

Pudet dicere, ad quot figurarum loca has computationes,
otiosus eo tempore, produxi. Nam tunc sane nimis delectabar
inventis hisce.

I. Newton, Letter to Oldenburg, October 24, 1676

Local Normal Forms of Functions

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To Jean-Pierre Serre

Introduction

This article contains lists of normal forms of functions in the neighbourhoods of critical points (the classification of all singularities with the number of modules $m=0, 1$ and 2 or with multiplicity $\mu \leq 16$ included) and the proofs of most of these classification theorems (the classification of unimodular singularities included).

For instance, the number ν of stable μ -equivalence classes (see the definitions below), is given for $\mu \leq 16$ by the following table:

μ	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
ν	1	1	1	2	2	3	3	4	4	7	11	15	14	17	22	32

The motivations for and the applications of the classification of critical points of functions, and their relation to Coxeter groups, braids, automorphic functions, platonic, caustics, wavefronts and stationary phase method are discussed in [1].

The technical results are formulated in §§ 1 and 2. We begin with some general remarks, arising from these empirical facts.

1. Normal Forms

For many classes of singularities there exist simple normal forms.

The following definitions give an exact meaning to these words.

The group of germs (or jets) of diffeomorphisms of \mathbb{C}^n , leaving O invariant, acts on the space of germs (jets) of functions at O . A *singularity class* is a subset of the space of germs of functions, invariant under this action. Each orbit is such a class. Two germs (jets) belonging to the same orbit are *equivalent*.

Another example of a class is the *μ -equivalence class*. The multiplicity (or the Milnor number) μ of a critical point $O \in \mathbb{C}^n$ of a function f is the Poincaré index of the vector field $\text{grad } f$ at the singular point O . Germs of functions f_0, f_1 at a critical point O are *μ -equivalent*, if there exist a holomorphical familie of functions f_t with critical point O of constant multiplicity μ (independent of t , $0 \leq t \leq 1$).

To define the *normal forms*, let us consider the space of polynomials, $M = \mathbb{C}[x_1, \dots, x_n]$, as a subset in the space of germs of functions $f(x_1, \dots, x_n)$ at O .

The *normal form for a given class K of functions* is a mapping $\Phi: B \rightarrow M$ from a finite dimensional linear “parameter space” B to the space of polynomials, satisfying three conditions:

1. $\Phi(B)$ intersects all the orbits, belonging to K ;
2. the counterimage of every orbit in B is a finite set;
3. the counterimage of the whole complement to K lies in some hypersurface

in B .

The normal form is *polynomial* (resp. *affine*) if Φ is a polynomial (resp. affine) mapping. An affine normal form is *simple*, if Φ is of the form:

$$\Phi(b_1, \dots, b_r) = \varphi + b_1 x^{m_1} + \dots + b_r x^{m_r},$$

where φ is a fixed polynomial, b_i are complex numbers and x^{m_i} are monomials. In most applications the polynomial φ itself is “simple”, that is—a sum of few monomials.

The existence of a unique normal form (even a polynomial one) for the whole μ -equivalence class is by no mean evident a priori. It is a rather surprising empirical fact, that such normal forms exist for all the singularities in the lists of § 1 (and hence for all the singularities with modules number $m=0, 1$ and 2).

Most of our normal forms are simple; perhaps, all singularities of § 1 have simple normal forms. It is not known, how vast is the class of singularities, whose μ -equivalence classes admit simple (or polynomial) normal forms (here it is natural to consider stable equivalence classes, see 5).

2. The Series of Singularities

All singularities in our lists are divided into some *series*, such as A_k or Z_k . We shall use light characters A, \dots, Z to denote μ -equivalence classes and bold characters $\mathbf{A}, \dots, \mathbf{Z}$ to denote singularity classes which contain more than one μ -equivalence classes.

While the series undoubtedly exist, they have yet no formal definition. Let us consider, for instance, series \mathbf{A} and \mathbf{D} , formed by the orbits of germs

$$A_k = x^{k+1} + y^2 (k \geq 1), \quad D_k = x^2 y + y^{k-1} (k \geq 4).$$

The A_k and D_k classes adjacences are

$$\begin{array}{ccccccc} A_1 & \leftarrow & A_2 & \leftarrow & A_3 & \leftarrow & A_4 & \leftarrow & \dots \\ & & & & \uparrow & & \uparrow & & \\ & & & & D_4 & \leftarrow & D_5 & \leftarrow & \dots \end{array}$$

[A class L of singularities is *adjacent* to class K , $K \leftarrow L$, if every germ (sufficient jet) of $f \in L$ can be deformed into a germ (sufficient jet) in K by an arbitrary small deformation.]

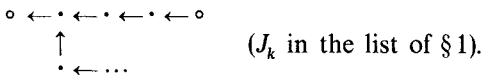
It is clear, that A_k and D_k are two different series. What is, however, the exact meaning of this?

To define series **A** is to define such a rule of reversing adjacency arrows, that leads from A_k to A_{k+1} and not to D_{k+1} . In the case of **A** singularities the rule is easy to find (**A** singularities have corank 1). In more complicated cases (see §§ 1, 2) one can also find the rules of reversing arrows (different in different cases). So, series with one or more subscripts and parameters occur (e.g., $T_{k,l,m} = axyz + x^k + y^l + z^m$).

In all these cases one can find the definition of the series *after* one have found the series, while no a priori general definition is known. It is clear, however, that the series are connected with singularities of infinite multiplicity (e.g. $\mathbf{D} \sim x^2 y$, $\mathbf{T} \sim xyz$). So the hierarchy of series reflects the hierarchy of nonisolated singularities.

3. Periodicity

The decomposition of many singularity classes into μ -equivalence classes is in some sence periodic. The whole decomposition (stratification) can be regarded as a chain of similar fragments (“animals”). Each animal is built of points (μ -equivalence classes), two of which are distinguished (the “head” and the “tail”). An animal can also contain some classes whose adjacency arrows join the tail to the head and some “legs” (infinite series of classes). The tail of each animal in the chain is adjacent to the head of the following one. For example, the stratification of the corank 2 singularities with the 3-jet equal to x^3 is a chain of animals each consisting of 5 points and 1 infinite leg:



The phenomenon of periodicity is only partially explained and for quasi-homogenous singularities only (the explanation is based upon some root technique for the quasihomogenous Lie algebra, related to that of Enriques and Demazure [2], see § 3). However this periodicity arises in all the calculations, connected with our classification theorems (so that it is sufficient to consider only first animal from every animal chain). This empirical periodicity have at present no satisfactory explanation. Like the existence of series, it suggests some algebraic structure of the set of all μ -equivalence classes.

4. Small Modules Number Classes

The main characteristic of a singularity class is from the point of view of applications its codimension c in the space of germs of functions with critical point O and critical value 0. Indeed, a generic function has only nondegenerate singularities (singularities with codimension $c=0$). Degenerate singularities appears inevitably only when *families* of functions depending upon parameters are considered. A class **K** of codimension c is unavoidable by small deformation if the number of parameters l is $\geq c$.

So in all the applications to l -parametrical problems we need classification of all singularities up to codimension l (that is, we need classify such a set of classes, that the complement to their union be of codimension larger than l).

This problem is not to be confused with the classification of singularities with orbit codimension $\leq l$ (that is, with $\mu \leq l+1$). The only value of the last problem for applications is its relation with the former one.

From the topological point of view the main characteristic of a singularity is the multiplicity μ of the critical point (equal to the number of nondegenerate critical points, to which a given point bifurcate after a small variation; not to be confused with the algebraico-geometrical multiplicity of the hypersurface $f^{-1}(0)$ at O).

An unexpected result of our calculation is the fact, that algebraically the most natural classification theorems are not those of singularities of small codimension c or of small multiplicity μ , but the classification of singularities with small modules number, m .

The modules number m of a germ of function $f: (\mathbb{C}^n, O) \rightarrow (\mathbb{C}, O)$ at critical point O is the minimal number M , such that some neighbourhood of a sufficient k -jet of the function f at O is covered by a finite number of no more than M -parametrical families of the orbits of the group of diffeomorphisms germs $(\mathbb{C}^n, O) \rightarrow (\mathbb{C}^n, O)$ acting on the space of functions with critical point O and critical value 0.

The modules number m is equal to the dimension of the $\mu = \text{const}$ stratum in the base space of the versal deformation minus 1 (see Gabriellov [3]). So the codimension c of this stratum in the space of germs of functions with critical value 0 at the critical point O , the multiplicity μ and the modules number m are related by the formula

$$\mu = c + m + 1.$$

At present, the complete classifications are known for:

- (i) all singularities with $c \leq 10$;
- (ii) all singularities with $\mu \leq 16$;
- (iii) all singularities with $m \leq 2$.

The singularities with the modules number $m=0, 1$ and 2 are called *simple*, *unimodular* and *bimodular* respectively. Lists of them are given in §1. A study of these lists leads to the following 4 conclusions:

1. The simple singularities are classified by the Coxeter groups A_k, D_k, E_6, E_7, E_8 (that is, by the platonic in the usual 3-space, see [4]).

2. The unimodular singularities form one infinite series $T_{k,l,m}$ and 14 "exceptional" families, generated by 14 quasihomogenous singularities ([5]).

All the quasihomogenous unimodular singularities can be constructed from 14 remarkable Lobatchevski triangles and 3 remarkable Euclidean triangles like simple singularities are constructed from platonic (Dolgatchev [6]).

3. The bimodular singularities form 8 infinite series and 14 "exceptional" families, generated by quasihomogenous singularities.

All the quasihomogenous bimodular singularities can be constructed from 6 Lobatchevski quadrangles and 14 Lobatchevski triangles. In the last case one

has to consider 2-, 3- and 5-fold coverings of the surface, constructed in the canonical way (Dolgachev).

4. All the singularities with the modules number $m=1$ and 2 are classified by the degenerations of elliptic curves, studied by Kodaira (Koulikov [7]). The Koulikov construction begins with a blowing up of 1, 2 or 3 points of the surface on the minimal resolution of the degenerate fiber; then one has to blow down this (resolved) fiber.

