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Euler Characteristics and Characters of Discrete Groups

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Contents

1. Introduction

This investigation began as an attempt to generalize to infinite groups the following theorem of Swan [26].

Theorem (Swan). *Let P be a finitely generated projective module over the group ring $\mathbb{Z}G$ of a finite group G. Then $\mathbb{Q}\otimes P$ is a free $\mathbb{Q}G$-module.*

The conjectural generalisation consists in: (i) interpreting the theorem in terms of the character χ_P of P; (ii) introducing a notion of rank, r_P, of P which, (a) makes sense even when G is infinite, and (b) determines χ_P when G is finite; and (iii) predicting in general that $r_P = r_F$ for some free module F. This conjecture is proved here for torsion free linear groups, and a weaker version of it is proved for all residually finite groups.

The definition of r_P uses the "universal trace functions" introduced by Stallings [25] and Hattori [13]. While writing up the relevant preliminaries it became apparent that the literature contains no systematic and comprehensive exposition of the basic properties of these trace functions, nor a unified account of the diverse types of problems which have been treated by what amount to trace function methods. I have attempted to approximate such an exposition here, particularly

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because available methods can be exploited to yield many results that are significantly stronger and more precise than those derived in the literature.

Following is a resumé of the paper.

Let k be a commutative ring. If M is a finitely generated projective k-module its trace function is a k-linear map $T_M\colon \mathrm{End}_k(M)\to k$ (cf. [4], II. 78). If A is a k-algebra and $\rho\colon A\to \mathrm{End}_k(M)$ defines a (right) A-module structure on M, its character $\chi_M\colon A\to k$ is defined by $\chi_M(a)=T_M(\rho(a))$. Since $\chi_M(ab)=\chi_M(ba)$ – χ_M is a "central function" on A – χ_M factors through the quotient map $T\colon A\to T(A)=A/[A,A]$, where $[A,A]$ denotes the subgroup of A generated by all $[a,b]=ab-ba$.

Let P be a finitely generated projective (right) A-module. Stallings [25] and Hattori [13] introduced a natural trace function $T_P\colon \mathrm{End}_A(P)\to T(A)$, whence a definition of the rank r_P of P as $T_P(1_P)\in T(A)$. These definitions lead to corresponding notions of "Lefschetz numbers" and "Euler characteristics" in $T(A)$ for appropriate endomorphisms and modules, respectively. These notions, and the indicated "duality" between characters and Euler characteristics, are recounted in §§ 2–4.

Suppose A is the k-algebra kG of a group G. If $s\in G$ one can identify $T(s)\in T(kG)$ with the G-conjugacy class of s, and $T(kG)$ with the free k-module having the set $T(G)$ of G-conjugacy classes as a basis. Thus every $r\in T(kG)$ is a finite linear combination, $r=\sum_{\tau\in T(G)} r(\tau)\cdot\tau$. This notation identifies r with a function $T(G)\to k$ with finite support, $\mathrm{supp}(r)\subset T(G)$. We can also view r thus as a "central function" $G\to k$ by putting $r(s)=r(T(s))$ for $s\in G$. This in turn identifies $T(kG)$ with an ideal in the ring $CF(G,k)$ of all central functions $G\to k$, where the characters χ_M of finitely generated k-projective kG-modules M live. If P is a finitely generated projective kG-module then so also is $Q=\mathrm{Hom}_k(M,P)$, and $r_Q=\chi_M\cdot r_P$. If G is finite then χ_P is defined and, as Hattori showed,

$$\chi_P(s)=|Z_G(s)|\cdot r_P(s^{-1}) \tag{1}$$

for $s\in G$ (Prop. (5.8)). Here we write $Z_G(s)$ for the centralizer in G of s and $|Z_G(s)|$ for its order. These and related facts are recounted in § 5.

When $P\cong(kG)^n$ one has $r_P(1)=n$ and $r_P(s)=0$ for $s\neq 1$. Thus, in view of (1) above, Swan's theorem is just the affirmation of the following conjecture when G is finite.

Strong Conjecture. Let G be a group and let P be a finitely generated projective $\mathbb{Z}G$-module. Then $r_P(s)=0$ for $s\neq 1$ in G.

The free $\mathbb{Z}$-module $P\otimes_{\mathbb{Z}G}\mathbb{Z}$ has rank $\sum_{\tau\in T(G)} r_P(\tau)$. This motivates the:

Weak Conjecture. $r_P(1)=\sum_{\tau\in T(G)} r_P(\tau)$.

The strong conjecture is in the same spirit as Serre's question ([24], p. 85) asking whether a group of "type (FP)" is automatically of "type (FL)". The weak conjecture was posed as a question by Dyer and Vasquez [10], and was the starting point of this investigation. We prove the strong conjecture (Prop. (9.2)), even over $\mathbb{C}$ in place of $\mathbb{Z}$, for a class of torsion free groups including those with faithful linear representations (Th. (9.6)). We prove the weak conjecture for all residually finite groups (Cor. (6.10)), this being an easy consequence of Swan's theorem. We give here a simple proof of a more general version of Swan's theorem (Th. (6.6)) which is derived from the following formula (Prop. (6.1)) for the rank $r_{P/H}$ of P restricted to a subgroup H of finite index in G:

$$r_{P/H}(s)=r_{P/G}(s)\cdot[Z_G(s)\colon Z_H(s)]$$

for all $s\in H$. Other results, examples, and questions concerning subgroups of finite index are discussed in § 6.

If k has prime characteristic p then $T(kG)$ admits a Frobenius endomorphism F sending $\sum r(\tau)\tau$ to $\sum r(\tau)^p\tau^p$, where $T(s)^p$ denotes $T(s^p)$, and such that $T_P(u^p)=T_P(u)^p$ if P is a finitely generated projective kG-module and $u\in\mathrm{End}_{kG}(P)$. In particular $r_P=T_P(1_P)$ is fixed by F. This imposes major constraints on $\mathrm{supp}(r_P)$ and on the values $r_P(s)\in k$. This was first observed and exploited by Zalesskii [27]. We present the properties of F in § 7.

The complex group algebra $\mathbb{C}G$ is treated in § 8, by Zalesskii's method of specialization to characteristic p, where Frobenius operates, and use of the Čebotarev density theorem. Specifically, let $P\neq 0$ be a finitely generated projective $\mathbb{C}G$-module, and put $r=r_P\in T(\mathbb{C}G)$. Kaplansky ([19]; see Th. (8.9) below) showed that $r(1)$ is a totally real algebraic number >0. He conjectured that $r(1)\in\mathbb{Q}$, and Zalesskii [27] proved this. Certainly the most interesting result obtained here is the following one (see Th. 18.1), proved with the aid of Zalesskii's methods.

a) The field E generated by all $r(s)$ $(s\in G)$ is a finite abelian extension of $\mathbb{Q}$.

b) For all but finitely many primes p, $\tau\mapsto\tau^p$ is a permutation of $S=\mathrm{supp}(r)$, and $r(\tau^p)=\sigma r(\tau)$ for $\tau\in S$, where σ is the Artin symbol $(p, E/\mathbb{Q})$.

c) Suppose $s\in G$ has finite order m and $r(s)\neq 0$. Let $w=\exp(2\pi i/m)$. Then $r(s)\in\mathbb{Q}(w)$, say $r(s)=f(w)$ where $f\in\mathbb{Q}[X]$, and we have $r(s^q)=f(w^q)$ for all q prime to m.

d) If G satisfies a certain "non-divisibility condition" (D) (see (9.1)) then $r(s)=0$ whenever s has infinite order (Prop. (9.2)).

e) Linear groups satisfy condition (D) (Th. (9.6)).

These methods have been used by Formanek [11], Passman and Sehgal [20], and others to investigate idempotents and units of finite order in $\mathbb{C}G$. Some of their results are recounted in § 8 to illustrate some nice applications of the present methods.

In the final § 10 we discuss groups of finite cohomological type. Specifically, suppose k (with trivial G action) has a finite resolution

$$0\to P_n\to\cdots\to P_0\to k\to 0$$

with each P_i a finitely generated projective kG-module; we then say G is of type (FP) over k, and put $r_G=\sum(-1)^i r_{P_i}\in T(kG)$, $\chi(G)=r_G(1)$, and $\tilde{\chi}(G)=\sum_{\tau\in T(G)} r_G(\tau)$, these being various kinds of Euler characteristics of G over k. If the k-modules $H_i(G,k)$ (resp., $H^i(G,k)$) are projective, say of rank h_i (resp., h^i) in k, then (Cor. (10.3)),

$$\tilde{\chi}(G)=\sum(-1)^i h_i=\sum(-1)^i h^i.$$

If $\chi(G) \neq 0$ then, as Stallings showed, the center $Z(G)$ is a finite group whose order is invertible in k (Prop. (10.4)). Further (Prop. (10.7)) $r_G(\alpha s) = r_G(s)$ for any automorphism α of G. If H is a subgroup of finite index in G then H is also of type (FP) and (Prop. (10.5)),

$$r_H(s) = r_G(s) \cdot [Z_G(s) : Z_H(s)]$$

for $s \in H$; in particular, for $s = 1$ we have

$$\chi(H) = \chi(G) \cdot [G : H].$$

Suppose k is a subring of $\mathbb{C}$ and $k \cap \mathbb{Q} = \mathbb{Z}$. Then we have the theorem of K. Brown (proved for $k = \mathbb{Z}$),

$$\tilde{\chi}(H) = \tilde{\chi}(G) \cdot [G : H].$$

The weak conjecture above, generalized to such rings k, predicts that $\chi(G) = \tilde{\chi}(G)$ when G is of type (FP) over such a k. Our results imply that this is so whenever G is residually finite (Prop. (10.5)).

Let $1 \to H \xrightarrow{\varepsilon} G \xrightarrow{\pi} G' \to 1$ be a group extension. One has natural homomorphisms $\varepsilon_*: T(kH) \to T(kG)$ and $\pi_*: T(kG) \to T(kG')$. Suppose H is of type (FP) over k. Then there is a naturally defined homomorphism $\pi^*: T(kG') \to T(kG)$ which remains slightly mysterious, but about which we can say the following (Th. (10.9)):

a) $\pi^* T_{G'}(1) = \varepsilon_* r_H$. If $\tau' \in T(G')$ then $\pi^* \tau'$ has support among those $\tau \in T(G)$ for which $\pi\tau = \tau'$, and $\pi_* \pi^* \tau' = L(\tau')\tau'$ for a certain $L(\tau') \in k$, with $L(1) = \tilde{\chi}(H)$.

b) If the k-modules $H_*(H, k)$ are projective, then $\tau' \mapsto L(\tau'^{-1})$ is the virtual character of the natural action of G' on $H_*(H, k)$.

c) Suppose further that G' is of type (FP) over k. Then G is likewise, and $r_G = \pi^*(r_{G'}) = \sum_{\tau' \in T(G')} r_{G'}(\tau') \cdot \pi^* \tau'$. Hence

$$\chi(G) = \chi(G') \cdot \chi(H). \tag{2}$$

d) If $\tau' \in T(G')$ then $L(\tau') \cdot r_{G'}(\tau') = \sum_{\pi\tau = \tau'} r_G(\tau')$. Hence

$$\tilde{\chi}(G) = \sum_{\tau' \in T(G')} L(\tau') \cdot r_{G'}(\tau').$$

In the course of preparing this manuscript I received a preprint of I.M. Chiswell's paper [8], where he also introduces the Euler characteristic $\chi(G) = r_G(1)$ when G is of type (FP) over k. (He writes $\chi(G, k)$ for r_G and $\mu(G)$ for $\chi(G)$.) Several of his results intersect with those here, as follows: His Lemma 5 is the value at $s = 1$ part of Proposition (6.2); his Theorem 1 is part of Proposition (10.5)(c); his Lemma 8 is part of Proposition (5.5)(c); his Lemma 10 and Theorem 3 are contained in Theorem (10.9). Chiswell conjectures formula (2) above. He also asks about the relation between $\chi(G)$ and $\tilde{\chi}(G)$ when $k = \mathbb{Z}$.

I have just learned also of work of John Stallings, "An extension theorem for Euler characteristics of groups", in which the Euler characteristic $\chi(G)$ is first introduced, and the essential substance of Theorem (10.9) is proved.

An announcement of some of the results here appears in the proceedings of the conference on commutative algebra held at Queens University, July, 1975.

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2. Hattori-Stallings Traces in $T(A)$

An exposition of most of this section can be found in Stallings [25], Hattori [13], or Gruenberg [12], § 8.10.

A denotes a ring; A-modules are understood to be right A-modules.

(2.1) *The Group* $T(A)$. It is the quotient $A/[A,A]$ of A by the additive group $[A,A]$ generated by all commutators $[a,b]=ab-ba$. We write T or T_A for the canonical projection $A \to T(A)$.

(2.2) *Coordinate Systems and* $\mathscr{P}(A)$. An A-module P is projective if and only if there is a family (x_i) in P and (f_i) in $P^* = \mathrm{Hom}_A(P,A)$ such that, for all $x \in P$, $f_i(x)=0$ for all but finitely many i, and $x=\sum_i x_i f_i(x)$ (cf. [4], II.46). The system (x_i), (f_i) will be called an A-coordinate system of P. The families (x_i) which can so occur are precisely the generating families in P, so we can choose a finite one if P is finitely generated. The category of finitely generated projective (right) A-modules will be denoted $\mathscr{P}(A)$.

(2.3) *Traces* T_P. Let $P \in \mathscr{P}(A)$. Define $t\colon P \times P^* \to T(A)$ by $t(x,f)=T(f(x))$. If $a \in A$ then $t(xa,f)=T(f(x)a)=T(af(x))=t(x,af)$. Thus t induces an additive map $P \otimes_A P^* \to T(A)$. The canonical map $P \otimes_A P^* \to \mathrm{End}_A(P)$, where the image of $x \otimes f$ sends $y \in P$ to $xf(y)$, is an isomorphism, since $P \in \mathscr{P}(A)$, which we view as an identification (cf. [4], II.77). We have thus defined a homomorphism

$$T_P\colon \mathrm{End}_A(P) \to T(A)$$

which is sometimes denoted $T_{P/A}$, and called the *trace* on the A-module P. It is characterized by, $T_P(x \otimes f)=T(f(x))$ for $x \in P$. $f \in P^*$. Writing $1_P=\sum x_i \otimes f_i$ amounts to choosing a finite coordinate system x_i, f_i of P. If $u \in \mathrm{End}_A(P)$, then $u \circ (x_i \otimes f_i)=u(x_i) \otimes f_i$ and $u=\sum u \circ (x_i \otimes f_i)$, so

$$T_P(u)=T\big(\textstyle\sum f_i(u(x_i))\big). \tag{1}$$

Note that if x_i is a free basis of P then f_i is the dual basis and the $u_{ji}=f_j(u(x_i))$ are the coefficients of the matrix representing $u\colon u(x_i)=\sum_j x_j u_{ji}$. Formula (1) then reads: $T_P(u)=T(\sum_i u_{ii})$.

(2.4) *Ranks.* If $P \in \mathscr{P}(A)$ its *rank*, denoted r_P or $r_{P/A}$, is the element

$$r_P = T_P(1_P) \in T(A).$$

If x_i, f_i is a finite coordinate system of P then $r_P=\sum_i f_i(x_i)$. If $P \cong A^n$, then $r_P=T(n)$.

(2.5) *Universal Property.* The functions $T_P\colon \mathrm{End}_A(P) \to T(A)$ constructed in (2.3) satisfy:

Additivity. If $u \in \mathrm{End}_A(P)$ and $u' \in \mathrm{End}_A(P')$, then $T_{P \oplus P'}(u \oplus u')=T_P(u)+T_{P'}(u')$.

Linearity. If $u, v \in \mathrm{End}_A(P)$ then $T_P(u+v) = T_P(u) + T_P(v)$.

Commutativity. If $u: P \to P'$ and $u': P' \to P$ then $T_P(u'u) = T_{P'}(uu')$.

Universality. Suppose $T'_P: \mathrm{End}_A(P) \to S$ is another collection of functions, defined for $P \in \mathscr{P}(A)$, with values in an abelian group S, and which is additive, linear, and commutative as above. Then there is a unique homomorphism $t: T(A) \to S$ such that $T'_P = t \circ T_P$ for all P.

These properties are established by Stallings in [25].

(2.6) *Modules of Type* (FP). An A-module M is said to be of type (FP) if it has a finite $\mathscr{P}(A)$-resolution, i.e., if there is an exact sequence

$$0 \to P_n \to \cdots \to P_0 \to M \to 0 \tag{2}$$

for some $n \geqq 0$ with each $P_i \in \mathscr{P}(A)$. If two of the three terms in a short exact sequence is of type (FP) so also is the third (cf. [1], Ch. I, Cor. (6.9)). A direct summand of a module of type (FP) is of type (FP) (cf. [8], Lemma 4).

(2.7) *Traces for Modules of Type* (FP). Consider a resolution (2) of an A-module M of type (FP). An endomorphism u of M can be lifted to an endomorphism (u_i) of the resolution. We then put

$$T_M(u) = \sum_{i \geqq 0} (-1)^i T_{P_i}(u_i).$$

This is independent of the resolution (2) of M and of the lifting (u_i) of u, and the resulting functions $T_M: \mathrm{End}_A(M) \to T(A)$ satisfy:

Additivity. If $0 \to (M', u') \to (M, u) \to (M'', u'') \to 0$ is an exact sequence of A-modules of type (FP) with endomorphisms, then $T_M(u) = T_{M'}(u') + T_{M''}(u'')$.

Linearity. If $u, v \in \mathrm{End}_A(M)$, then $T_M(u+v) = T_M(u) + T_M(v)$.

Commutativity. If $u: M \to M'$ and $u': M' \to M$ are homomorphisms of A-modules of type (FP) then $T_M(u'u) = T_{M'}(uu')$.

One can prove this using Grothendieck's resolution theorem ([1], Ch. VIII, Th. (4.2)) applied to the category C of pairs (M, u) as above and the subcategory C_0 of those for which $M \in \mathscr{P}(A)$. There is a universal family $t_M: \mathrm{End}_A(M) \to T(C)$ with the above three properties, and, in view of (2.5), one need only check that $T(A) = T(C_0) \to T(C)$ is an isomorphism. Grothendieck's theorem implies that $K_0(C_0) \to K_0(C)$ is an isomorphism. $T(C)$ is the quotient of $K_0(C)$ in which the commutative and linear relations are imposed, and similarly for $T(C_0)$. To obtain an induced isomorphism on these quotients of K_0, one uses the following observations:

1. If $u, v \in \mathrm{End}_A(M)$ lift to endomorphisms (u_i) and (v_i) of the resolution (2), then $(u_i + v_i)$ is such a lifting of $u+v$.

2. With the notation of the commutativity property, let

$$\cdots \to P'_i \to \cdots \to P'_0 \to M' \to 0$$

be a finite $\mathscr{P}(A)$-resolution of M'. Lift u to a morphism $(u_i: P_i \to P'_i)$ of resolutions, and u' to $(u'_i: P'_i \to P_i)$. Then $(u_i u'_i)$ is a lifting of uu' and $(u'_i u_i)$ is one of $u'u$.

(2.8) *Ranks of Modules of Type* (FP). If M is an A-module of type (FP) its rank, r_M or $r_{M/A}$, is the element

$$r_M = T_M(1_M) \in T(A).$$

The map $M \mapsto r_M$ is additive over exact sequences, so it defines a homomorphism $r\colon K_0(A) \to T(A)$.

(2.9) *Functoriality; Covariance.* Let $\alpha\colon A \to B$ be a ring homomorphism. It induces

$$\alpha_*\colon T(A) \to T(B), \qquad \alpha_* T_A(a) = T_B(\alpha a).$$

If $P \in \mathscr{P}(A)$ and $u \in \operatorname{End}_A(P)$ then $\alpha_* P = P \otimes_A B \in \mathscr{P}(B)$ and $\alpha_* u = u \otimes 1_B \in \operatorname{End}_B(\alpha_* P)$. The map $(P, u) \mapsto T_{\alpha_* P}(\alpha_* u) \in T(B)$ is clearly additive, linear, and commutative. By universality therefore it induces a homomorphism $T(A) \to T(B)$ which, on taking $P = A$, we see is just α_*. Explicitly,

$$T_{\alpha_* P}(\alpha_* u) = \alpha_* T_P(u).$$

The analogous formula when P is only of type (FP) holds if B is a flat left A-module, but not in general. When $u = 1_P$ we obtain the formula

$$r_{\alpha_* P} = \alpha_* r_P.$$

(2.10) *Contravariance.* Suppose B is a right A-module of type (FP). Then the same is true of every (right) B-module M of type (FP) (cf. (2.6)). If

$$u \in \operatorname{End}_B(M) \subset \operatorname{End}_A(M)$$

we can thus define $T_{M/A}(u)$. The map $(M, u) \to T_{M/A}(u)$ is clearly additive, linear, and commutative, so it defines a homomorphism

$$\alpha^* = \operatorname{Tr}_{B/A}\colon T(B) \to T(A)$$

such that

$$\operatorname{Tr}_{B/A}(T_{M/B}(u)) = T_{M/A}(u)$$

for all B-modules M of type (FP) and all $u \in \operatorname{End}_B(M)$.

