Inventiones math. 35, 233–259 (1976)

Inventiones mathematicae

Admissible Representations of a Semi-Simple Group over a Local Field with Vectors Fixed under an Iwahori Subgroup

Armand Borel (Princeton)

To Jean-Pierre Serre

Let G be the group of rational points of a connected semi-simple algebraic group $\mathscr{G}$ over a locally compact nonarchimedian local field k, and let (r, V) be an admissible representation of G in a vector space V over a field R of characteristic zero. If B is a compact open subgroup of G, then the fixed point set V^B of B in V is a finite dimensional space, acted upon in a natural way by the Hecke algebra $H(G, B)$ of compactly supported B-biinvariant R-valued functions on G. We shall be concerned here with the case where B is an Iwahori subgroup of G and prove that, in that case, every finite dimensional $H(G, B)$-module E occurs in this way. In fact, given E, there are two natural algebraic constructions of a smooth G-space of which the fixed point set of B is isomorphic to E as an $H(G, B)$-module: the analogues, in the context of smooth representations, of the induced and produced modules of [10], also to be denoted $I(E)$ and $P(E)$ (see 2.3)[1]. We shall prove more precisely that $I(E)$ and $P(E)$ are admissible (4.4), canonically isomorphic, irreducible if and only if E is, and that $E \mapsto P(E)$ defines an exact functor from finite dimensional $H(G, B)$-modules to admissible G-spaces (4.10). The proof of admissibility uses, besides standard facts on $H(G, B)$, mainly a lemma on buildings (4.1), due to F. Bruhat, which expresses a strong transitivity property of compact open subgroups of G, while that of 4.10 depends on results of [7] and on a lemma of Casselman (4.8).

In § 5 we consider the case where E is one-dimensional, i.e., when r is a character of $H(G, B)$, and $R = \mathbf{C}$, and determine in which case $P(E)$ belongs to the discrete series (or rather, is the space of smooth vectors of an element of the discrete series). There is always the special representation of Matsumoto [13] and Shalika [16]. In our setup, it corresponds to the special character σ, which assigns -1 to the standard generators e_s of $H(G, B)$ (see 3.5). However, if G is almost simple over k, of k-rank $l \geqq 2$ and $H(G, B)$ has more than two characters of degree one (see 3.4), then there is at least one other. Their list is derived in 5.8 from a criterion for a character σ to give rise to a square integrable representation (5.2) and results of MacDonald [12]. These representations are not cuspidal.

The special representation may be viewed as the natural representation of G in the space H_2^l of square integrable harmonic l-forms on the Bruhat-Tits building

---

[1] However, we shall use the word *coinduced* instead of *produced*.

X of G, where $l = rk_k \mathcal{G} = \dim X$, (6.1). There is a natural G-morphism η of the l-th cohomology group $H_c^l(X)$ of X with compact supports and complex coefficients into H_2^l. We shall prove that η is an isomorphism of $H_c^l(X)$ onto the space of smooth vectors in H_2^l, (6.2). One of the main results of [3] implies that the G-space $H_c^l(X)$ is admissible and that its character is the "Steinberg character": the alternating sum of induced characters from the trivial representations of the groups of rational points of the parabolic k-subgroups containing a given minimal one (see 6.3(2)). It follows then that this character is also the one of the special representation (6.4).

The above summarizes the contents of §§4, 5, 6, which contain the main results of this paper. §1 recalls some notions on admissible representations, Hecke algebras and convolution, mainly to fix some notation. §2 introduces induced and coinduced modules from a finite dimensional module over a Hecke algebra $H(G, B)$, where G is a locally compact totally disconnected group, B a compact open subgroup, and describes some of their properties. This is quite analogous to [10], where this is carried out for algebras, with some minor changes due to the fact that we work with smooth representations.

In §3, G and B are specialized as above, and we recall some standard properties of $H(G, B)$. Finally §7 is devoted to the proof of 4.1.

The fact that the Steinberg character is an irreducible unitary character was also announced by Casselman in [6] and proved in [7]. His argument is quite different from ours. In [14], Matsumoto also states, in a somewhat more general framework than ours, that every finite dimensional $H(G, B)$-module occurs as the fixed point set of B in an admissible representation of G.

In the course of this work, I have received precious help from several people, to whom I am glad to express my thanks. The starting point of this paper was a question raised to me by Harish-Chandra in December, 1970, namely whether the Steinberg character (6.3(2)) was an irreducible unitary character. The joint work with J-P. Serre summarized in [2] showed readily that this character was effective, and that it would be the character of the special representation if and only if the canonical map η: $H_c^l(X) \to H_2^l$ of 6.2 were injective. That was proved shortly afterwards. At that time, I benefited from a correspondance with J-P. Serre, where he notably emphasized the role of the Hecke algebra $H(G, B)$. This led me naturally to the more general questions studied in this paper. The results of §5 were proved in Fall, 1971, with the help of J. Tits, the admissibility of the coinduced modules $P(E)$ was realized a bit later and discussed in a seminar at the I.H.E.S. at Bures, in Fall, 1973; finally, the other results of §4 were obtained in Spring, 1974, with the help of P. Cartier, who suggested to use $I(E)$ and $P(E)$ concurrently, and of W. Casselman, who supplied Lemma 4.8.

I. Generalities

R denotes a field of characteristic zero[2], *G a locally compact totally disconnected unimodular group. Vector spaces are over R, and modules over groups or algebras are vector spaces.*

[2] In fact, without substantial change, the characteristic p of R can be allowed to be >0, provided it is assumed that G has no compact pro-p-subgroup $\neq \{1\}$.

§ 1. Convolution. Smooth and Admissible Representations

We recall here a few definitions and facts about smooth representations, in a form convenient for the sequel, mainly to fix some conventions and notation. See [7] for a general discussion.

1.1. Let E be a vector space, and X a set. Then $C(X, E)$ is the vector space of maps of X into E. If X is a locally compact totally disconnected space, then $C_c(X, E)$ is the space of elements of $C(X, E)$ with compact support, and $C^\infty(X, E)$ (resp. $C_c^\infty(X, E)$) is the space of smooth (i.e., locally constant) functions in $C(X, E)$, (resp. $C_c(X, E)$). If $E = R$, we write $C(X)$, $C_c(X)$, etc.

If $X = G$, and $f \in C(G, E)$, then $\check{f}$ denotes the function defined by $\check{f}(x) = f(x^{-1})$, $(x \in G)$. Fix a Haar measure on G. Assume E to be an algebra over R. Then $C_c^\infty(G, E)$ is an algebra with respect to the convolution product

$$(u * v)(x) = \int_G u(x \cdot y^{-1}) \cdot v(y)\, dy = \int_G u(y) \cdot v(y^{-1} \cdot x)\, dy, \tag{1}$$

which is in fact a finite sum. We have

$$(u * v)^\vee = \check{v} * \check{u}. \tag{2}$$

More generally, if E', E'' and E are vector spaces and $(e', e'') \mapsto e' \cdot e''$ is a bilinear map from $E' \times E''$ to E, then (1) defines a pairing $C_c^\infty(G, E') \times C_c^\infty(G, E'') \to C_c^\infty(G, E)$. These definitions are valid if R is only a commutative ring, and E', E'', E modules over R.

The right hand side already makes sense if one of the two factors has compact support. In particular, it allows one to give $C^\infty(G, E)$ a bimodule structure over $C_c^\infty(G)$.

Let B be a compact open subgroup of G, and assume the total measure of B to be 1. Then the space $C_c(B \backslash G/B)$ of compactly supported B-biinvariant R-valued functions on G is an algebra under convolution, the "Hecke algebra" of G with respect to B to be denoted $H_R(G, B)$, or $H(G, B)$ or simply H. For $g \in G$, we let e_g be the characteristic function of BgB. We have

$$\check{e}_g = e_{g^{-1}}.$$

The elements e_w, where w runs through a set of representatives of $B \backslash G/B$, form a vector space basis of $H(G, B)$, and the characteristic function e_1 of B is a left and right identity.

If E is a vector space, then $C^\infty(G, E)$ and $C_c^\infty(G, E)$ are H-bimodules under convolution, and $* e_1$ is a projector of $C^\infty(G, E)$ onto $C(G/B, E)$. Thus we may also view $C(G/B, E)$ and $C_c(G/B, E)$ as H-modules under right convolution. We have

$$(f * u)(x) = \int_G f(xy) \cdot u(y^{-1})\, dy = \sum_{y \in G/B} f(xy) \cdot u(y^{-1}), \tag{3}$$

$(f \in C(G/B, E), u \in H(G, B))$. In particular,

$$(f * \check{e}_w)(x) = \sum_{y \in BwB/B} f(x \cdot y), \qquad (x, w \in G; f \in C(G/B, E)). \tag{4}$$

Let us put

$$q_w = \mathrm{Card}(BwB/B) = \int_{BwB} dx. \tag{5}$$

We have then

$$(f*\check{e}_w)(1)=q_w\cdot f(w), \quad (f\in C(B\backslash G/B, E)), \tag{6}$$

$$(\check{e}_w*f)(1)=q_w\cdot f(w), \quad (f\in C(B\backslash G/B, E). \tag{7}$$

Note that $q_w=q_{w^{-1}}$ since G is unimodular.

Clearly, $H(G, B)$ commutes with G acting on $C(G/B, E)$ by left translations. In fact, it is easily seen that $H(G, B)$ is the full commuting algebra of G on $C_c(G/B, E)$.

1.2. A representation of G will be denoted either by the vector space V acted upon, or by the homomorphism $\pi\colon G\to GL(V)$, or by the pair (π, V). Let (π, V) be one. An element $v\in V$ is smooth if its isotropy group is open in G.

The set V^∞ of smooth vectors of V is a vector subspace (since the open subgroups of G form a fundamental system of neighborhoods of the identity) stable under G. The representation π is *smooth* if $V=V^\infty$, *admissible* if, in addition, V^U is finite dimensional for every open subgroup U of G. Every G-submodule or quotient module of a smooth (resp. admissible) representation is smooth (resp. admissible). If (π, V) is smooth, then it defines a representation of the convolution algebra $C_c^\infty(G)$ characterised by

$$\pi(f)\cdot v=\int_G f(x)\cdot\pi(x)\cdot v\,dx, \quad (f\in C_c^\infty(G)). \tag{1}$$

If B and dx are as above, $V=C(G/B, E)$ and π is given by translations, then it follows from the definitions that $\pi(f)\cdot v=f*v$.

1.3. Assume π to be smooth and G to be *compact*. Then $G\cdot v$ is finite for all $v\in V$, hence V is union of finite dimensional semi-simple G-modules. Therefore, V is semi-simple and if

$$0\to V'\to V\to V''\to 0 \tag{2}$$

is an exact sequence of smooth modules, then the sequence of isotypic submodules of a given type is exact; in particular,

$$0\to V'^G\to V^G\to V''^G\to 0 \tag{3}$$

is exact. If dx has total measure one, then $v\mapsto\int_G\pi(x)\cdot v\,dx$ is a projector of V onto V^G, whose kernel is the subspace $V(G)$ of V generated by the elements of the form $v-\pi(g)\cdot v$ $(g\in G; v\in V)$ [7: 3.2.1].