Unfortunately, all these coincidence of different classifications have only a posteriori proofs and thus depend on our calculations.

5. Stabilization

Two germs $f: (\mathbb{C}^n, O) \rightarrow (\mathbb{C}, 0)$ and $g: (\mathbb{C}^m, O) \rightarrow (\mathbb{C}, 0)$ are *stably equivalent*, if they become equivalent after a direct addition to both of nondegenerate quadratic forms (e.g., $f(x)=x^3$ is stably equivalent to $g(x, y)=x^3+y^2$, but not to $h(x, y)=x^3$). Two stably equivalent germs on equidimensional spaces are equivalent. In §1 we shall not show explicitly the arguments of functions when all the arguments are presented in the formulae.

The *corank* of a germ of a function at a critical point is the nullity of its second differential. Every germ of corank r is stably equivalent to a germ of a function in r variables. Classifying functions up to the stable equivalence one normally chooses a normal form with a minimal number of variables (equal to the corank).

Our calculations show, however, that sometimes *one can obtain more natural normal forms with more variables*.

The simplest case is the inclusion of the singularities $x^p+ax^2y^2+y^q$ in the series T ; series $\mathbf{W}^\# \sim (x^2+y^3)^2$ provides more interesting examples.

To formulate the general stabilisation problem we need some definitions. Let $f \in \mathbb{C}[[x_1, \dots, x_n]]$. The *support* of f is the set $\text{supp } f \subset \mathbb{Z}^n \subset \mathbb{R}^n$ of all indexes \mathbf{m} of the monomials $x^\mathbf{m}$ with nonzero coefficients in f . The *Newton polyhedron* $\Gamma = \Gamma(f)$ is the boundary of the convex hull of the union of all positive ortants $\mathbf{m} + \mathbb{R}_+^n$, $\mathbf{m} \in \text{supp } f$.

A function f is Γ -*nondegenerate* if the multiplicity μ of its critical point O has the minimal possible value ν , for functions with a given Newton polyhedron Γ . An explicit formula for this *Newton number* $\nu(\Gamma)$ is given by A. G. Kouchnirenko (see [1]).

Most of our normal forms are Γ -nondegenerate functions. It is not known, how vast is the class of functions stably equivalent to Γ -nondegenerate ones. For Γ -nondegenerate functions classification and reducing to normal forms can be done in a very explicit way: these problems are rather stereometrical (see [8]).

Our lists of normal forms are given in §1. They contain all the singularities with the modules number $m=0, 1$ and 2, all the singularities with $\mu \leq 16$, all the singularities of corank 2 with nonzero 4-jet, all the singularities of corank 3 with a 3-jet, which determine an irreducible cubic, and some other singularities.

§2 is an outline of the proofs of these results. It contains the statements of 105 theorems. These theorems together with the lists of §1 form a kind of singu-

larity determinant like plant determinators: an algorithm is described for finding every singularity's place in the lists of § 1.

Most of the theorems of § 2 are proved in § 3. All these proofs depend heavily on the geometrical technique, described in [8]. From this geometrical point of view, the formulae of §§ 1 and 2 are to be considered as descriptions of supports and Newton polyhedrons, and to prove the theorems one has rather to manipulate with these polyhedrons than to calculate.

§ 1. Lists of Normal Forms

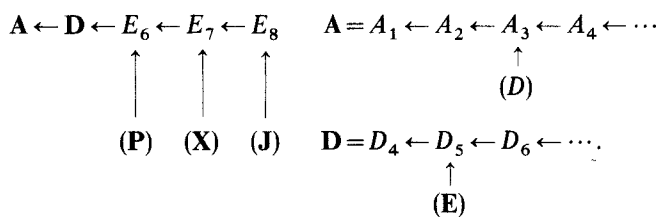
Letters A, \dots, Z stand here for stable equivalence classes of function germs (or families of function germs).

I. Singularities with Modules Number $m=0, 1$ and 2

I.0. Simple Singularities ($m=0$). There are 2 infinite series A, D , and 3 "exceptional" singularities E_6, E_7, E_8 :

A_k	$x^{k+1}, k \geq 1$
D_k	$x^2y + y^{k-1}, k \geq 4$
E_6	$x^3 + y^4$
E_7	$x^3 + xy^3$
E_8	$x^3 + y^5$

The adjacency diagram



The definitions of P, X, J are to be found below (see II).

I.1. Unimodular Singularities ($m=1$). There are 3 families of parabolical singularities, one series of hyperbolical singularities (with 3 subscripts), and 14 families of exceptional singularities.

The parabolical singularities

P_8	$x^3 + y^3 + z^3 + axyz$	$a^3 + 27 \neq 0$
X_9	$x^4 + y^4 + ax^2y^2$	$a^2 \neq 4$
J_{10}	$x^3 + y^6 + ax^2y^2$	$4a^3 + 27 \neq 0$

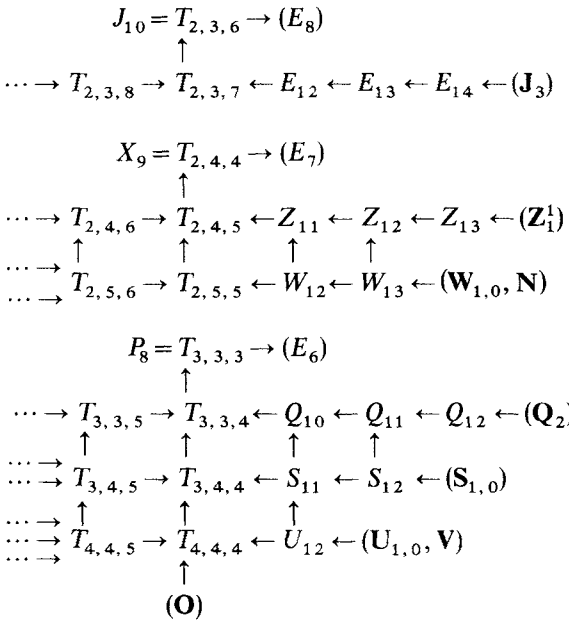
The hyperbolical singularities

$$T_{p,q,r}: x^p + y^q + z^r + axyz, \quad a \neq 0, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

The 14 exceptional families

E_{12}	$x^3 + y^7 + axy^5$	E_{13}	$x^3 + xy^5 + ay^8$
E_{14}	$x^3 + y^8 + axy^6$	Z_{11}	$x^3y + y^5 + axy^4$
Z_{12}	$x^3y + xy^4 + ax^2y^3$	Z_{13}	$x^3y + y^6 + axy^5$
W_{12}	$x^4 + y^5 + ax^2y^3$	W_{13}	$x^4 + xy^4 + ay^6$
Q_{10}	$x^3 + y^4 + yz^2 + axy^3$	Q_{11}	$x^3 + y^2z + xz^3 + az^5$
Q_{12}	$x^3 + y^5 + yz^2 + axy^4$	S_{11}	$x^4 + y^2z + xz^2 + ax^3z$
S_{12}	$x^2y + y^2z + xz^3 + az^5$	U_{12}	$x^3 + y^3 + z^4 + axyz^2$

Some of the adjacencies between the unimodular singularities



The singularity classes enclosed in brackets are not unimodular.

1.2. Bimodular Singularities ($m=2$). There are 8 infinite series and 14 exceptional families. In all the formulae of this section $\mathbf{a} = a_0 + a_1y$.

The 4 infinite series of bimodular singularities of corank 2

Notation	Normal form	Restrictions	Multiplicity μ
$J_{3,0}$	$x^3 + bx^2y^3 + y^9 + cxy^7$	$4b^3 + 27 \neq 0$	16
$J_{3,p}$	$x^3 + x^2y^3 + ay^{9+p}$	$p > 0, a_0 \neq 0$	$16 + p$
$Z_{1,0}$	$y(x^3 + dx^2y^2 + cxy^5 + y^6)$	$4d^3 + 27 \neq 0$	15
$Z_{1,p}$	$y(x^3 + x^2y^2 + ay^{6+p})$	$p > 0, a_0 \neq 0$	$15 + p$
$W_{1,0}$	$x^4 + ax^2y^3 + y^6$	$a_0^2 \neq 4$	15
$W_{1,p}$	$x^4 + x^2y^3 + ay^{6+p}$	$p > 0, a_0 \neq 0$	$15 + p$
$W_{1,2q-1}^\#$	$(x^2 + y^3)^2 + axy^{4+q}$	$q > 0, a_0 \neq 0$	$15 + 2q - 1$
$W_{1,2q}^\#$	$(x^2 + y^3)^2 + ax^2y^{3+q}$	$q > 0, a_0 \neq 0$	$15 + 2q$

The 4 infinite series of bimodular singularities of corank 3

Notation	Normal form	Restrictions	Multiplicity μ
$Q_{2,0}$	$x^3 + yz^2 + ax^2y^2 + xy^4$	$a_0^2 \neq 4$	14
$Q_{2,p}$	$x^3 + yz^2 + x^2y^2 + az^{6+p}$	$p > 0, a_0 \neq 0$	$14 + p$
$S_{1,0}$	$x^2z + yz^2 + y^5 + az^3y^3$	$a_0^2 \neq 4$	14
$S_{1,p}$	$x^2z + yz^2 + x^2y^2 + ay^{5+p}$	$p > 0, a_0 \neq 0$	$14 + p$
$S_{1,2q-1}^*$	$x^2z + yz^2 + zy^3 + ax^2y^{2+q}$	$q > 0, a_0 \neq 0$	$14 + 2q - 1$
$S_{1,2q}^*$	$x^2z + yz^2 + zy^3 + ax^2y^{2+q}$	$q > 0, a_0 \neq 0$	$14 + 2q$
$U_{1,0}$	$x^3 + xz^2 + xy^3 + ay^3z$	$a_0(a_0^2 + 1) \neq 0$	14
$U_{1,2q-1}$	$x^3 + xz^2 + xy^3 + ay^{1+q}z^2$	$q > 0, a_0 \neq 0$	$14 + 2q - 1$
$U_{1,2q}$	$x^3 + xz^2 + xy^3 + ay^{3+q}z$	$q > 0, a_0 \neq 0$	$14 + 2q$