(2.11) *Examples.* 1. Suppose A is commutative. Then $T(A) = A$ and, for $P \in \mathscr{P}(A)$, T_P is the usual trace (cf. [4], II.78). The image of $r\colon K_0(A) \to A$ is the subring generated by all idempotents.

2. Let $P \in \mathscr{P}(A)$ and $B = \operatorname{End}_A(P)$. If $\rho_P\colon A \to \operatorname{End}_B(P)$ is the canonical map, then $T_{P/B} \circ \rho_P\colon A \to T(B)$ induces a homomorphism $t_P\colon T(A) \to T(B)$. Suppose $P = A^n$; then we can identify B with the ring $M_n(A)$ of n by n matrices over A, ρ_{A^n} is an isomorphism of rings, and $t_{A^n}\colon T(A) \to T(M_n(A))$ is an isomorphism of groups which sends $T_A(a)$ to $T_{M_n(A)}(\operatorname{diag}(a, 0, \dots, 0))$. If A is commutative and we use t_{A^n} to identify $T(M_n(A))$ with A then $T_{A^n/M_n(A)}\colon M_n(A) \to A$ is the usual trace, $\operatorname{Tr}\colon (a_{ij}) \mapsto \sum a_{ii}$, whereas $T_{M_n(A)}\colon M_n(A) \to A$ is $n \cdot \operatorname{Tr} = \operatorname{Tr}_{M_n(A)/A}$.

3. If $A = A_1 \times A_2$, a product of rings, then $T(A) = T(A_1) \oplus T(A_2)$.

3. k-Algebras

Let k be a commutative ring. For any k-algebra A, $[A, A]$ is a sub-k-module of A; thus $T(A) = A/[A, A]$ is a k-module so that $T: A \to T(A)$ is k-linear.

(3.1) **Proposition** (cf. [12], § 8.10, Th. 7). *Let M be an A-module of type (FP). The map $T_M: \mathrm{End}_A(M) \to T(A)$ is k-linear. The rank $r_M \in T(A)$ is annihilated by* $\mathrm{ann}_k(M) = \{a \in k \mid aM = 0\}$.

Let $0 \to P_n \to \cdots \to P_0 \to M \to 0$ be a $\mathscr{P}(A)$-resolution of M, let $u \in \mathrm{End}_A(M)$, and let $a \in k$. Lift u to an endomorphism (u_i) of the resolution. Then (au_i) is a lifting of au, so $T_M(au) = \sum (-1)^i T_{P_i}(au_i)$. We are thus reduced to showing that $T_M(au) = a T_M(u)$ when $M \in \mathscr{P}(A)$. Let x_i, f_i be a coordinate system for M. Then $T_M(au) = T(\sum f_i(au(x_i))) = T(a \sum f_i(u(x_i))) = a T_M(u)$. The last assertion follows immediately from the first one.

(3.2) *Remarks.* 1. Any ring A may be viewed as an algebra over its center, C. Thus $T(A)$ is a C-module so that the maps T_M are C-linear, and r_M is annihilated by $\mathrm{ann}_C(M)$.

2. If X generates the k-algebra A as a k-module, then the elements $[a, b] = ab - ba$ where a, b run through X generate the k-module $[A, A]$, because the map $(a, b) \mapsto [a, b]$ is k-bilinear.

(3.3) **Proposition.** *Let A and B be k-algebras.*

a) *There is a (natural) isomorphism of k-modules $T(A \otimes_k B) \to T(A) \otimes_k T(B)$ sending $T(a \otimes b)$ to $T(a) \otimes T(b)$ for $a \in A$, $b \in B$. We identify the two modules via this isomorphism.*

b) *If $P \in \mathscr{P}(A)$, $Q \in \mathscr{P}(B)$, $u \in \mathrm{End}_A(P)$, and $v \in \mathrm{End}_B(Q)$ then $T_{P \otimes_k Q}(u \otimes v) = T_P(u) \otimes T_Q(v)$.*

Put $C = A \otimes_k B$. The projection $T_A \otimes T_B$: $C \to T(A) \otimes_k T(B)$ is surjective with kernel$=([A, A] \otimes B) + (A \otimes [B, B]) \subset [C, C]$. The elements $a \otimes b$ generate C as k-module so $[C, C]$ is generated as k-module by the elements $[a \otimes b, a' \otimes b'] = [a, a'] \otimes bb' + a'a \otimes [b, b']$. Thus $T_A \otimes T_B$ induces an isomorphism $T(C) \to T(A) \otimes T(B)$.

Let x_i, f_i and y_j, g_j be coordinate systems of P and Q, respectively. Then $x_i \otimes y_j, f_i \otimes g_j$ is a coordinate system of $P \otimes_k Q$, so

$$
\begin{aligned}
T_{P \otimes_k Q}(u \otimes v) &= T\Big(\sum_{i,j} (f_i \otimes g_j)(u x_i \otimes v y_j)\Big) = T\Big(\Big(\sum_i f_i(u x_i)\Big) \otimes \Big(\sum_j g_j(v y_j)\Big)\Big) \\
&= T_P(u) \otimes T_Q(v).
\end{aligned}
$$

Remarks. 1. Suppose B is a flat k-module. Then $Q \in \mathscr{P}(B)$ is also a flat k-module. If M is an A-module of type (FP) then $M \otimes_k Q$ is an $A \otimes_k B$-module of type (FP) and $T_{M \otimes_k Q}(u \otimes v) = T_M(u) \otimes T_Q(v)$ for $u \in \mathrm{End}_A(M)$. For if $P \to M$ is a finite $\mathscr{P}(A)$-resolution of M then $P \otimes_k Q \to M \otimes_k Q$ is a finite $\mathscr{P}(A \otimes_k B)$ resolution, thanks to the flatness assumption. If $\tilde{u} \in \mathrm{End}_A(P)$ is a lifting of u, then $\tilde{u} \otimes v$ is a lifting of $u \otimes v$, and the formula for $T_{M \otimes_k Q}(u \otimes v)$ follows from b) of the proposition.

2. Let α: $A \to A'$ and β: $B \to B'$ be homomorphisms of k-algebras. Then $(\alpha \otimes \beta)_* = \alpha_* \otimes \beta_*$: $T(A) \otimes_k T(B) \to T(A') \otimes_k T(B')$. Suppose A' (resp., B') is a

finitely generated projective right A-module (resp., B-module). Then $A'\otimes_k B'$ is a finitely generated projective right $(A\otimes_k B)$-module, and

$$\mathrm{Tr}_{A'\otimes B'/A\otimes B}=\mathrm{Tr}_{A'/A}\otimes\mathrm{Tr}_{B'/B}\colon\ T(A')\otimes_k T(B')\to T(A)\otimes_k T(B).$$

(3.4) **Corollary.** *Let k' be a commutative k-algebra. There is a natural isomorphism of k'-modules $k'\otimes_k T(A)\to T(k'\otimes_k A)$, sending $b\otimes T(a)$ to $T(b\otimes a)$ for $a\in A$, $b\in k'$. If $P\in\mathscr{P}(A)$ and $u\in\mathrm{End}_A(P)$ then $T_{k'\otimes_k P}(1\otimes u)=1\otimes T_P(u)$.*

This is the Proposition (3.3) in case when B is the commutative k-algebra k'.

Remarks. 1. If k' is a flat k-module the formula in the corollary remains valid if P is only assumed to be of type (FP).

2. Suppose k' is a finitely generated projective k-module, so that $k'\otimes_k A$ is a finitely generated projective right A-module. Then for $\alpha\in k'$ and $a\in A$ we have $\mathrm{Tr}_{k'\otimes_k A/A}(\alpha\otimes T_A(a))=\mathrm{Tr}_{k'/k}(\alpha)\cdot T_A(a)$.

(3.5) *Automorphisms.* Let A be a k-algebra. An automorphism α of A induces an automorphism $M\mapsto M^{(\alpha)}$ of the category of A-modules, where $M^{(\alpha)}$ has the same underlying k-module as M, but $a\in A$ acts on $x\in M^{(\alpha)}$ by $x\mapsto x\,\alpha(a)$. Suppose M is an A-module of type (FP). Then $M^{(\alpha)}$ is likewise, and if $u\in\mathrm{End}_A(M)=\mathrm{End}_A(M^{(\alpha)})$ we have

$$T_{M^{(\alpha)}}(u)=\alpha^{-1}\,T_M(u) \tag{1}$$

where α acts on $T(A)$ by $\alpha T(a)=T(\alpha a)$. In particular, for $u=1_M$,

$$r_{M^{(\alpha)}}=\alpha^{-1}(r_M). \tag{2}$$

It suffices to verify this when $M\in\mathscr{P}(A)$. Let $x_i\in M, f_i\colon M\to A$ be a finite A-coordinate system of M. Then it is easily checked that x_i, $\alpha^{-1}\circ f_i$ is a coordinate system of $M^{(\alpha)}$. Thus $T_{M^{(\alpha)}}(u)=T(\sum_i\alpha^{-1}(f_i(u(x_i))))=\alpha^{-1}\,T(\sum_i f_i(u(x_i)))=\alpha^{-1}\,T_M(u)$. From (2) we conclude that:

$$\text{If } M\cong M^{(\alpha)} \text{ then } \alpha(r_M)=r_M. \tag{3}$$

(3.6) Suppose k is an *algebraically closed field.* A k-algebra $E=k[u]$ generated by a single element u algebraic over k is isomorphic to $k[X]/(f(X))$ where $f(X)$ is the minimal polynomial of u. If $f(X)=(X-u_1)^{n_1}\dots(X-u_h)^{n_h}$, where $u_1,\dots,u_h$ are the distinct roots of f, then $E=E_1\times\cdots\times E_h$ where $E_i\cong k[X]/(X-u_i)^{n_i}$. If e_i are the (orthogonal) idempotents corresponding to the E_i, then we have

$$1=e_1+\cdots+e_h$$

$$u=u_1\,e_1+\cdots+u_h\,e_h+v$$

with v nilpotent. If u is semi-simple, i.e., if all $n_i=1$, then $v=0$.

Let A be a k-algebra, M a right A-module of type (FP), and suppose u is an A-endomorphism of M algebraic over k. We can apply the above discussion then to $E=k[u]\subset\mathrm{End}_A(M)$. Then M is the direct sum of its submodules $M_i=e_iM$, and $(M,u-v)$ decomposes into the direct sum of the $(M_i,u_i1_{M_i})$. Putting $r_i=r_{M_i/A}$ it follows then from Proposition (3.1) that with v nilpotent, and even zero when the minimal polynomial of u has no multiple roots. The above formula in some sense reduces the description of $T_{M/A}(u)$ for u algebraic over k to the two special cases when $u=1_M$ and when u is nilpotent.

(3.7) *Examples.* Suppose k is a field and A is a central simple (finite dimensional) k-algebra. The reduced trace $\mathrm{Tr}d\colon A\to k$ (cf. [5], §12, No. 3) induces a k-linear map $\mathrm{Tr}d\colon T(A)\to k$, and this is an isomorphism. In view of Corollary (3.4) it suffices to check this when k is algebraically closed, when it results from (2.11), Example 2. In view of (2.11), Example 3, this determines $T(A)$ whenever A is a finite dimensional absolutely semi-simple k-algebra, say with center C. Namely, $T(A)\cong C$, the isomorphism being given by the reduced trace on each simple factor.

4. Relation to Characters

As in §3, A denotes a k-algebra.

(4.1) *Characters.* The dual $\mathrm{Hom}_k(T(A),k)$ of $T(A)$ will be denoted $CF(A,k)$, and called the module of *central functions* from A to k. We shall often identify an element $\chi\in CF(A,k)$ with the composite function $A\xrightarrow{T}T(A)\xrightarrow{\chi}k$, vanishing on $[A,A]$. If $r\in T(A)$ we put $\langle r,\chi\rangle=\chi(r)$.

We denote by $\mathscr{R}_k(A)$ the category of A-modules M which are finitely generated and projective as k-modules. Its Grothendieck group (with short exact sequences furnishing defining relations) is denoted $R_k(A)$.

A module $M\in\mathscr{R}_k(A)$ gives rise to a *character* $\chi_M\in CF(A,k)$, defined by

$$\chi_M(a)=T_{M/k}(a_M)$$

where $a_M\colon x\mapsto xa$ for $a\in A$, $x\in M$. In particular

$$r_{M/k}=\chi_M(1).$$

The map $M\mapsto\chi_M$ defines a group homomorphism

$$\chi\colon R_k(A)\to CF(A,k).$$

(4.2) **Proposition.** *Let $P\in\mathscr{P}(A)$ and $M\in\mathscr{R}_k(A)$. Then* $\mathrm{Hom}_A(P,M)$ *is a finitely generated projective k-module, whose class in $K_0(k)$ is denoted $\langle P,M\rangle$. Its rank $r_{\langle P,M\rangle}$ is $\langle r_P,\chi_M\rangle$ $(=\chi_M(r_P))$.*

Let e be an idempotent A-endomorphism of $L=A^n$ with image isomorphic to P. Then $h=\mathrm{Hom}_A(e,M)$ is an idempotent k-endomorphism of $\mathrm{Hom}_A(L,M)\cong M^n$ with image isomorphic to $\mathrm{Hom}_A(P,M)$, whence the first assertion of the proposition. If e is represented by the matrix $(e_{ij})\in M_n(A)$ and if we identify

$$\mathrm{End}_k(\mathrm{Hom}_A(L,M))\quad\text{with}\quad M_n(\mathrm{End}_k(M))$$

then h is represented by the matrix (h_{ij}) where $h_{ij}=e_{jiM}$; $x\mapsto xe_{ji}$. Thus $r_{\langle P,M\rangle}=T_{M^n/k}(h)=\sum_i T_{M/k}(h_{ii})=\sum_i\chi_M(e_{ii})=\chi_M(\sum_i e_{ii})=\chi_M(r_P)$.

Remark. The element $r_{\langle P,M\rangle}=\langle r_P,\chi_M\rangle$ of k lies in the subring generated by all idempotents in k. If 0 and 1 are the only such idempotents, then $\langle r_P,\chi_M\rangle$ is thus an integer (multiple of $1\in k$).

(4.3) *Remark.* The function $(P, M) \mapsto \langle P, M \rangle$ is additive in each variable, so it defines a pairing $K_0(A) \times R_k(A) \to K_0(k)$. The Proposition (3.7) asserts the commutativity of the diagram,

$$\begin{array}{ccc} K_0(A) \times R_k(A) & \xrightarrow{\langle\, ,\, \rangle} & K_0(k) \\ \downarrow{\scriptstyle r \times \chi} & & \downarrow{\scriptstyle r} \\ T(A) \times CF(A, k) & \xrightarrow[\langle\, ,\, \rangle]{} & k. \end{array}$$

(4.4) *Functoriality.* Let $\mathcal{S}: A \to B$ be a homomorphism of k-algebras. The dual of the k-linear map $\mathcal{S}_*: T(A) \to T(B)$, $\mathcal{S}_* T(a) = T(\mathcal{S} a)$, is the map $\mathcal{S}^*: CF(B, k) \to CF(A, k)$, $\mathcal{S}^* \chi = \chi \circ \mathcal{S}$:

$$\langle \mathcal{S}^* \chi, r \rangle_A = \langle \chi, \mathcal{S}_* r \rangle_B$$

for $\chi \in CF(B, k)$ and $r \in T(A)$. If $M \in \mathcal{R}_k(B)$, then the underlying A-module $\mathcal{S}^* M$ belongs to $\mathcal{R}_k(A)$, whence a homomorphism $\mathcal{S}^*: R_k(B) \to R_k(A)$, $[M] \mapsto [\mathcal{S}^* M]$. Note that $\chi_{\mathcal{S}^* M} = \chi_M \circ \mathcal{S} = \mathcal{S}^* \chi_M$. If $P \in \mathcal{P}(A)$ then $\operatorname{Hom}_A(P, \mathcal{S}^* M) = \operatorname{Hom}_B(P \otimes_A B, M)$, whence

$$\langle P, \mathcal{S}^* M \rangle_A = \langle \mathcal{S}_* P, M \rangle_B.$$

Suppose B is a finitely generated projective right A-module. Then (see (2.10)) we have the commutative diagram

$$\begin{array}{ccc} K_0(B) & \xrightarrow{\mathcal{S}^* = \text{“Res”}} & K_0(A) \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\ T(B) & \xrightarrow[\operatorname{Tr}_{B/A}]{} & T(A). \end{array}$$

The k-linear map $\operatorname{Tr}_{B/A}$ has a dual $\operatorname{Tr}^*_{B/A}$: $CF(A, k) \to CF(B, k)$ defined by $\chi \mapsto \chi \circ \operatorname{Tr}_{B/A}$. In particular, if $P \in \mathcal{P}(B)$ then

$$\langle r_{P/B}, \operatorname{Tr}^*_{B/A} \chi \rangle_B = \langle r_{P/A}, \chi \rangle_A.$$

(4.5) **Proposition.** *Suppose B is a finitely generated projective right A-module. For $M \in \mathcal{R}_k(A)$ let $\mathcal{S}_* M$ denote the right B-module* $\operatorname{Hom}_A(B, M)$, *where $(hb)(b') = h(bb')$ for $b, b' \in B$, $h \in \mathcal{S}_* M$. Then $\mathcal{S}_* M \in \mathcal{R}_k(B)$ and $\chi_{\mathcal{S}_* M} = \chi_M \circ \operatorname{Tr}_{B/A}$. If $P \in \mathcal{P}(B)$ then $\langle P, \mathcal{S}_* M \rangle_B = \langle \mathcal{S}^* P, M \rangle_A$.*

Let $b_i, f_i: B \to A$ be a finite coordinate system of the right A-module B, and x_j, g_j: $M \to k$ one of the k-module M. Define $h_{ji} \in \mathcal{S}_* M$ and $\mathcal{S}_{ji}$: $\mathcal{S}_* M \to k$ by $h_{ji}(b) = x_j f_i(b)$ and $\mathcal{S}_{ji}(h) = g_j(h(b_i))$. Then if $h \in \mathcal{S}_* M$ and $b \in B$, one has

$$\begin{aligned} \Big(\sum_{j,i} h_{ji} \mathcal{S}_{ji}(h)\Big)(b) &= \sum_{j,i} x_j \cdot f_i(b) \cdot g_j(h(b_i)) = \sum_i x_j \cdot g_j(h(b_i))) \cdot f_i(b) \\ &= \sum_i h(b_i) f_i(b) = h\Big(\sum_i b_i f_i(b)\Big) = h(b). \end{aligned}$$

Thus h_{ji}, $\mathscr{S}_{ji}$ is a finite coordinate system of the k-module $\mathscr{S}_* M$. If $b \in B$ then

$$\chi_{\mathscr{S}_* M}(b) = \sum_{j,i} \mathscr{S}_{ji}(h_{ji} b) = \sum_{j,i} g_j((h_{ji} b)(b_i)) = \sum_{j,i} g_j(x_j f_i(b b_i)) = \sum_j g_j(x_j \beta) = \chi_M(\beta),$$

where $\beta = \sum_i f_i(b b_i) \in A$ is such that $T_A(\beta) = \mathrm{Tr}_{B/A}(T_B(b))$. Thus $\chi_{\mathscr{S}_* M}(b) = \chi_M(\mathrm{Tr}_{B/A}(T_B(b)))$, as claimed. The last assertion results from the canonical k-isomorphisms $\mathrm{Hom}_B(P, \mathrm{Hom}_A(B, M)) \cong \mathrm{Hom}_A(P \otimes_B B, M) \cong \mathrm{Hom}_A(P, M)$.

