has the homotopy type of a bouquet of μ spheres of dimension n.

iii) For all $0<\varepsilon<\varepsilon_0$ there exists $0<\eta<\eta_0$ such that the pair ε,η satisfies i), ii) and φ is homeomorphically equivalent to $\varphi_{\varepsilon,\eta}$.

From i)–iii) it follows that $\varphi\colon M-\varphi^{-1}(\Delta)\to B-\Delta$ is a fibre bundle intrinsically associated to $\varphi\colon(\mathbb{C}^{n+h},0)\to(\mathbb{C}^h,0)$.

For $\lambda\in B$ define $F_\lambda=\varphi^{-1}(\lambda)$ and let $\lambda_0\in B-\Delta$. Then, we have a representation $\sigma\colon\pi_1(B-\Delta,\lambda_0)\to\operatorname{Aut}(H_n(F_{\lambda_0},\mathbb{Z}))$ defined by following isotopically F_{λ_0} along γ for $\gamma\in\pi_1(B-\Delta,\lambda_0)$. Actually, $\pi_1(B-\Delta,\lambda_0)$ can be identified with the local fundamental group of $\mathbb{C}^h-\Delta$ at the origin. The above σ is called the full monodromy of φ.

Throughout this paper we assume that $(\Delta,0)\hookrightarrow(\mathbb{C}^h,0)$ is a reduced space. This assumption is equivalent to the following statement: λ is a simple point of Δ if and only if F_λ has one and only one singularity which is quadratic. In this case we shall say that φ is a regular deformation of the isolated singularity $\varphi^{-1}(0)$.

Let L designate a complex line through λ_0, passing near 0 and transversal to Δ. Then $L\cap\Delta$ consists of μ distinct simple points $\lambda_1,\dots,\lambda_\mu$. Let $\gamma_1,\dots,\gamma_\mu$ be μ loops in $L-\Delta$ constructed in the following manner: E_i denotes a small open disc around λ_i in L, d_i a point on the boundary of E_i; choose embeddings τ_i of $[0,1]$ into $L-\bigcup_{i=1}^{\mu}E_i$ with $\tau_i(0)=\lambda_0$ and $\tau_i(1)=d_i$, $i=1,\dots,\mu$, such that the images of two distinct τ_i intersect only at λ_0. Then define γ_i the loop obtained by describing τ_i first, then the counter-clockwise boundary of E_i and finally describing τ_i^{-1}.

The loops $\gamma_1,\dots,\gamma_\mu$ induce a free basis of $\pi_1(L-\Delta,\lambda_0)$. Since F_{λ_i} has an ordinary quadratic singularity, one knows (see for example Fary [2]) that to each γ_i can be associated a cycle $e_i\in H_n(F_{\lambda_0},\mathbb{Z})$ called the vanishing cycle at λ_i; which is uniquely determined, sign apart, by γ_i.

The action γ_i^* of γ_i on $H_n(F_{\lambda_0},\mathbb{Z})$ is described by the Picard-Lefschetz formula:

$$\gamma_i^*(z)=z-(-1)^{\frac{(n+1)(n+4)}{2}}\langle z,e_i\rangle\, e_i,\qquad z\in H_n(F_{\lambda_0},\mathbb{Z}),\tag{1.1}$$

where $\langle\ ,\ \rangle\colon H_n(F_{\lambda_0},\mathbb{Z})\times H_n(F_{\lambda_0},\mathbb{Z})\to\mathbb{Z}$ denotes the intersection product.

Moreover, one knows that

$$\langle e_i, e_i\rangle = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 2(-1)^{\frac{n(n+1)}{2}}, & \text{if } n \text{ is even.} \end{cases} \tag{1.2}$$

Note that the intersection product is symmetric if n is even and antisymmetric if n is odd.

It can be shown that $s\colon \pi_1(L-\Delta, \lambda_0) \to \pi_1(B-\Delta)$ is surjective, so that $\pi_1(B-\Delta)$ can be presented as the group generated by $\gamma_1, \ldots, \gamma_\mu$ modulo some set of relations $\{R_k\}_{k\in K}$ (such set of relations can be computed as in Lazzeri [5]).

Obviously, each relation R_k between the γ_i will be fulfilled by the γ_i^* and hence each R_k induces via (1.1) and (1.2) an equation $E_k(\langle e_i, e_j\rangle)$ among the intersection numbers $\langle e_i, e_j\rangle$, $i, j = 1, \ldots, \mu$.

For example, $\gamma_i = \gamma_j$ implies either $e_i = e_j$ or $\langle z, e_i\rangle = \langle z, e_j\rangle = 0$ for all $z \in H_n(F_{\lambda_0}, \mathbb{Z})$.

Assume now, that $e_i \neq e_j$ implies that there exists $z \in H_n(F_{\lambda_0}, \mathbb{Z})$ such that $\langle z, e_i\rangle \neq \langle z, e_j\rangle$. Note that this assumption is satisfied in our present case since $\varphi^{-1}(0)$ is an hypersurface singularity (see [6]).

Actually we can think of φ as a deformation of $\varphi^{-1}(0)$ (for the exact formulation and further details see Grauert [3]).

We say that φ determines the intersection matrix of $\varphi^{-1}(0)$ if and only if the following holds:

For each solution $(x_{ij})_{i,j}$ of $E_k(\langle e_i, e_j\rangle)$, $k \in K$, there exists a choice of orientation of the e_i such that $x_{ij} = \langle e_i, e_j\rangle$.

It has been conjectured by Brieskorn and by Pham that the semi-universal deformation of an hypersurface isolated singularity determines the intersection matrix.