The 14 exceptional families

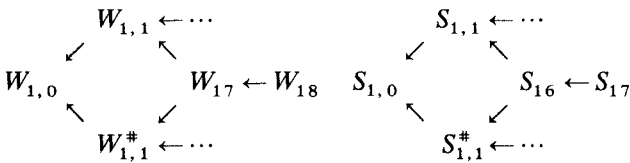
E_{18}	$x^3 + y^{10} + axy^7$	E_{19}	$x^3 + xy^7 + ay^8$
E_{20}	$x^3 + y^8 + axy^8$	Z_{17}	$x^3y + y^8 + axy^6$
Z_{18}	$x^3y + xy^6 + ay^9$	Z_{19}	$x^3y + y^9 + axy^7$
W_{17}	$x^4 + xy^5 + ay^7$	W_{18}	$x^4 + y^7 + ax^2y^4$
Q_{16}	$x^3 + yz^2 + y^7 + axy^5$	Q_{17}	$x^3 + yz^2 + xy^5 + ay^8$
Q_{18}	$x^3 + yz^2 + y^8 + axy^6$	S_{16}	$x^2z + yz^2 + xy^4 + ay^6$
S_{17}	$x^2z + yz^2 + y^6 + az^4y^4$	U_{16}	$x^3 + xz^2 + y^5 + ax^2y^2$

All the functions of all these families are bimodular (when the restrictions hold).

Some of the adjacencies between the bimodular singularities

$$J_{3,0} \leftarrow J_{3,1} \leftarrow E_{18} \leftarrow E_{19} \leftarrow E_{20}, \quad Z_{1,0} \leftarrow Z_{1,1} \leftarrow Z_{17} \leftarrow Z_{18} \leftarrow Z_{19},$$

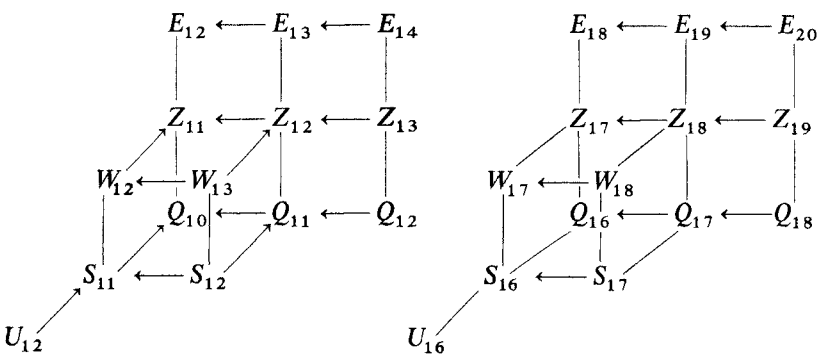
$$\begin{matrix} \uparrow & & \uparrow \\ \dots & & \dots \end{matrix}$$



$$Q_{2,0} \leftarrow Q_{2,1} \leftarrow Q_{16} \leftarrow Q_{17} \leftarrow Q_{18}, \quad U_{1,0} \leftarrow U_{1,1} \leftarrow U_{16}.$$

$$\begin{matrix} \uparrow & & \uparrow \\ \dots & & \dots \end{matrix}$$

The pyramids of exceptional singularities with modules numbers $m=1$ and 2



Each vertical joins the singularities of the same Kodaira class in the Koulikov's construction [7].

II. The Corank 2 Singularities with Nonzero 4-Jets

Through this section $\mathbf{a} = a_0 + \dots + a_{k-2}y^{k-2}$ for $k > 1$ and $\mathbf{a} = 0$ for $k = 1$.

II₁. The Corank 2 Singularities with Nonzero 3-Jets. These are the simple singularities **A**, **D**, E_6 , E_7 , E_8 and the singularities of the following infinite series of classes.

$$\mathbf{J} = \mathbf{J}_2 \leftarrow \mathbf{J}_3 \leftarrow \dots, \text{ where } \mathbf{J}_k = \leftarrow J_{k,0} \leftarrow J_{k,1} \leftarrow E_{6k} \leftarrow E_{6k+1} \leftarrow E_{6k+2} \leftarrow (\mathbf{J}_{k+1})$$

$$\uparrow$$

$$J_{k,2} \leftarrow J_{k,3} \leftarrow \dots$$

Notation	Normal form	Restrictions	Multiplicity μ	Modules number m
$J_{k,0}$	$x^3 + bx^2y^k + y^{3k} + cxy^{2k+1}$	$k > 1, 4b^3 + 27 \neq 0$	$6k - 2$	$k - 1$
$J_{k,i}$	$x^3 + x^2y^k + ay^{3k+i}$	$k > 1, i > 0, a_0 \neq 0$	$6k + i - 2$	$k - 1$
E_{6k}	$x^3 + y^{3k+1} + axy^{2k+1}$	$k \geq 1$	$6k$	$k - 1$
E_{6k+1}	$x^3 + xy^{2k+1} + ay^{3k+2}$	$k \geq 1$	$6k + 1$	$k - 1$
E_{6k+2}	$x^3 + y^{3k+2} + axy^{2k+2}$	$k \geq 1$	$6k + 2$	$k - 1$

Here $\mathbf{c} = c_0 + \dots + c_{k-3}y^{k-3}$ for $k > 2$; $\mathbf{c} = 0$ for $k = 2$.

II₂. The Corank 2 Singularities with Zero 3-Jets and Nonzero 4-Jets. These singularities form one infinite series of classes

$$\mathbf{X} = \mathbf{X}_1 \leftarrow \mathbf{X}_2 \leftarrow \dots$$

where

$$\mathbf{X}_k = X_{k,0} \leftarrow \mathbf{X}_k^* \leftarrow \begin{matrix} Y_k \\ W_k \\ Z_k \end{matrix} \leftarrow;$$

here

$$\begin{matrix} \Downarrow & & \Downarrow \\ \mathbf{X}_k^* = \leftarrow X_{k,1} \leftarrow X_{k,2} \leftarrow \dots, & \mathbf{Y}_k = \leftarrow Y_{1,1}^k \leftarrow Y_{2,1}^k \leftarrow \dots, \\ \mathbf{Z}_k = \leftarrow Z_0^k \leftarrow Z_1^k \leftarrow \dots, & \mathbf{Z}_0^k = \leftarrow Z_{12k-1}^k \leftarrow Z_{12k}^k \leftarrow Z_{12k+1}^k, \\ \uparrow & & \\ \mathbf{Z}_i^k = \leftarrow Z_{i,0}^k \leftarrow Z_{i,1}^k \leftarrow Z_{12k+6i-1}^k \leftarrow Z_{12k+6i}^k \leftarrow Z_{12k+6i+1}^k \leftarrow (\mathbf{Z}_{i+1}^k), & i > 0 \\ \uparrow & & \\ & Z_{i,2}^k \leftarrow Z_{i,3}^k \leftarrow \dots \end{matrix}$$

$$\mathbf{W}_k = \leftarrow W_{12k} \leftarrow W_{12k+1} \leftarrow W_{k,0} \leftarrow \begin{matrix} W_{k,1} \leftarrow W_{k,2} \leftarrow \dots \\ W_{12k+5} \leftarrow W_{12k+6} \leftarrow (\mathbf{X}_{k+1}) \\ W_{k,1}^* \leftarrow W_{k,2}^* \leftarrow \dots \end{matrix}$$

Classes X and Y

Notation	Normal Form	Restrictions	Multiplicity μ	Modules number m
$X_{k,0}$	$x^4 + \mathbf{b}x^3y^k + \mathbf{a}x^2y^{2k} + xy^{3k}$	$\Delta \neq 0, a_0b_0 \neq 9$ $k > 1$	$12k - 3$	$3k - 2$
$X_{k,p}$	$x^4 + \mathbf{a}x^3y^k + x^2y^{2k} + \mathbf{b}y^{4k+p}$	$a_0^2 \neq 4, b_0 \neq 0$ $p > 0, k > 1$	$12k - 3 + p$	$3k - 2$
$Y_{r,s}^k$	$[(x + \mathbf{a}y^k)^2 + \mathbf{b}y^{2k+s}](x^2 + y^{2k+r})$	$1 \leq s \leq 7, k > 1$ $a_0 \neq 0 \neq b_0$	$12k - 3 + r + s$	$3k - 2$

The case $k = 1$:

$X_{1,0}$	$x^4 + a_0x^2y^2 + y^4$	$a_0^2 \neq 4$	9	1
$X_{1,p}$	$x^4 + x^2y^2 + a_0y^{4+p}$	$a_0 \neq 0$	$9 + p$	1
$Y_{r,s}^1$	$x^{4+r} + a_0x^2y^2 + y^{4+s}$	$a_0 \neq 0$	$9 + r + s$	1

Of course, $X_{1,0} = X_9, X_{1,p} = T_{2,4,4+p}, Y_{r,s}^1 = T_{2,4+r,4+s}$. Here

$$\Delta = 4(a_0^3 + b_0^3) - a_0^2b_0^2 - 18a_0b_0 + 27,$$

$$\mathbf{b} = b_0 + \dots + b_{2k-2}y^{2k-2}.$$

Class Z

For singularities $Z_{i,0}^k$ and $Z_\mu^k (k > 1)$ normal forms are $f = (x + \mathbf{a}y^k)f_2$, where $a_0 \neq 0$ and f_2 is given in the following table:

Notation	f_2	Restrictions	Multiplicity, μ	Modules number, m
$Z_{i,0}^k$	$x^3 + dx^2y^{k+1} + \mathbf{c}xy^{2k+2i+1} + y^{3k+3i}$	$4d^3 + 27 \neq 0$ $k > 1, i \geq 0$	$12k - 3 + 6i$	$3k + i - 2$
$Z_{12k+6i-1}^k$	$x^3 + \mathbf{b}xy^{2k+2i+1} + y^{3k+3i+1}$	$k > 1, i \geq 0$	$12k + 6i - 1$	$3k + i - 2$
Z_{12k+6i}^k	$x^3 + xy^{2k+2i+1} + \mathbf{b}y^{3k+3i+2}$	$k > 1, i \geq 0$	$12k + 6i$	$3k + i - 2$
$Z_{12k+6i+1}^k$	$x^3 + \mathbf{b}xy^{2k+2i+2} + y^{3k+3i+2}$	$k > 1, i \geq 0$	$12k + 6i + 1$	$3k + i - 2$

Singularity $Z_{i,p}^k (k > 1, i > 0, p > 0)$ admits the normal form

$$(x^2 + \mathbf{a}xy^k + \mathbf{b}y^{2k+i}) \cdot (x^2 + y^{2k+2i+p}), \quad a_0 \neq 0, b_0 \neq 0,$$

its multiplicity is $\mu = 12k + 6i + p - 3$, the modules number $m = 3k + i - 2$.