(4.6) If $a, b \in A$ then L_a: $x \mapsto ax$ commutes with R_b: $x \mapsto xb$ $(x \in A)$, whence a k-algebra homomorphism ε: $A \otimes_k A^0 \to \mathrm{End}_k(A)$, $a \otimes b \mapsto L_a \circ R_b$. *Suppose A is a finitely generated projective k-module.* Then $T_{A/k} \circ \varepsilon$: $A \otimes A^0 \to k$ induces a k-linear map $T(A) \otimes_k T(A^0) \to k$, $T(a) \otimes T(b) \mapsto T_{A/k}(L_a \circ R_b)$. Identifying $T(A^0)$ with $T(A)$, the latter map defines a k-homomorphism

$$T(A) \to CF(A, k), \qquad r \mapsto r^t$$

defined by

$$T(a)^t(b) = T_{A/k}(L_a \circ R_b).$$

(4.7) **Proposition.** *Suppose A is a finitely generated projective k-module. Then the same is true of every $P \in \mathscr{P}(A)$, and $\chi_P = r_P^t$. Explicitly, if $r_P = T(a)$ then $\chi_P(b) = T_{A/k}(L_a \circ R_b) = T_{A/k}(x \mapsto a x b)$.*

Let a_i, f_i: $A \to k$ be a finite coordinate system of the k-module A, and let x_j, g_j: $P \to A$ be one of the A-module P. If $x \in P$ then $\sum_{i,j} x_j a_i f_i(g_j(x)) = \sum_j x_j g_j(x) = x$, so $x_j a_i, f_i g_j$: $P \to k$ is a finite coordinate system of the k-module P. If $b \in A$ then $\chi_P(b) = \sum_{i,j} f_i(g_j(x_j a_i b)) = \sum_i f_i(a a_i b) = T_{A/k}(L_a \circ R_b)$, where $a = \sum_j g_j(x_j)$ is such that $T(a) = r_P$. This proves (4.7).

Remark. Proposition (4.7) asserts the commutativity of the diagram

$$\begin{array}{ccc} K_0(A) & \longrightarrow & R_k(A) \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle \chi} \\ T(A) & \longrightarrow & CF(A, k) \end{array} \qquad \begin{array}{c} [P] \mapsto [P] \\ \\ r \mapsto r^t \end{array}$$

the upper arrow being what is in some cases called the *Cartan homomorphism.* In particular, if $P \in \mathscr{P}(A)$ then r_P determines χ_P.

5. Group Algebras

(5.1) Let G be a group, k a commutative ring, and A the group algebra kG. Since G is a k-basis for A it follows that the k-module $[A, A]$ is generated by the elements

$$[s, t] = st - ts = sus^{-1} - u = [su, s^{-1}]$$

where s, t, $u \in G$ and $u = ts$. Thus, for s, $t \in G$, one has $T(s) = T(t)$ *in* $T(kG)$ *if and only if s and t are conjugate in G.* We shall thus identify $T(s)$ (denoted also $T_G(s)$)

with the G-conjugacy class of s, and identify $T(kG)$ with the free k-module $k^{(T(G))}$ having the set $T(G)$ of conjugacy classes of G as a basis. Each $r \in T(kG)$ thus has a unique expression

$$r = \sum_{\tau \in T(G)} r(\tau) \cdot \tau \in T(kG)$$

where $\tau \mapsto r(\tau)$ is a function $T(G) \to k$ with finite support

$$\operatorname{supp}(r) = \{\tau \in T(G) \mid r(\tau) \neq 0\}.$$

If $s \in G$ we shall often write $r(s)$ for $r(T(s))$, and so identify r with a central function $G \to k$. The *involution* $a \mapsto \bar{a}$ of kG, defined by $\bar{s} = s^{-1}$ for $s \in G$, induces an involution $r \mapsto \bar{r}$ of $T(kG)$ so that $\bar{r}(s) = r(s^{-1})$, and $\overline{T(a)} = T(\bar{a})$.

(5.2) The dual $CF(kG, k)$ of $T(kG)$ can similarly be identified with the k-module $k^{T(G)}$ of *all* functions $T(G) \to k$, i.e., with the k-module $CF(G, k)$ of all central functions $G \to k$. Note that *$CF(G, k)$ is a commutative k-algebra* with multiplication

$$(\chi_1 \cdot \chi_2)(s) = \chi_1(s)\, \chi_2(s).$$

It further has an *involution* $\chi \mapsto \bar{\chi}$,

$$\bar{\chi}(s) = \chi(s^{-1}).$$

The remarks above furnish an identification of $T(kG)$ with the involution invariant ideal in $CF(G, k)$ of functions which vanish on all but finitely many conjugacy classes. In particular, this gives $T(kG)$ *the structure of a $CF(G, k)$-module:* $(\chi \cdot r)(s) = \chi(s)\, r(s)$ for $\chi \in CF(G, k)$, $r \in T(kG)$. Further $\overline{\chi \cdot r} = \bar{\chi} \cdot \bar{r}$.

(5.3) Let P and M be right kG-modules. Then $P^* = \operatorname{Hom}_{kG}(P, kG)$ and $M^v = \operatorname{Hom}_k(M, k)$ are right kG-modules with $(fs)(p) = s^{-1} f(p)$ and $(gs)(m) = g(ms^{-1})$ for $f \in P^*$, $p \in P$, $g \in M^v$, $m \in M$, and $s \in G$. Moreover $M \otimes_k P$ and $\operatorname{Hom}_k(M, P)$ are right kG-modules with $(m \otimes p)s = ms \otimes ps$ and $(fs)(m) = f(ms^{-1})s$ for $m \in M$, $p \in P$, f: $M \to P$, and $s \in G$. The canonical k-linear map $M^v \otimes_k P \to \operatorname{Hom}_k(M, P)$, $g \otimes p \mapsto (m \mapsto p\, g(m))$ is kG-linear. It is an isomorphism if $M \in \mathscr{R}_k(kG)$.

(5.4) A group homomorphism $\pi: G \to G'$ induces a k-linear map $\pi_*: T(kG) \to T(kG')$ such that $\pi_* T_G(s) = T_{G'}(\pi s)$ for $s \in G$. If $P \in \mathscr{P}(kG)$ then $\pi_* P = P \otimes_{kG} kG' \in \mathscr{P}(kG')$, and if $r_P = \sum_{\tau \in T(G)} r_P(\tau)\tau$ then $r_{\pi_* P} = \pi_* r_P = \sum_{\tau' \in T(G')} \left(\sum_{\pi_*(\tau) = \tau'} r_P(\tau)\right)\tau'$. When G' is the trivial group, we find that *the rank $r_{P \otimes_{kG} k/k}$ of $P \otimes_{kG} k \in \mathscr{P}(k)$ is* $\sum_{\tau \in T(G)} r_P(\tau)$.

(5.5) **Proposition.** *Suppose $P \in \mathscr{P}(kG)$ and $M \in \mathscr{R}_k(kG)$.*

(a) *$P^* \in \mathscr{P}(kG)$ and $r_{P^*} = \bar{r}_P$.*

(b) *$M^v \in \mathscr{R}_k(kG)$ and $\chi_{M^v} = \bar{\chi}_M$.*

(c) *The modules $M \otimes_k P$ and* $\operatorname{Hom}_k(M, P) \cong M^v \otimes_k P$ *belong to $\mathscr{P}(kG)$, and $r_{M \otimes_k P} = \bar{\chi}_M \cdot r_P$, i.e., $r_{M \otimes_k P}(s) = \chi_M(s^{-1})\, r_P(s)$ for $s \in G$.*

(a) Let x_i, f_i: $P \to kG$ be a finite kG-coordinate system of P. Define $\bar{x}_i$: $P^* \to kG$ by $\bar{x}_i(f) = \overline{f(x_i)}$. One checks easily that f_i, $\bar{x}_i$ is a kG-coordinate system of P^*. Thus $r_{P^*} = T(\sum \bar{x}_i(f_i)) = T(\sum \overline{f_i(x_i)}) = \overline{T(\sum f_i(x_i))} = \bar{r}_P$.

(b) Let $y_j, g_j\colon M \to k$ be a finite k-coordinate system of M. Define $y'_j\colon M^v \to k$ $y'_j(g) = g(y_j)$. Then g_j, y'_j is a finite k-coordinate system of M^v. If $s \in G$ we thus have $\chi_{M^v}(s) = \sum y'_j(g_j s) = \sum (g_j s)(y_j) = \sum g_j(y_j s^{-1}) = \chi_M(s^{-1})$.

In view of (b) and (5.3), part (c) will follow once we show that $M \otimes_k P \in \mathscr{P}(kG)$ and $r_{M \otimes_k P} = \bar{\chi}_M \cdot r_P$. We shall use the following well-known lemma.

(5.6) **Lemma.** *Let $H \to G$ be a group homomorphism, L a right kH-module, and M a right kG-module. Then*

$$\alpha\colon M \otimes_k (L \otimes_{kH} kG) \to (M \otimes_k L) \otimes_{kH} kG$$
$$y \otimes (x \otimes s) \mapsto (y s^{-1} \otimes x) \otimes s$$

is an isomorphism of kG-modules.

[On the right, $M \otimes_k L$ denotes the kH-module $\mathrm{Res}(M) \otimes_k L$, where $\mathrm{Res}(M)$ denotes M viewed as a kH-module.] This lemma is easily checked; see, for example, [1], Chapter XI, Proposition (1.5).

Proof of (c) *of* (5.5). We can identify P with the image of an idempotent endomorphism e of a free module $F \in \mathscr{P}(kG)$, and we can identify F with $L \otimes_k kG$ for some free module $L \in \mathscr{P}(k)$. Let $x_i \in L, f_i\colon L \to k$ be a k-coordinate system of L. Then $x_i \otimes 1, f_i \otimes 1_{kG}$ is a kG-coordinate system of $L \otimes_k kG$. Write $e(x \otimes 1) = \sum_{s \in G} e_s(x) \otimes s$ for $x \in L$. Then $r_P = T(\sum_i (f_i \otimes 1)(e(x_i \otimes 1))) = T(\sum_{i,s} f_i(e_s(s_i))\, s) = T(\sum_s a_s s)$, where $a_s = \sum_i f_i(e_s(x_i)) = T_{L/k}(e_s)$.

We can identify $M \otimes_k P$ with the image of $1_M \otimes_k e \in \mathrm{End}_{kG}(M \otimes_k (L \otimes_k kG))$. Under the natural kG-isomorphism

$$\alpha\colon M \otimes_k (L \otimes_k kG) \to (M \otimes_k L) \otimes_k kG$$

of Lemma (5.5), $e_1 = 1_M \otimes e$ is transformed to the endomorphism e' such that, for $y \in M$, $x \in L$, one has

$$\begin{aligned} e'((y \otimes x) \otimes 1) &= \alpha(e_1(\alpha^{-1}((y \otimes x) \otimes 1))) = \alpha(e_1(y \otimes (x \otimes 1)) = \alpha(y \otimes e(x \otimes 1)) \\ &= \sum_s \alpha(y \otimes (e_s(x) \otimes s)) = \sum_s (y s^{-1} \otimes e_s(x)) \otimes s. \end{aligned}$$

Thus $e'_s(y \otimes x) = y s^{-1} \otimes e_s(x)$. Since $M \otimes_k L \in \mathscr{P}(k)$ and $T_{M \otimes_k L}(e'_s) = \chi_M(s^{-1})\, T_{L/k}(e_s)$ we conclude that $(M \otimes_k L) \otimes_k kG \in \mathscr{P}(kG)$, and that $r_{M \otimes_k P} = T(\sum_s T_{(M \otimes_k L)/k}(e'_s)\, s)$, and therefore, for $\tau \in T(G)$, $r_{M \otimes_k P}(\tau) = \sum_{s \in \tau} \chi_M(s^{-1})\, T_{L/k}(e_s) = \chi_M(s^{-1})\, r_P(s)$. This completes the proof of (c).

(5.7) *Remark.* If $P \in \mathscr{P}(kG)$ there is a *finitely generated* subgroup H of G such that $P \cong L \otimes_{kH} kG$ for some $L \in \mathscr{P}(kH)$. Then for any $M \in \mathscr{R}_k(kG)$, $M \otimes_k P$ is isomorphic to $(\mathrm{Res}(M) \otimes_k L) \otimes_{kH} kG$, where $\mathrm{Res}(M)$ is M viewed as kH-module (Lemma (5.5)). Thus $r_{M \otimes_k P}$ is the image of $r_{\mathrm{Res}(M) \otimes_k L}$ under the map $T(kH) \to T(kG)$. This observation permits one to reduce certain questions raised here to the case of finitely generated groups.

(5.8) **Proposition** (Hattori [13]). *Suppose G is a finite group and $P \in \mathscr{P}(kG)$. Then*

$$\chi_P(s) = |Z_G(s)| \cdot r_P(s^{-1})$$

for $s \in G$. In particular $r_{P/k}(=\chi_P(1)) = |G| \cdot r_P(1)$.

Here $Z_G(s)$ denotes the centralizer of s in G. Let $a = \sum_s a_s s \in kG$ be such that $r_P = T(a)$. According to Proposition (4.7),

$$\chi_P(s) = T_{kG/k}(x \mapsto a x s) = \sum_t a_t T_{kG/k}(x \mapsto t x s).$$

Now $x \mapsto t x s$ permutes the k-basis G of kG, and $x = t x s$ for $x \in G$ if and only if $t = x s^{-1} x^{-1}$. The number of such x's is 0 if $t \notin T(s^{-1})$, and $|Z_G(s)|$ if $t \in T(s^{-1})$. Thus $\chi_P(s) = \sum_{t \in T(s^{-1})} a_t \cdot |Z_G(s)| = r_P(s^{-1}) \cdot |Z_G(s)|$.

Remark. Let $M = kG$ as k-module, on which G acts by $x \cdot s = s^{-1} x s$ for $x \in M$, $s \in G$. Then $M \in \mathscr{R}_k(kG)$ and $\chi_M(s) = |Z_G(s)|$. Therefore, for $P \in \mathscr{P}(kG)$, it follows from Proposition (5.8) and Proposition (5.5)(c) that $\chi_P = \chi_M \cdot \bar{r}_P = \bar{r}_{M \otimes_k P}$.

(5.9) **Corollary.** *Suppose the order $|G|$ of G is invertible in k. Let $P \in \mathscr{P}(kG)$ and put $a_P = \frac{1}{|G|} \sum_{s \in G} \chi_P(s^{-1}) s$. Then a_P belongs to the center of kG and $r_P = T(a_P)$.*

Clearly a_P is central. Fix $\tau \in T(G)$ and $s \in \tau$. The coefficient of τ in $T(a_P)$ is $\frac{1}{|G|} \sum_{t \in \tau} \chi_P(t^{-1}) = \frac{\chi(s^{-1}) \cdot |\tau|}{|G|} = \frac{\chi(s^{-1})}{|Z_G(s)|}$. The latter, by Proposition (5.7), is $r_P(\tau)$.

Remark. If k is a field and P is an absolutely irreducible kG-module, then $\chi_P(1) \cdot a_P = r_{P/k} \cdot a_P$ is the central idempotent in kG corresponding to P (see, for example, Curtis-Reiner [9], Ch. V, Th. (33.8)).

(5.10) **Corollary.** *Suppose k is an integral domain in which no prime divisor of $|G|$ is invertible. Then 0 and 1 are the only idempotents of kG.*

Let p be the characteristic of k. If $p > 0$ then G must be a p-group and then, if K is the field of fractions of k, KG is a local ring (its augmentation ideal is nilpotent), whence the result.

Suppose $p = 0$, and let e be an idempotent $\neq 0$ in kG. Put $P = ekG$; its rank $\chi_P(1)$ over k is $|G| \cdot r_P(1)$, by Proposition (5.8). Thus $\chi_P(1)/|G|$ is a rational number in k. Since no prime divisor of $|G|$ is invertible in k it follows that $\chi_P(1)/|G| \in \mathbb{Z}$. But if the direct summand $P \neq 0$ of kG has k-rank a multiple of $|G|$ we must have $P = kG$, i.e., $e = 1$.

6. Subgroups of Finite Index

As above, k denotes a commutative ring and G denotes a group.

(6.1) Let H be a subgroup of *finite index* in G. Then $kG = \oplus_s skH$ is a free right kH-module with finite basis consisting of representatives s of the cosets G/H. Thus we have (see (2.10)) a k-linear map $\mathrm{Tr} = \mathrm{Tr}_{kG/kH}$: $T(kG) \to T(kH)$, defined by $\operatorname{Tr}(T_G(a)) = T_{kG/kH}(L_a)$, where, for $a \in kG$, $L_a: b \mapsto ab$ $(b \in kG)$. If $t \in G$ then L_t permutes the direct summands skH of kG, and it stabilizes skH if and only if $tsH = sH$, i.e., $s^{-1}ts \in H$. In the latter case $tsb = s(s^{-1}ts)b$ for $b \in kH$. It follows that $T_{kG/kH}(L_t) = \sum_s T_H(s^{-1}ts)$ where s varies over representatives of G/H for which $s^{-1}ts \in H$. This proves part (a) of the next proposition.

(6.2) **Proposition.** *Let H be a subgroup of finite index in G, and put*

$$\operatorname{Tr} = \operatorname{Tr}_{kG/kH}: T(kG) \to T(kH).$$

(a) *If $t \in G$ then $\operatorname{Tr}(T_G(t)) = \sum_s T_H(s^{-1}ts)$, where s varies over representatives of the cosets G/H for which $s^{-1}ts \in H$.*

(b) *If $\tau \in T(G)$ then $\operatorname{Tr}(\tau) = \sum_\sigma z_\sigma \cdot \sigma$ where σ varies over elements of $T(H)$ such that $\sigma \subset \tau$, and where $z_\sigma = [Z_G(s) : Z_H(s)]$ for any $s \in \sigma$. In particular $\operatorname{Tr}(\tau) = 0$ if $\tau \cap H = \phi$.*

(c) *If $r \in T(kG)$ and $s \in H$ then*

$$\operatorname{Tr}(r)(s) = [Z_G(s) : Z_H(s)] \cdot r(s).$$

In particular $\operatorname{Tr}(r)(s) = [G : H] \cdot r(s)$ if $s \in Z(G)$, for example if $s = 1$.

It follows from (a) that, for $\tau = T_G(t) \in T(G)$, $\operatorname{Tr}(\tau) = \sum_{\substack{\sigma \in T(H) \\ \sigma \subset \tau}} z_\sigma \sigma$, where z_σ denotes the number of s as in (a) for which $s^{-1}ts \in \sigma$. Say $s_0^{-1}ts_0 \in \sigma$; then $s_1^{-1}ts_1 \in \sigma$ if and only if $s_1 h s_0^{-1} \in Z_G(t)$ for some $h \in H$, i.e., if and only if $s_1 \in Z_G(t) s_0 H$. Therefore z_σ is the number of H cosets in the double coset $Z_G(t) s_0 H$; this is the index in $Z_G(t)$ of $Z_G(t) \cap s_0 H s_0^{-1} = Z_{s_0 H s_0^{-1}}(t)$. Thus

$$z_\sigma = [Z_G(t) : Z_{s_0 H s_0^{-1}}(t)] = [Z_G(s_0^{-1} t s_0) : Z_H(s_0^{-1} t s_0)],$$

and, since $s_0^{-1} t s_0 \in \sigma$, this is the description of z_σ claimed in (b). Assertion (c) is immediate from (b), so Proposition (6.2) is proved.

(6.3) **Corollary.** *Let M be a kG-module of type (FP). Then M is a kH-module of type (FP) and, for $s \in H$, one has*

$$r_{M/kH}(s) = [Z_G(s) : Z_H(s)] \cdot r_{M/kG}(s).$$

In particular, $r_{M/kH}(1) = [G : H] \cdot r_{M/kG}(1)$.

In fact a finite $\mathscr{P}(kG)$-resolution of M is also a finite $\mathscr{P}(kH)$-resolution of M, and $r_{M/kH} = \operatorname{Tr}(r_{M/kG})$. Thus the corollary results from part (c) of Proposition (6.2).

(6.4) **Corollary.** *Suppose that, for all $s \in H$, $[Z_G(s) : Z_H(s)]$ is not a zero-divisor in k.*

(a) *The kernel of $\operatorname{Tr}: T(kG) \to T(kH)$ is the direct sum of those $k\tau$ $(\tau \in T(G))$ for which $\tau \cap H = \phi$.*

(b) *If $r \in T(kG)$ and $\sigma \in T(H)$ then $\sigma \in \operatorname{supp}(\operatorname{Tr}(r))$ if and only if $\sigma \subset \tau$ for some $\tau \in \operatorname{supp}(r)$.*

(6.5) **Proposition.** *Suppose H is a normal subgroup of finite index in G. Let $T_H(G)=\{\tau\in T(G)\,|\,\tau\subset H\}$. For each $\tau\in T_H(G)$ put $\tau_H=\sum_{\substack{\sigma\in T(H)\\ \sigma\subset\tau}}\sigma$, and put $z_\tau=[Z_G(t):Z_H(t)]$ for any $t\in\tau$ (this being independent of the choice of t).*

(a) *The action of G on H by conjugation induces an action on $T(kH)$. The module of invariants $T(kH)^G$ is k-free with basis the family of τ_H $(\tau\in T_H(G))$.*

(b) *If $\tau\in T(G)$ then $\operatorname{Tr}(\tau)=0$ if $\tau\notin T_H(G)$ and $\operatorname{Tr}(\tau)=z_\tau\cdot\tau_H$ if $\tau\in T_H(G)$.*

(c) *If $r\in T(kH)^G$ has image r' under $T(kH)\to T(kG)$ then $\operatorname{Tr}(r')=[G:H]\cdot r$.*

The action of G on $T(kH)$ permutes the basis $T(H)$, and the elements τ_H $(\tau\in T_H(G))$ are just the sums of basis elements in the various orbits, whence (a). Part (b) follows from (b) of Proposition (6.2). It suffices to prove (c) for $r=\tau_H$ for some $\tau\in T_H(G)$, in which case $r'=n\tau$, and $\operatorname{Tr}(r')=n\cdot z_\tau\tau_H=n\cdot z_\tau r$, where n denotes the number of $\sigma\in T(H)$ such that $\sigma\subset\tau$. If $\tau=T_G(t)$ then $n=[G:Z]$ where $Z=\{s\in G\,|\,sts^{-1}\in T_H(t)\}=H\cdot Z_G(t)$. Thus $[Z:H]=[Z_G(t):Z_H(t)]$, so

$$nz_\tau=[G:Z]\cdot[Z_G(t):Z_H(t)]=[G:Z][Z:H]=[G:H],$$

whence (c).