Various illustrative examples indicate that it is plausible that this conjecture actually holds true even when the deformations are regular.

In what follows we shall give an affirmative answer to the above generalised problem in the special case where the singularity considered arises from the Brieskorn polynomials $x_0^{a_0} + \cdots + x_n^{a_n}$, but under the weaker assumption that the deformations are linear. In these deformations, the discriminant Δ is expressed explicitely in a nice closed form and consequently the points of $L \cap \Delta$ are symmetrically distributed on the complex line L. This in turn, facilitates the choice of a set of geometrical generators of $\pi_1(L-\Delta, \lambda_0)$. The construction of the above convenient set of generators enables us to succesfully resolve the crucial step in our proof which is done by induction on the number of variables of the Brieskorn polynomial whose exponents a_j are greater then 2. Furthermore, we write the Picard-Lefschetz formula for the product $\gamma_1^* \ldots \gamma_\mu^*$ in a form which appears to be very efficient in exploiting the relevant informations from the above relations. This, not only solves the problem we have stated above, but also yields an explicit expression for the intersection matrix over a geometrical basis.

It should be remarked that Pham [9], using different methods, has also computed the intersection matrix between the vanishing cycles, however it is not clear whether the basis he has chosen for his computation is geometric. Nevertheless, it is surprisingly interesting to note that his intersection matrix coincides componentwise with ours. This leads us to suspect that Pham's basis is actually geometrical. It goes without saying, that it is important to know the intersection matrix over a geometrical basis.

Finally we present an explicit relationship between the matrix of linking numbers and the intersection matrix over a geometrical basis.

Since in general we don't have an explicit formula for the discriminant Δ, the method we have exhibited here is somewhat restrictive as it is apparently only applicable to the Brieskorn polynomials.

2. A Particular Choice of a Set of Generators

Before we present our construction let us note that if the hypersurface $\{x/f(x)=0\}$ has an isolated singularity at the origin then the moduli space and the discriminant of its semiuniversal deformation coincide with those of $\{(x, y)/f(x)+y^2=0\}$.

Actually, from our point of view the difference between them occurs in the Picard-Lefschetz formula. In particular, adding to $f(x)$ four squares will yield no change to the problem considered.

We make a particular choice of $L, \lambda_0, \gamma_1, \ldots, \gamma_\mu$ and compute directly the intersection matrix between the associated vanishing cycles. Note that for any other choice of these data, one can easily determine the new intersection matrix by expressing the new vanishing cycles in terms of the old ones. In fact, there is a braid action on the different choices, so that the invariance of any property needs only to be verified for the generators of this action.

This evidently shows that the intersection matrix is determined for any choice of $L, \lambda_0, \gamma_1, \ldots, \gamma_\mu$ if it is determined for a particular choice.

Suppose now that the singularity is given by $f(x)=\sum_{i=1}^{l} x_i^{a_i+1}=0$, $a_1 \geqq a_2 \geqq \cdots \geqq a_r > a_{r+1} = \cdots = a_l = 1$, with deformation

$$f(x)+\sum_{i=1}^{r} \alpha_i x_i = \beta \qquad \alpha_i, \beta \in \mathbb{C} \quad i=1, \ldots, r.$$

Define $L \equiv \{(\alpha_1, \alpha_2, \ldots, \alpha_r; \beta)/\alpha_i = \varepsilon_i, i=1, \ldots, r\}$, where the ε_i are real positive numbers such that $1 \gg \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_r > 0$.

The discriminant Δ of the above deformation is essentially (constants apart) given by

$$\alpha_1^{\frac{a_1+1}{a_1}}+\cdots+\alpha_r^{\frac{a_r+1}{a_r}}=\beta.$$

Then $\Delta\cap L$ can be described as the set of all

$$\omega_1^{(i_1)}+\omega_2^{(i_2)}+\cdots+\omega_r^{(i_r)}$$

where $\{\omega_j^{(i_j)}\}_{i_j=1,\ldots,a_j}$ are the a_j-th root of $\varepsilon_j^{a_j+1}$ ordered by increasing arguments. In this case we have $\mu=\prod_{j=1}^{r}a_j$.

We call $\gamma_{i_1 i_2\ldots i_r}$ the loop around $\omega_1^{(i_1)}+\cdots+\omega_r^{(i_r)}$ in $L-\Delta$ which satisfies the following inductive requirements.

(i) If $r=1$, choose $\gamma_1,\ldots,\gamma_{a_1}$ as illustrated in Fig. 1.

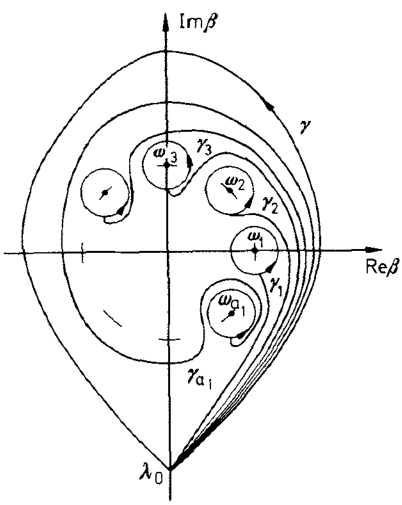

Fig. 1. The choice of loops in the case $r=1$

Obviously the loop γ is equal to the product $\gamma_1\gamma_2\cdots\gamma_{a_1}$.[2]

(ii) Supposing that the γ_σ, where σ stands for the $(r-1)$-tuple $(i_1,\ldots,i_{r-1})$, have been chosen around $\omega_\sigma=\omega_1^{(i_1)}+\cdots+\omega_{r-1}^{(i_{r-1})}$, choose γ_{σ,i_r} around $\omega_{\sigma,i_r}=\omega_\sigma+\omega_r^{(i_r)}$, for $i_r=1,\ldots,a_r$, as illustrated in Fig. 2a and b.