For $k = 1$ one modifies the above formulae in the following way:

1. One omits the index $k = 1$ in the notations.
2. Singularities $Z_{i,0}, Z_{6i+11}, Z_{6i+12}, Z_{6i+13} (i > 0)$ have normal forms $f = yf_2$, where f_2 is given by the table above.
3. $Z_{i,p}; y(x^3 + x^2y^{i+1} + \mathbf{b}y^{3i+p+3}), b_0 \neq 0, i > 0, p > 0$.

Through this section

$$\mathbf{b} = b_0 + \dots + b_{2k+i-2}y^{2k+i-2},$$

$$\mathbf{c} = c_0 + \dots + c_{2k+i-3}y^{2k+i-3}.$$

Class W

Notation	Normal form	Restriction	Multiplicity μ	Modules number m
W_{12k}	$x^4 + y^{4k+1} + \mathbf{a}xy^{3k+1} + \mathbf{c}x^2y^{2k+1}$	$k \geq 1$	$12k$	$3k - 2$
W_{12k+1}	$x^4 + xy^{3k+1} + \mathbf{a}x^2y^{2k+1} + \mathbf{c}y^{4k+2}$	$k \geq 1$	$12k + 1$	$3k - 2$
$W_{k,0}^*$	$x^4 + \mathbf{b}x^2y^{2k+1} + \mathbf{a}xy^{3k+2} + y^{4k+2}$	$k \geq 1, b_0^2 \neq 4$	$12k + 3$	$3k - 1$
$W_{k,i}^*$	$x^4 + \mathbf{a}x^3y^{k+1} + x^2y^{2k+1} + \mathbf{b}y^{4k+2+i}$	$i > 0, b_0 \neq 0$	$12k + 3 + i$	$3k - 1$
$W_{k,2q-1}^{\#}$	$(x^2 + y^{2k+1})^2 + \mathbf{b}xy^{3k+1+q} + \mathbf{a}y^{4k+2+q}$	$q > 0, b_0 \neq 0$	$12k + 2 + 2q$	$3k - 1$
$W_{k,2q}^{\#}$	$(x^2 + y^{2k+1})^2 + \mathbf{b}x^2y^{2k+1+q} + \mathbf{a}xy^{3k+2+q}$	$q > 0, b_0 \neq 0$	$12k + 3 + 2q$	$3k - 1$
W_{12k+5}	$x^4 + xy^{3k+2} + \mathbf{a}x^2y^{2k+2} + \mathbf{b}y^{4k+3}$	$k \geq 1$	$12k + 5$	$3k - 1$
W_{12k+6}	$x^4 + y^{4k+3} + \mathbf{a}xy^{3k+3} + \mathbf{b}x^2y^{2k+2}$	$k \geq 1$	$12k + 6$	$3k - 1$

Here $\mathbf{b} = b_0 + \dots + b_{2k-1}y^{2k-1}$, $\mathbf{c} = c_0 + \dots + c_{2k-2}y^{2k-2}$; as always, $\mathbf{a} = a_0 + \dots + a_{k-2}y^{k-2}$, for $k > 1$ and $\mathbf{a} = 0$ for $k = 1$.

III. The Corank 3 Singularities with Reduced 3-Jets

Besides the unimodular singularities T (see \mathbf{I}_1), there are 3 infinite series of classes $\mathbf{Q}, \mathbf{S}, \mathbf{U}$ of such singularities.

III₁. Series Q. The singularities with the 3-jet $x^3 + yz^2$ form one infinite series of classes:

$$\mathbf{Q} = \leftarrow \mathbf{Q}_1 \leftarrow \mathbf{Q}_2 \leftarrow \dots, \quad \text{where } \mathbf{Q}_1 = \leftarrow \mathbf{Q}_{1,0} \leftarrow \mathbf{Q}_{1,1} \leftarrow \mathbf{Q}_{1,2},$$

$$\mathbf{Q}_k = \leftarrow \mathbf{Q}_{k,0} \leftarrow \mathbf{Q}_{k,1} \leftarrow \mathbf{Q}_{6k,4} \leftarrow \mathbf{Q}_{6k,5} \leftarrow \mathbf{Q}_{6k,6} \leftarrow (\mathbf{Q}_{k,1})$$

$$k > 1 \quad \quad \quad \uparrow$$

$$\quad \quad \quad \mathbf{Q}_{k,2} \leftarrow \mathbf{Q}_{k,3} \leftarrow \dots.$$

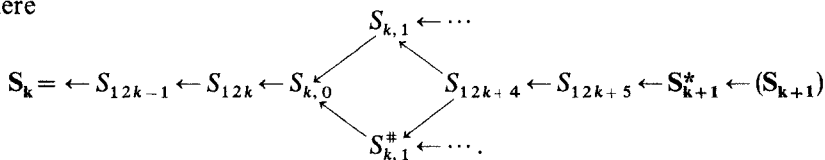
Notation	Normal form	Restrictions	Multiplicity μ	Modules number m
$Q_{k,0}$	$\varphi + \mathbf{b}x^2y^k + xy^{2k}$	$k > 1, b_0^2 \neq 4$	$6k + 2$	k
$Q_{k,i}$	$\varphi + x^2y^k + \mathbf{b}y^{3k+i}$	$k > 1, b_0 \neq 0$	$6k + 2 + i$	k
Q_{6k+4}	$\varphi + y^{3k+1} + \mathbf{b}xy^{2k+1}$	$k \geq 1$	$6k + 4$	k
Q_{6k+5}	$\varphi + xy^{2k+1} + \mathbf{b}y^{3k+2}$	$k \geq 1$	$6k + 5$	k
Q_{6k+6}	$\varphi + y^{3k+2} + \mathbf{b}xy^{2k+2}$	$k \geq 1$	$6k + 6$	k

Here $\varphi = x^3 + yz^2$, $\mathbf{b} = b_0 + \dots + b_{k-1}y^{k-1}$.

III₂. Series S. The singularities with the 3-jet $x^2z + yz^2$ form one infinite series of classes:

$$\mathbf{S} = \leftarrow \mathbf{S}_1 \leftarrow \mathbf{S}_2 \leftarrow \dots,$$

where

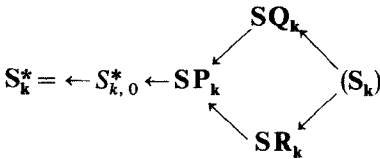


Notation	Normal form	Restrictions	Multiplicity μ	Modules number m
S_{12k-1}	$\varphi + y^{4k} + axy^{3k} + cz y^{2k+1}$	—	$12k-1$	$3k-2$
S_{12k}	$\varphi + xy^{3k} + cy^{4k+1} + az y^{2k+1}$	—	$12k$	$3k-2$
$S_{k,0}$	$\varphi + y^{4k+1} + ax y^{3k+1} + bz y^{2k+1}$	$b_0^2 \neq 4$	$12k+2$	$3k-1$
$S_{k,i}$	$\varphi + x^2 y^{2k} + ax^3 y^k + by^{4k+1+i}$	$i > 0, b_0 \neq 0$	$12k+2+i$	$3k-1$
$S_{k,2q-1}^*$	$\varphi + zy^{2k+1} + bx y^{3k+q-1} + ay^{4k+q}$	$q > 0, b_0 \neq 0$	$12k+2q+1$	$3k-1$
$S_{k,2q}^*$	$\varphi + zy^{2k+1} + bx^2 y^{2k+q} + ax y^{3k+q+1}$	$q > 0, b_0 \neq 0$	$12k+2q+2$	$3k-1$
S_{12k+4}	$\varphi + xy^{3k+1} + az y^{2k+2} + by^{4k+2}$	—	$12k+4$	$3k-1$
S_{12k+5}	$\varphi + y^{4k+2} + ax y^{3k+2} + bz y^{2k+2}$	—	$12k+5$	$3k-1$

Here $\varphi = x^2 z + yz^2$;

$$\begin{aligned} \mathbf{a} &= a_0 + \dots + a_{k-2} y^{k-2} \quad \text{for } k > 1, \quad \mathbf{a} = 0 \quad \text{for } k = 1; \\ \mathbf{b} &= b_0 + \dots + b_{2k-1} y^{2k-1}; \\ \mathbf{c} &= c_0 + \dots + c_{2k-2} y^{2k-2}. \end{aligned}$$

The S_k^* ($k > 1$) singularities are:



$$\mu(S_{k,0}^*) = 12k - 4, \quad m(S_{k,0}^*, \dots, SR_k) = 3k - 2, \quad \text{codim } S_k^* = 9k - 3.$$