1.4. *Contragredient Representation.* Assume (π, V) to be smooth. The natural representation π of G in the space $\tilde{V}=V'^\infty$ of smooth elements in the dual V' of V is the contragredient representation $\tilde{\pi}$ of π. It is admissible if and only if π is. For every open subgroup K of G, the space $\tilde{V}^K$ is the orthogonal subspace to $V(K)$, and may be canonically identified to the dual of V^K [7: 2.1.9]. We have

$$\langle v, \tilde{\pi}(g)\cdot\tilde{v}\rangle=\langle\pi(g^{-1})\cdot v, \tilde{v}\rangle; \quad \langle v, \tilde{\pi}(f)\cdot\tilde{v}\rangle=\langle\pi(\check{f})\cdot v, \tilde{v}\rangle, \tag{1}$$

$(v\in V; \tilde{v}\in\tilde{V}; g\in G; f\in C_c^\infty(G))$. In particular

$${}^t\pi(f)=\tilde{\pi}(\check{f}), \quad (f\in C_c^\infty(G)). \tag{2}$$

1.5. *Coefficients.* Given $v \in V$, $\tilde{v} \in \tilde{V}$, we denote by $c_{v,\tilde{v}}$ the function on G defined by

$$c_{v,\tilde{v}}(g) = \langle v, \tilde{\pi}(g) \cdot \tilde{v} \rangle = \langle \pi(g^{-1}) \cdot v, \tilde{v} \rangle, \quad (g \in G). \tag{1}$$

We let l_g (resp. r_g) denote the effect of the left (resp. right) translation by $g \in G$ on functions on G. Thus

$$l_g f(x) = f(g^{-1} \cdot x), \quad r_g f(x) = f(x \cdot g), \quad (x \in G), \tag{2}$$

where f is a map of G in some set. Straightforward computations yield the formulae

$$c_{\pi(x) \cdot v, \tilde{\pi}(y) \cdot \tilde{v}} = l_x \cdot r_y \cdot c_{v,\tilde{v}}, \quad (v \in V, \tilde{v} \in \tilde{V}; x, y \in G), \tag{3}$$

$$c_{\pi(f) \cdot v, \tilde{\pi}(h) \cdot \tilde{v}} = f * c_{v,\tilde{v}} * \check{h}, \quad (v \in V, \tilde{v} \in \tilde{V}; f, h \in C_c^\infty(G)) \tag{4}$$

(of which (3) differs slightly from [7:2.5.1] since our $c_{v,\tilde{v}}$ is the V-transform of the $c_{v,\tilde{v}}$ there).

§ 2. Induced and Coinduced Modules from a Representation of a Hecke Algebra

In this section, B is a compact open subgroup of G, and convolution is taken with respect to a Haar measure dx giving B total measure one.

2.1. Let E be a vector space. The group G acts by left translations on $C(G/B, E)$ and $C_c(G/B, E)$. Clearly, the map

$$C_c(G/B) \otimes_R E \to C_c(G/B, E), \tag{1}$$

which assigns to $f \otimes e$ the function $x \mapsto f(x) \cdot e$ $(x \in G)$ is an isomorphism. Since the intersection of finitely many conjugates of B is open in G, the representation of G on $C_c(G/B, E)$ is smooth. On the other hand, $C(G/B, E)$ is not smooth in general. The canonical bilinear form $\langle\ \rangle$ on $E' \times E$ allows one to define a pairing of $C(G/B, E')$ and $C_c(G/B, E)$ by

$$\langle \varphi, \psi \rangle = \sum_{x \in G} \langle \varphi(x), \psi(x) \rangle, \quad (\varphi \in C(G/B, E'), \psi \in C_c(G/B, E)), \tag{2}$$

or, equivalently, by

$$\langle \varphi, \psi \rangle = (\varphi * \check{\psi})(1). \tag{3}$$

The map j: $C(G/B, E') \to C_c(G/B, E)'$ is readily seen to be an isomorphism, and therefore yields an isomorphism

$$C(G/B, E')^\infty \xrightarrow{\sim} C_c(G/B, E)^\sim. \tag{4}$$

2.2. In this paper a representation (r, E) of $H(G, B)$ in a vector space E is a homomorphism r: $H(G, B) \to \mathrm{End}(E)$ which maps e_1 onto the identity. By definition, the contragredient representation to r is the representation $\tilde{r}$ in the dual space E' to E characterised by

$${}^t r(u) = \tilde{r}(\check{u}), \quad (u \in H(G, B)). \tag{1}$$

We have therefore

$$\langle r(u) \cdot e, e' \rangle = \langle e, \tilde{r}(\check{u}) \cdot e' \rangle, \quad (e \in E, e' \in E'; u \in H(G, B)). \tag{2}$$

If V is a smooth G-module, then V^B is stable under $H(G, B)$ acting via the natural representation of $C_c^\infty(G)$, and is an $H(G, B)$-module. If V is admissible, then $\tilde{V}^B$ is the contragredient H-module to V^B. If W is a smooth G-module, the restriction to the fixed points under B yields a natural map

$$\mathrm{Hom}_G(V, W) \to \mathrm{Hom}_{H(G,B)}(V^B, W^B). \tag{3}$$

2.3. Let (r, E) be a representation of $H(G\ B)$. We let

$$I_{B,G}(E) = I_{B,G}(r) = C_c(G/B) \otimes_H E, \tag{1}$$

$$P^0_{B,G}(E) = P^0_{B,G}(r) = \{f \in C(G/B, E) \mid f * u = r(u) \cdot f, (u \in H)\}. \tag{2}$$

Both $I_{B,G}(r)$ and $P^0_{B,G}(r)$ will be viewed as G-modules via left translations. The former is smooth, but the latter is not in general, and we let

$$P_{B,G}(E) = P_{B,G}(r) = (P^0_{B,G}(r))^\infty. \tag{3}$$

We shall sometimes suppress the indices B, G if they are clear from the context.

We note that $I(E)$ is the quotient of $C_c(G/B) \otimes_R E$ by the subspace M spanned by elements of the form

$$(f * h) \otimes e - f \otimes r(h) \cdot e \qquad (f \in C_c(G/B);\, e \in E;\, h \in H),$$

and that the canonical projection $C_c(G/B) \otimes_R E \to I(E)$ commutes with G.

Remark. In view of the elementary relation

$$C(G/B, E) = \mathrm{Hom}(C_c(G/B), E), \tag{4}$$

and of the fact that e_1 is a projector of $C_c^\infty(G, E)$ onto $C_c(G/B, E)$, we have

$$I_{B,G}(E) = C_c^\infty(G) \otimes_H E, \qquad P^0_{B,G}(E) = \mathrm{Hom}_H(C_c^\infty(G), E), \tag{5}$$

where

$$\mathrm{Hom}_H(C_c^\infty(G), E)$$
$$= \{f \in \mathrm{Hom}(C_c^\infty(G), E) \mid f(\varphi * u) = r(u) \cdot f(\varphi), (u \in H;\, \varphi \in C_c^\infty(G))\}. \tag{6}$$

Therefore $I_{B,G}(E)$ (resp. $P^0_{B,G}(E)$) is the induced (resp. produced) module from H to $C_c^\infty(G)$, as defined by G.D. Higman in [10].

2.4. Proposition. *We keep the notation of* 2.3.

(i) *The map* μ_0: $e \mapsto e_1 \otimes e$ *induces an* H*-isomorphism of* E *onto* $I(E)^B$.

(ii) *The map* ν_0: $f \mapsto f(1)$ *is an* H*-morphism of* $P(E)$ *onto* E, *which maps* $P(E)^B$ *isomorphically onto* E. *For* $e \in E$, *the element* $f_e \in P(E)^B$ *mapped onto* e *by* ν_0 *is the function*

$$f_e = \sum_{w \in B \backslash G/B} e_w \cdot q_w^{-1} \cdot r(\check{e}_w) \cdot e,$$

which assigns $q_w^{-1} \cdot r(\check{e}_w) \cdot e$ *to* $x \in BwB$.

(iii) *Every non-zero G-submodule of P(E) contains a non-zero element fixed under B. The G-module I(E) is generated by* $I(E)^B$.

(i) Left convolution by e_1 on $C_c(G/B)$ defines projectors of $I(E)$ onto $I(E)^B$ and of $C_c(G/B)$ onto $H = C_c(G/B)^B$. Therefore

$$I(E)^B = H \otimes_H E = E,$$

and (i) follows.

(ii) In the notation of 1.1(4)(5), we have

$$q_w \cdot f(x) = r(\check{e}_w) \cdot f(1), \qquad (f \in P(E)^B; w \in B \backslash G/B; x \in BwB). \tag{1}$$

In fact, $f(x) = f(w)$; by 1.1(6), the left-hand side is equal to $(f * \check{e}_w)(1)$, which equals $r(\check{e}_w) \cdot f(1)$ by definition of $P(E)$. This implies that v_0: $P(E)^B \to E$ is injective. Given $e \in E$, let us now define $f \in C(G/B, E)$ by (1) and the condition $f(1) = e$. This function is left-invariant under B; by 1.1(6), it satisfies

$$(f * e_w)(1) = r(e_w) \cdot f(1), \qquad (w \in B \backslash G/B). \tag{2}$$

Since the e_w ($w \in B \backslash G/B$) span H, this implies by linearity

$$(f * u)(1) = r(u) \cdot f(1), \qquad \text{for all } u \in H. \tag{3}$$

Together with 1.1(6), it yields, for $x \in BwB$:

$$q_w \cdot (f * u)(x) = (f * u * \check{e}_w)(1) = r(u) \cdot r(\check{e}_w) \cdot f(1) = r(u) \cdot (f * \check{e}_w)(1),$$

$$q_w (f * u)(x) = r(u) \cdot q_w \cdot f(w) = r(u) \cdot q_w \cdot f(x).$$

This shows that $f \in P(E)^B$, hence that v_0 is surjective. Taking (3) and 1.1(7) into account, we have, for $f \in P(E)^B$ and $w \in G$,

$$v_0(\check{e}_w * f) = (\check{e}_w * f)(1) = q_w \cdot f(w) = r(\check{e}_w) \cdot f(1) = r(\check{e}_w) \cdot v_0(f).$$

Since the e_w's span H, it follows that v_0 induces an H-isomorphism of $P(E)^B$ onto E.

Let M be the kernel of $e_1 *$ on $P(E)$. Then $P(E)$ is the direct sum of M and $P(E)^B$ and M is also annihilated by all elements of H. For any $f \in C(G/B, E)$, we have

$$f(1) = (e_1 * f)(1), \tag{4}$$

hence $M = \ker v_0$, which completes the proof of the first part of (ii). The second one then follows from (1).

(iii) Let V be a non-zero G-submodule of $P(E)$. Being invariant under left translations, it contains an element f such that $f(1) \neq 0$. But then $(e_1 * f)(1) \neq 0$, hence $V^B \neq 0$.

For $x \in G$, let ε_x be the characteristic function of xB on G/B. The tensor product $C_c(G/B) \otimes_R E$ is spanned by the subspaces

$$\varepsilon_x \otimes E = l_x^{-1}(e_1 \otimes E), \qquad (x \in G),$$

hence $C_c(G/B)\otimes_R E$ is generated, as a G-module, by $e_1 \otimes E$. Since $I(E)$ is a quotient of $C_c(G/B)\otimes_R E$, this is *a fortiori* true for $I(E)$, whence the second assertion of (iii).

In the sequel, we shall often identify E with $I(E)^B$ or $P(E)^B$ by means of μ_0 or ν_0.

2.5. Proposition. *Let* (r, E) *be a finite dimensional* H*-module, and* (π, V) *a smooth* G*-module. Then the natural restriction maps* (2.2(3))

$$\rho_P\colon \operatorname{Hom}_G(V, P(E)) \to \operatorname{Hom}_H(V^B, E) \quad \textit{and} \quad \rho_I\colon \operatorname{Hom}_G(I(E), V) \to \operatorname{Hom}_H(E, V^B)$$

are bijective.