[2] In the product $\gamma_i\gamma_j$ we first let γ_j operate and then γ_i.

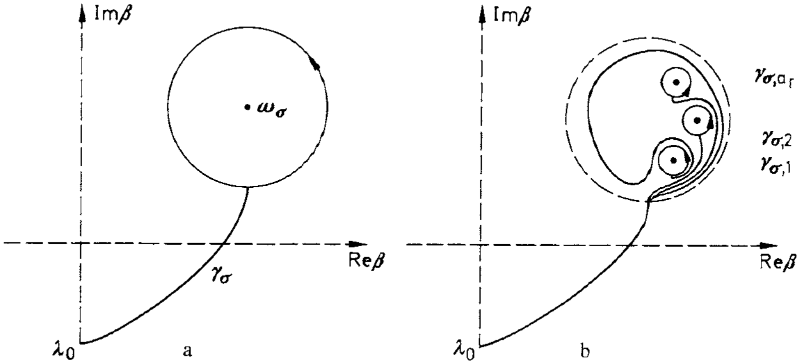

Fig. 2a and b. The inductive choice of loops

This construction gives the following equalities:

$$\gamma_{i_1 i_2 \ldots i_{r-1}} = \prod_{i=1}^{a_r} \gamma_{i_1 i_2 \ldots i_{r-1} i}$$

$$\gamma_{i_1 i_2 \ldots i_{r-2}} = \prod_{i=1}^{a_{r-1}} \gamma_{i_1 i_2 \ldots i_{r-2} i}$$

....

$$\gamma_{i_1} = \prod_{i=1}^{a_2} \gamma_{i_1 i}$$

$$\gamma = \prod_{i=1}^{a_1} \gamma_i . \text{ [3]}$$

The following lemma is fundamental.

Lemma 2.1. *For $i_1 = 1, \ldots, a_1 - 1$ if we make α_1 describe the the circle $S(\varepsilon_1) = \{\alpha_1 \in \mathbb{C} / |\alpha_1| = \varepsilon_1\}$ we get, among others, the following relations*

$$\gamma^*_{i_1 i_2 \ldots i_r} = (\gamma^*)^{-1} \gamma^*_{i_1+1} \gamma^*_{i_1+1\, i_2 \ldots i_r} (\gamma^*_{i_1+1})^{-1} \gamma^* . \tag{2.1}$$

Proof. When α_1 describes $S(\varepsilon_1)$, $\omega_1^{(i_1)}$ is transformed into $\omega_1^{(i_1+1)}$; from the above construction, when α_1 describes the a_1-th part of $S(\varepsilon_1)$, we have after $a_1 + 1$ steps the following sequence of transformations

$$\gamma_{i_1 i_2 \ldots i_r} \mapsto \gamma_{i_1+1\, i_2 \ldots i_r} \mapsto \cdots \mapsto \gamma_{a_1 i_2 \ldots i_r} \mapsto \gamma^{-1} \gamma_1 \gamma_{1 i_2 \ldots i_r} \gamma_1^{-1} \gamma$$
$$\mapsto \gamma^{-1} \gamma_2 \gamma_{2 i_2 \ldots i_r} \gamma_2^{-1} \gamma \mapsto \cdots \mapsto \gamma^{-1} \gamma_{i_1+1} \gamma_{i_1+1\, i_2 \ldots i_r} \gamma_{i_1+1}^{-1} \gamma .$$

The only step which is not trivial is $\gamma_{a_1 i_2 \ldots i_r} \mapsto \gamma^{-1} \gamma_1 \gamma_{1 i_2 \ldots i_r} \gamma_1^{-1} \gamma$ and can be verified by superposition of Fig. 3a, b and c.

[3] The products are to be developed from left to right, for example $\prod_{i=1}^{a_1} \gamma_i = \gamma_1 \gamma_2 \ldots \gamma_{a_1}$.