(6.6) **Theorem.** *Let k be an integral domain of characteristic 0. Suppose that if $s\in G$ has finite order which is invertible in k then $s=1$. Let $P\in\mathscr{P}(kG)$ and let n denote the rank $r_{P\otimes_{kG}k/k}$ of the k-module $P\otimes_{kG}k$. Assume further that $G/Z(G)$ is finite. Then $r_P=r_{(kG)^n}$; i.e., $r_P(1)=n$ and $r_P(s)=0$ for $s\neq 1$.*

Since $n=\sum_{\tau\in T(G)} r_P(\tau)$ (see (5.4)) it suffices to show that $r_P(s)=0$ for $s\neq 1$. Suppose $r_P(s)\neq 0$. The subgroup H of G generated by s and $Z(G)$ is abelian and has finite index in G. Moreover, Corollary (6.3) shows that $r_{P/kH}(s)=[Z_G(s):H]\cdot r_{P/kG}(s)$. Thus it suffices to prove the theorem for the kH-module P; in other words we have reduced to the case of an *abelian* group.

When G is abelian the theorem amounts to the well-known assertion that the commutative ring kG has no idempotents except 0 and 1. If G is torsion, this follows from Corollary (5.10). In the general case it therefore follows from the next lemma.

(6.7) **Lemma.** *Let G be an abelian group with torsion subgroup H. Let k be any commutative ring. Then all idempotents in kG belong to kH.*

It suffices to treat the case when G is finitely generated, hence the direct product of H with a free abelian group F. Then $k[G]=k[H][F]$, so, replacing k by kH and G by F, we may assume G is free. By induction on the rank of G we finally reduce similarly to the case when G is cyclic, so kG is a ring of Laurent polynomials $k[t,t^{-1}]$. Let e be idempotent in $k[t,t^{-1}]$. Then $e\in k$ if k is an integral domain. In general, therefore e lands in k modulo each prime ideal of k, hence modulo the nil radical N of k. Therefore, there is an idempotent $e_0\in k$ congruent modulo $N\cdot k[t,t^{-1}]$ to e. But then $e-ee_0$ and e_0-ee_0 are nilpotent idempotents, so $e=e_0\in k$.

(6.8) **Corollary** (Swan). *With k, G, P as in Theorem* (6.6), *let K be the field of fractions of k, and assume G is finite. Then $K \otimes_k P$ is a free KG-module.*

It suffices to show that χ_P is the character of a free module. But r_P determines χ_P (Prop. (5.8)) and $r_P = r_{(kG)^n}$ (Theorem (6.6)).

Remark. Corollary (6.8) can be strengthened to say that $P_{\mathfrak{p}}$ is a free $k_{\mathfrak{p}}G$-module for all primes $\mathfrak{p}$ of k. This can be deduced from Corollary (6.8) using the non-singularity of the Cartan matrices of the modular group algebras $(k_{\mathfrak{p}}|\mathfrak{p}k_{\mathfrak{p}})[G]$ (see [1], Chap. X, Cor. (1.2)). Another proof of this is given by Hattori [13], by showing that $r_{P_{\mathfrak{p}}/k_{\mathfrak{p}}G}$ determines the isomorphism class of $P_{\mathfrak{p}}$.

(6.9) *Remark.* Suppose k is an integral domain of characteristic $p>0$. Suppose G is a group such that $G/Z(G)$ has finite order prime to p and all torsion in G is p-torsion (necessarily therefore in $Z(G)$). Then if $P \in \mathscr{P}(kG)$ one has $r_P(s)=0$ for $s \neq 1$. The above proof of Theorem (6.6) applies without essential change.

(6.10) **Corollary.** *Let k be an integral domain of characteristic zero in which no rational prime is invertible. Let G be a group and let $P \in \mathscr{P}(kG)$. If G is residually finite then*

$$r_P(1) = \sum_{\tau \in T(G)} r_P(\tau) \qquad (= r_{P \otimes_{kG} k/k}).$$

Let $\pi: G \to G'$ be the projection to a finite quotient G' of G in which all $\tau \in T(G)$ for which $r_P(\tau) \neq 0$ and $\tau \neq 1$ remain distinct from 1 in $T(G')$. Put $P' = P \otimes_{kG} kG' \in \mathscr{P}(kG')$. By Theorem (6.6) $r_{P'}(s')=0$ for all $s' \neq 1$ in G'. Therefore

$$0 = \sum_{\substack{\tau' \in T(G') \\ \tau' \neq 1}} r_{P'}(\tau') = \sum_{\substack{\tau' \in T(G') \\ \tau' \neq 1}} \sum_{\substack{\tau \in T(G) \\ \pi(\tau) = \tau'}} r_P(\tau) = \sum_{\substack{\tau \in T(G) \\ \tau \neq 1}} r_P(\tau),$$

by the choice of G'. This proves the corollary.

(6.11) *Remark.* It is tempting to use the method of proof of Corollary (6.10) to prove even that $r_P(s)=0$ for $s \neq 1$ in G. To do this one needs a stronger property than residual finiteness, viz., that distinct conjugacy classes in G can be distinguished in finite quotients of G. In fact even the following condition suffices:

(*) The characters χ_M of modules $M \in \mathscr{R}_k(G)$ separate the conjugacy classes in G.

At the (innocent) cost of adjoining some roots of unity to k, condition (*) holds whenever the previous condition holds. To show that $r_P(\tau)=0$ for $\tau \neq 1$ when (*) holds, choose a k-linear combination $\chi = \sum a_M \chi_M$ of characters of modules $M \in \mathscr{R}_k(kG)$, such that $\chi(\tau) \neq 0$ but $\chi(\tau')=0$ for all $\tau' \neq \tau$ in $\operatorname{supp}(r_P)$. Then $\chi \cdot r_P = \sum a_M \chi_M \cdot r_P = \sum a_M \chi_{M^v \otimes_k P}$ and $M^v \otimes_k P \in \mathscr{P}(kG)$ (Prop. (5.5)(c)). Therefore Corollary (6.10) implies that $0 = \chi(1) \cdot r_P(1) = \sum_{\tau' \in T(G)} \chi(\tau') r_P(\tau') = \chi(\tau) r_P(\tau)$, whence $r_P(\tau)=0$.

(6.12) *Example.* Condition (*) in (6.11) may well fail even for finitely generated linear groups, as we shall now indicate. Let $f(X) \in \mathbb{Z}[X]$ be an irreducible monic polynomial of degree $n \geq 2$ with constant term 1 such that the field $\mathbb{Q}(\alpha)$ generated by a root α of f has class number >1. Then in $A = \mathbb{Z}[\alpha]$ there exist non-isomorphic invertible ideals L_1 and L_2. Relative to $\mathbb{Z}$-bases of L_1 and L_2 the action of α is represented by matrices s_1 and s_2, respectively, in $SL_n(\mathbb{Z})$. Since L_1 and L_2 are non-isomorphic, s_1 and s_2 are not conjugate in $GL_n(\mathbb{Z})$. However, if k is a commutative ring such that $k\otimes_{\mathbb{Z}} L_1$ and $k\otimes_{\mathbb{Z}} L_2$ are isomorphic $(k\otimes_{\mathbb{Z}} A)$-modules then the images s_{1k} and s_{2k} of s_1 and s_2 in $SL_n(k)$ are conjugate in $GL_n(k)$. Since L_1 and L_2 are both principal after any $\mathbb{Z}$-localisation, it follows that s_{1k} and s_{2k} are conjugate in $GL_n(k)$, say $s_{2k}=u_k s_{1k} u_k^{-1}$, for $k=\mathbb{Q}$ or $k=\mathbb{Z}/q\mathbb{Z}$ with q any integer $\neq 0$. Let $\Gamma=SL_{2n}(\mathbb{Z})$ and $S_i=\begin{pmatrix} s_i & 0\\ 0 & I\end{pmatrix}\in\Gamma$, $i=1,2$. Then with k and u_k as above, we have $S_{2k}=U_k S_{1k} U_k^{-1}$ where $U_k=\begin{pmatrix} u_k & 0\\ 0 & u_k^{-1}\end{pmatrix}\in SL_{2n}(k)$. Moreover S_1 and S_2 are not conjugate in Γ, since they define modules L'_1 and L'_2 over $\mathbb{Z}[X]/(f(X)\cdot(X-1))=\mathbb{Z}[\alpha']$, α' the residue class of X, which are non-isomorphic. In fact, since $L'_i\cong L_i\oplus\mathbb{Z}^n$ with α' acting like 1 on $\mathbb{Z}^n$ and like α on L_i, an isomorphism $L'_1\to L'_2$ would induce an isomorphism $L_1\to L_2$, for $L_i=\{x\in L'_i\mid f(\alpha')x=0\}$, $i=1,2$.

Claim. For all finite dimensional $(\mathbb{C}\Gamma)$*-modules* M, $\chi_M(S_1)=\chi_M(S_2)$.

It suffices to show this when M is irreducible. In that case it follows from [2] and [23], pages 501–504, that $M=V\otimes_{\mathbb{C}} W$ where the action of Γ on V factors through a finite quotient $SL_{2n}(\mathbb{Z}/q\mathbb{Z})$ of Γ (q an integer $\neq 0$) and where W is obtained by restriction to Γ of a representation of $SL_n(\mathbb{Q})$ (in fact of the algebraic group SL_n). Since, as observed above, the images S_{1k} and S_{2k} are conjugate in $SL_{2n}(k)$ for $k=\mathbb{Z}/q\mathbb{Z}$ and $k=\mathbb{Q}$, the claim follows.

(6.13) **Corollary.** (cf. K. Brown [6], Th. 3). *Let k be an integral domain of characteristic zero in which no rational prime is invertible. Let G be a group, H a subgroup of finite index, and $P\in\mathscr{P}(kG)$. Then*

$$r_{P\otimes_{kH}k/k}=[G:H]\cdot r_{P\otimes_{kG}k/k}.$$

Let N be a normal subgroup of finite index in G contained in H. It clearly suffices to verify the corollary with G, H, P replaced by G/N, H/N, and $P\otimes_{kN}k\in\mathscr{P}(k[G/N])$. Thus we may assume G is finite. In this case it follows from Corollary (6.10) that $r_{P\otimes_{kG}k/k}=r_{P/kG}(1)$ and $r_{P\otimes_{kH}k/k}=r_{P/kH}(1)$, so the present corollary follows from Proposition (6.2)(c).

(6.14) Let H be a *normal* subgroup of G, with quotient $G'=G/H$. The natural projection $\pi\colon kG\to kG'$ induces $\pi_*\colon T(kG)\to T(kG')$, $\pi_* T_G(s)=T_{G'}(s')$, where we write s' for $\pi(s)$. In general,

$$\pi_*\Big(\sum_{\tau\in T(G)} r(\tau)\tau\Big)=\sum_{\tau'\in T(G')}\Big(\sum_{\pi_*\tau=\tau'} r(\tau)\Big)\tau'.$$

Let P,M be (right) kG-modules. Then $\operatorname{Hom}_k(P,M)$ is a kG-module $((ft)(x)=f(xt^{-1})t)$ whose H-invariants, $\operatorname{Hom}_{kH}(P,M)$, form a kG'-module.

(6.15) **Proposition.** *Assume the quotient group $G'=G/H$ is finite. Let $P\in\mathscr{P}(kG)$ and $M\in\mathscr{R}_k(kG)$, and put $Q=\operatorname{Hom}_{kH}(P,M)$. Then $Q\in\mathscr{P}(kG')$ and*

$$r_{Q/kG'}=\pi_*(\overline{\chi_M\cdot r_P})=\pi_*(r_{M\otimes_k P^*}).$$

Explicitly, if $\tau'\in T(G')$ then $r_Q(\tau')=\sum_{\pi_*\tau=\tau'^{-1}}\chi_M(\tau)\cdot r_P(\tau)$.

(6.16) **Corollary.** *If* $r_P(s)=0$ *for all* $s \neq 1$ *in* G *then* $r_Q(s')=0$ *for all* $s' \neq 1$ *in* G'.

We shall prove a more general form of Proposition (6.15) which applies *without the assumption that* G' *is finite*. To do so, we replace $\mathrm{Hom}_{kH}(P, M)$ by its submodule $Q = \mathrm{Hom}'_{kH}(P, M)$, defined as follows: Choose a finite set X of generators of the kG-module P; then $f \in Q$ if and only if f vanishes on Xs for all but finitely many coset representatives s of G/H. Note that this is indeed a sub-kG'-module, independent of the choice of X, and that $\mathrm{Hom}'_{kH}(\cdot, M)$ is a functor on the category of finitely generated kG-modules. We shall prove that $Q \in \mathscr{P}(kG')$ and $r_Q = \pi_* (\overline{\chi_M \cdot r_P})$.

Identify P with the image of an idempotent endomorphism e of a free finitely generated kG-module, which we may represent as $L \otimes_{kH} kG$ for some $L \in \mathscr{P}(kH)$. Then $L \otimes_k kG'$ is a right kH-kG'-bimodule, so $\mathrm{Hom}_{kH}(L \otimes_k kG', M)$ is a right kG'-module: $(f' t')(x \otimes s') = f(x \otimes s' t'^{-1})$ for $x \in L$, $s', t' \in G'$. One checks easily that

$$\mathrm{Hom}_{kH}(L \otimes_{kH} kG, M) \to \mathrm{Hom}_{kH}(L \otimes_k kG', M) \tag{1}$$
$$f \mapsto f' : x \otimes s' \mapsto f(x \otimes s) s^{-1}$$

for $x \in L$ and $s \in G$, is a well-defined isomorphism of kG'-modules. (We recover f from f' by, $f(x \otimes s) = f'(x \otimes s') s$.) For $f \in \mathrm{Hom}_{kH}(L \otimes_{kH} kG, M)$, $x \in L$, $s \in G$, put $f_s(x) = f(x \otimes s)$ and $f'_{s'}(x) = f'(x \otimes s')$. Then $f_s(x) = f'_{s'}(x) s$. Note that

$$f \in \mathrm{Hom}'_{kH}(L \otimes_{kH} kG, M)$$

if and only if $f_s = 0$ for all but finitely many s mod H, i.e., if and only if $f'_{s'} = 0$ for all but finitely many $s' \in G'$. Thus the isomorphism (1) carries $\mathrm{Hom}'_{kH}(L \otimes_{kH} kG, M)$ to $\mathrm{Hom}'_{kH}(L \otimes_k kG', M)$ consisting of f' with $f'_{s'} = 0$ for most s'.

The image $Q = \mathrm{Hom}'_{kH}(P, M)$ of the endomorphism $f \mapsto f \circ e$ of

$$\mathrm{Hom}'_{kH}(L \otimes_{kH} kG, M)$$

is thus isomorphic to the image of the idempotent endomorphism $\varepsilon : f' \mapsto (f \circ e)'$ of $\mathrm{Hom}'_{kH}(L \otimes_k kG', M)$. For $x \in L$ put $e(x \otimes 1) = \sum_s e_s(x) \otimes s$ where s varies over representatives of G/H. Then for $s \in G$ we have

$$\begin{aligned}(\varepsilon f')_{s'}(x) &= (\varepsilon f')(x \otimes s') = (f \circ e)(x \otimes s) s^{-1} = f\Big(\sum_{t \bmod H} e_t(x) \otimes ts\Big) s^{-1} \\ &= \Big(\sum_{t \bmod H} f'(e_t(x) \otimes t' s') ts\Big) s^{-1} = \sum_{t \bmod H} f'(e_t(x) \otimes t' s') t,\end{aligned}$$

whence

$$(\varepsilon f')_{s'} = \sum_{t \bmod H} f'_{t's'}(e_t(x)) t. \tag{2}$$

To prove the proposition we shall construct a finite kG'-coordinate system of $U = \mathrm{Hom}'_{kG'}(L \otimes_k kG', M)$ and use it to compute $r_Q = T_{U/kG'}(\varepsilon)$. Let $x_i, f_i : L \to kH$ be a finite kH-coordinate system of L. Then $x_i \otimes 1$, $f_i \otimes 1_{kG} : L \otimes_{kH} kG \to kG$ is a finite kG-coordinate system, so $r_P = T_{L\otimes_{kH}kG}(e) = T_G(\sum_i (f_i \otimes 1)(e(x_i \otimes 1))) = T_G(\sum_i (f_i \otimes 1)(\sum_{t \bmod H} e_t(x_i) \otimes t)) = T_G(\sum_{t \bmod H} \sum_i f_i(e_t(x_i))t)$, so

$$r_P = T_G(\sum_{t \bmod H} a_t t), \tag{3}$$

where $a_t = \sum_i f_i(e_t(x_i)) \in kH$. (Note that a_t would be the element defining $T_{L/kH}(e_t)$ except that $e_t\colon L \to L$ is not kH-linear, but rather semi-linear with respect to the automorphism $u \mapsto tut^{-1}$ of H.)

Let $y_j, g_j\colon M \to k$ be a finite k-coordinate system of M. Define

$$z_{ji} \in U = \mathrm{Hom}'_{kH}(L \otimes_k kG', M)$$

$$h_{ji}\colon U \to kG'$$

by $(z_{ji})_{s'}(x) = y_j f_i(x)$ if $s' = 1$, and 0 if $s' \neq 1$, and $h_{ji}(f') = \sum_{s' \in G'} g_j(f'_{s'}(x_i))s'$, the latter sum being finite since $f'_{s'} = 0$ for most s'. To show z_{ji}, h_{ji} is a kG'-coordinate system of U, let $f' \in U$, $x \in L$, and $t \in G$. Then

$$\begin{aligned}
&(\sum_{j,i} z_{ji} h_{ji}(f'))(x \otimes t') \\
&\quad = (\sum_{j,i} z_{ji} \sum_{s' \in G'} g_j(f'_{s'}(x_i))s')(x \otimes t') \\
&\quad = \sum_{j,i,s'} (z_{ji}s')(x \otimes t') \cdot g_j(f'_{s'}(x_i)) \\
&\quad = \sum_{j,i,s'} z_{ji}(x \otimes t's'^{-1}) \cdot g_j(f'_{s'}(x_i)) \\
&\quad = \sum_{j,i} y_j f_i(x) \cdot g_j(f'_{t'}(x_i)) \\
&\quad = \sum_i f'_{t'}(x_i) f_i(x) = \sum_i f'(x_i \otimes t') f_i(x) \\
&\quad = \sum_i f'(x_i f_i(x) \otimes t') = f'(x \otimes t').
\end{aligned}$$

Now with the kG'-coordinate system z_{ji}, h_{ji} we compute $r_{Q/kG'} = T_{U/kG'}(\varepsilon)$ as follows:

$$\begin{aligned}
&\sum_{ji} h_{ji}(\varepsilon(z_{ji})) \\
&\quad = \sum_{j,i} \sum_{s' \in G'} g_j(\varepsilon(z_{ji})_{s'}(x_i))s' \\
&\quad = \sum_{j,i} \sum_{s' \in G'} g_j(\sum_{t \bmod H} (z_{ji})_{t's'}(e_t(x_i))t)s' \qquad \text{(see (2))} \\
&\quad = \sum_{j,i} \sum_{s' \in G'} g_j(y_j f_i(e_{s^{-1}}(x_i))s^{-1})s' \qquad (s' = \pi s) \\
&\quad = \sum_{s' \in G'} \sum_j g_j(y_j a_{s^{-1}} s^{-1})s' \qquad (a_s = \sum_i f_i(e_s(x_i))) \\
&\quad = \sum_{s' \in G} \chi_M(a_{s^{-1}} s^{-1})s'.
\end{aligned}$$

Thus $r_{Q/kG'} = \sum_{s' \in G'} \chi_M(a_s s) T_{G'}(s'^{-1})$ and $r_{P/kG} = T_G(\sum_{s \bmod H} a_s s)$, by (3). Define a k-linear map χ from $kG = \bigoplus_{s \bmod H} kHs$ to $T(kG')$ by $\chi(b) = \chi_M(b)\, T_{G'}(s'^{-1})$ for $b \in kHs$. If $s, t \in G$ then $\chi(st) = \chi_M(st) T_{G'}((st)'^{-1}) = \chi_M(ts) T_{G'}((ts)'^{-1})$. Thus χ induces a k-linear map χ: $T(kG) \to T(kG')$,

$$\chi(T_G(s)) = \chi_M(s) \cdot T_{G'}(s'^{-1}).$$

We have $\chi(r_{P/kG}) = \chi(\sum_{s \bmod H} a_s s) = \sum_{s \bmod H} \chi_M(a_s s) T_{G'}(s'^{-1}) = r_{Q/kG'}$, by the formulas derived above. Writing $r_P = \sum_{\tau \in T(G)} r_P(\tau)\tau$, we conclude that

$$\begin{aligned} r_{Q/kG'} = \chi(r_P) &= \sum_{\tau \in T(G)} r_P(\tau)\, \chi_M(\tau)\, \pi_* \tau^{-1} = \sum_{\tau' \in T(G')} \Big(\sum_{\substack{\tau \in T(G) \\ \pi_* \tau = \tau'^{-1}}} r_P(\tau)\, \chi_M(\tau) \Big) \tau' \\ &= \pi_* \Big(\sum_{\tau \in T(G)} r_P(\tau)\, \chi_M(\tau)\, \tau^{-1} \Big) = \pi_* (\overline{\chi_M \cdot r_P}). \end{aligned}$$

By Proposition (5.5), $\overline{\chi_M \cdot r_P} = r_{M \otimes_k P^*}$. This concludes the proof of Proposition (6.15).