III₃. Series U. The singularities with the 3-jet $x^3 + xz^2$ form one infinite series of classes

$$U = \leftarrow U_1 \leftarrow U_2 \leftarrow \dots,$$

where

$$\begin{aligned} U_k = \leftarrow U_{12k} \leftarrow U_{k,0} \leftarrow U_{k,1} \leftarrow U_{12k+4} \leftarrow U_{k+1}^* \leftarrow (U_{k+1}) \\ \uparrow \\ U_{k,2} \leftarrow \dots \end{aligned}$$

Notation	Normal form	Restrictions	Multiplicity μ	Modules number m
U_{12k}	$\varphi + y^{3k+1} + ax y^{2k+1} + bz y^{2k+1} + dx^2 y^{k+1}$	—	$12k$	$4k-3$
$U_{k,2q}$	$\varphi + xy^{2k+1} + ax^2 y^{k+1} + by^{3k+2+q} + cz y^{2k+1+q}$	$q \geq 0; c_0 \neq 0$	$12k+2+2q$	$4k-2$
$U_{k,2q-1}$	$\varphi + xy^{2k+1} + ax^2 y^{k+1} + bz y^{2k+1+q} + cz^2 y^{k+q}$	$q > 0; c_0 \neq 0$	$12k+1+2q$	$4k-2$
U_{12k+4}	$\varphi + y^{3k+2} + ax y^{2k+2} + bz y^{2k+2} + cx^2 y^{k+1}$	—	$12k+4$	$4k-2$

Here $\varphi = x^3 + xz^2$; $c_0^2 + 1 \neq 0$ for $q = 0$; and through the table

$$\begin{aligned} \mathbf{a} &= a_0 + \dots + a_{k-2} y^{k-2} \quad \text{for } k > 1, \quad \mathbf{a} = 0 \quad \text{for } k = 1, \\ \mathbf{b} &= b_0 + \dots + b_{k-2} y^{k-2} \quad \text{for } k > 1, \quad \mathbf{b} = 0 \quad \text{for } k = 1, \end{aligned}$$

$$\mathbf{c} = c_0 + \dots + c_{2k-1} y^{2k-1},$$

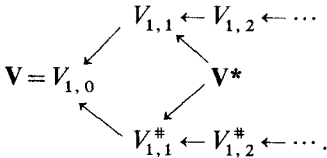
$$\mathbf{d} = d_0 + \dots + d_{2k-2} y^{2k-2}.$$

The \mathbf{U}_k^* singularities ($k > 1$) are

$$\begin{array}{ccccccc} \mathbf{U}_k^* = & \leftarrow & U_{k,0}^* & \leftarrow & \mathbf{U}\mathbf{P}_k & \leftarrow & \mathbf{U}\mathbf{R}_k & \leftarrow & \mathbf{U}\mathbf{T}_k \\ & & & & \uparrow & & \uparrow & & \uparrow \\ & & & & \mathbf{U}\mathbf{Q}_k & \leftarrow & \mathbf{U}\mathbf{S}_k & \leftarrow & (\mathbf{U}_k), \end{array}$$

$$\mu(\mathbf{U}_{k,0}^*) = 12k - 4, \quad m(\mathbf{U}_{k,0}^*, \dots, \mathbf{U}\mathbf{T}_k) = 4k - 3, \quad \text{codim}(\mathbf{U}_k^*) = 8k - 2.$$

IV. Series V. The corank 3 singularities with the 3-jet x^2y belong to the classes



Notation	Normal form	Restrictions	Multiplicity μ	Modules number m
$V_{1,0}$	$x^2y + z^4 + az^3y + \mathbf{b}z^2y^2 + zy^3$	$\Delta(a, b_0) \neq 0$	15	3
$V_{1,p}^*$	$x^2y + z^4 + bz^3y + z^2y^2 + \mathbf{a}y^{4+p}$	$b^2 \neq 4, a_0 \neq 0$	$15 + p$	3
$V_{1,2q-1}^*$	$x^2y + z^3y + ay^2z^2 + y^4 + \mathbf{b}xz^2 + a^q$	$4a^3 + 27 \neq 0, b_0 \neq 0$	$15 + 2q - 1$	3
$V_{1,2q}^*$	$x^2y + z^3y + ay^2z^2 + y^4 + \mathbf{b}z^4 + a^q$	$4a^3 + 27 \neq 0, b_0 \neq 0$	$15 + 2q$	3

Here $p > 0, q > 0, \mathbf{a} = a_0 + a_1y, \mathbf{b} = b_0 + b_1z$. For the \mathbf{V}^* singularities $\mu(\mathbf{V}^*) \geq 17, m(\mathbf{V}^*) \geq 3, c(\mathbf{V}^*) = 13$.

V. Other Singularities. All the singularities, which normal forms are not given in this article, belong to the following 7 classes:

Notation	Corank	Adjacency	Definition	$c \geq$	$\mu \geq$	$m \geq$	Theorem
\mathbf{N}	2	$\mathbf{N} \rightarrow W_{13}$	$j_4 = 0$	12	16	3	47-49
\mathbf{S}^*	3	$\mathbf{S}_k^* \rightarrow S_{12k-9}$	th. 77	15	20	4	77-81
\mathbf{U}^*	3	$\mathbf{U}_k^* \rightarrow U_{12k-8}$	th. 90	14	20	5	90-96
\mathbf{V}^*	3	$\mathbf{V}^* \rightarrow V_{1,1} \cup V_{1,1}^*$	th. 98	13	17	3	97-102
\mathbf{V}'	3	$\mathbf{V}' \rightarrow \mathbf{V}$	$j_3 = x^3$	13	18	4	103
\mathbf{V}''	3	$\mathbf{V}'' \rightarrow \mathbf{V}'$	$j_3 = 0$	16	27	10	104
\mathbf{O}	> 3	$\mathbf{O} \rightarrow T_{4,4,4}$	corank ≥ 4	10	16	5	105

Here $k > 1$.

Remark. For every arrow $\mathbf{K} \leftarrow \mathbf{L}$ in §1 (say, $\mathbf{J}_2 \leftarrow \mathbf{J}_3$), every singularity in class \mathbf{K} has a neighbourhood which does not intersect \mathbf{L} (e.g there is no adjacency $J_{2,i} \rightarrow J_{3,0}$ for any i).

§ 2. Singularity Determinator

Notations

f	a germ of a holomorphic function at an <i>isolated</i> critical point O , with finite multiplicity μ ; its Taylor series at O ; a formal series in x, y or in x, y, z , with multiplicity μ .
$f \sim g$	germs (or series) f and g at O are equivalent (there exists a germ of a diffeomorphism h (or a formal series h) such that $f = g \circ h$).
\Rightarrow	"implies"
\Rightarrow	"see" (references \Rightarrow i are not parts of the theorem statements; they give the number of the theorem containing the classification of the given singularity classes).
$j_k f$	k -jet of f at O (or the Taylor k -polynomial).
A, \dots, Z	stable equivalence classes defined in § 1.
$m(f)$	modules number of the germ of f at O .
$c(f)$	codimension of the $\mu = \text{const}$ stratum of a function germ f in the space of germs of functions having critical value 0 at the critical point O .
$c(\mathbf{K})$	codimension of the singularity class \mathbf{K} in the same space.
$j\{\mathbf{x}^{\mathbf{m}_i}\}f$	quasijet (or quasi Taylor polynomial of f at O determined by the monomials $\mathbf{x}^{\mathbf{m}_i}$, see the explanation below).
$j\{\mathbf{x}^{\mathbf{m}_i}\}f \approx g$	quasihomogenous equivalence of two quasijets or of two quasi Taylor polynomials, see the explanation below.
j^*, φ	abbreviations used in theorems 58–65, 66–81, 82–89, 98–102 ; their meaning is to be found above the first theorem of each group.
Δ	discriminant. In theorems 36, 37, 47, 48, 98, 99 , $\Delta = 4(a^3 + b^3) + 27 - a^2 b^2 - 18 ab$.

Explanation. A system $\{\mathbf{x}^{\mathbf{m}_i}\}$ of n monomials in n variables x_1, \dots, x_n with linearly independent exponents $\mathbf{m}_i \in \mathbb{Z}^n \subset \mathbb{R}^n$ defines an hyperplane $\Gamma \subset \mathbb{R}^n$, $\Gamma = \{\mathbf{m}: (\boldsymbol{\alpha}, \mathbf{m}) = 1\}$. If all the components α_i of the vector $\boldsymbol{\alpha}$ are positive, $\boldsymbol{\alpha}$ is called the *quasihomogeneity type*. The number $(\boldsymbol{\alpha}, \mathbf{m})$ is then called the *order* of the monomial $\mathbf{x}^{\mathbf{m}}$.

A polynomial $f = \sum f_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}$ is *quasihomogenous*, of degree d and of type $\boldsymbol{\alpha}$ if $(\boldsymbol{\alpha}, \mathbf{m}) = d$ for all \mathbf{m} with $f_{\mathbf{m}} \neq 0$.

The type $\boldsymbol{\alpha}$ defines a graded ring structure in $\mathbb{C}[[x_1, \dots, x_n]]$ and a ring filtration $\mathcal{E}_0 \supset \dots, \mathcal{E}_d = \{f: (\boldsymbol{\alpha}, \mathbf{m}) \geq d \forall \mathbf{m}: f_{\mathbf{m}} \neq 0\}$.

The factor space $\mathcal{E}_0/\mathcal{E}_d$, $d > 1$ is by definition the *quasijet space defined by the monomials* $\{\mathbf{x}^{\mathbf{m}_i}\}$ (or by the type $\boldsymbol{\alpha}$). For a fixed coordinate system one can identify quasijets with the polynomials having only monomials of order ≤ 1 (that is, their exponents lie on Γ or the same side of Γ as O).

Quasihomogenous diffeomorphisms are diffeomorphisms of \mathbb{C}^n which preserve the degrees in the graded ring $\mathbb{C}[[x_1, \dots, x_n]]$. The Lie group of the quasihomogenous diffeomorphisms acts on the space of quasijets and on the space of quasihomogenous polynomials. Two quasijets (or two polynomials) are *quasihomogenously equivalent* if they belong to the same orbit of this action.