Given $\alpha \in \operatorname{Hom}_G(V, P(E))$, let us denote by α_0 the restriction of α to V^B. If $\alpha_0 = \beta_0$, then $\alpha - \beta$ is a G-morphism which is zero on V^B. Since $P(E)$ has no non-zero G-module F with $F^B = 0$ (2.4), we have $\alpha = \beta$, hence ρ_P is injective. Let β_0 be an H-morphism of V^B into E. We define a map $\beta\colon V \to C(G/B, E)$ by the rule

$$\beta(v)(x) = \beta_0(e_1(x^{-1} \cdot v)), \qquad (v \in V; x \in G). \tag{1}$$

Let (v_i) be a basis of V^B, and (v^i) be the dual basis of $\tilde{V}^B$ (see 1.4). Then, in the notation of 1.5, we have

$$\beta(v) = \sum c_{v, v^i} \cdot \beta_0(v_i). \tag{2}$$

The relation 1.5(3) shows that β is a G-morphism of V into $C(G/B, E)$. Let us show that $\beta(V) \subset P(E)$. Let $h \in H$. The transformation $\pi(h)$ leaves V^B stable and we may write uniquely

$$\pi(h) \cdot v_i = \sum_j a_i^j \cdot v_j, \qquad (a_i^j \in R). \tag{3}$$

By 1.4(2), ${}^t\pi(h) = \pi(\check{h})$, hence

$$\tilde{\pi}(\check{h}) \cdot v^j = \sum_i a_i^j \cdot v^i. \tag{4}$$

Using 1.5(4), we get

$$\beta(v) * h = \sum_i (c_{v, v^i} * h) \cdot \beta_0(v_i) = \sum_i c_{v, \tilde{\pi}(\check{h}) v^i} \cdot \beta_0(v_i),$$

$$\beta(v) * h = \sum_{i,j} c_{v, v^j} \cdot a_j^i \cdot \beta_0(v_i).$$

Since β_0 is an H-isomorphism, we also have

$$r(h) \cdot \beta_0(v_i) = \beta_0(\pi(h) \cdot v_i)) = \sum_j a_i^j \cdot \beta_0(v_j),$$

whence

$$\beta(v) * h = \sum_i c_{v, v^i} r(h) \cdot \beta_0(v_i) = r(h) \cdot \beta(v), \tag{5}$$

which shows that $\beta(v) \in P(E)$. If $v \in V^B$, then $\beta(v) \in P(E)^B$ and moreover $\nu_0(\beta(v)) = \beta(v)(1) = \sum \langle v, v^i \rangle \beta_0(v_i) = \beta_0(v)$, hence $\beta_0 = \rho_P(\beta)$.

Since $I(E)$ is generated by E as a G-module, the map ρ_I is injective. Let $\alpha_0\colon E \to V^B$ be an H-morphism. It extends uniquely to an R-linear map

$$\alpha_1\colon \; C_c(G/B) \otimes_R E \to V$$

given by

$$\alpha_1(f\otimes e)=\pi(f)\cdot\alpha_0(e), \qquad (f\in C_c(G/B); e\in E).$$

If $m\in C_c^\infty(G)$, then

$$\alpha_1((m*f)\otimes e)=\pi(m*f)\cdot\alpha_0(e)=\pi(m)\cdot\pi(f)\cdot\alpha_0(e)=\pi(m)\cdot\alpha_1(f\otimes e),$$

hence α_1 is $C_c^\infty(G)$-equivariant, and therefore also G-equivariant [7:2.2.1]. If $h\in H$, then

$$\alpha_1((f*h)\otimes e)=\pi(f*h)\cdot\alpha_0(e)=\pi(f)\cdot\pi(h)\cdot\alpha_0(e)=\pi(f)\cdot\alpha_0(r(h)\cdot e)$$
$$=\alpha_1(f\otimes r(h)\cdot e),$$

hence α_1 annihilates the kernel of the canonical projection $C_c(G/B)\otimes_R E\to I(E)$ (see 2.3). The map α: $I(E)\to V$ obtained from α_1 by going over to the quotient is then a G-morphism which extends α_0.

2.6. Proposition. *Let (r, E) be an H-module. The canonical isomorphism*

$$C(G/B, E')^\infty \xrightarrow{\sim} (C_c(G/B, E))^\sim$$

of 2.1(4) *induces an isomorphism of $P(E')$ onto $I(E)^\sim$.*

By definition, $P(E')$ is a G-submodule of $C(G/B, E')$. It suffices to show that it is the orthogonal subspace to the kernel M of the projection

$$q\colon\ C_c(G/B)\otimes_R E=C_c(G/B, E)\to C_c(G/B)\otimes_H E=I(E).$$

Let $f\in C(G/B, E')$, $\varphi\in C_c(G, B)$, $e\in E$, $h\in H$. From 2.1, we get

$$\langle f, (\varphi*h)\otimes e\rangle=\langle f*(\varphi*h)^\vee(1), e\rangle=\langle f*\check{h}*\check{\varphi})(1), e\rangle,$$

$$\langle f, (\varphi*h)\otimes e\rangle=\langle f*\check{h}, \varphi\otimes e\rangle. \tag{1}$$

$$\langle f, \varphi\otimes r(h)e\rangle=\langle (f*\check{\varphi})(1), r(h)\cdot e\rangle=\langle \tilde{r}(\check{h})(f*\check{\varphi})(1), e\rangle,$$

$$\langle f, \varphi\otimes r(h)e\rangle=\langle \tilde{r}(\check{h})f, \varphi\otimes e\rangle. \tag{2}$$

The proposition follows immediately from (1) and (2) and 2.3.

II. Semi-Simple Groups Over a Local Field

From now on, k is a non-archimedian local field with a finite residue field, q the number of elements of the residue field, $\mathscr{G}$ a connected semi-simple k-group, $\tilde{\mathscr{G}}$ its universal covering, ω: $\tilde{\mathscr{G}}\to\mathscr{G}$ the canonical central isogeny, and l the k-rank of $\mathscr{G}$. We assume $l\geq 1$.

§ 3. The Hecke Algebra with Respect to an Iwahori Subgroup

3.1. Algebraic groups over k will usually be denoted by script letters, and the groups of their k-rational points by the corresponding Roman letters. In particular

$$G=\mathscr{G}(k), \qquad \tilde{G}=\tilde{\mathscr{G}}(k).$$

We let $\tilde{T}=(\tilde{G},\tilde{B},\tilde{N},S)$ be the Tits system of $\tilde{G}$ considered in [5], where $\tilde{B}$ is an Iwahori subgroup of $\tilde{G}$, and X the associated building. X is a polysimplicial chamber complex of dimension l. We let C_0 be the chamber fixed by B and A_0 the apartment fixed by $\tilde{B}\cap\tilde{N}$. Therefore $A_0=\tilde{N}\cdot C_0$. The Weyl group $\tilde{N}/(\tilde{N}\cap\tilde{B})$ of $\tilde{T}$ will be denoted $\tilde{W}$. The apartment A_0 has a canonical structure of affine euclidean space with respect to which $\tilde{W}$ acts faithfully as a euclidean reflection group. S is the set of reflections to the hyperplanes containing the faces of co-dimension 1 of C_0.

The group G maps naturally into $(\operatorname{Aut}\tilde{G})(k)$, hence operates on $\tilde{G}$ and on X. The latter action is also continuous and proper. We let B be the subgroup of G fixing C_0 (pointwise), N be the stabilizer of A_0 in G, and G_0 (resp. N_0) the biggest subgroup of G (resp. N) which acts on X by special automorphisms, i.e., automorphisms which preserve the type of a face [5:2.1.1]. Let L be the intersection of N with the normalizer $\mathcal{N}_G(B)$ of B in G. Then $G=L\cdot G_0$, the group G_0 is normal, of finite index, in G. The group $G/G_0=L/(L\cap G_0)$ may be identified with a finite commutative subgroup ψ of the group $\operatorname{Aut}\operatorname{Cox}S$ of automorphisms of the Coxeter diagram of (W,S) which leaves stable every connected component of the latter. The isogeny ω induces an isomorphism

$$\tilde{W}\xrightarrow{\sim} N_0/(N_0\cap B), \tag{1}$$

and bijections

$$\tilde{G}/\tilde{B}=G_0/B,\qquad \tilde{B}\diagdown\tilde{G}/\tilde{B}=B\diagdown G_0/B. \tag{2}$$

We have semi-direct product decompositions

$$G=\Psi\ltimes G_0,\qquad W=N/(B\cap N)=\Psi\ltimes\tilde{W}, \tag{3}$$

and $T_0=(G_0,B,N,S)$, $T=(G,B,N)$ are respectively a Tits system [4: IV, § 2] and a generalized Tits system [4: IV, § 2, Exer. 8] or [11]. We have the decompositions

$$G_0=\coprod_{w\in\tilde{W}} BwB,\qquad G=\coprod_{w\in W} BwB. \tag{4}$$

Moreover

$$BwBw'B=Bww'B,\quad \text{if } w,w'\in\tilde{W}\quad\text{and}\quad l(ww')=l(w)+l(w'), \tag{5}$$

where $l(\ )$ denotes the length in $\tilde{W}$ with respect to S [4: IV, § 1]. Let

$$q_w=\operatorname{Card}(BwB/B),\qquad (w\in W). \tag{6}$$

Then

$$q_w=\operatorname{Card}(\tilde{B}w\tilde{B}/\tilde{B}),\qquad \text{if } w\in\tilde{W}, \tag{7}$$

$$q_{ww'}=q_w\cdot q_{w'},\quad \text{if } w,w'\in\tilde{W}\quad\text{and}\quad l(ww')=l(w)+l(w'). \tag{8}$$

We have

$$L\cap G_0=B, \tag{9}$$

so that the elements of Ψ can be viewed as cosets modulo B, whose elements normalize B. We have

$$B\psi B=\psi B=B\psi,\quad B\psi B\psi' B=B\psi\psi' B,\qquad (\psi,\psi'\in\Psi), \tag{10}$$

$$\psi BwB=B\psi wB,\quad q_{\psi w}=q_w,\qquad (w\in\tilde{W};\psi\in\Psi). \tag{11}$$

For all this and the facts recalled in 3.2 below, see [4: IV, § 2, Exer. 8], or § 3 of [11], where the underlying groups are split but the arguments are general.

3.2. The Hecke algebra $H(G_0, B)$ will often be denoted by H_0. It follows from 3.1(2) that ω yields an isomorphism of $H(\tilde{G},\tilde{B})$ with H_0. The algebra H_0 is spanned over R by the characteristic functions e_w of the cosets BwB $(w\in\tilde{W})$. The e_w's have the following properties:

1) $e_w\cdot e_{w'}=e_{w\cdot w'}$, if $w, w'\in\tilde{W}$ and $l(w\cdot w')=l(w)+l(w')$

2) $e_s^2=(q_s-1)\cdot e_s+q_s\cdot e_1$, $(s\in S)$.

3) Let $s, s'\in S$ be distinct and let $m=m(s,s')$ be the order of $s\cdot s'$. Then m is finite, except if $|S|=2$, and we have

$$(e_s\cdot e_{s'})^r\cdot e_s=e_{s'}\cdot(e_s\cdot e_{s'})^r,\qquad \text{if } m(s,s')=2r+1,$$

$$(e_s\cdot e_{s'})^r=(e_{s'}\cdot e_s)^r,\qquad \text{if } m(s,s)'=2r.$$

4) Let $w\in\tilde{W}$, $s\in S$ and assume that $l(s\cdot w)<l(w)$. Then

$$e_s\cdot e_w=(q_s-1)\,e_w+q_s\cdot e_{sw}.$$

5) The algebra H_0 is generated, as an R-algebra, by 1 and the elements e_s $(s\in S)$. The relations (2), (3) form a presentation of H_0.