I had hoped the formula of Proposition (6.15) would yield a simple direct proof of the following theorem of Ken Brown ([6], Th. 7).

(6.17) **Theorem** (K. Brown). *Assume the quotient $G' = G/H$ is finite and that k is a field. If $P \in \mathscr{P}(\mathbb{Z}G)$ and $M \in \mathscr{R}_k(kG)$ then* $\mathrm{Hom}_{\mathbb{Z}H}(P, M)$ *is a free kG'-module.*

If $\mathrm{char}(k) = 0$ this follows from Corollary (6.16) whenever P satisfies the strong conjecture ($r_P(s) = 0$ for $s \neq 1$ in G).

Brown proves this theorem under the assumption that G is finitely generated. In fact the general case can be reduced to that when G is finitely generated. It suffices (see Remark (5.7)) to observe that if G is a subgroup of a group G_1 with normal subgroup H_1 such that $G_1 = H_1 \cdot G$ and $H = H_1 \cap G$, and if $P \in \mathscr{P}(\mathbb{Z}G)$ and $M \in \mathscr{R}_k(kG_1)$ then the kG'-modules $\mathrm{Hom}_{\mathbb{Z}H}(P, M)$ and $\mathrm{Hom}_{\mathbb{Z}H_1}(P \otimes_{\mathbb{Z}G} \mathbb{Z}G_1, M)$ are isomorphic. (We identify $G' = G/H$ with G_1/H_1.) One first identifies $\mathrm{Hom}_{\mathbb{Z}H}(P\, M)$ with $\mathrm{Hom}_{\mathbb{Z}H_1}(P \otimes_{\mathbb{Z}H_1} \mathbb{Z}H_1, M)$ and then notes that the natural map from $P \otimes_{\mathbb{Z}H} \mathbb{Z}H_1$ to $P \otimes_{\mathbb{Z}G} \mathbb{Z}G_1$ is bijective.

7. Characteristic p: Frobenius

Let k be a commutative ring of prime characteristic p, i.e. an $\mathbb{F}_p$-algebra.

(7.1) **Proposition.** *Let A be a k-algebra.* (a) *There is an additive endomorphism F: $t \mapsto t^p$* (called the "Frobenius endomorphism") *of $T(A)$ such that $T(a)^p = T(a^p)$ for $a \in k$ and $T(\alpha a)^p = \alpha^p T(a)^p$ for $\alpha \in k$.*

(b) *Let M be an A-module of type* (FP) (see (2.6)) *and let $u \in \mathrm{End}_A(M)$. Then $T_M(u^p) = T_M(u)^p$. In particular, $r_M^p = r_M$.*

Let $a, b \in A$. According to Jacobson ([15], V, No. 7)

(*) $(a+b)^p = a^p + b^p + s(a, b)$, with $s(a, b) \in [A, A]$.

Suppose, for some $u, v \in A$, we have $a = uv$ and $b = -vu$. Then $a+b = [u, v]$ and

$$a^p + b^p = uv(uv)^{p-1} - (vu)^{p-1} vu = [u, v(uv)^{p-1}],$$

whence

$$[u, v]^p = [u, v(uv)^{p-1}] + s(u, v) \in [A, A].$$

From this and (*) we conclude that: If $c \in [A, A]$ then $c^p \in [A, A]$ and so $(a+c)^p \equiv a^p \bmod [A, A]$. It follows that $T(a) \mapsto T(a^p)$ is a well defined additive endomorphism of $T(A)$; clearly $T((\alpha a)^p) = \alpha^p T(a^p)$ for $\alpha \in k$, since T is k-linear; this proves (a).

The preceeding considerations, applied to $\mathrm{End}_A(M)$ in place of A, show that the maps $u \mapsto T_M(u^p)$ for M an A-module of type (FP), and $u \in \mathrm{End}_A(M)$, are additive, linear, and commutative in the sense of § 2. The universal property of $T(A)$ implies therefore the existence of an endomorphism φ of $T(A)$ such that $\varphi T_M(u) = T_M(u^p)$ for all (M, u) as above. Taking $M = A$, we see that φ is the p-th power map constructed above. This proves Proposition (7.1).

(7.2) Let G be a group. Then the iterated Frobenius F^m: $T(kG) \to T(kG)$ is given by

$$\begin{aligned} F^m\Big(\sum_{\tau \in T(G)} r(\tau)\tau\Big) &= \sum_{\tau \in T(G)} r(\tau)^{p^m} \tau^{p^m} \\ &= \sum_{\sigma \in T(G)} \Big(\sum_{\tau^{p^m} = \sigma} r(\tau)\Big)^{p^m} \sigma \end{aligned}$$

where, for $\tau = T(s)$ with $s \in G$, $F^m \tau = \tau^{p^m} = T(s^{p^m})$. The latter shows that F^m is the semi-linear extension of the set map F^m: $T(G) \to T(G)$, $\tau \mapsto \tau^{p^m}$.

(7.3) **Lemma.** *Let $s \in G$ and $n \geqq 1$ be such that $F^n T(s) = T(s)$, i.e. such that s is conjugate in G to s^{p^n}. If s has finite order m then m is prime to p. If s has infinite order then s belongs to a subgroup H of G isomorphic to the additive group of the ring $\mathbb{Z}[1/p]$.*

The first assertion is obvious, so assume s has infinite order and $s = t s^{p^n} t^{-1}$. Put $s_r = t^r s t^{-r}$ for $r \geqq 0$. Then $s_r^{p^n} = t^r s^{p^n} t^{-r} = t^{r-1} s t^{1-r} = s_{r-1}$, so the group H generated by $s_0 = s, s_1, s_2, \ldots$ is isomorphic to $\mathbb{Z}[1/p]$, say by sending s_r to p^{-nr}.

(7.4) **Lemma** (7.3) suggests the interest of the following condition on G.

(D_p) If $s \in G$ is conjugate to s^{p^n} for some $n \geqq 1$ then s has finite order (necessarily prime to p).

Equivalently, if F^n fixes $T(s)$ for some $n \geqq 1$ then s has finite order.

(7.5) **Proposition.** *Let $r = \sum r(\tau)\tau$ be an element of $T(kG)$ fixed by F^m, $m \geqq 1$. Put $S = \mathrm{supp}(r) = \{\tau \mid r(\tau) \neq 0\}$, and let R denote the subring of k generated by all $r(\tau)$.*

(a) *F^m permutes S, and $r(\tau^{p^m}) = r(\tau)^{p^m}$ for $\tau \in S$. Moreover $r(1)^{p^m} = r(1)$.*

(b) *Suppose $s \in G$ and $T(s) \in S$. There is an integer n, $1 \leqq n \leqq \mathrm{Card}(S)$, such that s is conjugate to $s^{p^{mn}}$. If s has finite order, its order is prime to p. If s has infinite order, s belongs to a subgroup of G isomorphic to the additive group of $\mathbb{Z}[1/p]$. (Condition (D_p) on G forbids the latter possibility.)*

(c) *The ring R is a finite product of finite fields. Its dimension over $\mathbf{F}_p$ is $\leq p^{m\cdot\mathrm{Card}(S)}$.*

We have $r=r^{p^m}=\sum_\tau r(\tau)^{p^m}\tau^{p^m}$, so, for all $\sigma\in S$, $r(\sigma)=\sum_{\tau^{p^m}=\sigma} r(\tau)^{p^m}$. It follows that the finite set $F^m S$ must contain S, so, by the box principle, F^m must induce a bijection $S\to S$. It follows that $r(\tau^{p^m})=r(\tau)^{p^m}$ for $\tau\in S$. In particular

$$r(1)=r(1^{p^m})=r(1)^{p^m} \quad \text{if } r(1)\neq 0,$$

and likewise if $r(1)=0$, whence (a).

Assertion (b) follows from Lemma (7.3) with n the cardinal of the orbit of $T(s)$ under F^m.

Let $n_1,\dots,n_u$ be the cardinals of the orbits of F^m in S, so that $\mathrm{Card}(S)=n_1+\dots+n_u$. If τ_i belongs to the orbit of cardinal n_i, and if $r_i=r(\tau_i)$, then $r_1,\dots,r_u$ generate the ring R, and $r_i^{p^{mn_i}}=r_i$. Thus R is a quotient of the tensor product R' of the rings $\mathbf{F}_p[X]/(X^{p^{mn_i}}-X)$, of dimension p^{mn_i} over $\mathbf{F}_p$. Since $\mathbf{F}_p[X]/(X^{p^n}-X)\cong\prod_{d|n}\mathbf{F}_{p^d}$ and $\dim R\leq\dim R'=\prod_i p^{mn_i}=p^{m\cdot\mathrm{Card}(S)}$, assertion (c) follows.

Remark. Suppose K is a field. With the notation of the last part of the proof, we then have $\mathbb{F}_p[r_i]=\mathbb{F}_{p^{f_i}}$, where f_i divides mn_i; hence $f_i|n_i$ if $m=1$, i.e. if F fixes r. Moreover, $R=\mathbb{F}_{p^f}$ where f is the least common multiple of the f_i.

(7.6) **Corollary.** *Let $P\in\mathscr{P}(kG)$ and let u be an automorphism of P of finite order n prime to p. Let m denote the order of p in $(\mathbb{Z}/n\mathbb{Z})^*$. Then $r=T_{P/kG}(u)$ is fixed by F^m, so the conclusions of Proposition (7.5) apply. In particular $r_P=T_{P/kG}(1_P)$ is fixed by F, so Proposition (7.5) applies to r_P with $m=1$.*

In fact, since $p^m\equiv 1 \bmod n$ we have $u^{p^m}=u$, so Proposition (7.1)(6) implies that $r=T_P(u)=T_P(u^{p^m})=T(u)^{p^m}$.

Remark. The corollary applies equally well if P is only a kG-module of type (FP).

(7.7) **Corollary.** *Suppose k is an algebraically closed field and $P\in\mathscr{P}(kG)$. There is a finite field k' contained in k and a $P'\in\mathscr{P}(k'G)$ such that $r_P=r_{P'}\in T(k'G)$. Moreover $r_P(1)\in\mathbf{F}_p$.*

It clearly suffices to prove the analogue of the corollary where we take for k' the algebraic closure of $\mathbf{F}_p$ in k; the P' then obtained will be defined over a finite subfield of k'. Let k_1 be a finitely generated k'-algebra in k such that $P\cong k\otimes_{k_1}P_1$ for some $P_1\in\mathscr{P}(k_1G)$. Choose a retraction $f\colon k_1\to k'$ (Nullstellensatz), and put $P'=k'\otimes_{k_1}P_1\in\mathscr{P}(k'G)$. We have $r_{P_1}=r_P=\sum r_P(\tau)\tau$ and $r_{P'}=\sum f(r_P(\tau))\tau$. But Proposition (7.5) and Corollary (7.6) imply that $r_P(\tau)\in k'$ so $f(r_P(\tau))=r_P(\tau)$ for all $\tau\in T(G)$. Thus $r_{P'}=r_P$, as required. The fact that $r_P(1)^p=r_P(1)$ (Proposition (7.5)(a)) implies finally that $r_P(1)\in\mathbf{F}_p$.

(7.8) **Proposition** (cf. Passman [28], Th. 2.2). *Let $r=\sum r(\tau)\,\tau\in\tau(kG)$ and suppose $F^m r=0$ for some $m\geq 1$.*

(a) *For all $\sigma\in T(G)$, $(\sum_{\tau^{p^m}=\sigma} r(\tau))^{p^m}=0$.*

(b) *Suppose $r(s)=0$ whenever s has finite order divisible by p. Then $r(s)^{p^m}=0$ whenever s has finite order; in particular $r(1)^{p^m}=0$.*

We have $0=F^m r=\sum_\sigma(\sum_{\tau^{p^m}=\sigma} r(\tau))^{p^m}\sigma$ (see (7.2)), whence (a). Let $S_0\subset S=\operatorname{supp}(r)$ consist of those $T(s)\in S$ for which s has finite order. Then $F^m S_0$ and $F^m(S-S_0)$ are clearly disjoint, and the hypothesis of (b) implies that the restriction of F^m to S_0 is injective; in fact some power of F fixes each element of S_0. Thus it follows from (a) that $r(\tau)^{p^m}=0$ for all $\tau\in S_0$, whence (b).

(7.9) *Remarks.* 1. Suppose, in Proposition (7.8), that k is reduced, i.e. has no nilpotent elements $\neq 0$. Then (a) and (b) of (7.8) can be strengthened as follows:

(a′) For all $\sigma\in T(G)$, $\sum_{\tau^{p^m}=\sigma} r(\tau)=0$.

(b′) Suppose $r(s)=0$ whenever s has finite order divisible by p. Then $r(s)=0$ whenever s has finite order; in particular $r(1)=0$.

2. The following conditions on G are clearly equivalent:

(i) $F\colon T(G)\to T(G)$ is injective.

(ii) G contains no element of order p, and, if s, t are elements of infinite order in G such that s^p is conjugate to t^p, then s is conjugate to t.

Under these conditions, if $r\in T(kG)$ then $F^m r=0$ if and only if $r(s)^{p^m}=0$ for all $s\in G$. Hence $F\colon T(kG)\to T(kG)$ is injective if k is reduced.

(7.10) **Corollary.** *Let M be a kG-module of type (FP), and let v be a nilpotent endomorphism of M, say $v^{p^m}=0$. Then $r=T_M(v)$ is annihilated by F^m, so the conclusions of Proposition* (7.8) *apply to r.*

In fact $0=T_M(0)=T_M(v^{p^m})=T_M(v)^{p^m}=F^m r$, by Proposition (7.1)(b).

(7.11) *Remarks.* Let M be a kG-module of type (FP) and let t be an automorphism of finite order of M. Then we can write $t=sv=vs$ where v has order a power p^m of p and s has order n prime to p. We have $v=1_M+u$ with $1_M=v^{p^m}=1_M+u^{p^m}$, so $u^{p^m}=0$. Moreover $t=sv=s+su=s+us$ and $(su)^{p^m}=0$. It follows that $T_M(t)=T_M(s)+T_M(us)$. Information about $T_M(s)$ furnished by Corollary (7.6); for example the conjugacy classes of finite order in $\operatorname{supp}(T_M(s))$ have order prime to p, and these exhaust $\operatorname{supp}(T_M(s))$ if G satisfies condition (D_p) of (7.4). On the other hand information on $T_M(us)$ is furnished by Corollary (7.10). For example if G has no p-torsion then there is no torsion in $\operatorname{supp}(T_M(us))$ if k is reduced (Remark (7.9)).

Suppose, more generally, that u is an endomorphism of M which is integral over k, i.e. $k[u]\subset\operatorname{End}_{kG}(M)$ is a finitely generated k-module. Then $T_M(u)$ belongs to the finitely generated k-module $T_M(k[u])$ which is stable under F. It follows that, for some *finite* set $S\subset T(G)$, $\operatorname{supp}(r)\subset S$ for all $r\in T_M(k[u])$, in particular for all $F^m T_M(u)$, $m\geqq 1$.

8. The Complex Group Algebra

Here we investigate the ranks r_P of finitely generated projective modules P over a complex group algebra $\mathbf{C}G$. The main result, Theorem (8.1), gives some detailed information about arithmetic properties of the numbers $r_P(s)$, including Zaleskii's theorem that $r_P(1)\in\mathbf{Q}$. The method we use was originated by Zaleskii. Kaplansky's theorem, that $r_P(1)>0$ if $P\neq 0$, is recalled, and applications of these results are given here and in § 9.

(8.1) **Theorem.** *Let G be a group. Let $r=r_M=\sum_{\tau\in T(G)} r(\tau)\tau$ be the rank of a $\mathbf{C}G$-module M of type (FP). Put $S=\operatorname{supp}(r)=\{\tau|r(\tau)\neq 0\}$* and *$R=\{r(\tau)|\tau\in T(G)\}$. Let $E=\mathbf{Q}(R)$, the subfield of $\mathbf{C}$ generated by R.*

(a) *E is a finite abelian extension of $\mathbf{Q}$; put $\Gamma=\operatorname{Gal}(E/\mathbf{Q})$.*

There is a finite set Π of rational primes, including those ramified in E, with the following properties.

(b) *If p is a prime not in Π then $\tau\mapsto\tau^p$ is a bijection $S\to S$, and $r(\tau^p)=\sigma r(\tau)$ for $\tau\in S$, where σ is the Artin symbol $(p, E/\mathbf{Q})$* (cf. [22], Ch. I, § 8).

Let $s\in G$ be such that $T(s)\in S$.

(c) *If p is a prime not in Π then s is conjugate in G to s^{p^n} for some n, $1\leqq n\leqq \operatorname{Card} S$.*

(d) *Suppose s has finite order m, and put $w=\exp(2\pi i/m)$. Then $r(s)\in\mathbb{Q}(w)$, say $r(s)=f(w)$ for some polynomial $f\in\mathbb{Q}[X]$. We have $r(s^q)=f(w^q)$ for all q prime to m. In particular $r(1)\in\mathbb{Q}$* (Zaleskii [27]).

(e) *Suppose s has infinite order. Then s belongs to a subgroup H of G isomorphic to the additive group of all rational numbers with denominator prime to all primes in Π.*

Let $A=\mathbb{Z}[R]$, the subring of $\mathbb{C}$ generated by R. Choose $P_0, P_1\in\mathscr{P}(\mathbb{C}G)$ *so that* $r=r_{P_0}-r_{P_1}$. We can choose a subring B of $\mathbb{C}$ with the following properties (i)–(iv):

(i) B is a finitely generated $\mathbb{Z}$-algebra.

(ii) The module P_i is isomorphic to $\mathbb{C}\otimes_B P_i'$ for some finitely generated projective BG-module P_i'.

This condition permits us to identify r with $r_{P_0'}-r_{P_1'}\in T(BG)$; in particular $A\subset B$.

(iii) If $\tau\in T(G)$ and $r(\tau)\neq 0$ then $r(\tau)$ is invertible in B. If $\tau'\in T(G)$ and $r(\tau)\neq r(\tau')$ then $r(\tau)-r(\tau')$ is invertible in B.

(iv) B is integrally closed.

To achieve (ii) B need only contain all coefficients of elements of G in the entries of idempotent matrices defining P_0 and P_1, and (iii) requires inversion of a finite number of non-zero elements. The integral closure of the $\mathbb{Z}$-algebra generated by this finite collection of elements is still finitely generated (cf. [18], (31.H), Th. 72). Note that condition (iii) implies:

(iii′) If $\tau, \tau'\in T(G)$ and $r(\tau)\neq 0$ then, for any proper ideal I of B, $r(\tau)\not\equiv 0 \bmod I$, and, if $r(\tau)\neq r(\tau')$, then $r(\tau)\not\equiv r(\tau') \bmod I$.