Determinator

1. $\mu(f) < \infty \Rightarrow$ one of the four possibilities holds:
 $\text{corank } f \leq 1 \Rightarrow \mathbf{2},$
 $\quad \quad \quad = 2 \Rightarrow \mathbf{3},$
 $\quad \quad \quad = 3 \Rightarrow \mathbf{50},$
 $\quad \quad \quad > 3 \Rightarrow \mathbf{105}.$
2. $\text{corank } f \leq 1 \Rightarrow f \in A_k \ (k \geq 1).$

Through theorems 3–49, $f \in \mathbb{C}[[x, y]]$.

3. $j_2 f = 0 \Rightarrow$ one of the four possibilities holds:
 $j_3 f \approx x^2 y + y^3 \Rightarrow \mathbf{4},$
 $\quad \quad \approx x^2 y \quad \Rightarrow \mathbf{5},$
 $\quad \quad \approx x^3 \quad \quad \Rightarrow \mathbf{6_1},$
 $\quad \quad = 0 \quad \quad \quad \Rightarrow \mathbf{13}.$
4. $j_3 f = x^2 y + y^3 \Rightarrow f \in D_4.$
5. $j_3 f = x^2 y \Rightarrow f \in D_k \ (k > 4).$

Through theorems 6–9, $k \geq 1$.

- 6_k. $j_{x^3, y^{3k}} f(x, y) = x^3 \Rightarrow$ one of the four possibilities holds:
 $j_{x^3, y^{3k+1}} f \approx x^3 + y^{3k+1} \Rightarrow \mathbf{7_k},$
 $j_{x^3, x y^{2k+1}} f \approx x^3 + x y^{2k+1} \Rightarrow \mathbf{8_k},$
 $j_{x^3, y^{3k+2}} f \approx x^3 + y^{3k+2} \Rightarrow \mathbf{9_k},$
 $j_{x^3, y^{3k+2}} f = x^3 \quad \quad \quad \Rightarrow \mathbf{10_{k+1}}.$
- 7_k. $j_{x^3, y^{3k+1}} f = x^3 + y^{3k+1} \Rightarrow f \in E_{6k}.$
- 8_k. $j_{x^3, x y^{2k+1}} f = x^3 + x y^{2k+1} \Rightarrow f \in E_{6k+1}.$
- 9_k. $j_{x^3, y^{3k+2}} f = x^3 + y^{3k+2} \Rightarrow f \in E_{6k+2}.$

Through theorems 10–12, $k > 1$.

- 10_k. $j_{x^3, y^{3k-1}} f = x^3 \Rightarrow$ one of the three possibilities holds:
 $j_{x^3, y^{3k}} f \approx x^3 + a x^2 y^k + y^{3k}, \quad 4a^3 + 27 \neq 0 \Rightarrow \mathbf{11_k};$
 $\quad \quad \approx x^3 + x^2 y^k \quad \quad \quad \Rightarrow \mathbf{12_k};$
 $\quad \quad \approx x^3 \quad \quad \quad \quad \quad \quad \quad \Rightarrow \mathbf{6_k}.$
- 11_k. $j_{x^3, y^{3k}} f = x^3 + a x^2 y^k + y^{3k}, \quad 4a^3 + 27 \neq 0 \Rightarrow f \in J_{k,0}.$
- 12_k. $j_{x^3, y^{3k}} f = x^3 + x^2 y^k \Rightarrow f \in J_{k,p} \ (p > 0).$

Series X

13. $j_3 f(x, y) = 0 \Rightarrow$ one of the six possibilities holds:
 $j_4 \approx x^4 + a x^2 y^2 + y^4, \quad a^2 \neq 4 \Rightarrow \mathbf{14};$
 $\quad \quad \approx x^4 + x^2 y^2 \quad \quad \quad \Rightarrow \mathbf{15};$
 $\quad \quad \approx x^2 y^2 \quad \quad \quad \quad \quad \Rightarrow \mathbf{16};$
 $\quad \quad \approx x^3 y \quad \quad \quad \quad \quad \quad \Rightarrow \mathbf{17};$
 $\quad \quad \approx x^4 \quad \quad \quad \quad \quad \quad \quad \Rightarrow \mathbf{25};$
 $\quad \quad = 0 \quad \quad \quad \quad \quad \quad \quad \Rightarrow \mathbf{47};$
14. $j_4 f = x^4 + a x^2 y^2 + y^4, \quad a^2 \neq 4 \Rightarrow f \in X_9 = X_{1,0} = T_{2,4,4}.$
15. $j_4 f = x^4 + x^2 y^2 \quad \quad \quad \Rightarrow f \in X_{1,p} = T_{2,4,4+p} \ (p > 0).$

$$16. j_4 f = x^2 y^2 \Rightarrow f \in Y_{p,q}^1 = T_{2,4+p,4+q} \quad (p > 0, q > 0).$$

$$17. j_4 f = x^3 y \Rightarrow j_{x^3 y, y^4} f = x^3 y \Rightarrow 18_1.$$

Through theorems 18–21, $p \geq 1$.

$$18_p. j_{x^3 y, y^{3p+1}} f = x^3 y \Rightarrow \text{one of the four possibilities holds:}$$

$$j_{x^3 y, y^{3p+2}} f \approx x^3 y + y^{3p+2} \Rightarrow 19_p,$$

$$j_{x^3 y, x y^{2p+2}} f \approx x^3 y + x y^{2p+2} \Rightarrow 20_p,$$

$$j_{x^3 y, y^{3p+3}} f \approx x^3 y + y^{3p+3} \Rightarrow 21_p,$$

$$j_{x^3 y, y^{3p+3}} f = x^3 y \Rightarrow 22_{p+1}.$$

$$19_p. j_{x^3 y, y^{3p+2}} f = x^3 y + y^{3p+2} \Rightarrow f \in Z_{6p+5}.$$

$$20_p. j_{x^3 y, x y^{2p+2}} f = x^3 y + x y^{2p+2} \Rightarrow f \in Z_{6p+6}.$$

$$21_p. j_{x^3 y, y^{3p+3}} f = x^3 y + y^{3p+3} \Rightarrow f \in Z_{6p+7}.$$

Through theorems 22–24, $p > 1$.

$$22_p. j_{x^3 y, y^{3p}} f = x^3 y \Rightarrow \text{one of the three possibilities holds:}$$

$$j_{x^3 y, y^{3p+1}} f = y(x^3 + b x^2 y^p + y^{3p}), \quad 4b^3 + 27 \neq 0 \Rightarrow 23_p;$$

$$y(x^3 + x^2 y^p) \Rightarrow 24_p;$$

$$x^3 y \Rightarrow 18_p.$$

$$23_p. j_{x^3 y, y^{3p+1}} f = y(x^3 + b x^2 y^p + y^{3p}), \quad 4b^3 + 27 \neq 0 \Rightarrow f \in Z_{p-1,0}.$$

$$24_p. j_{x^3 y, y^{3p+1}} f = y(x^3 + x^2 y^p) \Rightarrow f \in Z_{p-1,r} \quad (r > 0).$$

Series W

$$25. j_4 f(x, y) = x^4 \Rightarrow j_{x^4, y^4} f = x^4 \Rightarrow 26_1.$$

Through theorems 26–34, $k \geq 1$.

$$26_k. j_{x^4, y^{4k}} f = x^4 \Rightarrow \text{one of the three possibilities holds:}$$

$$j_{x^4, y^{4k+1}} f \approx x^4 + y^{4k+1} \Rightarrow 27_k,$$

$$j_{x^4, x y^{3k+1}} f \approx x^4 + x y^{3k+1} \Rightarrow 28_k,$$

$$j_{x^4, x y^{3k+1}} f = x^4 \Rightarrow 29_k.$$

$$27_k. j_{x^4, y^{4k+1}} f = x^4 + y^{4k+1} \Rightarrow f \in W_{12k}.$$

$$28_k. j_{x^4, x y^{3k+1}} f = x^4 + x y^{3k+1} \Rightarrow f \in W_{12k+1}.$$

$$29_k. j_{x^4, x y^{3k+1}} f = x^4 \Rightarrow \text{one of the four possibilities holds:}$$

$$j_{x^4, y^{4k+2}} f \approx x^4 + b x^2 y^{2k+1} + y^{4k+2}, \quad b^2 \neq 4 \Rightarrow 30_k;$$

$$\approx x^4 + x^2 y^{2k+1} \Rightarrow 31_k;$$

$$\approx (x^2 + y^{2k+1})^2 \Rightarrow 32_k;$$

$$= x^4 \Rightarrow 33_k.$$

$$30_k. j_{x^4, y^{4k+2}} f = x^4 + b x^2 y^{2k+1} + y^{4k+2}, \quad b^2 \neq 4 \Rightarrow f \in W_{k,0}.$$

$$31_k. j_{x^4, y^{4k+2}} f = x^4 + x^2 y^{2k+1} \Rightarrow f \in W_{k,i} \quad (i > 0).$$

$$32_k. j_{x^4, y^{4k+2}} f = (x^2 + y^{2k+1})^2 \Rightarrow f \in W_{k,i}^\# \quad (i > 0).$$

$$33_k. j_{x^4, y^{4k+2}} f = x^4 \Rightarrow \text{one of the three possibilities holds:}$$

$$j_{x^4, x y^{3k+2}} f \approx x^4 + x y^{3k+2} \Rightarrow 34_k;$$

$$j_{x^4, y^{4k+3}} f \approx x^4 + y^{4k+3} \Rightarrow 35_3;$$

$$j_{x^4, y^{4k+3}} f = x^4 \Rightarrow 36_{k+1}.$$

$$34_k. j_{x^4, x y^{3k+2}} f = x^4 + x y^{3k+2} \Rightarrow f \in W_{12k+5}.$$

$$35_k \cdot j_{x^4, y^{4k+3}} f = x^4 + y^{4k+3} \Rightarrow f \in W_{12k+6}.$$

Through theorems 36-46, $k > 1$.