Let $\psi\in\Psi$. It defines a permutation of $(e_s)_{s\in S}$ given by

$$e_s\mapsto\psi\cdot e_s=e_{\psi(s)}.$$

By the above, this permutation extends to an automorphism of H_0. Let $R[\psi]$ be the group algebra of Ψ over R and let $R[\Psi]\tilde{\otimes}H_0$ be the ordinary tensor product $R[\Psi]\otimes H_0$ of modules, endowed with the product defined by

6) $(\psi\otimes h)\cdot(\psi'\otimes h')=\psi\cdot\psi'\otimes(\psi'^{-1}\cdot h)\cdot h'$.

Then [11: § 3] or [4: IV, § 2, Ex. 25]

7) $H\cong R[\Psi]\tilde{\otimes}H_0$.

3.3. *Characters of Degree One.* Two elements s, s' of S are conjugate in $\tilde{W}$ if and only if there exists a chain of elements of S: $s=s_0, s_1, \ldots, s_q=s'$ such that $s_i\cdot s_{i+1}$ is of finite odd order for $i=0,1,\ldots,q-1$ [4: IV, 1.3, Prop. 3]. The intersections S_i of S with the conjugacy classes of $\tilde{W}$ are therefore the connected components of the graph obtained by erasing the multiple edges in the Coxeter graph Cox S of S. The number m of such classes in equal to

1 if Cox S is of type $\mathbf{A}_n$ $(n\geqq 2)$, $\mathbf{D}_n$ $(n\geqq 3)$, $\mathbf{E}_i$ $(i=6,7,8)$,

2 if Cox S is of type $\mathbf{A}_1$, $\mathbf{B}_n$ $(n\geqq 2)$, $\mathbf{G}_2$, $\mathbf{F}_4$,

3 if Cox S is of type $\mathbf{C}_n$ $(n\geqq 2)$.

3.4. Proposition. *Let K be a commutative algebra over R. Let S_i $(1 \leqq i \leqq m)$ be the intersections of S with the conjugacy classes in $\tilde{W}$. A map $(e_s)_{s\in S} \to K$ is the restriction of a homomorphism $H_0 = H(G_0, B) \to K$, if and only if it is constant on each S_i and equal to -1 or q_s $(s \in S_i)$ on S_i. In particular, $H(G_0, B)$ has 2^m characters of degree one.*

The necessity of the condition follows from 3.2(2), (3) and its sufficiency from 3.2(5).

3.5. Proposition. *Let $\tau \in \hat{\Psi}$ be a character of Ψ, and σ a character of H_0. Then $\chi = \sigma \otimes \tau$ is a character of H if and only if σ is constant on the orbits of Ψ in S.*

This is obvious.

Special Cases. a) The algebra H_0 has a unique character σ_0 which is equal to -1 on every e_s, to be called *the special character* of H_0. If $\tau \in \hat{\Psi}$, then $\tau \otimes \sigma_0$ is a character of H, also to be called special.

b) Clearly, $q_s = q_{s'}$ if s and s' are conjugate in $\tilde{W}$. The map $e_s \mapsto q_s$ extends therefore to a character σ_1 of degree 1 of H_0; it satisfies $\sigma_1(e_w) = q_w$ for all $w \in \tilde{W}$. Given $\psi \in \hat{\Psi}$, the map $s \mapsto \psi(s)$ extends to an automorphism of G_0; therefore q_s is constant on the orbits of Ψ, and, for any character τ of $\hat{\Psi}$, the product $\tau \otimes \sigma_1$ is a character of H.

3.6. Proposition. *Let (r, E) be a finite dimensional representation of $H(G, B)$. Then the elements $r(e_g)$ $(g \in G)$ are invertible.*

In view of 3.1(3), (4) it suffices to show this for $g = \psi \cdot w$ $(\psi \in \Psi;\ w \in \tilde{W})$. From 3.1(10), it follows that $\psi \mapsto r(e_\psi)$ is a representation of Ψ, hence $r(\psi)$ is invertible. If $w \in \tilde{W}$ and $w = s_1 \dots s_q$ $(q = l(w))$ is a reduced decomposition of w, then $r(e_w) = r(e_{s_1}) \dots r(e_{s_q})$ by 3.2(1), hence there remains only to show that $r(e_s)$ is invertible for $s \in S$. But this follows from 3.2(2), which shows more precisely that the only possible eigenvalues of $r(e_s)$ are q_s and -1.

§ 4. Admissible Representations

The following lemma and its proof are due to F. Bruhat. The proof will be given in § 7.

4.1. Lemma (F. Bruhat). *Let U be a compact open subgroup of G_0 (cf. 3.1). There exists a number $d_0 > 0$ with the following property: given a chamber C of the building X of $\tilde{G}$ such that $d_s(C, C_0) > d_0$, there exists a chamber D adjacent to C satisfying the two following conditions:*

(i) $d_s(D, C_0) = d_s(C, C_0) - 1$.

(ii) *the group U is transitive on the set (C, D) of chambers C' such that $C' \cap D = C \cap D$.*

If C_1 is a chamber of X, then $d_s(C_1, C_0)$ is the integer d such that the minimal galleries connecting C_1 and C_0 have $d+1$ elements. We recall that two distinct chambers C_1, C_2 are said to be adjacent if their intersection is a face of codimension one. If so, we let (C_1, C_2) be the set of chambers C such that $C \cap C_2 = C_1 \cap C_2$.

4.2. To use 4.1, we give first an interpretation in X of the convolution formula

$$(f * e_s)(x) = \sum_{y \in BsB/B} f(x \cdot y), \quad (x \in G_0,\ s \in S,\ f \in C(G_0/B, E)), \tag{1}$$

which is a special case of 1.1(4) since $e_s = \check{e}_s$. Let C_{0s} be the face of C_0 which is fixed (pointwise) under the reflection s. As is well-known, the isotropy group of C_{0s} is the group $B_{\{s\}} = BsB \cup B$ and is transitive on the set of chambers of X containing C_{0s}.

Therefore, if

$$BsB = \coprod_{i \leqq i \leqq q_s} x_i B, \tag{2}$$

then $x_i \cdot C_0$ runs through all chambers of X which contain C_{0s} and are $\neq C_0$, i.e.

$$\bigcup_i x_i \cdot C_0 = \{C' \mid C' \cap C_0 = C_{0s}\}, \tag{3}$$

and if $x \in G$ and $x \cdot C_0 = C_1$, then

$$\bigcup_i x \cdot x_i C_0 = \{C' \mid C' \cap C_1 = x \cdot C_{0s}\}; \tag{4}$$

with the notation introduced at the end of 4.1, (4) can also be written

$$\bigcup_i x x_i \cdot C_0 = (C, C_1), \tag{5}$$

if C is any chamber such that $C \cap C_1 = x \cdot C_{0s}$. We recall further that if $x, x' \in G_0$ map C_0 onto C_1, they define the same isomorphism of C_0 onto C (since G_0 consists by definition of special automorphisms): hence $x \cdot C_{0s} = x' \cdot C_{0s}$. This is the face of type s of $x \cdot C_0$.

In view of this, (1) translates into the following: let C, D be adjacent chambers of X and s the type of $C \cap D$. Then

$$(f * e_s)(D) = \sum_{C' \in (C, D)} f(C'), \quad (f \in C(G_0/B, E)). \tag{6}$$

$$\operatorname{Card}(C, D) = q_s. \tag{7}$$

4.3. Proposition. *Let U be a compact open subgroup of G_0. Then $C_c(U \backslash G_0/B)$ is an H_0-module of finite type, with respect to convolution on the right.*

Let d be a positive number, and $D(d) = \{C \mid d_s(C, C_0) \leqq d\}$. The latter is a finite set. We may replace U by a smaller open subgroup, hence assume $U \subset B$. Then $D_0(d)$ is stable under U.

Let L_d be the H_0-submodule of $C_c(U \backslash G_0/B)$ generated by the elements of $C_c(U \backslash G_0/B)$ with support in $D(d)$. Since $D(d)$ consists of finitely many chambers, L_d is finitely generated. It suffices therefore to show that if d is equal to the d_0 of Lemma 4.1, then $L_d = C_c(U \backslash G_0/B)$.

By 2.1, the G_0-module $C(G_0/B)^\infty$ is the contragredient module to $C_c(G_0/B)$, the pairing being given by

$$\langle \varphi, \psi \rangle = (\varphi * \check{\psi})(1) = \sum_{x \in G_0/B} \varphi(x)\psi(x) \quad (\varphi \in C(G_0/B)^\infty,\ \psi \in C_c(G_0/B)). \tag{1}$$

We have then also

$$\langle \varphi * h, \psi \rangle = \langle \varphi, \psi * \check{h} \rangle \qquad (\varphi \in C(G_0/B)^\infty,\ \psi \in C_c(G_0/B),\ h \in H_0). \tag{2}$$

Under the pairing (1), the space

$$C(U \backslash G_0/B) = C(G_0/B)^U$$

is the dual to

$$C_c(U \backslash G_0/B) = C_c(G_0/B)^U.$$

To prove our assertion, it is enough therefore to show that if $\lambda \in C(U \backslash G_0/B)$ is zero on L_{d_0} then $\lambda = 0$.

We prove first that $\lambda(C) = 0$ if $C \in D(d_0)$. Let χ_C be the characteristic function of $U \cdot C$ in G_0/B. Then (1) shows that

$$c \cdot \lambda(C) = \langle \lambda, \chi_C \rangle, \qquad \text{where } c = \operatorname{Card} U \cdot C; \tag{3}$$

but, since $U \cdot D(d_0) = D(d_0)$, the function χ_C belongs to L_{d_0} hence $\lambda(C) = 0$.

Let now $C \notin D(d_0)$. Let D be as in 4.1 and s be the type of $C \cap D$ (4.2). Then U is transitive on (C, D), hence λ is constant on (C, D) and 4.2(6)(7) yield

$$(\lambda * e_s)(D) = \sum_{C' \in (C, D)} \lambda(C') = q_s \cdot \lambda(C). \tag{4}$$

Using induction on $d_s(C, C_0)$ and 3.1(8), we see that, given a chamber C not in $D(d_0)$, there exist $w \in \check{W}$ and a chamber $C' \in D(d_0)$ such that

$$(\lambda * e_w)(C') = q_w \cdot \lambda(C). \tag{5}$$

Let $c' = \operatorname{Card} U \cdot C'$. By (2), (3) and (5)

$$\langle \lambda, \chi_{C'} * \check{e}_w \rangle = \langle \lambda * e_w, \chi_{C'} \rangle = c' \cdot (\lambda * e_w)(C') = c' \cdot q_w \cdot \lambda(C).$$

Since $\chi_{C'} * \check{e}_w \in L_{d_0}$ this shows that $\lambda(C) = 0$.

4.4. Theorem. *Let* (r, E) *(resp.* (r_0, E_0)*) be a finite dimensional* $H = H(G, B)$*-module (resp.* $H_0 = H(G_0, B)$*-module). Then* $P_{B,G}(E)$ *and* $I_{B,G}(E)$ *(resp.* $P_{B,G_0}(E_0)$ *and* $I_{B,G_0}(E_0)$*) are admissible G-modules (resp.* G_0*-modules).*

By 2.6, $P_{B,G}(E)$ and $P_{B,G_0}(E)$ are the contragredient modules to $I_{B,G}(E')$ and $I_{B,G_0}(E'_0)$ respectively. It suffices therefore to prove our statement for the induced modules $I_{B,G}(E)$ and $I_{B,G_0}(E_0)$. We next reduce the proof to the case of G_0.