According to the generic freeness lemma of Hochster-Roberts ([14], No. 8) there is an integer $u\neq 0$ such that A_u and B_u/A_u are free $\mathbb{Z}_u$-modules, where the subscript denotes localization by inverting u. Let p be a rational prime. The condition

(v) $p\nmid u$

implies that the homomorphism $A/pA\to B/pB$ is unchanged by inverting u. Since A_u is a direct summand (as $\mathbf{Z}_u$-module) of B_u it follows that $A/pA\to B/pB$ is injective, so we can identify $\bar{A}=A/pA$ with the subring of $\bar{B}=B/pB$ generated by the image $\bar{R}$ of R. But if $\bar{P}_i$ denotes the $\bar{B}G$-module P_i'/pP_i' (see (ii)) then $r_{\bar{P}_0}-r_{\bar{P}_1}\in T(\bar{B}G)$ is just the reduction mod pB, $\bar{r}$, of r. Thus $\bar{A}$ is the $\mathbf{F}_p$-algebra generated by the coefficients of $r_{\bar{\mathfrak{p}}_0} - r_{\bar{\mathfrak{p}}_1}$. It follows therefore from Proposition (7.5) and Corollary (7.6) that $\bar{A}$ is *finite* and *reduced*. Moreover

$$\dim_{\mathbf{F}_p}(\bar{A}) = \dim_{\mathbf{F}_p}(A_u/pA_u) = \operatorname{rank}_{\mathbf{Z}_u}(A_u),$$

so we conclude that A_u has finite rank over $\mathbf{Z}_u$. Thus the field of fractions $E = \mathbf{Q}(R)$ of A is a *finite* extension of $\mathbf{Q}$. Let E' be the smallest galois extension of $\mathbf{Q}$ in $\mathbf{C}$ containing E, and put $\Gamma = \operatorname{Gal}(E'/\mathbf{Q})$.

Continuing the discussion above, condition (iii') implies that $S = \operatorname{supp}(\bar{r})$. Thus Proposition (7.5) and Corollary (7.6) further imply that:

(vi) $\tau \mapsto \tau^p$ defines a bijection $S \to S$ and $r(\tau^p) \equiv r(\tau)^p \bmod pB$ for $\tau \in S$.

Let $\sigma \in \Gamma$. By the Čebotarev density theorem (cf. [17], Ch. XV, § 5, Th. 6, Ex. 1), there are infinitely many primes $\mathfrak{p}$ of E' unramified over $\mathbf{Q}$ such that σ is the Frobenius automorphism $(\mathfrak{p}, E'/\mathbf{Q})$. In all but finitely many cases we may identify $\mathfrak{p}$ with a prime ideal of $B' = B \cap E'$, and then σ is characterized by the condition: $\sigma(x) \equiv x^p \bmod \mathfrak{p}$ for all $x \in B'$, where p is the rational prime over which $\mathfrak{p}$ lies. For all but finitely many $\mathfrak{p}$'s, p will satisfy condition (v), and hence also condition (vi). From the congruences $\sigma r(\tau) \equiv r(\tau)^p \bmod \mathfrak{p}$ and $r(\tau^p) \equiv r(\tau)^p \bmod pB$, we conclude that $\sigma r(\tau) \equiv r(\tau^p) \bmod \mathfrak{p}$. Since S is finite condition (vi) implies that, for infinitely many $\mathfrak{p}$ and p as above, τ^p takes the same value, say $\tau_1 \in S$, whence $\sigma r(\tau) - r(\tau_1) \in \mathfrak{p}$ for infinitely many primes $\mathfrak{p}$ of B'. It follows that $\sigma r(\tau) = r(\tau_1) \in R$, so that Γ stabilizes R, whence $E = \mathbf{Q}(R)$ is galois over $\mathbf{Q}$, i.e. $E = E'$. Further, condition (v) implies that $A_u = B'_u$. In view of condition (iii') the congruence $\sigma r(\tau) \equiv r(\tau^p) \bmod \mathfrak{p}$ now further implies that $\sigma r(\tau) = r(\tau^p)$. In view of (vi) we then have $\sigma r(\tau) \equiv r(\tau)^p \bmod pB$, whence, $\sigma(a) \equiv a^p \bmod pA_u$ for all $a \in A_u = \mathbb{Z}_u[R]$. Consequently σ is the Frobenius automorphism $(\mathfrak{p}', E/\mathbf{Q})$ for *all* primes $\mathfrak{p}'$ above p (satisfying (v)). It follows that σ coincides with its conjugates, so is central in Γ. Since σ was arbitrary we conclude that Γ is abelian and that $\sigma = (p, E/\mathbf{Q})$, the Artin symbol. This proves assertions (a) and (b) of the theorem, where Π is taken to be the set of primes dividing u. Note that if $p \nmid u$, i.e. if p satisfies (v), then we have seen that $A/pA = A_u/pA_u$ is reduced, so p is unramified in E. Assertion (c) follows from (b).

For q an integer not divisible by any primes in Π we can define the Artin symbol $\sigma_q = (q, E/\mathbf{Q})$ in Γ so that it is multiplicative in q; it then follows from (b) that, whenever $T(s) \in S$, we have $\sigma_q r(s) = r(s^q)$. Suppose s has finite order m. Since $\tau \mapsto \tau^q$ is a permutation of S (by (b)), q must be prime to m. Further $r(s)$ is fixed by the Artin symbols σ_q with $q \equiv 1 \bmod m$, so it lies in $E \cap \mathbb{Q}(w)$, where $w = \exp(2\pi i/m)$, say $r(s) = f(w)$ with f a polynomial in $\mathbb{Q}[X]$. We then have $r(s^q) = \sigma_q f(w) = f(w^q)$ for q as above. Note that such q's represent every invertible element of $\mathbb{Z}/m\mathbb{Z}$. This proves (d).

To prove (e), consider, for each $s \in G$, the multiplicative monoid $C(s)$ of positive integers q such that s is conjugate in G to s^q. If $T(s) \in S$ then, according to (b), for each prime $p \notin \Pi$, $C(s)$ contains p^n for some n such that $1 \leqq n \leqq \operatorname{Card} S$. Let M denote the multiplicative monoid generated by all primes $p \notin \Pi$. Putting $N = (\operatorname{Card} S)!$, we conclude that if $T(s) \in S$ then $q^N \in C(s)$ for all $q \in M$. Writing $s = t s^{q^N} t^{-1} = (t s t^{-1})^{q^N}$ we see that $s = u^{q^N}$ with $T(u) = T(s) \in S$. List the primes $p \notin \Pi$: $p_1, p_2, \ldots, p_n, \ldots$. Put $a_n = p_1^N \ldots p_n^N$ and $b_n = a_1 \ldots a_n$. By induction on n we can construct a sequence $s_0, s_1, \ldots, s_n, \ldots$ in $T(s)$ so that $s_0 = s$ and $s_n^{b_n} = s_{n-1}$. The subgroup H of G generated by this sequence is clearly isomorphic to the direct limit of the sequence of groups and homomorphisms

$$\mathbb{Z} \xrightarrow{b_1} \mathbb{Z} \xrightarrow{b_2} \mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \xrightarrow{b_n} \cdots$$

which can be identified with the additive group of rational numbers with denominator in M.

This concludes the proof of Theorem (8.1).

(8.2) *Remark.* Keep the notation of Theorem (8.1). It seems reasonable to conjecture that $r(s)=0$ for all s of infinite order. Conditions under which this is the case are described in §9, Proposition (9.2) and Theorem (9.6).

(8.3) **Corollary.** *Let P be a finitely generated projective $\mathbf{C}G$-module. There is a finite field extension L of $\mathbf{Q}$ and a finitely generated projective LG-module $\bar{P}$ such that $r_{\bar{P}}=r_P$.*

It clearly suffices to prove the analogous assertion with L replaced by the algebraic closure $\bar{\mathbf{Q}}$ of $\mathbf{Q}$. Let A be a finitely generated $\bar{\mathbf{Q}}$-algebra in $\mathbf{C}$ so that P arises by base change $A \to \mathbf{C}$ from a finitely generated projective AG-module P_1. Then $r_P = r_{P_1} \in T(AG)$. Choose a $\bar{\mathbf{Q}}$-algebra homomorphism $f\colon A \to \bar{\mathbf{Q}}$ (Nullstellensatz), and put $\bar{P} = \bar{\mathbf{Q}} \otimes_A P_1$. If $r_P = \sum_{\tau\in T(G)} r(\tau)\tau = r_{P_1}$ then $r_{\bar{P}} = \sum_{\tau\in T(G)} f(r(\tau))\tau$. Part (a) of Theorem (8.1) asserts that $r(\tau)\in\bar{\mathbf{Q}}$ for all τ, whence $r_{\bar{P}}=r_P$, as required.

(8.4) We next propose to show, in certain cases, that traces (over $\mathbf{C}G$) of nilpotent endomorphisms vanish. The method of proof involves the following condition on the group G:

(C) If $s_1, s_2 \in G$ are such that s_1^p is conjugate to s_2^p for all but finitely many primes p, then s_1 is conjugate to s_2.

(8.5) **Theorem.** *Let M be a $\mathbf{C}G$-module of type (FP), let v be a nilpotent endomorphism of M, and put $r = T_M(v) = \sum r(\tau)\tau$.*

(a) *$r(s)=0$ for all s of finite order.*

(b) *If G satisfies condition* (C) *of* (8.4) *then $r=0$.*

(c) *If G has a faithful linear representation over some field then G satisfies* (C).

Remark. Assertion (a) for $s=1$ and $M=\mathbb{C}G$ is Corollary (2.3) of Passman [28]. We use the same method of proof here.

According to [1], Chapter XII, Proposition (6.2), there is a finite resolution $P \to M$ of M, $P=(\cdots 0 \to P_n \to \cdots \to P_0 \to 0)$ with each $P_i \in \mathscr{P}(\mathbf{C}G)$, and a lifting of v to a nilpotent endomorphism (v_i) of P. Then $r = T_M(v) = \sum (-1)^i T_{P_i}(v_i)$, so it suffices to prove the theorem when M is a projective module P.

Choose a finitely generated subring B of $\mathbf{C}$ such that there is a module $P' \in \mathscr{P}(BG)$ and a nilpotent $v' \in \mathrm{End}_{BG}(P')$ such that (P, v) is isomorphic to $(\mathbf{C}\otimes_B P', \mathbf{C}\otimes_B v')$; this is clearly possible, and we then have $r = T_{P'}(v') \in T(BG)$. We can further assume that all non-zero $r(\tau)$ are invertible in B.

Let p be a prime not invertible in B (this excludes only finitely many). Put $\bar{B} = B/pB$, $\bar{P} = \bar{B}\otimes_B P' \in \mathscr{P}(\bar{B}G)$, and $\bar{v} = \bar{B}\otimes_B v'$. Then $T_{\bar{P}}(\bar{v}) \in T(\bar{B}G)$ is the reduction mod Bp, $\bar{r}$, of r, to which we may apply Proposition (7.8). Suppose p does not divide the order of any s of finite order for which $r(s) \neq 0$; this excludes only finitely many primes. Then Proposition (7.8)(b) and Corollary (7.10) imply that, for some $m > 0$, $r(s)^{p^m} \equiv 0 \bmod Bp$ whenever s has finite order. But if $r(s) \neq 0$ then $r(s)$ is invertible in B, by construction of B. Thus the congruence above implies that $r(s) = 0$ for all s of finite order, whence (a).

Suppose G satisfies condition (C). Then, for almost all primes p, the map $\tau \mapsto \tau^p$ is injective on $\operatorname{supp}(r)$. Choosing such a p large enough so that $v'^p = 0$ we conclude from Corollary (7.10) and Proposition (7.8)(a), applied to $\bar{r} = T_{\bar{p}}(\bar{v})$, that $r(\tau)^p \equiv 0 \bmod Bp$ for all τ. But if $r(\tau) \neq 0$, it is invertible in B, so the congruence implies that $r(\tau) = 0$ for all τ, whence (b).

Assertion (c) follows from the next result.

(8.6) **Proposition.** *Let F be a field and let G be a subgroup of $GL_n(F)$ for some $n \geqq 1$. Suppose $g_1, g_2 \in G$ are such that for infinitely many primes p, there is a power $q_p > 1$ of p such that $g_1^{q_p}$ and $g_2^{q_p}$ are conjugate in G. Then g_1 and g_2 are conjugate in G.*

There is no loss in assuming F is algebraically closed. Let $g_i = s_i u_i = u_i s_i$ be the Jordan decomposition of g_i in $GL_n(F)$, with s_i diagonalisable and u_i unipotent (cf. [3], Chap. I, Cor. 1 of Prop. (4.2)). Let Π denote the infinite set of primes alluded to in the statement of the proposition, but from which we exclude the characteristic of F, if it is > 0. For each $p \in \Pi$ choose $h_p \in G$ such that $h_p g_2^{q_p} h_p^{-1} = g_1^{q_p}$. From the uniqueness of the Jordan decomposition, we conclude that $h_p s_2^{q_p} h_p^{-1} = s_1^{q_p}$ and $h_p u_2^{q_p} h_p^{-1} = u_1^{q_p}$. Since $p \neq \operatorname{char}(K)$ the map $u \mapsto u^{q_p}$ is injective (even bijective) on the set of unipotents u in $GL_n(F)$. (If $\operatorname{char}(K) = l > 0$ then u has order a power of l, prime to q_p. If $\operatorname{char}(K) = 0$ then $u = \exp(1/q_p \log(u^{q_p}))$.) Thus from $(h_p u_2 h_p^{-1})^{q_p} = u_1^{q_p}$ we conclude that $h_p u_2 h_p^{-1} = u_1$. Thus, after replacing g_2 by one of its conjugates in G, we can arrange that $u_1 = u_2$; call this element u. Then the preceeding discussion shows that, for all $p \in \Pi$, we have

$$h_p s_2^{q_p} h_p^{-1} = s_1^{q_p}, \qquad h_p u h_p^{-1} = u.$$

We may further conjugate G by an element of $GL_n(F)$ to put s_1 in the diagonal form $s_1 = \operatorname{diag}(a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_m, \ldots, a_m)$ where $a_1, \ldots, a_m$ are the distinct eigenvalues of s_1. A power s_1^q of s_1 has the same centralizer $Z_{GL_n(F)}(s_1^q)$ in $GL_n(F)$ as s_1 provided $a_i^q \neq a_j^q$ whenever $i \neq j$, in other words if $(a_i/a_j)^q \neq 1$ for $i \neq j$. Let Π_0 denote the set of primes dividing the order of some a_i/a_j of finite order. Then if q is divisible by no prime in Π_0 the discussion above shows that $Z_{GL_n(F)}(s_1^q) = Z_{GL_n(F)}(s_1)$. Choose two primes p_1, p_2 in $\Pi - \Pi_0$ and put $q_i = q_{p_i}$ and $h_i = h_{p_i}$, $i = 1, 2$. Put $g_3 = h_1 g_2 h_1^{-1} = s_3 u$, so that $g_3^{q_1} = g_1^{q_1}$, and so $s_3^{q_1} = s_1^{q_1}$. Put $h = h_2 h_1^{-1}$. Then h commutes with u and $h g_3^{q_2} h^{-1} = h_2 g_2^{q_2} h_2^{-1} = g_1^{q_2}$, so $h s_3^{q_2} h^{-1} = s_1^{q_2}$. Thus $s_1^{q_2 q_1} = (h s_3^{q_2} h^{-1})^{q_1} = h(s_3^{q_1})^{q_2} h^{-1} = h s_1^{q_1 q_2} h^{-1}$, so $h \in Z_{GL_n(F)}(s_1^{q_1 q_2})$. By the choice of p_1 and p_2 outside of Π_0, therefore, h centralizes s_1, and hence $s_3^{q_i} = s_1^{q_i}$ for $i = 1$ and 2. Writing $1 = t_1 q_1 + t_2 q_2$ with $t_1, t_2 \in \mathbb{Z}$ we see that $s_i = (s_i^{q_1})^{t_1} \cdot (s_i^{q_2})^{t_2}$ is the same for $i = 1$ or 3, i.e. $s_1 = s_3$. But then $g_1 = g_3 = h_1 g_2 h_1^{-1}$, so g_1 and g_2 are conjugate in G, as claimed.

(8.7) **Proposition.** *Let M be a $\mathbb{C}G$-module of type (FP). Let $u \in \operatorname{End}_{\mathbb{C}G}(M)$ be algebraic over $\mathbb{C}$, say $f(u) = 0$ where $f(X) = \prod_{i=1}^{h} (X - u_i)^{n_i}$, $u_1, \ldots, u_h$ being the distinct roots of f.*

(a) *M is the direct sum of the modules* $M_i = \mathrm{Ker}\,(u_i \cdot 1_M - u)^{n_i}$ $(1 \leqq i \leqq h)$. *Put* $r_i = r_{M_i} \in T(\mathbb{C}G)$.

(b) $T_M(u) = u_1 r_1 + \cdots + u_h r_h + T_M(v)$ *where v is a nilpotent endomorphism of M, which is zero if all* $n_i = 1$.

(c) *If G satisfies condition* (C) *of* (8.2), *for example if G admits a faithful linear representation over some field, then* $T_M(v) = 0$.

Assertions (a) and (b) just formulate the conclusions of (3.6) in the present setting. Assertion (c) results from Theorem (8.5)(b) and (c).

(8.8) *Remark.* Suppose u is even algebraic over $\mathbf{Q}$, so that $u_1, \ldots, u_h$ are algebraic numbers. Then it folloes from Theorem (8.1) that, if $r = u_1 r_1 + \cdots + u_h r_h$, then $r(s)$ is an algebraic number for all $s \in G$. For example suppose $u^m = 1_M$. Then $T_M(u) = r$ by Proposition (8.7)(b), and $r(s)$ belongs to a cyclotomic field for all $s \in G$.

To draw further conclusions we shall invoke the following well known theorem of Kaplansky.

(8.9) **Theorem** (Kaplansky). *Let G be a group and let $P \in \mathscr{P}(\mathbf{C}G)$. If $P \neq 0$ then the rational number $r_P(1)$* (cf. Th. (8.1)(d)) *is* > 0.

In fact this was proved before Zaleskii's result on the rationality of $r_P(1)$, and it was shown by Kaplansky that $r_P(1)$ is a totally real algebraic number > 0. The proof is phrased in terms of the trace of the coefficient of $1 \in G$ of an idempotent n by n matrix over $\mathbf{C}G$. A published proof, with details only for $n = 1$, can be found in [19]. See Passman [28], Theorem (2.6) for further references.

(8.10) **Corollary.** *Let P be a projective $\mathbf{C}G$-module with n generators. Then* $0 \leqq r_P(1) \leqq n$. *If* $r_P(1) = 0$ *then* $P = 0$. *If* $r_P(1) = n$ *then* $P \cong (\mathbf{C}G)^n$.

Writing $P \otimes P' \cong (\mathbf{C}G)^n$ we have $r_P(1) + r_{P'}(1) = n$, and the corollary follows from Theorem (8.9).

(8.11) **Corollary.** *Let k be a subring of $\mathbf{C}$ in which no rational prime is invertible. Then any idempotent e in kG equals* 0 *or* 1.

Put $r = T(e)$ and $r' = T(1-e)$. Then $r(1)$ and $r'(1)$ belong to $k \cap \mathbf{Q} = \mathbb{Z}$, they are $\geqq 0$, and $r(1) + r'(1) = 1$. Hence either $r(1) = 0$, so $e = 0$, or $r(1) = 1$, so $e = 1$, by Corollary (8.10) (with $P = e\mathbf{C}G$).

(8.12) **Corollary.** *Let* $e = \sum_{s \in G} e(s) s$ *be an idempotent in* $\mathbf{C}G$. *If* $e(1) = \sum_s e(s)$ *then* $e = 0$ *or* $e = 1$.

In fact $\sum_s e(s)$, being the augmentation of e, is an idempotent of $\mathbb{C}$, hence either 0 or 1. Thus the corollary follows, as above, from Corollary (8.10).

(8.13) *Remarks.* 1. The condition that $\mathbb{C}G$ contain no idempotents except 0 and 1 is very strong. Indeed let K be any algebraically closed field and let A be a K-algebra. Then the following conditions on A are equivalent.

(a) A contains no idempotents except 0 and 1.

(b) Every finite dimensional subalgebra B of A is of the form $B = K + N$ with N a nilpotent ideal.

(c) Every element of A algebraic over K is a scalar plus a nilpotent element.

(d) Every unit of finite order not divisible by char(K) in A is a scalar.

(a)$\Rightarrow$(b): Let N be the radical of B. Since idempotents of B/N lift to idempotents of B, the semi-simple K-algebra B/N has no non-trivial idempotents, hence is just K.

(b)$\Rightarrow$(c): If $u \in A$ is algebraic over K we apply (b) to $B=K[u]$.

(c)$\Rightarrow$(d): If u has finite order not divisible by char(K) then $K[u]$ is semi-simple, so (c) implies u is a scalar.

(d)$\Rightarrow$(a): If e is a non-trivial idempotent and z is a non-trivial root of unity in K then $u=ze+(1-e)$ is not scalar and has finite order not divisible by char(K).

2. If G contains non-trivial elements of finite order then clearly $\mathbf{C}G$ contains non-trivial idempotents. If, on the other hand, G is torsion free, than it is believed that $\mathbf{C}G$ should contain no non-trivial idempotents (nor even any divisors of zero). We shall indicate in §9, (Prop. (9.2)) a large class of torsion free groups G, including all those with faithful linear representations (Theorem (9.6)), for which $\mathbb{C}G$ contains no non-trivial idempotents.