Series X_k

36_k. $j_{x^4, y^{4k-1}} f = x^4 \Rightarrow$ one of the five possibilities holds:

$$\begin{aligned} j_{x^4, y^{4k}} f &\approx x^4 + b x^3 y^k + a x^2 y^{2k} + x y^{3k}, & \Delta \neq 0, & \quad ab \neq 9 \Rightarrow 37_k; \\ &\approx x^2 (x^2 + a x y^k + y^{2k}), & a^2 \neq 4 & \quad \Rightarrow 38_k; \\ &\approx x^2 (x + y^k)^2 & & \quad \Rightarrow 39_k; \\ &\approx x^3 (x + y^k) & & \quad \Rightarrow 40_k; \\ &\approx x^4 & & \quad \Rightarrow 26_k. \end{aligned}$$

$$37_k \cdot j_{x^4, y^{4k}} f = x^4 + b x^3 y^k + a x^2 y^{2k} + x y^{3k}, \quad \Delta \neq 0 \Rightarrow f \in X_{k,0}.$$

$$38_k \cdot j_{x^4, y^{4k}} f = x^2 (x^2 + a x y^k + y^{2k}), \quad a^2 \neq 4 \Rightarrow f \in X_{k,p} (p > 0).$$

$$39_k \cdot j_{x^4, y^{4k}} f = x^2 (x^2 + y^k)^2 \Rightarrow f \in Y_{r,s} (1 \leq s \leq r).$$

$$40_k \cdot j_{x^4, y^{4k}} f = (x + y^k) x^3 \Rightarrow f = f_1 f_2, \quad \text{where } j_{x, y^k} f_1 \approx x + y^k; \\ j_{x^3, y^{3k}} f_2 = x^3 \Rightarrow 41_k.$$

Through theorems 41-44, $i \geq 0, p > 0$.

41_k. $j_{x^3, y^{3k}} f_2 = x^3 \Rightarrow$ one of the five possibilities holds:

$$\begin{aligned} f_2 \in E_{6(k+i)} &\Rightarrow 42_{k,i}; \\ f_2 \in E_{6(k+i)+1} &\Rightarrow 43_{k,i}; \\ f_2 \in E_{6(k+i)+2} &\Rightarrow 44_{k,i}; \\ f_2 \in J_{k+i+1,0} &\Rightarrow 45_{k,i+1}; \\ f_2 \in J_{k+i+1,p} &\Rightarrow 46_{k,i+1,p}. \end{aligned}$$

Through theorems 42-46, $f(x, y) = f_1 f_2$, where

$$j_{x, y^k} f_1 \approx x + y^k, \quad j_{x^3, y^{3k}} f_2 = x^3.$$

$$42_{k,i} \cdot f_2 \in E_{6(k+i)} \Rightarrow f \in Z_{12k+6i-1}^k.$$

$$43_{k,i} \cdot f_2 \in E_{6(k+i)+1} \Rightarrow f \in Z_{12k+6i}^k.$$

$$44_{k,i} \cdot f_2 \in E_{6(k+i)+2} \Rightarrow f \in Z_{12k+6i+1}^k.$$

Through theorems 45-46, $i \geq 1, p > 0$.

$$45_{k,i} \cdot f_2 \in J_{k+i,0} \Rightarrow f \in Z_{i,0}^k.$$

$$46_{k,i,p} \cdot f_2 \in J_{k+i,p} \Rightarrow f \in Z_{i,p}^k.$$

47. $j_4 f = 0 \Rightarrow$ one of the two possibilities holds:

$$\begin{aligned} j_5 f &\approx x^4 y + a x^3 y^2 + b x^2 y^3 + x y^4, & \Delta \neq 0, & \quad ab \neq 9 \Rightarrow 48; \\ j_5 f &\text{ is degenerate} & & \quad \Rightarrow 49. \end{aligned}$$

$$48. j_5 f = x^4 y + a x^3 y^2 + b x^2 y^3 + x y^4, \quad \Delta \neq 0 \Rightarrow f \in N_{16}, \text{ i.e.} \\ f \sim x^4 y + a x^3 y^2 + b x^2 y^3 + x y^4 + c x^3 y^3, \quad ab \neq 9, \quad \Delta(a, b) \neq 0; \\ \mu(f) = 16, \quad m(f) = 3, \quad c(f) = 12.$$

$$49. j_5 f \text{ is degenerate} \Rightarrow \mu(f) > 16, \quad m(f) > 2, \quad c(f) > 12.$$

Corank 3 Singularities

Through theorems 50-104, $f \in \mathbb{C}[[x, y, z]]$.

50. $j_2 f(x, y, z) = 0 \Rightarrow$ one of the ten possibilities holds.

$$j_3 f \approx x^3 + y^3 + z^3 + a x y z, \quad a^3 + 27 \neq 0 \Rightarrow 51;$$

$$\begin{aligned}
&\approx x^3 + y^3 + x y z \Rightarrow \mathbf{52} \text{ (series P);} \\
&\approx x^3 + x y z \quad \Rightarrow \mathbf{54} \text{ (series R = } T_{3,p,q}\text{);} \\
&\approx x y z \quad \Rightarrow \mathbf{56} \text{ (series T),} \\
&\approx x^3 + y z^2 \quad \Rightarrow \mathbf{58} \text{ (series Q);} \\
&\approx x^2 z + y z^2 \quad \Rightarrow \mathbf{66} \text{ (series S);} \\
&\approx x^3 + x z^2 \quad \Rightarrow \mathbf{82} \text{ (series U);} \\
&\approx x^2 y \quad \Rightarrow \mathbf{97} \text{ (class V);} \\
&\approx x^3 \quad \Rightarrow \mathbf{103;} \\
&= 0 \quad \Rightarrow \mathbf{104.}
\end{aligned}$$

Series T

51. $j_3 f = x^3 + y^3 + z^3 + a x y z, \quad a^3 + 27 \neq 0 \Rightarrow f \in P_8 = T_{3,3,3}$.
52. $j_3 f = x^3 + y^3 + x y z \Rightarrow f \sim x^3 + y^3 + x y z + \alpha(z), \quad j_3 \alpha = 0, \quad \Rightarrow \mathbf{53}$.
53. $f = x^3 + y^3 + x y z + \alpha(z), \quad j_3 \alpha = 0 \Rightarrow f \in P_{p+5} = T_{3,3,p} (p > 3)$.
54. $j_3 f = x^3 + x y z \Rightarrow f = x^3 + x y z + \alpha(y) + \beta(z), \quad j_3 \alpha = j_3 \beta = 0, \quad \Rightarrow \mathbf{55}$.
55. $f = x^3 + x y z + \alpha(y) + \beta(z), \quad j_3(\alpha, \beta) = 0 \Rightarrow f \in R_{p,q} = T_{3,p,q} \quad (q \geq p > 3)$.
56. $j_3 f = x y z \Rightarrow f \sim x y z + \alpha(x) + \beta(y) + \gamma(z), \quad j_3(\alpha, \beta, \gamma) = 0, \quad \Rightarrow \mathbf{57}$.
57. $f = x y z + \alpha(x) + \beta(y) + \gamma(z), \quad j_3(\alpha, \beta, \gamma) = 0 \Rightarrow f \in T_{p,q,r} \quad (r \geq q \geq p > 3)$.

Series Q

Through theorems 58–65 $\varphi = x^3 + y z^2, j_\lambda^* = j_{y z^2, x^3, \lambda}$ (λ is a monomial).

58. $j_3 f = \varphi \Rightarrow f \sim \varphi + \alpha(y) + x \beta(y), \quad j_3(\alpha, x \beta) = 0, \quad \Rightarrow \mathbf{59}_1$.

Through theorems 59–62, $k \geq 1$.

- 59_k. $f = \varphi + \alpha(y) + x \beta(y), \quad j_{y^3 k}^* f = \varphi \Rightarrow$ one of the four possibilities holds:

$$\begin{aligned}
j_{y^{3k+1}}^* f &\approx \varphi + y^{3k+1} \Rightarrow \mathbf{60}_k; \\
j_{x y^{2k+1}}^* f &\approx \varphi + x y^{2k+1} \Rightarrow \mathbf{61}_k; \\
j_{y^3 k+2}^* f &\approx \varphi + y^{3k+2} \Rightarrow \mathbf{62}_k; \\
j_{y^3 k+2}^* f &= \varphi \quad \Rightarrow \mathbf{63}_{k+1}.
\end{aligned}$$

- 60_k. $j_{y^3 k+1}^* f = \varphi + y^{3k+1} \Rightarrow f \in Q_{6k+4}$.

- 61_k. $j_{x y^{2k+1}}^* f = \varphi + x y^{2k+1} \Rightarrow f \in Q_{6k+5}$.

- 62_k. $j_{y^3 k+2}^* f = \varphi + y^{3k+2} \Rightarrow f \in Q_{6k+6}$.

Through theorems 63–65, $k > 1$.

- 63_k. $f = \varphi + \alpha(y) + x \beta(y), \quad j_{y^3 k-1}^* f = \varphi \Rightarrow$ one of the three possibilities holds:

$$\begin{aligned}
j_{y^3 k}^* f &\approx \varphi + a x^2 y^k + x y^{2k}, \quad a^2 \neq 4 \Rightarrow \mathbf{64}_k; \\
&\approx \varphi + x^2 y^k \quad \Rightarrow \mathbf{65}_k; \\
&= \varphi \quad \Rightarrow \mathbf{59}_k.
\end{aligned}$$

- 64_k. $j_{y^3 k}^* f = \varphi + a x^2 y^k + x y^{2k}, \quad a^2 \neq 4 \Rightarrow f \in Q_{k,0}$.

- 65_k. $j_{y^3 k}^* f = \varphi + x^2 y^k \Rightarrow f \in Q_{k,i} \quad (i > 0)$.

Series S

Through theorems 66–81 $\varphi = x^2 z + y z^2, j^* = j_{x^2 z, y z^2, \lambda}$ (λ is a monomial).