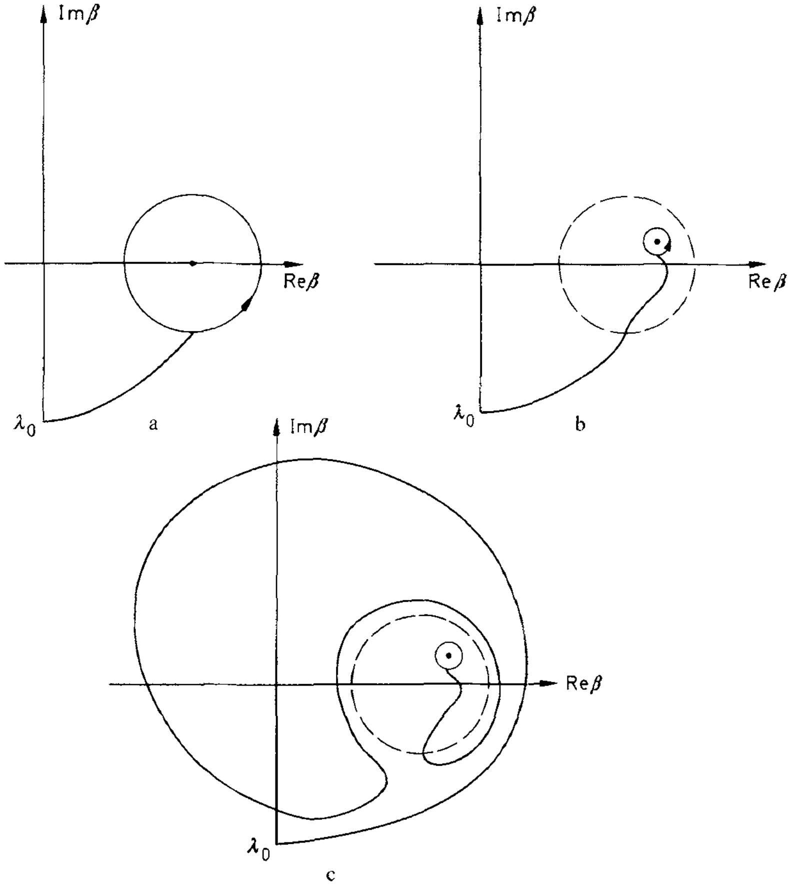

Fig. 3. a The loop γ_1. b The loop $\gamma_{1 i_2 \dots i_r}$; c The loop $\gamma^{-1}\gamma_1\gamma_{1 i_2 \dots i_r}\gamma_1^{-1}\gamma$

3. The Picard-Lefschetz Formula

We consider again the general case of an isolated hypersurface singularity, with the notations of § 1, denote by γ^* the product $\gamma_1^* \dots \gamma_\mu^*$ in $\mathrm{Aut}(H_n(F_{\lambda_0}, \mathbb{Z}))$.

One can easily derive, by induction, from the Picard-Lefschetz formula that:

$$\begin{aligned}\gamma^*(z) = z \pm \sum_{\sigma} \langle z, e_\sigma \rangle e_\sigma + \sum_{\sigma_2 > \sigma_1} \langle z, e_{\sigma_2} \rangle \langle e_{\sigma_2}, e_{\sigma_1} \rangle e_{\sigma_1} \\ \pm \sum_{\sigma_3 > \sigma_2 > \sigma_1} \langle z, e_{\sigma_3} \rangle \langle e_{\sigma_3}, e_{\sigma_2} \rangle \langle e_{\sigma_2}, e_{\sigma_1} \rangle e_{\sigma_1} + \cdots\end{aligned} \tag{3.1}$$

where the sign $+$ is for $n \equiv 1, 2 \pmod 4$ and sign $-$ for $n \equiv 0, 3 \pmod 4$.

Moreover $\langle e_\sigma, e_\sigma\rangle$ equals 2 if $n \equiv 0 \pmod 4$, 0 if $n \equiv 1, 3 \pmod 4$ and -2 if $n \equiv 2 \pmod 4$.

From the above formulas we derive the following:

$$\gamma^*(z)_\sigma = z_\sigma \pm \langle z, e_\sigma\rangle \pm \sum_{\sigma_1 > \sigma} (\gamma^*(z)_{\sigma_1} - z_{\sigma_1}) \langle e_{\sigma_1}, e_\sigma\rangle \tag{3.2}$$

where z_σ denotes the σ-component of z and sign $+$ is for $n \equiv 1, 2 \pmod 4$, sign $-$ is for $n \equiv 0, 3 \pmod 4$.

The proof is as follows; from (3.1) we have for $n \equiv 0, 3 \pmod 4$

$$\gamma^*(z)_\sigma = z_\sigma - \langle z, e_\sigma\rangle + \sum_{\sigma_1 > \sigma} \langle z, e_{\sigma_1}\rangle \langle e_{\sigma_1}, e_\sigma\rangle$$
$$- \sum_{\sigma_2 > \sigma_1 > \sigma} \langle z, e_{\sigma_2}\rangle \langle e_{\sigma_2}, e_{\sigma_1}\rangle \langle e_{\sigma_1}, e_\sigma\rangle + \cdots$$

so

$$\gamma^*(z)_\sigma = z_\sigma - \langle z, e_\sigma\rangle + \sum_{\sigma_1 > \sigma} \Big(\langle z, e_{\sigma_1}\rangle - \sum_{\sigma_2 > \sigma_1} \langle z, e_{\sigma_2}\rangle \langle e_{\sigma_2}, e_{\sigma_1}\rangle + \cdots\Big) \langle e_{\sigma_1}, e_\sigma\rangle$$

the result follows by observing that the expression between parentheses is equal to $-(\gamma^*(z)_{\sigma_1} - z_{\sigma_1})$. The other case is similar

Lemma 3.1. *For an isolated singularity of an n-dimensional hypersurface, the following data determine each other*

(i) $\gamma^*(e_i)$ *for all* i

(ii) $\langle e_i, e_j\rangle$ *for all* i, j; $i < j$.

Proof. Obviously by (3.1) (ii) determines (i). Conversely assume that (i) is known; $\langle e_{\mu-1}, e_\mu\rangle$ is determined by (3.2) as follows

$$\langle e_{\mu-1}, e_\mu\rangle = \pm \gamma^*(e_{\mu-1})_\mu.$$

Suppose now, by induction, that $\langle e_i, e_j\rangle$ is determined for all i, j; $\nu < i < j$, then again by formula (3.2) we get for $\nu < j$

$$\pm \langle e_\nu, e_j\rangle = \gamma^*(e_\nu)_j \mp \sum_{\sigma > j} \gamma^*(e_\nu)_\sigma \langle e_\sigma, e_j\rangle$$

but for $\nu < j < \sigma$ the inductive assumption gives the intersection numbers $\langle e_\sigma, e_j\rangle$ so $\langle e_\nu, e_j\rangle$ is also determined.