3. A general conjecture is that, if k is an integral domain, then kG contains non-trivial idempotents only if G contains an element of finite order $n>1$ invertible in k.

4. It was shown by Burns [7] that, for an arbitrary commutative ring k and group G, if $e=\sum_{s\in G} e(s)s$ is a *central* idempotent in kG, then the (normal) subgroup generated by $\operatorname{supp}(e)=\{s \mid e(s)\neq 0\}$ is *finite*.

The following nice result of Passman is announced by Sehgal in [20]. We include a proof since it is a pleasant application of the results at hand, and then present some interesting consequences drawn by Sehgal.

(8.14) **Proposition** (Passman). *Let k be a subring of $\mathbb{C}$ in which no rational prime is invertible. Let $u=\sum_{s\in G} u(s)s$ be a unit of finite order $n>1$ in kG and of augmentation $\sum_s u(s)$ equal to* 1. *Then $u(1)=0$.*

(Note that the group of units of kG is the direct product of those of k with those of augmentation 1.)

The decomposition of the semi-simple subalgebra $\mathbb{C}[u]$ of $\mathbf{C}G$ leads to an expression of u in the form $u=u_1e_1+\cdots+u_he_h$ where the u_i are distinct n^{th} roots of unity, and where the e_i are non-zero pairwise orthogonal idempotents with sum 1. Put $r=T(u)$ and $r_i=T(e_i)$ $(1\leq i\leq h)$, in $T(\mathbb{C}G)$. Then $r=u_1r_1+\cdots+u_hr_h$ so $u(1)=r(1)=u_1r_1(1)+\cdots+u_hr_h(1)$. By Theorem (8.9) each $r_i(1)$ is a rational number >0, and their sum is 1. Moreover $h>1$ for otherwise u would be a scalar of augmentation 1, hence equal to 1, contrary to our hypothesis that $n>1$. Thus $u(1)$ is a proper convex rational linear combination of roots of unity, so it, as well as all of its conjugates over $\mathbf{Q}$, lies in the interior of the unit circle. Therefore $u(1)$ is an element of k algebraic over $\mathbf{Q}$ whose norm from $\mathbf{Q}(u(1))$ to $\mathbf{Q}$ has absolute value <1. Since no rational prime is invertible in k, $u(1)$ must be integral over $\mathbb{Z}$, so its norm is an integer, of absolute value <1, hence 0. The proposition now follows.

(8.15) **Corollary** (Sehgal [20]). *If $n=p^h$ for some prime p then there is an element $s\in G$ of order n such that $u(s)\notin kp$.*

We have $u^{p^h}=1$ whereas $u^{p^{h-1}}\neq 1$, so Proposition (8.14) implies $u^{p^{h-1}}(1)=0$. For $i\geq 0$ let S_i denote the set of $\tau\in\operatorname{supp}(r)$ for which $\tau^{p^i}=1$. Then, applying Frobenius in $T((k/kp)G)$, we conclude that $1\equiv(\sum_{\tau\in S_h} r(\tau))^{p^h} \bmod kp$, whereas $0\equiv(\sum_{\tau\in S_{h-1}} r(\tau))^{p^{h-1}} \bmod kp$. It follows that $1\equiv(\sum_{\tau\in S_h-S_{h-1}} r(\tau))^{p^h} \bmod kp$. The corollary follows by taking a suitable $s\in G$ such that $T(s)\in S_h-S_{h-1}$.

(8.16) **Corollary** (Sehgal [20]). *If G is torsion free* 1 *is the only unit of finite order and augmentation* 1 *in kG.*

9. Non-Divisibility Conditions on G

(9.1) Theorem (8.1) suggests introducing the following condition (D) on a group G (cf. Formanek [11], No. 4).

(D) Suppose H is a finitely generated subgroup of G, $s\in H$, N is an integer >0, and, for all but finitely many primes p, s is conjugate in H to s^{p^N}; then s has finite order.

(9.2) **Proposition.** *Let M be a $\mathbb{C}G$-module of type (FP). If G satisfies condition* (D) *then $r_M(s)=0$ for all s of infinite order in G.*

It clearly suffices to prove this when M is a projective $\mathbb{C}G$-module P. Then there is a finitely generated subgroup H of G and a $Q\in\mathscr{P}(\mathbb{C}H)$ such that $P\cong Q\otimes_{\mathbb{C}H}\mathbb{C}G$. If $s\in H$ has infinite order and $r_Q(s)$ $(=r_P(s))\neq 0$ then there is an integer $N>0$ such that s is conjugate in H to s^{p^N} for almost all primes p, by Theorem (8.1)(c). (One can take $N=c!$ where $c=\operatorname{Card}(\operatorname{supp}(r_Q))$.) But this contradicts condition (D) on G.

(9.3) **Corollary** (cf. Formanek [11], Th. 9). *Let G be a torsion free group satisfying condition* (D). *Then $\mathbb{C}G$ contains no idempotents except* 0 *and* 1.

Let $e=\sum_{s\in G} e(s)s$ be an idempotent in $\mathbb{C}G$. Then $T(e)=r_P=\sum_{\tau\in T(G)} r(\tau)\tau$ where $P=(\mathbb{C}G)e$ and $r(\tau)=\sum_{s\in\tau} e(s)$. Proposition (9.2) implies that $r(\tau)=0$ for $\tau\neq 1$; whence $r_P=e(1)\cdot 1$, and $e(1)=\sum_{s\in G} e(s)$. Now it follows from Corollary (8.12) that $e=0$ or 1.

(9.4) **Theorem.** *Let G be a group satisfying condition* (D), *and let P be a finitely generated projective $\mathbf{Q}G$-module. Then r_P is a $\mathbf{Q}$-linear combination of the elements r_{P_H} where $P_H=\mathbf{Q}[G/H]=\mathbf{Q}G\otimes_{\mathbf{Q}H}\mathbf{Q}_H$, H being any finite cyclic subgroup of G, and $\mathbf{Q}_H$ the $\mathbf{Q}H$-module $\mathbf{Q}$ with trivial H-action.*

It follows from Theorem (8.1)(d) and Proposition (9.2) that $r_P(s)=0$ if s has infinite order, and that r_P takes the same value on two generators of the same finite cyclic subgroup H of G. For each such H let $\sigma_H\in T(\mathbf{Q}G)$ be the sum of the conjugacy classes that contain a generator of H, i.e. $\sigma_H(s)=1$ if s generates a conjugate of H, and $\sigma_H(s)=0$ otherwise. The preceeding remarks show that r_P

is a $\mathbf{Q}$-linear combination of the elements σ_H, so it suffices to express the σ_H's in terms of the r_H's. But both σ_H and r_H are images in $T(\mathbf{Q}G)$ of the corresponding elements of $T(\mathbf{Q}H)$. Thus we reduce to the case when $H=G$. In this case, for each finitely generated $\mathbf{Q}G$-module V, we have $\chi_V=|G|\cdot r_V$, and it is well known that σ_G is a $\mathbf{Q}$-linear combination of the characters $\chi_{\mathbf{Q}[G/H]}$ where H varies over subgroups of G (cf. Serre [21], Chap. II, § 8). This proves Theorem (9.4).

Remark. Theorem (9.4) is analogous to Artin's theorem on induced rational characters of finite groups, to which the last part of the proof refers.

(9.5) *Condition* (D): *Remarks and Examples.*

1. Condition (D) is clearly stable under passage to subgroups and to filtered direct limits.

2. Let H be a normal subgroup of a group G such that G/H satisfies condition (D). Then G satisfies (D) if either H is a torsion group (e.g. if H is finite), or if each element of H has a finite G-conjugacy class.

3. Condition (D) is a consequence of the following condition on a group G:

(D′) An element $s\in G$ has finite order if, in some finitely generated subgroup H of G containing s, s is a p$^{\text{th}}$ power for infinitely many primes p.

The next theorem shows that *linear groups satisfy* (D′), *and hence* (D).

4. We do not known whether all residually finite groups satisfy condition (D).

(9.6) **Theorem.** *Let F be a field, and let G be a finitely generated subgroup of $GL_n(F)$. Suppose $g\in G$ is such that, for infinitely many primes p, the equation $g=x^p$ is solvable with $x\in G$. Then g has finite order.*

Since G is finitely generated there is a finitely generated subring A of F such that $G\subset GL_n(A)$. In fact this is the only finiteness property of G we use, so we may even assume $G=GL_n(A)$. After extending F and A slightly we may further assume that A contains the eigenvalues of g. There is no harm in supposing also that F is the field of fractions of A, and that A is integrally closed in F. Let $\bar{F}$ be an algebraic closure of F. According to Proposition (A.4) of the Appendix, there is an integer $N>0$ such that if $\alpha\in\bar{F}$ is such that $[F(\alpha):F]\leqq n$ and $\alpha^m\in F$ for some m prime to N then $\alpha\in F$.

Let $S\subset A$ denote the set of eigenvalues of g. Let $a\in S$. If $g=x^p$ with $x\in GL_n(A)$ then $a=\alpha^p$ for some eigenvalue $\alpha\in\bar{F}$ of x and $[F(\alpha):F]\leqq n$ (since α is a root of the characteristic polynomial of x). If p is prime to N (as above) then $\alpha\in F$, and so $\alpha\in A$ since α is integral over A, and A is integrally closed in F. It follows that, for such p, a is a p^{th} power in the group A^* of units of A. According to a theorem due essentially to Rosenlicht (see [16], II, § 4, Cor. of Th. 5), A^* is a finitely generated abelian group. Hence an element of A^* which is a p^{th} power for infinitely many primes p must have finite order. The hypothesis on g thus implies that all eigenvalues a of g are roots of unity. Some power $u=g^r$ of g then has all eigenvalues equal to 1, i.e. u is unipotent. If char$(F)=q$ then for $q>0$ we have $u^{q^n}=1$, so the theorem is proved in this case. Suppose $q=0$. Clearly u inherits the hypothesis made on g: For infinitely many primes p, $u=x^p$ is solvable for some $x\in GL_n(A)$. The eigenvalues of such an x are then p^{th} roots of unity of degree $\leqq n$

over F, whence equal to 1 once p is sufficiently large, by Proposition (A.3) of the Appendix. For such p, x itself is unipotent, whence $x=\exp\left(\frac{1}{p}\log(u)\right)$. Put $u=1+X$, so that $X^n=0$, and put $L=\log(u)=X-\frac{X^2}{2}+\frac{X^3}{3}-\cdots(-1)^{n-1}\frac{X^n}{n}$. Then $E(t)=\exp(tL)=1+tL+\frac{(tL)^2}{2!}+\cdots+\frac{(tL)^n}{n!}$ is a polynomial in t with matrix coefficients over the ring $B=A[1/n!]$. Further, we know that $E(1/p)$ is a matrix in $GL_n(A)$ for infinitely many primes p. The next lemma therefore implies that the polynomial $E(t)$ is constant, whence $u=E(1)=E(0)=1$, and the theorem is proved.

(9.7) **Lemma.** *Let B be a finitely generated subring of a field F of characteristic* 0. *Let $f(t)=b_0+b_1t+\cdots+b_nt^n$, $b_n\neq 0$, be a polynomial in $B[t]$ such that $f(1/p)\in B$ for infinitely many primes p. Then f is a constant.*

Let C be a finitely generated integrally closed subring of F containing B and the inverse of the product of all non-zero coefficients of f. All but finitely many primes p are not invertible in C. Such a p belongs to finitely many height one prime ideals $\mathfrak{p}$ of C. For such a $\mathfrak{p}$ the ultrametric inequality for the $\mathfrak{p}$-adic valuation v implies that $v(f(1/p))=-n\cdot v(p)$. But for infinitely many p's as above $v(f(1/p))\geqq 0$ by hypothesis. Since $v(p)>0$ it follows that $n=0$, i.e. f is constant.

Appendix on Cyclotomic Extensions

Let F be a field, $\bar{F}$ an algebraic closure of F, and, for each integer $m\geqq 1$, let μ_m denote the group of m^{th} roots of unity in $\bar{F}$. Put

$$\varphi_F(m)=[F(\mu_m):F].$$

(A.1) *Examples.* 1. $F=\bar{F}$: $\varphi_F(m)=1$ for all m.

2. $F=\mathbf{Q}$: $\varphi_{\mathbf{Q}}(m)=\varphi(m)=\operatorname{Card}(\mathbb{Z}/m\mathbb{Z})^*$.

3. Let p be the characteristic exponent of F and write $m=p^r m'$ with m' prime to p. Then $\varphi_F(m)=\varphi_F(m')$. In fact $\mu_m=\mu_{m'}$.

4. $F=\mathbf{F}_q$ with $q=p^n$: For m prime to p, $\varphi_{\mathbf{F}_q}(m)=$ the order of q in $(\mathbb{Z}/m\mathbb{Z})^*$.

Let F_0 denote the prime field in F, and F_1 the algebraic closure of F_0 in F.

(A.2) **Lemma.** $\varphi_F(m)=\varphi_{F_1}(m)$ *for all* m.

Since the field F_1 is perfect and algebraically closed in F, F is a regular extension of F_1. Consequently the F-algebra $F\otimes_{F_1}F_1(\mu_m)$ is a field, isomorphic to $F(\mu_m)$; whence the lemma.

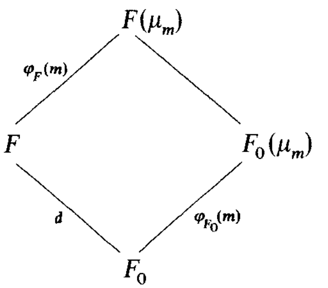
Remark. If F is a finitely generated extension of F_0 then so also are its subfields, whence F_1 is a finite extension of F_0.

(A.3) **Proposition.** *Suppose F is a finitely generated extension of its prime field F_0; let p denote its characteristic exponent.*

(a) *Given an integer* $n>0$, *there are only finitely many integers* m *prime to* p *such that* $\varphi_F(m) \leqq n$.

(b) *Suppose* $F_0 = \mathbf{Q}$. *There is an integer* $m_1 \geqq 1$ *such that for all integers* m *prime to* m_1, *one has* $\varphi_F(m) = \varphi_{\mathbf{Q}}(m)$ $(=\varphi(m))$.

In view of Lemma (A.2) and the Remark above, it suffices to treat the case when F is a *finite* extension of F_0, say of degree $d=[F:F_0]$. Then $d \cdot \varphi_F(m) = [F(\mu_m):F_0]$ is a multiple of $\varphi_{F_0}(m)$, so $\varphi_F(m) \geqq \varphi_{F_0}(m)/d$, and assertion (a) follows from the special case: $F=F_0$. When $F_0=\mathbf{Q}$ the assertion is an obvious property of the Euler φ-function. When $F_0=F_p$ and $p \nmid m$ we have $m \leqq |\mathbf{F}_p(\mu_m)| = p^{\varphi_{\mathbf{F}_p}(m)}$, whence $\varphi_{\mathbf{F}_p}(m) \geqq \dfrac{\log(m)}{\log(p)}$, thus proving (a) also in this case.



To prove (b) we may, after enlarging F if necessary, assume F is galois over $\mathbf{Q}$. Let $F^{ab} = F \cap \mathbf{Q}(\mu_\infty)$, where μ_∞ denotes the group of all roots of unity. Choose m_1 so that $F^{ab} \subset \mathbf{Q}(\mu_{m_1})$. The claim is that, for m prime to m_1, $\varphi_F(m) = \varphi_{\mathbf{Q}}(m)$, in other words that the extensions F and $\mathbf{Q}(\mu_m)$ of $\mathbf{Q}$ are linearly disjoint. Since they are galois over $\mathbf{Q}$ it suffices to show that $F \cap \mathbf{Q}(\mu_m) = \mathbf{Q}$. Since

$$F \cap \mathbf{Q}(\mu_m) \subset F \cap \mathbf{Q}(\mu_\infty) = F^{ab} \subset \mathbf{Q}(\mu_{m_1}),$$

it suffices to observe that $\mathbf{Q}(\mu_m) \cap \mathbf{Q}(\mu_{m_1}) = \mathbf{Q}$; the latter is a well known property of cyclotomic extensions of $\mathbf{Q}$ (cf. [17], Chap. IV, p. 75).

(A.4) **Proposition.** *Suppose* F *is a finitely generated extension of its prime field; let* p *denote its characteristic exponent; let* n *be an integer* >0. *The integers* $m \geqq 1$ *prime to* p *for which* $\varphi_F(m) \leqq n$ *are finite in number* (Proposition (A.3)(a)); *let* m_0 *denote their product. If* $\alpha \in \bar{F}$ *is such that* $[F(\alpha):F] \leqq n$ *and* $\alpha^m \in F$ *for some* m *prime to* $p \cdot m_0 \cdot n!$ *then* $\alpha \in F$.

All conjugates of α over F are of the form $w\alpha$ with $w \in \mu_m$. It follows that $N_{F(\alpha)/F}(\alpha) = w\alpha^r$ for some $w \in \mu_m$ and $r = [F(\alpha):F] \leqq n$. Note that r is prime to m so we can write $1 = ar + bm$ with $a, b \in \mathbb{Z}$. Then F contains the element $(w\alpha^r)^a \cdot (\alpha^m)^b = w^a \alpha$. It follows that $F(\alpha) = F(w^a)$ so that $r = \varphi_F(m')$ where m' = the order of w^a, a divisor of m, hence prime to $pm_0 n!$. But $\varphi_F(m') = r \leqq n$ so, by definition of m_0, we must have $m'=1$. Thus $w^a = 1$ so $\alpha \in F$ as claimed.

10. Euler Characteristics of Groups of Type (*FP*)

As usual k denotes a commutative ring and G a group.

(10.1) *Definition.* We say G is of *type* (*FP*) *over* k if k (with trivial G action) is a kG-module of type (*FP*). In this case we call $r_k = r_{k/kG} \in T(kG)$ the *complete Euler characteristic* of G over k, and $r_k(1) \in k$, which we denote $\chi(G)$, the *Euler characteristic* of G over k. We also define the *homological Euler characteristic* of G over k to be $\tilde{\chi}(G) = \sum_{\tau \in T(G)} r_k(\tau) \in k$.

Remarks. 1. Let $0 \to P_n \to \cdots \to P_0 \to k \to 0$ be a resolution with each $P_i \in \mathscr{P}(kG)$, so that $r_k = \sum (-1)^i r_{P_i}$. If k' is a commutative k-algebra then tensoring with k' over k yields a similar resolution of k' over $k'G$, so G is of type (*FP*) over k', and $r_{k'} \in T(k'G)$ is the image under base change of $r_k \in T(kG)$. Similarly for $\chi(G)$ and $\tilde{\chi}(G)$.

2. If G is of type (*FP*) over any $k \neq 0$ then, since the augmentation ideal of kG is finitely generated, so also is the group G (cf. [6], Proof of Th. 4).

3. If G is *finite* then G is of type (*FP*) over k if and only if its order $|G|$ is invertible in k. Then k is isomorphic to $(kG)e$ where e is the central idempotent $e = \frac{1}{|G|} \sum_{s \in G} s$, and $r_k(s) = \frac{1}{|Z_G(s)|}$ for all $s \in G$ (cf. Prop. (5.8)). We have $\chi(G) = \frac{1}{|G|}$ and $\tilde{\chi}(G) = 1$.

4. Suppose G is *abelian* and of type (*FP*) over k. From 2. we conclude that $G = H \times F$ with H finite and F free abelian. Since H has finite cohomological dimension over k, its order $|H|$ must be invertible in k. As in 3. therefore $kH = k \times R$ for a certain ring R. Similarly $kG = (kH)[F] = k[F] \times R[F]$, and the kG-module k is annihilated by $0 \times R[F]$. Therefore $r_k \in k[F] \times 0$. A free resolution of k over kF can be obtained from the Koszul complex associated to the sequence $1 - s_1, \ldots, 1 - s_n$ where $s_1, \ldots, s_n$ is a free basis of F. From this one sees that $r_k = 0$ if $n > 0$, i.e. if $F \neq \{1\}$. In conclusion, $r_k = 0$ *unless G is finite*, the case discussed in 3.

(10.2) **Proposition.** *If k is a kG-module of type (FP) the same is true of every kG-module $M \in R_k(kG)$, and*

$$r_M = \bar{\chi}_M \cdot r_k \in T(kG). \tag{1}$$

If k is a field the k-modules $H_i(G, M)$ and $H^i(G, M)$ are finite dimensional, and one has

$$\sum_{i \geq 0} (-1)^i \dim H_i(G, M) = \sum_{\tau \in T(G)} \chi_M(\tau^{-1}) r_k(\tau) \tag{2}$$

$$\sum_{i \geq 0} (-1)^i \dim H^i(G, M) = \sum_{\tau \in T(G)} \chi_M(\tau) r_k(\tau). \tag{3}$$

[The left sides of (2) and (3) should be interpreted as elements of k.]