We have $G = L \cdot G_0$, where L normalizes B (3.1). Since B is its own normalizer in G_0, it follows that each orbit of G_0 on G/B contains exactly one fixed point under B. If x is such a point, then $g \mapsto g \cdot x$ provides an isomorphism of G_0/B onto $G_0 \cdot x$, whence a canonical G_0-equivariant bijection of G/B onto the disjoint union of $m = [G: G_0]$ copies of G_0/B. This yields canonical G_0-equivariant isomorphisms

$$C_c(G/B, E) \xrightarrow{\sim} C_c(G_0/B, E) \oplus \cdots \oplus C_c(G_0/B, E), \qquad (m \text{ summands}), \tag{1}$$

$$C(G/B, E) \xrightarrow{\sim} C(G_0/B, E) \oplus \cdots \oplus C(G_0/B, E), \qquad (m \text{ summands}). \tag{2}$$

These isomorphisms are furthermore H_0-equivariant, H_0 being identified to a subalgebra of $H \cong R[\Psi] \tilde{\otimes} H_0$ (see 3.2(7)) by the map $h \mapsto 1 \otimes h$ $(h \in H_0)$.

Since G_0 is open, normal, of finite index in G, a G-module is smooth (resp. admissible) if and only it is so with respect to G_0. Thus we have to show that $I_{B,G}(E)$ is admissible as a G_0-module. Since H_0 is a subalgebra of H, the module $I(E) = C_c(G/B) \otimes_H E$ is a quotient of $C_c(G/B) \otimes_{H_0} E$. But, by (1), the latter is the direct sum of m copies of $I_{B,G_0}(E)$, hence it is admissible if and only if $I_{B,G_0}(E)$ is.

Let U be a compact open subgroup of G_0. Then

$$I_{B,G_0}(E_0)^U = C_c(U \setminus G_0/B) \otimes_{H_0} E_0.$$

Since $C_c(U \setminus G_0/B)$ is a H_0-module of finite type (4.3), this shows that the left-hand side is finite dimensional, hence $I_{B,G_0}(E_0)$ is admissible.

4.5. *Remark.* Theorem 4.4 can be proved directly for the coinduced modules without using 2.6. In fact this was my original argument. P. Cartier suggested that it would also yield 4.2 and that I consider the modules $I(E)$ as well.

We outline here the argument for the coinduced modules. First, it is easily seen that $P_{B,G}(E)$ is a G_0-invariant subspace of the direct sum of m copies of $P_{B,G_0}(E)$, whence the reduction to G_0.

Let U be a compact open subgroup of G_0, and d_0 be as in Lemma 4.1. To show that $P_{B,G_0}(E_0)^U$ is finite dimensional, it suffices to prove that if $f \in P_{B,G_0}(E_0)^U$ is zero on $D(d_0)$, then it is identically zero. Let C be a chamber such that $d_s(C, C_0) > d_0$. Let D be as in 4.1, and s the type of $C \cap D$. We have then again, using 4.2(6), (7):

$$(f * e_s)(D) = q_s \cdot f(C), \tag{1}$$

whence, by induction, the existence of $w \in \tilde{W}$ and $C' \in D(d_0)$ such that

$$(f * e_w)(C') = q_w \cdot f(C); \tag{2}$$

this yields

$$q_w \cdot f(C) = r_0(e_w) \cdot f(C'), \tag{3}$$

and our assertion.

4.6. Let $\mathscr{P}; \mathscr{P}^-$ be two opposite minimal parabolic k-subgroups of $\mathscr{G}, \mathscr{M} = \mathscr{P} \cap \mathscr{P}^-$ their common Levi subgroup and $\mathscr{N}, \mathscr{N}^-$ their respective unipotent radicals. We may (and do) choose them in such a way that B admits the "Iwahori factorization" [7: 1.4.4]:

$$B = N_0^- \cdot M_0 \cdot N_0, \quad (N_0^- = N^- \cap B;\ M_0 = M \cap B;\ N_0 = N \cap B), \tag{1}$$

and that M_0 is the greatest compact subgroup of $G_0 \cap M$. In particular, M_0 is normal in M.

We shall use some notation and results of [7]. If V is a smooth N-module, V_N denotes the greatest quotient of V on which N acts trivially [7: 3.2]. If V is a P-module, V_N is viewed as an M- or P-module in the obvious way and the projection $V \to V_N$ commutes with P. The assignment $V \mapsto V_N$ defines an exact functor from smooth N-modules to vector spaces [7: 3.2.3].

The following lemma is a slight strengthening of Theorem 3.3.3 in [7], valid for Iwahori subgroups:

4.7. Lemma. *Let (π, V) be an admissible G-module. Then the canonical projection $V \to V_N$ induces an isomorphism of V^B onto $(V_N)^{M_0}$.*

Let α be the restriction of $V \to V_N$ to V^B. Obviously $\alpha(V^B) \subset (V_N)^{M_0}$.

Recall that for $a \in G$, $e_a \in H(G, B)$ is the characteristic function of the double coset BaB and q_a its volume (1.1). It follows from the definitions that we have:

$$\pi(e_a) \cdot v = q_a \cdot \pi(e_1) \cdot \pi(a) \cdot v, \quad (v \in V^B). \tag{1}$$

By [7: 4.1.4], there exists $a \in G$ such that moreover:

$$\alpha \text{ induces an isomorphism of } \pi(e_1) \cdot \pi(a) \cdot V^B \text{ onto } (V_N)^{M_0}. \tag{2}$$

However, the restriction of $\pi(e_a)$ to V^B is an invertible automorphism (3.6), hence, by (1), $\pi(e_1) \cdot \pi(a) \cdot V^B = V^B$.

4.8. Lemma (Casselman). *Let E be an admissible G-module. Assume that E is generated by E^B as a G-module and that, if E' is a non-zero G-submodule of E, then $E'^B \neq 0$. Then*

(i) *Any exact sequence of admissible G-modules*

$$0 \to E \to F \to F' \to 0$$

where $F'^B = 0$, splits.

(ii) *if E' is a G-submodule of E, then E' is generated by E'^B as a G-module.*

Proof. By 4.6, 4.7:

$$(F'_N)^{M_0} = 0, \quad (E_N)^{M_0} = (F_N)^{M_0} \quad \text{and} \quad \alpha: E^B \to (E_N)^{M_0} \text{ is an isomorphism.} \tag{1}$$

The exact sequence [7: 3.2.3]

$$0 \to E_N \to F_N \to F'_N \to 0 \tag{2}$$

gives then rise to a direct sum decomposition

$$F_N = (E_N)^{M_0} \oplus L \tag{3}$$

where L is the sum of the isotypic subspaces of M_0 in F_N for the irreducible representations of M_0 different from the trivial representation. Since M_0 is normal in M, both spaces are stable under M. Thus (3) is also a direct sum decomposition of M- or P-modules.

Let V be the space of the unnormalized induced representation $\mathbf{Ind}((E_N)^{M_0} | P, G)$ [7: 2.2]. By Frobenius reciprocity [7: Thm 2.4.1], the projection $\beta_N: F_N \to (E_N)^{M_0}$ is the map canonically associated to a G-morphism $\beta: F \to V$. We want to prove:

$$E \cap \ker \beta = 0, \quad \beta(E) = \beta(F). \tag{4}$$

β_N is an isomorphism of $E_N = (F_N)^{M_0}$ onto itself. Hence, by 4.7, the homomorphism $\beta: F^B = E^B \to V^B$ is injective. Our second assumption on E then implies the first part of (4).

The G-module $\beta(F)/\beta(E)$ is isomorphic to a decomposition factor of V. The space $(\beta(F)/\beta(E))^B$ is a quotient of $(F/E)^B$ hence is zero. That $\beta(F)=\beta(E)$ then follows from the following known fact (see 4.9 below)

(*) *Let σ be a finite dimensional representation of M trivial on M_0, and L a non-zero decomposition factor of* $\mathbf{Ind}(\sigma|P, G)$. *Then* $L^B \neq 0$.

This proves (4). It follows that $\ker\beta$ is isomorphic to F' as a G-module and is a supplement to E whence (i).

Let now E' be a G-submodule of E and E'' the G-submodule generated by E'^B. By (i), the exact sequence

$$0 \to E'' \to E' \to E'/E'' \to 0,$$

splits, hence E'/E'' may be identified with a G-submodule of E'. However $(E'/E'')^B = E'^B/E''^B = 0$, whence $E'=E''$.

4.9. The assertion (*) above is essentially proved in [7]. A similar one is also announced in [14: p. 19]. We indicate briefly how to derive it from [7]. To refer freely to [7] we remark first that, since the modulus δ_P of P is trivial on the compact group M_0 and since, by definition [7: 3.1] the normalized induced representation $\mathrm{Ind}(\sigma|P, G)$ is $\mathbf{Ind}(\sigma\delta_P^{\frac{1}{2}}|P, G)$, we may shift from one to the other.

By [7: Thm. 3.3.3], it suffices to prove that $L_N \neq 0$. Since the functor $\sigma \mapsto \mathrm{Ind}\,\sigma$ is exact, as follows from its definition, we may, by an easy induction on the length of a Jordan-Hölder series of σ, assume that σ is irreducible. The G-module L has finite length [7: 6.3.8], so we may assume it to be irreducible. But then L embeds in an induced module ($\mathrm{Ind}(\tau|P, G)$, where τ is a finite dimensional irreducible representation of M trivial on M_0 [7: 6.3.9] and $L_N \neq 0$ by [7: 3.2.5].

Here, B need not be an Iwahori subgroup. It suffices that it be a compact open subgroup of G admitting an Iwahori factorization (4.6(1)).

4.10. Theorem. *Let (r, E) be a finite dimensional $H(G, B)$-module. Then the canonical morphism $j: I_{B,G}(E) \to P_{B,G}(E)$ is an isomorphism. The G-module $I_{B,G}(E)$ is irreducible if and only if E is an irreducible $H(G, B)$-module. The assignment $E \mapsto I(E)$ (resp. $E \mapsto P(E)$) is an exact functor from finite dimensional H-modules to admissible G-modules. The H-module $C_c(G/B)$ is flat.*

Let M be the G-submodule of $P(E)$ spanned by $G \cdot P(E)^B$, and $V = P(E)/M$. Every non-zero G-submodule of $P(E)$ contains a non-zero B-fixed vector (2.4). On the other hand, $V^B = P(E)^B/M^B$ is zero. By 4.8, the exact sequence of G-modules

$$0 \to M \to P(E) \to V \to 0,$$

splits. Thus V may be viewed as a G-submodule of $P(E)$ and the equality $V^B=0$ implies $V=0$ by 2.4. Therefore $P(E)$ is generated by E as a G-module and j is surjective. This also applies to $P(E')$. Since $P(E')$ is the contragredient to $I(E)$ (2.6), it follows that every non-zero G-submodule of $I(E)$ has non-zero B-fixed vectors [7: 2.2.3], hence j is injective; furthermore, if V is a G-submodule of $I(E)$, then V is spanned by $G \cdot V^B$ (4.8); if E' is an H-submodule of E, then the smallest G-submodule F of $I(E)$ containing E' is a quotient of $I(E')$, hence (2.4) satisfies

$F^B = E'$. These two remarks imply the irreducibility statement. Let

$$E' \xrightarrow{\alpha} E \xrightarrow{\beta} E'', \tag{1}$$

be an exact sequence of finite dimensional H-modules, and

$$I(E') \xrightarrow{I(\alpha)} I(E) \xrightarrow{I(\beta)} I(E''), \tag{2}$$

the corresponding sequence of induced modules. Let $F = \ker I(\beta)$. By assumption and 2.5, $F^B = \ker \beta = \operatorname{Im} \alpha$. Since, by the above, F is generated by F^B as a G-module, we have $F = \operatorname{Im} I(\alpha)$. This proves the third assertion for induced modules. In view of the first one, it also follows for coinduced modules. By the definition of induced modules, the last one is just a reformulation of the third one (for induced modules).