4. The Computation

Let us fix some notations:

$$I = (i_1, i_2, \ldots, i_r), \qquad J = (j_1, j_2, \ldots, j_r), \qquad K = (k_1, k_2, \ldots, k_r)$$

and so on;

$$I' = (i_2, \ldots, i_r),$$

a symbol like $(i_1 + 1, K')$ means $(i_1 + 1, k_2, \ldots, k_r)$.

Lemma 4.1. *For* $1 \leqq i_1 \leqq a_1 - 1$, *we have*

$$\gamma^*(e_I) = \lambda_I \gamma^*_{i_1+1}(e_{i_1+1,I'}), \qquad \text{where } \lambda_I = \pm 1.$$

Proof. At a point $z \in H_n(F_{\lambda_0})$ we have, by (2.1) and (1.1), that

$$\langle (\gamma^*)^{-1}(z), e_I \rangle \gamma^*(e_I) = \langle (\gamma^*_{i_1+1})^{-1}(z), e_{i_1+1,I'} \rangle \gamma^*_{i_1+1}(e_{i_1+1,I'}).$$

Now there exist points $z', z'' \in H_n(F_{\lambda_0})$ (see [5]) such that

$$\langle (\gamma^*)^{-1}(z'), e_I \rangle = \langle \gamma^*_{i_1+1})^{-1}(z''), e_{i_1+1,I'} \rangle = 1.$$

Writing

$$\langle (\gamma^*_{i_1+1})^{-1}(z'), e_{i_1+1,I'} \rangle = \lambda_I \in \mathbb{Z}$$

and

$$\langle (\gamma^*)^{-1}(z''), e_I \rangle = \beta_I \in \mathbb{Z},$$

we get

$$\gamma^*(e_I) = \lambda_I \gamma^*_{i_1+1}(e_{i_1+1,I'}) \quad \text{and} \quad \beta_I \gamma^*(e_I) = \gamma^*_{i_1+1}(e_{i_1+1,I'});$$

combining together these two formulas we obtain $\beta_I \cdot \lambda_I = 1$ and then $\lambda_I = \pm 1$.

We fix the orientation of the vanishing cycles in the following way: Suppose, inductively, that the vanishing cycles of the form $e_{1,I'}$ have been oriented; for all I' choose the orientation of $e_{2,I'}$ in such a way that

$$\gamma^*(e_{1,I'}) = \lambda \gamma^*_2(e_{2,I'})$$

where

$$\lambda = \begin{cases} 1, & \text{if } (l-1) \equiv 0, 1 \pmod 4 \\ -1, & \text{if } (l-1) \equiv 2, 3 \pmod 4). \end{cases}$$

Iterating this process we have fixed the orientation of the cycles $e_{i_1,I'}$ for all i_1.

Proposition 4.1. (Pham [8]) *for* $r=1$ *we have:*

(i) $\langle e_i, e_j \rangle = 0$ *if* $|i-j| > 1$,

(ii) $\langle e_i, e_{i+1} \rangle = 1$ *if* $i < a_1$.

Proof. We use essentially (3.2) and Lemma (4.1) restated for $r=1$

(i) in this case it is sufficient to prove the assertion for $j > i+1$:

$$0 = \gamma^*_{i+1}(e_{i+1})_j = \lambda \gamma^*(e_i)_j = \pm \langle e_i, e_j \rangle,$$

(ii) if $(l-1) \equiv 0 \pmod 4$,

$$\langle e_i, e_{i+1} \rangle = -\gamma^*(e_i)_{i+1} = -\gamma^*_{i+1}(e_{i+1})_{i+1} = -(1 - \langle e_{i+1}, e_{i+1} \rangle) = -(1-2) = 1$$

the other cases are similar.

At this point let us make the inductive assumption:

(I.A) *Let* $(i_1, i_2, \ldots, i_r) < (i_1, j_2, \ldots, j_r)$,[4] *then* $\langle e_{i_1 i_2 \ldots i_r}, e_{i_1 j_2 \ldots j_r}\rangle = 0$ *unless for every* $\eta, \nu, |i_\eta - j_\eta| \leqq 1$ *and* $(i_\nu - j_\nu)(i_\eta - j_\eta) \geqq 0$ *in which case*

$$\langle e_{i_1 i_2 \ldots i_r}, e_{i_1 j_2 \ldots j_r}\rangle = \begin{cases} 1, & \text{if } (l-1) \equiv 0, 1 \pmod 4 \\ (-1)^{j_2 - i_2 + \cdots + j_r - i_r - 1}, & \text{if } (l-1) \equiv 2, 3 \pmod 4. \end{cases}$$

Proposition 4.2[5]. *Let* $I < J$. *If for every* $\eta, \nu, |i_\eta - j_\eta| \leqq 1$ *and*

$$(i_\nu - j_\nu)(i_\eta - j_\eta) \geqq 0,$$

then

$$\langle e_I, e_J\rangle = \begin{cases} 1 & \text{for } (l-1) \equiv 0, 1 \pmod 4 \\ (-1)^{j_1 - i_1 + j_2 - i_2 + \cdots + j_r - i_r - 1} & \text{for } (l-1) \equiv 2, 3 \pmod 4 \end{cases}$$

otherwise $\langle e_I, e_J\rangle = 0$.