Let $0 \to P_n \to \cdots \to P_0 \to k \to 0$ be a resolution with each $P_i \in \mathscr{P}(kG)$, so that $r_k = \sum_i (-1)^i r_{P_i}$. Then (Prop. (5.5)) $0 \to P_n \otimes_k M \to \cdots \to P_0 \otimes_k M \to M \to 0$ is a resolution with each $P_i \otimes_k M \in \mathscr{P}(kG)$ and $r_{P_i \otimes_k M} = \bar{\chi}_M \cdot r_{P_i}$. Thus $r_M = \sum_i (-1)^i \bar{\chi}_M \cdot r_{P_i}$ $= \bar{\chi}_M \cdot r_k$. The augmentation $\sum_\tau r_M(\tau) = \sum_\tau \chi_M(\tau^{-1}) r_k(\tau) \in k$ is the alternating sum of the ranks of the k-modules $(P_i \otimes_k M) \otimes_{kG} k$, these being the components of a complex whose homology is $H_i(G, M)$, whence (2). Similarly $H^*(G, M)$ is the homology of the complex $\mathrm{Hom}_{kG}(P_i, M)$, so (3) will follow if we show that the k-module $\mathrm{Hom}_{kG}(P_i, M)$ has rank $\sum_{\tau} \chi_M(\tau) \cdot r_{P_i}(\tau)$. But this results from Proposition (6.15) (with $H = G$).

Remark. It is clear from the above proof that the assumption that k is a field is stronger than necessary; one only needs to know that the k-modules $H_i(G, M)$ and $H^i(G, M)$ are of type (FP), in which case we use their ranks over k in place of "dim" in formulas (2) and (3). This remark applies as well to the next corollary.

(10.3) **Corollary.** *If k is a field and a kG-module of type (FP) then*

$$\tilde{\chi}(G) = \sum_{i \geqq 0} (-1)^i \dim H_i(G, k) = \sum_{i \geqq 0} (-1)^i \dim H^i(G, k). \tag{4}$$

This is the case $M = k$ of formulas (2) and (3). [Of course the members of (4) must be interpreted as elements of k.]

Corollary (10.3) shows that our definition of $\tilde{\chi}(G)$ agrees with that of K. Brown [6], §4, in case $k = \mathbb{Z}$. It also motivates the terminology "homological Euler characteristic" for $\tilde{\chi}(G)$.

(10.4) Recall ((3.1) and (3.2)) that the center $Z(kG)$ of kG acts on $T(kG)$ so that $T: kG \to T(kG)$ is $Z(kG)$ linear.

Proposition (cf. Stallings [25], or [12], §8, 10, Th. 7). *Suppose k is a kG-module of type (FP).*

(a) *If $c \in Z(kG)$ has augmentation $c_0 \in k$ then $c r_k = c_0 r_k$. In particular $z r_k = r_k$ for all $z \in Z(G)$.*

(b) *If $\chi(G) \neq 0$ then $Z(G)$ is a finite group whose order is invertible in k.*

Part (a) results from Proposition (3.1) (cf. Remark (3.2)). It implies that $r_k(zs) = r_k(s)$ for all $s \in G$ and $z \in Z(G)$; in particular $r_k(z) = r_k(1) = \chi(G)$ for all $z \in Z(G)$. But only finitely many elements of $Z(G)$ can belong to $\mathrm{supp}(r_k)$. Thus, if $\chi(G) \neq 0$, then $Z(G)$ is finite. Since k has finite projective dimension as a $kZ(G)$-module (because G is of type (FP)) the finite group $Z(G)$ must have order invertible in k.

Remarks. 1. When $k = \mathbb{Z}$ the conclusion of (b) implies that $Z(G) = \{1\}$.

2. The hypothesis $r_k(1) \neq 0$ in (b) can be relaxed to suppose only that $\mathrm{supp}(r_k)$ contains some *finite* conjugacy class of G.

3. If $k = \mathbb{Z}$ and G is residually finite then $\tilde{\chi}(G) \neq 0$ implies $Z(G) = \{1\}$ (cf. [12], §8.9, Prop. 13). For in this case Proposition (10.5)(d) below implies that $\chi(G) = \tilde{\chi}(G)$.

(10.5) **Proposition.** *Suppose k is a kG-module of type (FP). Let H be a subgroup of finite index in G.*

(a) *k is a kH-module of type (FP), and*

$$r_{k/kH}(t) = r_{k/kG}(t) \cdot [Z_G(t) : Z_H(t)]$$

for all $t \in H$.

(b) $\chi(H) = \chi(G) \cdot [G : H]$.

Suppose k is an integral domain of characteristic zero in which no rational prime is invertible.

(c) $\tilde{\chi}(H) = \tilde{\chi}(G) \cdot [G:H]$.

(d) *If G is residually finite then $\chi(G) = \tilde{\chi}(G)$ (and $\chi(H) = \tilde{\chi}(H)$).*

(a) is a special case of Corollary (6.3), and (b) is the case $t=1$ of (a); (c) results from Corollary (6.13) and (d) from Corollary (6.10).

(10.6) *Comparison with Definitions of Serre* [24] *and Brown* [6]. Let H be a subgroup of finite index in G, and suppose $\mathbb{Z}$ is a $\mathbb{Z}H$-module of type (FP) (thus G is of type (VFP) in the notation of [24]). Then $\mathbb{Q}$ is a $\mathbb{Q}G$-module of type (FP), since $\mathbb{Q}$ is a $\mathbb{Q}G$-direct summand of $\mathbb{Q}[G/H] = \mathbb{Q}G \otimes_{\mathbb{Q}H} \mathbb{Q}$. Over $\mathbb{Q}$ then $\chi(G)$ is defined and we have

$$\chi(G) = \frac{\chi(H)}{[G:H]} \qquad \text{(Prop. (10.5)(b))}. \tag{5}$$

It follows that this $\chi(G)$ coincides with that defined by Serre [24] when G is of "type (VFL)", say when H is of "type (FL)" over $\mathbb{Z}$. As observed in [6], page 218, this extension of the definition of $\chi(G)$ so that $\chi(G) \in \mathbb{Z}$ when G is of type (FP) over $\mathbb{Z}$ affirmatively answers a question of Serre ([24], p. 101).

Let $\chi_B(G)$ denote the Euler characteristic defined by Ken Brown ([6], §4). It follows from Corollary (10.3) that $\chi_B(H) = \tilde{\chi}(H)$, and so $\tilde{\chi}_B(G) = \tilde{\chi}(H)/[G:H]$. This definition is justified by the fact that

$$\tilde{\chi}(G) = \frac{\tilde{\chi}(H)}{[G:H]} \qquad \text{(Prop. (10.5)(b))} \tag{6}$$

whenever G is of type (FP) over $\mathbb{Z}$. Conjecturally $\chi_B(G) = \chi(G)$, equivalently $\tilde{\chi}(G) = \chi(G)$ whenever G is of type (FP) over $\mathbb{Z}$. Proposition (10.5)(d) affirms this whenever G is residually finite.

If G is of type (FP) over a k different from $\mathbb{Z}$, $\chi(G)$ and $\tilde{\chi}(G)$ need no longer coincide. Formula (5) remains valid, but not (6) in general.

(10.7) **Proposition.** *Suppose k is a kG-module of type (FP).*

(a) *If $\alpha \in \mathrm{Aut}(G)$ and $s \in G$ then $r_k(\alpha s) = r_k(s)$.*

(b) *If $\tau \in T(G)$ and $r_k(\tau) \neq 0$ the orbit of τ under $\mathrm{Aut}(G)$ acting on $T(G)$ is finite.*

Extending α to an automorphism of kG, we have $k^{(\alpha)} = k$, with the notation of (3.5), so (3.5)(3) implies r_k is fixed by α acting on $T(kG)$, whence (a). This implies that the finite set $\mathrm{supp}(r_k) \subset T(G)$ is stable under $\mathrm{Aut}(G)$, whence (b).

(10.8) *Normal Subgroups.* Let H be a normal subgroup of G and put $G' = G/H$. The inclusion $\varepsilon\colon H \to G$ and projection $\pi\colon G \to G'$ induce homomorphisms $\varepsilon_*\colon T(kH) \to T(kG)$ and $\pi_*\colon T(kG) \to T(kG')$. G acts by conjugation on H, therefore also on $T(kH)$, $H_*(H, k)$, etc. In fact this makes $H_*(H, k)$ a kG'-module.

(10.9) **Theorem.** *Suppose k is a kH-module of type (FP); put $r_H = r_{k/kH}$.*

(a) *Let $t \in H$. We have $r_H(sts^{-1}) = r_H(t)$ for all $s \in G$. Put $n_t = [G: H \cdot Z_G(t)]$. Then $r_H(t) = 0$ and $(\varepsilon_* r_H)(t) = 0$ unless n_t is finite, in which case $(\varepsilon_* r_H)(t) = n_t \cdot r_H(t)$. In particular, $(\varepsilon_* r_H)(1) = r_H(1) = \chi(H)$.*

(b) *kG' is a kG-module of type (FP), and $r_{kG'/kG}=\varepsilon_* r_H$. Hence* (see (2.10)) *there is a natural homomorphism $\pi^*: T(kG')\to T(kG)$ and $\pi^* T_{G'}(1)=\varepsilon_* r_H$.*

(c) *Let $\tau'\in T(G')$. Then $\pi^*\tau'$ has support among those $\tau\in T(G)$ for which $\pi\tau=\tau'$. Hence $\pi_*\pi^*\tau'=L(\tau')\tau'$ for a certain element $L(\tau')\in k$.*

(d) *Let $s'\in G'$. There is a finite complex $P'=(P_i')$ of finitely generated projective k-modules, whose homology is $H_*(H,k)$, and an endomorphism $g'=(g_i')$ of P' such that,* (i) *$L(s')=\sum_i(-1)^i T_{P_i'/k}(g_i')$, and* (ii) *$H_*(g')$ is the natural action of s'^{-1} on $H_*(H,k)$. We have $L(1)=\tilde{\chi}(H)$. If the k-modules $H_i(H,k)$ are projective then $\bar{L}$ is the virtual character of the natural action of G' on $H_*(H,k)$.*

(e) *Suppose further that k is a kG'-module of type (FP); put $r_{G'}=r_{k/kG'}$. Then k is also a kG-module of type (FP); put $r_G=r_{k/kG}$. We have*

$$r_G=\pi^*(r_{G'})=\sum_{\tau'\in T(G')} r_{G'}(\tau')\cdot\pi^*\tau'.$$

Explicitly, if $s\in G$ and $\pi s\in\tau'$ then $r_G(s)=r_{G'}(\pi s)\cdot(\pi^*\tau')(s)$. If $s\in H$ then $r_G(s)=\chi(G')\cdot(\varepsilon_* r_H)(s)$, which is evaluated in (a). *For $s=1$ we obtain $\chi(G)=\chi(G')\cdot\chi(H)$.*

(f) *$\pi_* r_G=L\cdot r_{G'}$. Explicitly, if $\tau'\in T(G')$ then $L(\tau')\cdot r_{G'}(\tau')=\sum_{\pi\tau=\tau'} r_G(\tau)$. Hence $\tilde{\chi}(G)=\sum_{\tau'\in T(G')} L(\tau')r_{G'}(\tau')$.*

If $t\in H$ and $\tau=T_G(t)$ then $n_t=[G:H\cdot Z_G(t)]$ is the number of H-conjugacy classes σ contained in τ. If n_t is finite let $\tilde{\tau}$ denote the sum of these σ's in $T(kH)$. Applying Proposition (10.7) to the action of G by conjugation on H, we conclude that r_H is a linear combination of such $\tilde{\tau}$'s. Since $\varepsilon_*\tilde{\tau}=n_t\cdot\tau$, this proves (a).

Let

$$0\to P_n\to\cdots\to P_0\to k\to 0 \tag{8}$$

be a kH-resolution with each $P_i\in\mathscr{P}(kH)$. Then

$$0\to P_n\otimes_{kH}kG\to\cdots\to P_0\otimes_{kH}kG\to kG'\to 0 \tag{9}$$

is a kG-resolution with each $P_i\otimes_{kH}kG\in\mathscr{P}(kG)$, whence (b).

Let $s\in G$. If M is a kH-module we shall write $M^{(s)}$ for the kH-module with underlying k-module M, but where $a\in kH$ acts on $x\in M$ by $x\mapsto xsas^{-1}$. One can identify $M^{(s)}$ with $M\otimes_{kH}kH$ with respect to the change of rings $kH\to kH$, $a\mapsto s^{-1}as$. The functor $M\mapsto M^{(s)}$ induces k-isomorphisms $H_*(H,M)\to H_*(H,M^{(s)})$. We have $k^{(s)}=k$, and the corresponding automorphism of $H_*(H,k)$ is just the natural action on $H_*(H,k)$ of s'^{-1}, where $s'=\pi s\in G'$. We can compute this automorphism from a kH-homomorphism of resolutions

$$\begin{tikzcd}
0 \arrow[r] & P_n \arrow[r] \arrow[d, "g_n"] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "g_0"] & k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_n^{(s)} \arrow[r] & \cdots \arrow[r] & P_0^{(s)} \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

Here g_i is a k-linear endomorphism of P_i such that $g_i(xa)=g_i(x)sas^{-1}$ for $x\in P_i$ and $a\in kH$. The k-module $P_i'=P_i\otimes_{kH}k\in\mathscr{P}(k)$ can be canonically identified with

$P_i^{(s)} \otimes_{kH} k$, so that $g'_i = g_i \otimes_{kH} k$ is then a k-linear endomorphism of P'_i. The complex (P'_i) has homology $H_*(H, k)$, and its endomorphism (g'_i) induces in $H_*(H, k)$ the natural action of s'^{-1}. Put

$$L'(s') = \sum_{i \geq 0} (-1)^i \, T_{P'_i}(g'_i) \in k.$$

(It is not difficult to see that this depends only on s'.) The arguments above show that, if the k-modules $H_*(H, k)$ are projective, then $\bar{L}'$ is the virtual character of the action of G' on $H_*(H, k)$. Further, if $s = 1$ we can take all g_i to be the identity. Then we find that $L'(1) = \sum (-1)^i r_{P'_i/k}$. Since $r_H = \sum (-1)^i r_{P_i/kH}$ we conclude that $L'(1) = \sum_{\sigma \in T(H)} r_H(\sigma) = \tilde{\chi}(H)$.

Consider now the endomorphism $g_i \otimes s$ of $P_i \otimes_{kH} kG$ sending $x \otimes t$ to $g_i(x) \otimes st$ for $x \in P_i$, $t \in G$. It is well defined since, for $h \in H$,

$$g_i(xh) \otimes st = g_i(x) s h s^{-1} \otimes st = g_i(x) \otimes sht.$$

Moreover it is kG-linear, and $(g_i \otimes s)$ is an endomorphism of the resolution (9) covering the endomorphism $a \mapsto s' a$ of kG'. By definition of π^*, therefore,

$$\pi^* T_{G'}(s') = \sum_{i \geq 0} (-1)^i \, T_{P_i \otimes_{kH} kG}(g_i \otimes s).$$

To evaluate this more explicitly, fix an i, put $P = P_i$, $g = g_i$, and let $x_j \in P$, $f_j \colon P \to kH$ be a finite kH-coordinate system. Then $x_j \otimes 1$, $f_j \otimes 1_{kG}$ is a finite kG-coordinate system of $P \otimes_{kH} kG$, so $T_{P \otimes_{kH} kG}(g \otimes s) = T_{kG}(\alpha_i)$, where

$$\alpha_i = \sum_j (f_j \otimes 1)((g \otimes s)(x_j \otimes 1)) = \sum_j (f_j \otimes 1)(g(x_j) \otimes s) = a_i s,$$

where $a_i = \sum_j f_j(g(x_j)) \in kH$. We thus have $\pi^* T_{G'}(s') = T_{kG}(as)$ where $a = \sum_{i \geq 0} (-1)^i a_i \in kH$. It follows that $\pi^* T_{G'}(s')$ has support among those τ for which $\pi\tau = T_{G'}(s')$, and so $\pi_* \pi^* T_{G'}(s') = L(s') \cdot T_{G'}(s')$ for a certain $L(s') \in k$. This proves (c).

We compute $\pi_* \pi^* T_{G'}(s')$ directly by applying $- \otimes_{kG} kG'$ to the resolution (9) and its endomorphism $(g_i \otimes s)$. We can identify

$$(P_i \otimes_{kH} kG) \otimes_{kG} kG' \quad \text{with} \quad (P_i \otimes_{kH} k) \otimes_k kG' = P'_i \otimes_k kG',$$

so that the endomorphism $(g_i \otimes s) \otimes_{kG} kG'$ becomes $g'_i \otimes s'$, sending $x' \otimes t'$ to $g'_i(x') \otimes s' t'$ for $x' \in P'_i$, $t' \in G'$. Thus

$$\begin{aligned} \pi_* \pi^* T_{G'}(s') &= \sum_{i \geq 0} (-1)^i \, T_{P'_i \otimes kG'}(g'_i \otimes s') \\ &= \Big(\sum_{i \geq 0} (-1)^i \, T_{P'_i/k}(g'_i)\Big) \cdot T_{G'}(s') = L'(s') \cdot T_{G'}(s'). \end{aligned}$$

This shows that $L'(s') = L(s')$, and thus incidentally that $L'(s')$ depends only on s'. Now the claims made about L in assertion (d) follow from the properties of L' established above.

Since kG' is a kG-module of type (FP) so also is every kG'-module of type (FP), in particular k itself in case k is a kG'-module of type (FP). We then further have

$r_{k/kG} = \pi^*(r_{k/kG'})$, whence $r_G = \sum_{\tau' \in T(G')} r_{G'}(\tau')\,\pi^*\tau'$. According to (c) the elements $\pi^*\tau'$ have pairwise disjoint supports, whence the formulas for $r_G(s)$ given in (e). The formulas in (f) are immediate from (c) and (e). This proves Theorem (10.9).

(10.10) *Examples.* 1. Suppose, in the setting of Theorem (10.9), that $G' = G/H$ is finite, of order invertible in k. Then (see (10.1), Remark 3) k is a kG'-mocule of type (FP), and $r_{G'}(s') = 1/|Z_{G'}(s')|$ for $s' \in G'$. From Theorem (10.9)(f) we thus find that

$$L(s')/|Z_{G'}(s')| = \sum_{\pi\tau = T_{G'}(s')} r_G(\tau)$$

and

$$\tilde{\chi}(G) = \frac{1}{|G'|} \sum_{s' \in G'} L(s').$$

K. Brown points out that the latter translates into virtual characters the isomorphism of $H^*(G, k)$ with $H^*(H, k)^{G'}$.

2. Suppose that $G = H \cdot Z_G(H)$, for example that H is central in G, or that the k-modules $H_i(H, k)$ are projective and G' acts trivially on them. In any of these cases we have $L(s') = L(1) = \tilde{\chi}(H)$ for all $s' \in G'$, and so $\pi_* \pi^* r' = \tilde{\chi}(H) \cdot r'$ for all $r' \in T(kG')$. Suppose further that G' is of type (FP) over k. Then Theorem (10.9)(f) implies that

$$\tilde{\chi}(G) = \tilde{\chi}(H)\,\tilde{\chi}(G').$$

In case H is abelian and infinite then (see (10.1), Rem. 4) $r_H = 0$, so we conclude that $\chi(G) = \tilde{\chi}(G) = 0$. This strengthens Stallings' theorem (Prop. (10.4)) in special cases.

(10.11) **Corollary.** *Suppose, in Theorem* (10.9), *that k is an integral domain of characteristic zero, G' is finite, and no prime divisor of $|G'|$ is invertible in k. Suppose k is a kG-module of type (FP), and put $r_G = r_{k/kG}$.*

(a) $\pi_* r_G = (\tilde{\chi}(H)/|G'|) \cdot T_{G'}(1)$

(b) $L = \tilde{\chi}(H) \cdot \chi_{kG'}$.

It follows from Swan's Theorem (Cor. (6.8)) that

$$\pi_* r_G = n \cdot T_{G'}(1), \qquad \text{where } n = \pi_* r_G(1). \tag{10}$$

The field of fractions K of k is a KG'-module of type (FP), and $r_{G'} = r_{K/KG'}$ is given by $r_{G'}(s') = 1/|Z_{G'}(s')|$ (see (10.10)). From (10.9)(f) we have $\pi_* r_G = L \cdot r_{G'}$, so $\pi_* r_G(s') = L(s')/|Z_{G'}(s')|$. It follows now from (10) that $L(s') = 0$ for $s' \neq 1$, and that $n = L(1)/|G'|$. By (10.9)(d), $L(1) = \tilde{\chi}(H)$, whence (a) and (b).

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