4.11. Corollary. *Let V be an admissible G-module and assume that V is generated by $G \cdot V^B$. Then the canonical morphisms $I(V^B) \xrightarrow{\mu} V \xrightarrow{\nu} P(V^B)$ are isomorphisms. Every G-submodule of V is generated as a G-module by its B-fixed vectors.*

The composition $\nu \circ \mu$ is the morphism j of 4.10 hence is an isomorphism. Moreover, μ is surjective since V is generated by $G \cdot V^B$. This proves the first assertion. The second one follows from 2.4 and 4.8.

4.12. Corollary. *Let V be an admissible G-module. Let M be the G-submodule generated by V^B. Then V is the direct sum of M and of a G-submodule N such that $N^B = 0$.*

By 4.11, $M = I(V^B)$ and every G-submodule of M is generated by its B-fixed vectors. Since $(V/M)^B = V^B/V^B = 0$, 4.12 follows from 4.8.

§ 5. Square Integrable Representations Induced from Characters of Degree One

In this section, R is the field $\mathbf{C}$ of complex numbers; G' stands for G or G_0 (3.1).

5.1. Let $L^2(G'/B)$ be the space of elements $f \in C(G'/B)$ such that

$$\|f\|^2 = \sum_{x \in G'/B} |f(x)|^2 < \infty.$$

Endowed with the scalar product

$$(f, g) = \sum_{x \in G'/B} f(x) \cdot \overline{g(x)}, \qquad (f, g \in L^2(G'/B)),$$

it is a Hilbert space on which G' acts by unitary transformations via left translations. It contains $C_c(G'/B)$ as a dense subspace. It follows from 4.4(2) that $L^2(G/B)$, viewed as a G_0-module, is the direct sum of finitely many copies of $L^2(G_0/B)$. Since B is compact, $L^2(G'/B)$ may be identified with a closed G'-submodule of $L^2(G'/B)$, hence the closed non-zero irreducible G'-submodules of $L^2(G'/B)$ are members of the discrete series of G'. We want to determine those which are of the form $P_{B,G'}(\chi)$, where χ is a character of degree one of the Hecke algebra $H(G', B)$.

5.2. Proposition. *Let σ be a character of degree 1 of H_0 and τ a character of Ψ such that $\chi=\sigma\circ\tau$ is a character of H (see 3.5). Let $P_\chi=P_{B,G}(\chi)$, $L_\chi=L^2(G/B)\cap P_\chi$, $P_\sigma=P_{B,G_0}(\sigma)$ and $L_\sigma=L^2(G_0/B)\cap P_\sigma$. Then*

(i) $L_\chi\neq 0 \Leftrightarrow L_\chi^B\neq 0 \Leftrightarrow P_\chi=L_\chi$,

(ii) $L_\sigma\neq 0 \Leftrightarrow L_\sigma^B\neq 0 \Leftrightarrow P_\sigma=L_\sigma$,

and these conditions are equivalent to

(iii) $C_\sigma=\sum_{w\in\tilde{W}} q_w^{-1}\cdot\sigma(w)^2<\infty$.

The convolution on the right by an element of $H(G',B)$ obviously preserves $L^2(G'/B)$ and is continuous. Hence L_σ and L_χ are closed subspaces of $L^2(G_0/B)$ and $L^2(G/B)$ respectively.

Let M be a closed invariant subspace of $L^2(G'/B)$ and pr_M the orthogonal projection of $L^2(G'/B)$ onto M. Then $\mathrm{pr}_M(C_c(G'/B))$ is dense in M. Since $C_c(G'/B)$ is the space spanned by the G'-transforms of e_1, it follows that $\mathrm{pr}_M(e_1)\neq 0$ if $M\neq 0$. This yields the first equivalence in (i), (ii); the second one follows from the irreducibility of P_σ and P_χ (4.10). We have $e_s=\check{e}_s$, hence $\sigma(e_w)=\sigma(\check{e}_w)$, since the $\sigma(e_s)$'s commute. Therefore, by 2.4(ii), the space $(P_\sigma)^B$ is spanned by

$$f_\sigma=\sum_{w\in\tilde{W}} q_w^{-1}\sigma(e_w)e_w, \tag{1}$$

and in view of 3.1(3)(4)(10)(11), $(P_\chi)^B$ is spanned by

$$f_\chi=\Big(\sum_{\psi\in\Psi}\tau(\psi)\Big)\cdot f_\sigma. \tag{2}$$

We have

$$\|f_\chi\|^2=|\Psi|\cdot\|f_\sigma\|^2, \qquad (|\Psi|=\mathrm{Card}\,\Psi). \tag{3}$$

Since G_0 is the disjoint union of the BwB $(w\in\tilde{W})$, we have

$$\|f_\sigma\|^2=\sum_{w\in\tilde{W}}\ \sum_{x\in BwB/B} f_\sigma(x)^2; \tag{4}$$

f_σ being constant on BwB $(w\in\tilde{W})$, this gives

$$\|f_\sigma\|^2=\sum_{w\in\tilde{W}} q_w\cdot|f_\sigma(w)|^2=\sum_{w\in\tilde{W}} q_w^{-1}\cdot\sigma(w)^2=C_\sigma. \tag{5}$$

Therefore

$$L_\sigma^B\neq 0\Leftrightarrow \|f_\sigma\|^2<\infty\Leftrightarrow C_\sigma<\infty\Leftrightarrow\|f_\chi\|^2<\infty\Leftrightarrow L_\chi^B\neq 0,$$

whence the equivalence of (iii) with the conditions in (i), (ii).

5.3. Proposition. *We keep the notation of 5.2 and assume that $L_\sigma\neq 0$. Let d_σ (resp. d_χ) be the formal degree of the representation of G_0 in L_σ (resp. G in L_χ), computed with respect to the Haar measure dx for which B has volume 1. Then*

$$d_\chi=|\Psi|^{-1}d_\sigma, \qquad d_\sigma=C_\sigma^{-1}.$$

The orthogonal projection of e_1 onto L_σ is non-zero (see proof of 5.2), invariant under B, hence is a non-zero multiple of f_σ. We may therefore write:

$$f_\sigma=c\cdot e_1+u$$

with $u \in L^2(G_0/B)$ orthogonal to L_σ, whence

$$C_\sigma = (f_\sigma, f_\sigma) = c \cdot (f_\sigma, e_1) = c \cdot f_\sigma(1) = c,$$
$$f_\sigma = C_\sigma \cdot e_1 + u \qquad (u \in L^2(G_0/B),\ u \perp L_\sigma). \tag{1}$$

By definition of the formal degree, we have

$$\int_{G_0} |(f_\sigma, l_x \cdot f_\sigma)|^2 \, dx = d_\sigma^{-1} \cdot \|f_\sigma\|^4. \tag{2}$$

From (1) we get

$$(f_\sigma, l_x \cdot f_\sigma) = C_\sigma (f_\sigma, l_x \cdot e_1) = C_\sigma^2 \cdot f_\sigma(x^{-1}), \qquad (x \in G_0),$$

hence

$$(f_\sigma, l_x \cdot f_\sigma) = C_\sigma \cdot f_\sigma(w) = C_\sigma \cdot q_w^{-1} \cdot \sigma(w), \qquad (w \in \tilde{W};\ x \in B \cdot w^{-1} \cdot B), \tag{3}$$

and, by (2)

$$\int_{G_0} |(f_\sigma, l_x \cdot f_\sigma)|^2 \, dx = C_\sigma^2 \cdot \|f_\sigma\|^2 = C_\sigma^3 = d_\sigma^{-1} \|f_\sigma\|^4 = d_\sigma^{-1} \cdot C_\sigma,$$

which yields our assertion for d_σ. The proof for d_χ is the same.

5.4. We now discuss the finiteness of C_σ, assuming that $\mathscr{G}$ is almost simple over k. As in 3.3, we let S_i $(1 \leqq i \leqq m)$ be the intersections of S with the conjugacy classes in $\tilde{W}$ and put $q_i = q_s$ for $s \in S_i$ (3.5). For $w \in \tilde{W}$, the number of elements of S_i occuring in a reduced decomposition of w depends only on w (as follows from Prop. 5, p. 16 of [4]) and will be denoted $l_i(w)$. Thus

$$l(w) = \sum_{1 \leqq i \leqq m} l_i(w). \tag{1}$$

We let t_i be an indeterminate,

$$\mathbf{l}(w) = (l_1(w), \ldots, l_m(w)), \qquad \mathbf{t} = (t_1, \ldots, t_m),$$
$$\mathbf{t}^{\mathbf{l}(w)} = \prod_{1 \leqq i \leqq m} t_i^{l_i(w)} \tag{2}$$

and consider the formal power series

$$W(\{t_i\}) = \sum_{w \in \tilde{W}} \mathbf{t}^{\mathbf{l}(w)}. \tag{3}$$

Given a character σ of degree 1 of H_0, let us put

$$\varepsilon_i = \begin{cases} 1 & \text{if } \sigma(e_s) = q_s \quad \text{for } s \in S_i, \\ -1 & \text{if } \sigma(e_s) = -1 \ \text{ for } s \in S_i. \end{cases} \tag{4}$$

We have

$$\sigma(e_w)^2 = \prod_i q_i^{(1+\varepsilon_i) l_i(w)}, \tag{5}$$

$$\sigma(e_w)^2 / q_w = \prod_i q_i^{\varepsilon_i l_i(w)}, \tag{6}$$

whence the

5.5. Lemma. *The sum C_σ is finite if and only if $W(\{t_i\})$ converges for $t_i = q_i^{\varepsilon_i}$, and then we have*

$$C_\sigma = W(\{q_i^{\varepsilon_i}\}).$$

5.6. The series $W(\mathbf{t})$ represents a rational function [15: Prop. 26], which, according to [12] can be written in the form

$$W(\mathbf{t}) = P(\mathbf{t}) \cdot \left(\prod_1^l Q_j(\mathbf{t}) \right)^{-1},$$

where $P(\mathbf{t})$ is a polynomial with positive coefficients and $Q_j(\mathbf{t})$ is of the form

$$Q_j(\mathbf{t}) = 1 - t_1^{q_{j1}} \dots t_m^{q_{jm}}, \qquad (q_{ji} \geqq 0, \text{ integral}).$$

Therefore

$$C_\sigma < \infty \Leftrightarrow \prod_{i=1}^m q_i^{\varepsilon_i q_{ji}} < 1 \qquad \text{for } j = 1, \dots, l. \tag{1}$$

We have $q_i = q^{a_i}$ with $q \geqq 2$, and $a_i \geqq 1$, integral, hence (1) may be written

$$C_\sigma < \infty \Leftrightarrow \sum_{i=1}^m a_i \cdot \varepsilon_i q_{ji} < 0, \qquad (j = 1, \dots, l). \tag{2}$$

5.7. If σ is the special character, then $\varepsilon_i = -1$ for all i and the conditions of 5.6(2) are fulfilled. The representations L_σ or L_χ are the special representations constructed by Matsumoto [13] and Shalika [16], to which we shall come back in §6. In fact, in this case

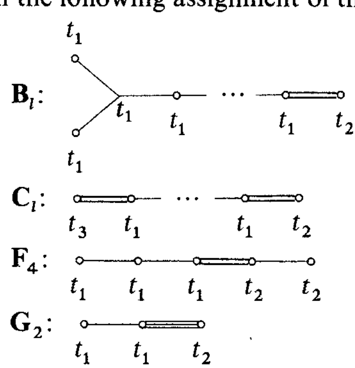
$$C_\sigma = \sum q_w^{-1}. \tag{1}$$

If $m = 1$, this is then the only square integrable representation we obtain in this way.