Proof. We have only to prove the assertion when $j_1 > i_1$. From Lemma 4.1 and formula (3.2) we have

$$\begin{aligned}\gamma^*(e_I)_J = \lambda \gamma^*_{i_1+1}(e_{i_1+1, I'})_J = \lambda\big[&(e_{i_1+1, I'})_J \pm \langle e_{i_1+1, I'}, e_J\rangle \\ &\pm \sum_{K > J} \big(\gamma^*_{i_1+1}(e_{i_1+1, I'})_K - (e_{i_1+1, I'})_K\big)\langle e_K, e_J\rangle\big]\end{aligned}$$

so

$$\begin{aligned}&\gamma^*(e_I)_J \mp \sum_{K > J} \gamma^*(e_I)_K \langle e_K, e_J\rangle \\ &\quad = \lambda\big[(e_{i_1+1, I'})_J \pm \langle e_{i_1+1, I'}, e_J\rangle \mp \sum_{K > J} (e_{i_1+1, I'})_K \langle e_K, e_J\rangle\big]\end{aligned} \tag{4.1}$$

on the other hand

$$\gamma^*(e_I)_J = (e_I)_J \pm \langle e_I, e_J\rangle \pm \sum_{K > J} \big(\gamma^*(e_I)_K - (e_I)_K\big)\langle e_K, e_J\rangle$$

but $(e_I)_J = (e_I)_K = 0$ because $I < J < K$ so

$$\pm \langle e_I, e_J\rangle = \gamma^*(e_I)_J \mp \sum_{K > J} \gamma^*(e_I)_K \langle e_K, e_J\rangle. \tag{4.2}$$

(4.1) and (4.2) give

$$\langle e_I, e_J\rangle = \pm \lambda\big[(e_{i_1+1, I'})_J \pm \langle e_{i_1+1, I'}, e_J\rangle \mp \sum_{K > J} (e_{i_1+1, I'})_K \langle e_K, e_J\rangle\big]. \tag{4.3}$$

There are three cases to consider (i) suppose that $j_1 \geqq i_1 + 2$, then from (4.2) we get

$$\pm \langle e_I, e_J\rangle = 0$$

[4] Here the multi-indexes have the lexicographic order.

[5] E. Brieskorn communicated to us that A.M. Gabrielov, using different methods, has computed the intersection matrix, over a geometrical basis, of $f(x) + g(y)$.
Gabrielov, A.M.: Funkcionalnij analiz i jewo priloženija, Vol. 7, No 3, 18–32 (1973).

because

$$\gamma^*(e_I)_K = \lambda \gamma^*_{i_1+1}(e_{i_1+1,I'})_K = 0 \qquad \text{for } K \geqq J$$

the last equality holds because $k_1 \geqq i_1 + 2$ and by the Picard-Lefschetz formula $\gamma^*_{i_1+1}(e_{i_1+1,I'})$ is a linear combination of vanishing cycles of the type $e_{i_1+1,h_2\ldots h_r}$.

(ii) Suppose that $j_1 = i_1 + 1$ and there exists $\eta \geqq 2$ for which $j_\eta = i_\eta - 1$ or $|j_\eta - i_\eta| > 1$; in this case we have from (4.3) that $\langle e_I, e_J \rangle = 0$ because if $J' < I'$ then $(e_{i_1+1,I'})_J = 0$ and

$$\pm \langle e_{i_1+1,I'}, e_J \rangle \mp \sum_{K>J} (e_{i_1+1,I'})_K \langle e_K, e_J \rangle$$
$$= \pm \langle e_{i_1+1,I'}, e_J \rangle \mp \langle e_{i_1+1,I'}, e_J \rangle = 0$$

if $J' > I'$ then $(e_{i_1+1,I'})_J = 0$, $\pm \langle e_{i_1+1,I'}, e_J \rangle = 0$ by (I.A) and

$$(e_{i_1+1,I'})_K = 0 \qquad \text{for } K > J > (i_1 + 1, I').$$

(iii) Suppose that $j_1 = i_1 + 1$ and for every $\eta \geqq 2, j_\eta = i_\eta + 1$ or $j_\eta = i_\eta$; in this case $(e_{i_1+1,I'})_K = 0$ for $K > J$ the from (4.3) we get

$$\langle e_I, e_J \rangle = \pm \lambda [(e_{i_1+1,I'})_J \pm \langle e_{i_1+1,I'}, e_J \rangle]$$

if $I' = J'$ we have

$$\langle e_I, e_J \rangle = \pm \lambda [1 \pm \langle e_J, e_J \rangle] = 1$$

if $I' \neq J'$ we have

$$\langle e_I, e_J \rangle = \lambda \langle e_{i_1+1,I'}, e_J \rangle$$
$$= \begin{cases} 1, & \text{if } (l-1) \equiv 0, 1 \pmod 4 \\ (-1)^{1+j_2-i_2+\cdots+j_r-i_r-1}, & \text{if } (l-1) \equiv 2, 3 \pmod 4 \end{cases}$$

where the last equality follows from (I.A).

5. The Linking Matrix

We consider now another matrix associated to a set of vanishing cycles of an isolated hypersurface singularity, which corresponds to the bilinear form, on $H_n(F_{\lambda_0}, \mathbb{Z})$, defined in the following way:

Let f: $(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be the germ of an analytic function with an isolated critical point at $0 \in \mathbb{C}^{n+1}$. The fibration f: $f^{-1}(S_\varepsilon(1)) \cap B_\delta(n+1) \to S_\varepsilon(1)$ for $0 < \varepsilon \ll \delta$ is equivalent to the fibration χ: $S_\delta(n+1) - f^{-1}(0) \to S_1(1)$, defined by $\chi(z) = \dfrac{f(z)}{\|f(z)\|}$ (see Milnor [7]). It follows that each F_λ can be thought of as embedded in $S_\delta(n+1)$.