5.8. We now consider the cases where $m \geqq 2$. By [12], the polynomials $Q_j(\mathbf{t})$ $(1 \leqq j \leqq l)$ are

$(1 - t_1 t_2)$	for the type $\mathbf{A}_1$
$(1 - t_1^{l-2+j} \cdot t_2)$	for the type $\mathbf{B}_l$ $(l \geqq 3)$
$(1 - t_1^{l-2+j} t_2 t_3)$	for the type $\mathbf{C}_l$ $(l \geqq 2)$
$(1 - t_1^3 t_2^2)$, $(1 - t_1^4 t_2^3)$, $(1 - t_1^5 t_2^3)$, $(1 - t_1^6 t_2^5)$	for the type $\mathbf{F}_4$
$(1 - t_1^2 t_2)$, $(1 - t_1^3 t_2^2)$,	for the type $\mathbf{G}_2$

with the following assignment of the t_i's to the conjugacy classes in S



In the sequel we denote a character by the sequence of the ε_i defined by 5.4(4). For each type, and possible set of values of the q_i, we shall indicate the σ such that L_σ belongs to the discrete series, as given by application of 5.6(2) and 5.8.

(i) *Assume first the q_i's to be equal.* This is the case if G is split over k or, more generally, if G is quasi-split over k and splits over a totally ramified extension of k. The characters different from the special one which yield a square integrable representation are then

$(-1,-1,1)$, $(-1,1,-1)$, $(-1,1,1)$ for the types $\mathbf{C}_l$ $(l \geqq 4)$,
$(-1,-1,1)$, $(-1,1,-1)$ for the types $\mathbf{C}_2, \mathbf{C}_3$,
$(-1,1)$ for the types $\mathbf{B}_l$ $(l \geqq 3)$, $\mathbf{F}_4$, $\mathbf{G}_2$.

(ii) *Assume the q_i's are not equal.* If $l=1$, and say, $q_1 < q_2$, then $(-1,1)$ gives rise to a square integrable representation, whereas $(1,-1)$ does not.

Let now $l \geqq 2$. The following table gives for each type, the possible values of the q_i, which were communicated to me by J. Tits, and the characters other than the special one which yield a square integrable representation.

Type of $\tilde{W}$	(q_1, q_2)	Characters	Type of $\tilde{W}$	(q_1, q_2)	Characters
$\mathbf{B}_l$ $(l \geqq 4)$	(q, q^2)	$(-1,1)$	$\mathbf{F}_4$	(q, q^2)	$(1,-1)$
$\mathbf{B}_l$ $(l \geqq 2)$	(q^2, q)	$(-1,1)$	$\mathbf{F}_4$	(q^2, q)	$(-1,1)$
$\mathbf{B}_l$ $(l \geqq 3)$	(q^2, q^3)	$(-1,1)$	$\mathbf{G}_2$	(q, q^3)	$(1,-1)$
			$\mathbf{G}_2$	(q^3, q)	$(-1,1)$

For the type $\mathbf{C}_l$, the list is as follows.

(q_1, q_2, q_3)	l	Characters	(q_1, q_2, q_3)	l	Characters
	2	$(1,-1,-1), (-1,1,-1)$		2	$(1,-1,-1)$
(q, q, q^2)	$\geqq 3$	$(-1,-1,1), (-1,1,-1)$	(q^2, q, q^4)	$\geqq 2$	$(-1,-1,1), (-1,1,-1)$
	$\geqq 5$	$(-1,1,1)$		$\geqq 4$	$(-1,1,1)$
	2	$(1,-1,-1)$		2	$(1,-1,-1)$
(q, q^2, q^2)	$\geqq 2$	$(-1,-1,1), (-1,1,-1)$	(q^2, q^2, q^3)	$\geqq 2$	$(-1,-1,1), (-1,1,-1)$
	$\geqq 6$	$(-1,1,1)$		$\geqq 4$	$(-1,1,1)$
(q^2, q, q)	$\geqq 2$	$(-1,-1,1), (-1,1,-1)$		2	$(1,-1,-1)$
	$\geqq 3$	$(-1,1,1)$	(q^2, q^3, q^3)	$\geqq 2$	$(-1,-1,1), (-1,1,-1)$
(q^2, q, q^2)	$\geqq 2$	$(-1,-1,1), (-1,1,-1)$		$\geqq 5$	$(-1,1,1)$
	$\geqq 3$	$(-1,1,1)$			
	$\geqq 2$	$(-1,1,-1)$		2	$(1,-1,-1)$
(q^2, q, q^3)	$\geqq 3$	$(-1,-1,1)$	(q^2, q^3, q^4)	$\geqq 2$	$(-1,-1,1), (-1,1,-1)$
	$\geqq 4$	$(-1,1,1)$		$\geqq 5$	$(-1,1,1)$

5.9. *Remarks.* (1) Let σ be a character of H_0 and π the representation of G_0 in L_σ. We use the notation of 4.6, 4.7. By 4.7 and 4.7(1), the space $(L_\sigma)_N$ is one-dimensional and we have

$$\pi(a) \cdot v = \sigma(e_a) \cdot q_a^{-1} \cdot v, \qquad (v \in (L_\sigma)_N;\ a \in A^-), \tag{1}$$

where A^- is a suitable negative Weyl chamber in a k-split torus of M. Moreover,

$$q_a = |\delta_P(a)|^{-1}, \quad (a \in A^-). \tag{2}$$

It follows that (π, L_σ) embeds in the induced representation $\mathrm{Ind}(\chi|P, G)$ where χ is the character of P which is equal to $\sigma(a)\,\delta_P(a)$ on A^-.

(2) [7: 6.5.1] gives a criterion for the square integrability of an irreducible submodule of an induced representation. In view of the previous remark it yields in our case

$$|\sigma(a)| < (q_a)^{\frac{1}{2}}, \quad (a \in A^-). \tag{3}$$

It can be checked directly that this is equivalent to 5.6(2). In fact, the first determination of the square integrable L_σ, obtained with the help of J. Tits, was based on a criterion quite similar to (3), proved by some geometric considerations on euclidean reflection groups.

(3) I understand that the results stated without proof or reference in 5.8 will eventually be found somewhere in the sequel to [5].

5.10. *Remarks on the formal degree.* It follows from 5.3 and 5.5, 5.6 that d_σ is a rational number. If σ_0 is the special character and $\sigma \neq \sigma_0$, then $|\sigma(w)| \geqq |\sigma_0(w)|$ for all $w \in \tilde{W}$, and $|\sigma(w)| \neq |\sigma_0(w)|$ for at least one w, hence $d_\sigma < d_{\sigma_0}$. Examples show that in general neither of d_σ and d_{σ_0} is an integral multiple of the other. We note that the elements of the discrete series constructed here are not cuspidal, since the coefficient $x \mapsto (f_\sigma, l_x \cdot f_\sigma)$ is not compactly supported in view of 5.2(1).

5.11. The algebra $H = H_{\mathbf{C}}(G, B)$ is in a natural way an involutive Lie algebra, the involution $h \mapsto h^*$ being defined by $f \mapsto f^*$ where $f^*(x) = \overline{f(x^{-1})}$ ($f \in C_c(B \backslash G/B)$, $x \in G$). In particular $e_w^* = e_{w^{-1}}$ ($w \in G$). Let E be a finite dimensional Hilbert space. A representation r of H into $\mathrm{End}\, E$ is self-adjoint if, for every $h \in H$, $r(h^*)$ is the adjoint of $r(h)$. In particular, every one-dimensional representation is so. Clearly, if (π, V) is an admissible unitary representation of G, then the associated representation of H into V^B is self-adjoint. It is not known whether conversely, if (r, E) is self-adjoint, the representation $P(E)$ is unitary. If $l = 1$, however this is the case according to [14: p. 20].

§ 6. The Special Representation and Cohomology of Buildings

In this section $\mathscr{G}$ is simply connected and almost simple over k.

6.1. As before, X is the Bruhat-Tits buildung of G. It is a locally finite simplicial complex of dimension l, union of its simplices of dimension l, which are called the chambers. We let $C^j(X)$ (resp. $C_c^j(X)$) be the space of j-dimensional complex valued cochains with arbitrary (resp. compact) supports on X. In particular $C^l(X) = C(G/B)$ and $C_c^l = C_c(G/B)$. We let $d\colon C^j \to C^{j+1}$ be the coboundary operator and $\delta\colon C^j(X) \to C^{j-1}(X)$ its adjoint with respect to a suitable scalar product [1; 9] which, for $j = l$, coincides with the one introduced on $C(G/B)$ in § 5. A element c of $C^j(X)$ is *harmonic* if it is annihilated by d and δ. If c is square integrable, this is equivalent to being annihilated by the Laplace operator $\Delta = d\delta + \delta d$ [1; 9].

$H_c^j(X)$ (resp. $H_j(X)$) is the j-th cohomology group *with compact supports* (resp. j-th homology group *with arbitrary supports*) of X with complex coefficients. In particular

$$H_c^l(X) = C_c^l(X)/dC_c^{l-1}, \qquad H_l(X) = C^l(X) \cap \ker\delta = Z_l. \tag{1}$$

$C_2^j(X)$ is the space of square integrable j-cochains, and H_2^j the space of square integrable harmonic cochains. In particular, $C_2^l \cong L^2(G/B)$; there is an orthogonal decomposition

$$L^2(G/B) = C_2^l = H_2^l \oplus \overline{dC_2^{l-1}} \tag{2}$$

where $\overline{}$ denotes closure in the Hilbert space $L^2(G/B)$ [1; 9]. An element $u \in C^l(X)$ is a cycle if and only if, viewed as an element of $C^l(G/B)$, it satisfies the relations

$$f * e_s = -f, \qquad (s \in S). \tag{3}$$

It follows that f is a cycle if and only if

$$f * h = \sigma_0(h) \cdot f, \qquad \text{for all } h \in H(G, B), \tag{4}$$

where σ_0 is the special character (3.5). By 5.2, the identification $C^l(X) = C(G/B)$ then provides isomorphisms of G-modules

$$H_l(X) = P^0(\sigma_0), \qquad H_2^l = P^0(\sigma_0) \cap L^2(G/B), \tag{5}$$

where $P^0(\sigma_0)$ is as in 2.3 and

$$H_l(X)^\infty = P(\sigma_0) = L_{\sigma_0}. \tag{6}$$

Let pr_h be the orthogonal projection of C_2^l onto H_2^l. Since $C_c^l(X) \subset C_2^l$, ker pr_h contains dC_2^{l-1} (see (2)) and this projection induces a homomorphism of G-modules

$$\eta\colon H_c^l(X) \to H_2^l = L_{\sigma_0}. \tag{7}$$

6.2. Theorem. *The homomorphism $\eta\colon H_c^l(X) \to H_2^l$ is injective. Its image is the space of smooth elements of H_2^l.*

Since $H_c^l(X)$ is a quotient of $C_c^l(X) = C_c(G/B)$, it is generated by $G \cdot e_1$ and is smooth, hence $\mathrm{Im}\,\eta$ is generated by $G \cdot \eta(e_1)$ and contained in L_{σ_0}. As remarked in the proof of 5.2, $\eta(e_1) \neq 0$, hence $\mathrm{Im}\,\eta$ is a non-zero G-submodule of L_{σ_0}. Since the latter is irreducible (4.10; it also follows from the irreducibility of the special representation and [7: 2.1.5]), $\mathrm{Im}\,\eta$ is equal to L_{σ_0}. There remains to show that η is injective.