Let 1^+ be an arbitrary point in $S_1(1)$ different from 1. Define $F=\chi^{-1}(1)$, $F^+=\chi^{-1}(1^+)$ and denote by $x\mapsto x^+$ the isomorphism $H_n(F,\mathbb{Z})\to H_n(F^+,\mathbb{Z})$ obtained by an isotopy that brings F into F^+ turning counterclockwise on $S_1(1)$.

For $x, y\in H_n(F,\mathbb{Z})$ define $L(x,y)=$ linking number of the pair (x, y^+).

$L(x,y)$ can be defined also in the following way.

Let $T=B_\delta(n+1)\cap f^{-1}(B_\varepsilon^*(1))$, T is contractible, so from the exact sequence of the pair T, F_λ, with $|\lambda|=\varepsilon$, one gets an isomorphism

$$\psi_\lambda\colon\ H_{n+1}(T, F_\lambda;\mathbb{Z})\simeq H_n(F_\lambda,\mathbb{Z}).$$

Define $\Delta(x)=\psi_\lambda^{-1}(x)$ for $x\in H_n(F_\lambda,\mathbb{Z})$; there is, for $\lambda\neq\lambda'$, an intersection product

$$\langle\ ,\ \rangle\colon\ H_{n+1}(T, F_\lambda;\mathbb{Z})\times H_{n+1}(T, F_{\lambda'};\mathbb{Z})\to\mathbb{Z}$$

such that one has, for $x, y\in H_n(F,\mathbb{Z})$, that $L(x,y)=\langle\Delta(x),\Delta(y^+)\rangle$.

Make a little deformation $\tilde f$ of f such that the critical value 0 of f is decomposed into a set Λ of $\mu=\operatorname{rank} H_n(F,\mathbb{Z})$ distinct critical values.

Choose a set of smooth embeddings of $[0,1]$ into $B_\varepsilon^*(1)$, which connect the point ε to the points of Λ and are disjoint outside the point ε.

Remark that there is a natural ordering $\gamma_1,\dots,\gamma_\mu$ of these arcs, induced by the clockwise order in which they intersect the boundary of a small disk around ε.

Let $\Lambda=\{\lambda_1,\dots,\lambda_\mu\}$, where λ_i denotes the end point of γ_i.

Proposition 5.1[6]

$$L(e_i,e_j)=\begin{cases}0, & \text{if } i<j\\ \langle e_j, e_i\rangle, & \text{if } i>j\\ (-1)^{\frac{n(n+1)}{2}}, & \text{if } i=j.\end{cases}$$

Proof. Let ε^+ be a complex number of modulus ε and small positive argument. One can construct (by deforming $\gamma_1,\dots,\gamma_\mu$; see Fig. 4) a set

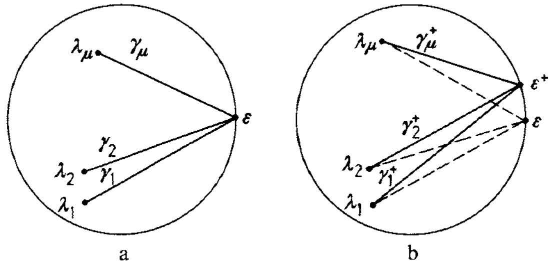

Fig. 4. a The vanishing arcs. b The deformed vanishing arcs

[6] The Reviewer communicated to us that this result has also been obtained independently by A.H. Durfee in a preprint entitled "Fibered Knots and Algebraic Singularities", Univ. of Calif., Berkeley June 1973.

of embeddings $\gamma_1^+, \ldots, \gamma_\mu^+$ of $[0, 1]$ into $B_\varepsilon^*(1)$ which connect ε^+ to $\lambda_1, \ldots, \lambda_\mu$ and such that

(i) the vanishing cycle associated to γ_i^+ is e_i^+,

(ii) the images of γ_i and γ_j^+, are disjoint for $i<j$, they intersect transversally at one point if $i>j$ and intersect only at λ_i for $i=j$.

Remark that $\Delta(e_i)$ (respectively $\Delta(e_j^+)$) can be represented by a cone over the sphere that represents e_i (respectively e_j^+) with vertex in the critical point of f over λ_i (respectively λ_j). This can be seen following the non singular fiber along γ_i (respectively γ_j^+).

It follows that if $i<j$, $\Delta(e_i)$ and $\Delta(e_j^+)$ can be represented by cycles with disjoint supports, so that

$$0=\langle \Delta(e_i), \Delta(e_j^+)\rangle = L(e_i, e_j).$$

If $i>j$, let t denote the intersection point of γ_i and γ_j^+; choose a trivialization of f over an open set U which contains the image of γ_i, from ε to t and the image of γ_j^+, from ε^+ to t. Suppose that e_i, e_j are represented by cycles in F which intersect each other transversally at a finite set of points.