The scalar product on $C^l(X)$ defines a pairing between $C^l(X)$ and $C_c^l(X)$. It is well-known, and elementary, that $Z_l = C^l \cap \ker\delta$ is the annihilator of dC_c^{l-1}, so that we may identify $H_l(X)$ with the dual of $H_c^l(X)$ in such a way that the canonical pairing is defined by our scalar product. By [3: 5.6, 5.10] the G-space $H_c^l(X)$ is admissible. Therefore $H_l(X)^\infty$ is admissible and is the contragredient to $H_c^l(X)$ [7: 2.1.9] in particular, if U is a compact open subgroup of G, then $H_l(X)^U$ is canonically isomorphic to the dual of $H_c^l(X)^U$ and we have

$$\dim H_l(X)^U = \dim H_c^l(X)^U. \tag{1}$$

On the other hand, since η is surjective, $L_{\sigma_0}^U$ is a quotient of $H_c^l(X)^U$.

However, by 6.1(6), $L_{\sigma_0}=H_l(X)^\infty$, hence by (1), $\dim L^U_{\sigma_0}=\dim H^l_c(X)^U$ and η is injective on $H^l_c(X)^U$. Since $H^l_c(X)$ is the union of such subspaces, our assertion follows.

6.3. Let $\mathscr{P}$ be a minimal parabolic k-subgroup of G, Φ the set of roots of $\mathscr{G}$ with respect to a maximal k-split torus contained in $\mathscr{P}$ and Δ the set of simple roots associated to $\mathscr{P}$. Let $\mathscr{P}_I$ $(I\subset\Delta)$ be the parabolic k-subgroups of $\mathscr{G}$ containing $\mathscr{P}$, and $C^\infty(G/P_I)$ the space of locally constant complex valued functions on G/P_I. It is an admissible G-module (with respect to left translations), the unnormalized induced representation $\mathbf{Ind}(1|P_I,G)$. Let θ_I be its character. In the Grothendieck group of admissible G-modules, we have, by [3: 5.6, 5.10], the equality

$$H^l_c(X)=\sum_{I\subset\Delta}(-1)^{|I|}\,C^\infty(G/P_I), \tag{1}$$

where $|I|=\operatorname{Card} I$. The character of the admissible G-module $H^l_c(X)$ is then the alternating sum

$$\sum_{I\subset\Delta}(-1)^{|I|}\,\theta_I, \tag{2}$$

the "Steinberg character" of G. In view of 6.2, we have

6.4. Corollary. Let θ_I be the character of $\mathbf{Ind}(1|P_I,G)$ $(I\subset\Delta)$. Then the character of the special representation is equal to

$$\sum_{I\subset\Delta}(-1)^{|I|}\,\theta_I.$$

6.5. *Remark.* As we saw, the fact that in 6.2 $\eta(H^l_c(X))=(H^l_2)^\infty$ is rather elementary, so that the main point is the injectivity of η, which is equivalent to $H^l_c(X)$ being an irreducible G-module. If the isomorphism 6.3(1) is granted, this is in turn equivalent to the right hand side of 6.3(1) representing the class of an irreducible admissible G-module. This is included in a more general Theorem announced in [8].

§ 7. Proof of Lemma 4.1

7.1. In this section, we use freely some terminology and known facts on buildings, to be found in [5]. Much of what we need is also reviewed in [3: §§ 4, 5].

As in 3.1, C_0 is the chamber fixed under $\tilde{B}$ and A_0 the apartment of X fixed under $\tilde{H}=\tilde{B}\cap\tilde{N}$. We recall that an apartment (resp. a wall) in X is the transform by some element of $\tilde{G}$ of A_0 (resp. of the fixed point set of a reflection in $\tilde{W}$). Any two chambers are contained in an apartment [5: 2.3.1]. A half-apartment is a closed half-space in an apartment whose boundary is a wall. If α is one, then $\partial\alpha$ denotes the wall which is its boundary (in any apartment containing it) and U_α the greatest unipotent subgroup of $\tilde{G}$ which fixes it. Thus U_α is normal in the fixer $\tilde{G}_\alpha$ of α in $\tilde{G}$, and $\tilde{G}_\alpha$ is the semidirect product of $\tilde{H}$ and U_α.

We let $d(\,,)$ be the canonical metric on X [5: § 2]. It is invariant under $\tilde{G}$ and induces α euclidean metric on each apartment. If M is closed and N compact in X, then

$$d(M, N) = \min_{x \in M, y \in N} d(x, y).$$

The ball of radius a and center $x \in X$ is the set of $y \in X$ such that $d(x, y) \leqq a$.

7.2. Lemma. (i) *Let C be a chamber of X. Let A be an apartment containing C and C_0 and α a half-apartment of A such that $C \not\subset \alpha$ and $C \cap \partial\alpha$ is a face of codimension* 1 *of C. Let D be the chamber contained in α such that $D \cap C = C \cap \partial\alpha$. Then U_α is transitive on the set of chambers C' of X such that $C' \cap D = C \cap D$.*

(ii) *There exists a constant $\lambda > 0$ with the following property: if a is a half-apartment and $x \in \alpha$, then U_α fixes the ball of radius $\lambda \cdot d(x, \partial\alpha)$ and center x.*

(i) follows from 5.1.10, p. 86, and (ii) from 7.4.33, p. 179 in [5]. We note that it suffices to prove the existence of λ for one given half-apartment, since the half-apartments form finitely many orbits under $\tilde{G}$.

7.3. Lemma. *Let a_0 be a strictly positive real number. There exists a number $a_1 > 0$ with the following property: if C is a chamber of X, $d_s(C, C_0) \geqq a_1$ and A is an apartment containing C, C_0, then there exists a half-apartment $\alpha \subset A$ such that $d(C_0, \partial\alpha) \geqq a_0$, $C_0 \subset \alpha$, $C \subset (A - \alpha) \cup \partial\alpha$ and $C \cap \partial\alpha$ is a face of codimension one of C.*

Since $\tilde{G}$ is transitive on the set of pairs consisting of an apartment A' and a chamber $C' \subset A'$ [5: 2.26, p. 36], it suffices to prove this in a given apartment, say A_0. It follows there by an elementary argument in euclidean geometry, using the fact that the chambers are all congruent polysimplices and hence that the set of angles of two faces of codimension one of all chambers is finite.

7.4. Lemma. *Let U be a compact open subgroup of $\tilde{G}$. There exists a constant b_0 with the following property: let C be a chamber of X such that $d_s(C, C_0) \geqq b_0$. Then there exists a minimal gallery $\gamma = \{C_0, C_1, \ldots, C_m = C\}$, $(m = d_s(C, C_0) + 1)$, connecting C_0 and C such that U is transitive on the set (C, C_{m-1}) of chambers C' in X such that $C' \cap C_{m-1} = C \cap C_{m-1}$.*

We may replace U by a smaller group, hence assume that $U \subset B$ fixes C_0. Let D be a ball in X, with center a point $x_0 \in C_0$, big enough so that the subgroup of $\tilde{G}$ which fixes it is contained in U. Let r be the radius of D, $a_0 = r/\lambda$, and choose a_1 as in 7.3. Let C be a chamber such that $d_s(C, C_0) \geqq a_1$. Let A and α be as in 7.3. Then the wall $\partial\alpha$ contains a face of codimension 1 of C and separates C and C_0. There exists therefore in A a minimal gallery $\gamma = (C_0, C_1, \ldots, C_m = C)$ such that

$$C_i \subset \alpha \, (0 \leqq i < m), \qquad C \cap C_{m-1} = C \cap \partial\alpha = C_{m-1} \cap \partial\alpha \tag{1}$$

[4: IV, §1, Ex. 16]. The gallery γ, being minimal in A, is also minimal in X [5: 2.3.6, p. 38] hence

$$m = d_s(C, C_0) + 1, \qquad d_s(C_{m-1}, C_0) = d_s(C, C_0) - 1. \tag{2}$$

By 7.2(ii), the group U_α fixes the ball of radius $\lambda \cdot d(x_0, \partial\alpha)$. Since

$$d(x_0, \partial\alpha) \geqq d(C_0, \partial\alpha) \geqq a_0 = r/\lambda,$$

the group U_α fixes the ball D chosen above hence,

$$U_\alpha \subset U. \tag{3}$$

Finally, the chamber C_m, C_{m-1} satisfy, with respect to A and α, the assumptions imposed on C, D in 7.2(i), therefore U_α is transitive on the set of (C, C_{m-1}). This proves 7.4, with $b_0 = a_1$.

7.5. *Proof of Lemma* 4.1. Since the isogeny $\omega: \tilde{G} \to G$ induces a homomorphism $\tilde{G} \to G_0$ which commutes with the actions of $\tilde{G}$ and G_0 on X, it suffices to consider the case where $G = \tilde{G}$, hence $G_0 = \tilde{G}$.

Let U be a compact open subgroup of $\tilde{G}$. Choose b_0 as in 7.4. Then, by 7.4, all our conditions are fulfilled if we take $d_0 = b_0$ and choose for D the element C_{m-1} of the gallery γ (see 7.4(1), (2)).

References

1. Borel, A.: Cohomologie de certains groupes discrets et laplacien p-adique. Sém. Bourbaki 26ème année (1973/74), Exp. 437, 24 p
2. Borel, A., Serre, J-P.: Cohomologie à supports compacts des immeubles de Bruhat-Tits; application à la cohomologie des groupes S-arithmétiques. C.R. Acad. Sci. Paris **272**, 110–113 (1971)
3. Borel, A., Serre, J-P.: Cohomologie d'immeubles et de groupes S-arithmétiques. A paraître dans Topology
4. Bourbaki, N.: Groupes et algèbres de Lie, Chap. IV, V, VI, Act. Sci. Ind. 1337. Paris: Hermann 1968
5. Bruhat, F., Tits, J.: Groupes réductifs sur un corps local. Chap. I. Publ. Math. I.H.E.S. **41**, 1–251 (1972)
6. Casselman, W.: The Steinberg character as a true character. Harmonic analysis on homogeneous spaces. Proc. Symp. pur. math. **26**, 413–417 (1974)
7. Casselman, W.: Introduction to the theory of admissible representations of $\mathfrak{P}$-adic reductive groups. To appear
8. Casselman, W.: On a p-adic vanishing theorem of Garland. Bull. A.M.S. **80**, 1001–1004 (1974)
9. Garland, H.: p-adic curvature and the cohomology of discrete subgroups of p-adic groups. Annals of Math. **97** (2), 375–423 (1973)
10. Higman, D.G.: Induced and produced modules. Canadian J.M. **VII**, 490–508 (1955)
11. Iwahori, N., Matsumoto, H.: On some Bruhat decompositions and the structure of the Hecke rings of p-adic Chevalley groups. Publ. Math. I.H.E.S. **25**, 5–48 (1965)
12. MacDonald, I.G.: The Poincaré series of a Coxeter group. Math. Annalen **199**, 161–174 (1972)
13. Matsumoto, H.: Fonctions sphériques sur un groupe semi-simple p-adique. C.R. Acad. Sci. Paris **269**, 829–832 (1969)
14. Matsumoto, H.: Analyse harmonique dans certains systèmes de Coxeter et de Tits. Analyse sur les harmonique et groupes de Lie. Lecture Notes in Math. **497**, 257–276. Berlin-Heidelberg-New York: Springer 1975
15. Serre, J-P.: Cohomologie des groupes discrets. Prospects in Mathematics, Annals of Math. Studies 70. Princeton: Princeton University Press 1970
16. Shalika, J.A.: On the space of cusp forms of a p-adic Chevalley group. Annals of Math. **92** (2), 262–278 (1970)

Armand Borel
The Institute for Advanced Study
School of Mathematics
Princeton, N.J. 08540
USA

Received December 23, 1975