Then $\Delta(e_i)$ and $\Delta(e_j^+)$ can be represented by cones which, over U, are products of e_i and e_j with lines l_i, l_j^+. It follows that $\Delta(e_i)$ and $\Delta(e_j^+)$ intersect transversally in $f^{-1}(t)$ at the points of $e_i \cap e_j$. Taking care of signs by remarking that the lines l_i, l_j^+ have a positive intersection, it follows that

$$L(e_i, e_j) = \langle \Delta(e_i), \Delta(e_j^+)\rangle = (-1)^n \langle e_i, e_j\rangle = \langle e_j, e_i\rangle.$$

Finally we compute $L(e_i, e_i)$. The same argument as before reduces us to computing $\langle \Delta(e_i), \Delta(e_i^+)\rangle$ in a neighborhood of $f^{-1}(\lambda_i)$, so that one may suppose that

$$\mu=1, \quad f(z)=\sum_0^n z_i^2.$$

We do the calculation explicitely: let $\Sigma=\{z\in\mathbb{C}^{n+1}/|f(z)|=1\}$. then $f\colon \Sigma \to S_1(1)=S$ is equivalent to the fibration of the singularity. If $\theta\in S$, define $F(\theta)=f^{-1}(\theta)$ and let $z_\alpha = x_\alpha + i\, y_\alpha$, $\alpha=0, \ldots, n$. Then the vanishing cycle e in $F(1)$ is represented by

$$\left\{(x_0, y_0, \ldots, x_n, y_n)\in\mathbb{R}^{2n+2} \Big/ \sum_0^n x_i^2 = 1,\ y_j=0\right\},$$

and

$$\Delta(e)=\left\{(x_0, y_0, \ldots, x_n, y_n)\in\mathbb{R}^{2n+2} \Big/ \sum_0^n x_i^2 \leqq 1,\ y_j=0\right\}.$$

We can choose $1^+ = -1$. The isotopy $F(1) \to F(\theta^2)$ is obtained by $(z_0, \dots, z_n) \mapsto (\theta z_0, \dots, \theta z_n)$ so that $F(1) \to F(-1)$ is just

$$(x_0, y_0, \dots, x_n, y_n) \mapsto (-y_0, x_0, \dots, -y_n, x_n).$$

It follows that $\Delta(e^+) = \left\{(x_0, y_0, \dots, x_n, y_n) \in \mathbb{R}^{2n+2} \Big/ \sum_0^n y_i^2 \leqq 1, x_j = 0\right\}$ and $\Delta(e)$, $\Delta(e^+)$ intersect transversally at one point.

If $(x_0, \dots, x_n)$ are positively ordered coordinates for $\Delta(e)$, $(y_0, \dots, y_n)$ will be positively ordered for $\Delta(e^+)$. Since the orientation of $\mathbb{C}^{n+1}$ is given by the ordering $(x_0, y_0, \dots, x_n, y_n)$ which can be obtained from $(x_0, \dots, x_n, y_0, \dots, y_n)$ by means of $\frac{n(n+1)}{2}$ transpositions, it follows that

$$L(e, e) = \langle \Delta(e), \Delta(e^+) \rangle = (-1)^{\frac{n(n+1)}{2}}.$$

Each bilinear form B over $H = H_n(F, \mathbb{Z})$ can be interpreted as an element of $\mathrm{Hom}(H, H^*)$, where H^* is the dual module of H over $\mathbb{Z}$. In particular L, ${}^tL \in \mathrm{Hom}(H, H^*)$.

Corollary 1. *L, ${}^tL$ are isomorphisms.*

Proof. By Proposition 5.1, on a geometrical basis, one has $\det L = \det {}^tL = (-1)^{\frac{n(n+1)}{2}\mu}$.

From the above corollary it follows that ${}^tL^{-1} \cdot L$ can be interpreted as an automorphism of H.

Corollary 2. *$(-1)^{n+1}\, {}^tL^{-1} \cdot L$ coincides with the automorphism M of $H_n(F, \mathbb{Z})$ associated to the fibration χ over $S_1(1)$.*

Proof. This equality can be checked directly, on a geometrical basis, using formula 3.1 and the expression of $L(e_i, e_j)$ in Proposition 5.1. We give an alternative proof which uses only the fact that L is invertible: for $x, y \in H_n(F, \mathbb{Z})$ one has $L(x, y) = (-1)^{n+1} L(M(y), x)$; because

$$\langle \Delta(x), \Delta(y^+) \rangle = \langle \Delta(x^+), \Delta(M(y)) \rangle = (-1)^{n+1} \langle \Delta(M(y)), \Delta(x^+) \rangle.$$

This can be expressed by the formula $L = (-1)^{n+1}\, {}^tL \cdot M$.
Since L is invertible one gets the assertion.

Remark. As noticed by Brieskorn, on a geometrical basis the equation $M = (-1)^{n+1}\, {}^tL^{-1} \cdot L$ can be solved in L; because by Proposition 5.1 on such a basis L has a representative matrix that is triangular, so that the coefficients of L can be determined recursively from those of M. In view of 5.1, this is only another way of stating Lemma 3.1.

On the other hand it is not clear whether the bilinear form L is determined by giving M as an automorphism of the $\mathbb{Z}$-module $H_n(F, \mathbb{Z})$.

By Proposition 5.1 it follows that the intersection pairing I on $H_n(F, \mathbb{Z})$ satisfies $I = {}^tL + (-1)^n\,{}^tL$.

So we have

on an ordered geometrical basis it is equivalent to know L, I or M
in general L determines I and M.

It might be interesting to study the obstruction given by L to the ordering of a geometrical basis; in other words let a geometrical basis $e_1, \ldots, e_\mu$ be given, and let $(\sigma_1, \ldots, \sigma_\mu)$ be any permutation of $(1, \ldots, \mu)$ such that $L(e_{\sigma_i}, e_{\sigma_j}) = 0$ for $i < j$; is it true that $(e_{\sigma_1}, \ldots, e_{\sigma_\mu})$ is an ordered geometrical basis for a suitable choice of $\gamma_1, \ldots, \gamma_\mu$? This seems to be of interest when comparing the monodromy group with the group of isometries of the intersection pairing I.

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(Received October 13, 1973)