66. $j_3 f = \varphi \Rightarrow f = \varphi + \alpha(y) + x \beta(y) + z \gamma(y), \quad j_3(\alpha, x \beta, z \gamma) = 0, \quad \Rightarrow \mathbf{67}_1$.

Through theorems 67–76, $k \geq 1$.

- 67_k**. $f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y)$,
 $j_{y^{4k-1}}^* f = \varphi \Rightarrow$ one of the three possibilities holds:
 $j_{y^{4k}}^* f \approx \varphi + y^{4k} \Rightarrow$ **68_k**;
 $j_{xy^{3k}}^* f \approx \varphi + xy^{3k} \Rightarrow$ **69_k**;
 $j_{xy^{3k}}^* f = \varphi \Rightarrow$ **70_k**.
- 68_k**. $j_{y^{4k}}^* f = \varphi + y^{4k} \Rightarrow f \in S_{12k-1}$.
- 69_k**. $j_{xy^{3k}}^* f = \varphi + xy^{3k} \Rightarrow f \in S_{12k}$.
- 70_k**. $f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y)$, $j_{xy^{3k}}^* f = \varphi \Rightarrow$ one of the four possibilities holds:
 $j_{y^{4k+1}}^* f \approx \varphi + y^{4k+1} + bz y^{2k+1}$, $b^2 \neq 4 \Rightarrow$ **71_k**;
 $\approx \varphi + x^2 y^{2k} \Rightarrow$ **72_k**;
 $\approx \varphi + z y^{2k+1} \Rightarrow$ **73_k**;
 $= \varphi \Rightarrow$ **74_k**.
- 71_k**. $j_{y^{4k+1}}^* f = \varphi + y^{4k+1} + bz y^{2k+1}$, $b^2 \neq 4 \Rightarrow f \in S_{k,0}$.
- 72_k**. $j_{y^{4k+1}}^* f = \varphi + x^2 y^{2k} \Rightarrow f \in S_{k,i}$ ($i > 0$).
- 73_k**. $j_{y^{4k+1}}^* f = \varphi + z y^{2k+1} \Rightarrow f \in S_{k,i}^{\#}$ ($i > 0$).
- 74_k**. $f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y)$,
 $j_{y^{4k+1}}^* f = \varphi \Rightarrow$ one of the three possibilities holds:
 $j_{xy^{3k+1}}^* f \approx \varphi + xy^{3k+1} \Rightarrow$ **75_k**;
 $j_{y^{4k+2}}^* f \approx \varphi + y^{4k+2} \Rightarrow$ **76_k**;
 $j_{y^{4k+2}}^* f = \varphi \Rightarrow$ **77_{k+1}**.
- 75_k**. $j_{xy^{3k+1}}^* f = \varphi + xy^{3k+1} \Rightarrow f \in S_{12k+4}$.
- 76_k**. $j_{y^{4k+2}}^* f = \varphi + y^{4k+2} \Rightarrow f \in S_{12k+5}$.
- Through theorems **77-81**, $k > 1$.
- 77_k**. $f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y)$, $j_{y^{4k-2}}^* f = \varphi \Rightarrow$ one of the five possibilities holds:
 $j_{y^{4k-1}}^* f \approx \varphi + ax^2 y^{k-1} + bxy^k z + xy^{3k-1}$, $\Delta \neq 0 \Rightarrow$ **78_k**.
 $\approx \varphi + xy^k z + ax^3 y^{k-1}$, $a^2 \neq a \Rightarrow$ **79_k**.
 $\approx \varphi + x^3 y^{k-1} \Rightarrow$ **80_k**.
 $\approx \varphi + xy^k z \Rightarrow$ **81_k**.
 $= \varphi \Rightarrow$ **67_k**.
- 78_k**. $j_{y^{4k-1}}^* f = ax^2 y^{k-1} + bxy^k z + xy^{3k-1}$, $\Delta \neq 0 \Rightarrow f \in S_{k,0}^*$, $\mu(f) = 12k - 4$,
 $m(f) = 3k - 2$, $c(S_{k,0}^*) = 9k - 3$.
- 79_k**. $j_{y^{4k-1}}^* f = \varphi + xy^k z + ax^3 y^{k-1}$, $a^2 \neq a \Rightarrow f \in \mathbf{SP}_k$, $\mu(f) \geq 12k - 3$,
 $m(f) = 3k - 2$, $c(\mathbf{SP}_k) = 9k - 2$.
- 80_k**. $j_{y^{4k-1}}^* f = \varphi + x^3 y^{k-1} \Rightarrow f \in \mathbf{SQ}_k$, $\mu(f) \geq 12k - 2$, $m(f) = 3k - 2$,
 $c(\mathbf{SQ}_k) = 9k - 1$.
- 81_k**. $j_{y^{4k-1}}^* f = \varphi + xy^k z \Rightarrow f \in \mathbf{SR}_k$, $\mu(f) \geq 12k - 2$, $m(f) = 3k - 2$,
 $c(\mathbf{SR}_k) = 9k - 1$.

Series U

Through theorems **82-89**, $\varphi = x^3 + xz^2$, $j^* = j_{x^3, z^3, \lambda}$ (λ is a monomial).

- 82**. $j_3 f = \varphi \Rightarrow f \sim \varphi + \alpha(y) + x\beta(y) + z\gamma(y) + x^2 \delta(y)$, $j_3(\alpha, x\beta, z\gamma, x^2 \delta) = 0 \Rightarrow$ **83₁**.
 Through theorems **83-89**, $k \geq 1$.

83_k. $f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y) + x^2\delta(y)$, $j_{y^{3k}}^* f = \varphi \Rightarrow$ one of the two possibilities holds:

$$\begin{aligned} j_{y^{3k+1}}^* f &\approx \varphi + y^{3k+1} \Rightarrow \mathbf{84}_k, \\ &= \varphi \Rightarrow \mathbf{85}_k. \end{aligned}$$

84_k. $j_{y^{3k+1}}^* f = \varphi + y^{3k+1} \Rightarrow f \in U_{12k}$.

85_k. $f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y) + x^2\delta(y)$, $j_{y^{3k+1}}^* f = \varphi \Rightarrow$ one of the three possibilities holds:

$$\begin{aligned} j_{xy^{2k+1}}^* f &\approx \varphi + xy^{2k+1} + cz y^{2k+1}, \quad c(c^2+1) \neq 0 \Rightarrow \mathbf{86}_k; \\ &\approx \varphi + xy^{2k+1} \Rightarrow \mathbf{87}_k; \\ &= \varphi \Rightarrow \mathbf{88}_k. \end{aligned}$$

86_k. $j_{xy^{2k+1}}^* f = \varphi + xy^{2k+1} + cz y^{2k+1}$, $c(c^2+1) \neq 0 \Rightarrow f \in U_{k,0}$.

87_k. $j_{xy^{2k+1}}^* f = \varphi + xy^{2k+1} \Rightarrow f \in U_{k,p}$ ($p > 0$).

88_k. $f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y) + x^2\delta(y)$, $j_{xy^{2k+1}}^* f = \varphi \Rightarrow$ one of the two possibilities holds:

$$\begin{aligned} j_{y^{3k+2}}^* f &\approx \varphi + y^{3k+2} \Rightarrow \mathbf{89}_k; \\ &= \varphi \Rightarrow \mathbf{90}_{k+1}. \end{aligned}$$

89_k. $j_{y^{3k+2}}^* f = \varphi + y^{3k+2} \Rightarrow f \in U_{12k+4}$.

Through theorems **90–96**, $k \geq 2$;

$\varphi = x^2z + xz^2$; $j^* = j_{x^3, z^3, \lambda}$ (λ is a monomial).

90_k. $f = x^3 + xz^2 + \alpha(y) + x\beta(y) + z\gamma(y) + x^2\delta(y)$, $j_{y^{3k-1}}^* f = x^3 + xz^2 \Rightarrow$ one of the seven possibilities holds:

$$\begin{aligned} j_{y^{3k}}^* f &\approx \varphi + ax^2y^k + bx y^k z + y^k z^2 + cx y^{2k}, \quad \Delta \neq 0 \Rightarrow \mathbf{91}_k; && \Rightarrow \mathbf{91}_k; \\ &\approx \varphi + ax^2y^k + bx y^k z + y^k z^2, \quad 4a \neq b^2, \quad a(a+1-b) \neq 0 \Rightarrow \mathbf{92}_k; \\ &\approx x^3 + ax^2z + xz^2 + z^2y^k, \quad a^2 \neq 4 && \Rightarrow \mathbf{93}_k; \\ &\approx \varphi + x^2y^k + ax y^k z, \quad a^2 = a && \Rightarrow \mathbf{94}_k; \\ &\approx \varphi + x^2y^k && \Rightarrow \mathbf{95}_k; \\ &\approx \varphi + x y^k z && \Rightarrow \mathbf{96}_k; \\ &= \varphi && \Rightarrow \mathbf{83}_k. \end{aligned}$$

91_k. $j_{y^{3k}}^* f = \varphi + ax^2y^k + bx y^k z + y^k z^2 + cx y^{2k}$, $\Delta \neq 0 \Rightarrow f \in U_{k,0}^*$,
 $\mu(f) = 12k - 4$, $m(f) = 4k - 3$, $c(U_{k,0}^*) = 8k - 2$.

92_k. $j_{y^{3k}}^* f = \varphi + ax^2y^k + bx y^k z + y^k z^2$, $4a \neq b^2$, $a(a+1-b) \neq 0 \Rightarrow f \in \mathbf{UR}_k$,
 $\mu(f) \geq 12k - 3$, $m(f) = 4k - 3$, $c(\mathbf{UR}_k) = 8k - 1$.

93_k. $j_{y^{3k}}^* f = x^3 + ax^2z + xz^2 + z^2y^k$, $a^2 \neq 4 \Rightarrow f \in \mathbf{UQ}_k$, $\mu(f) \geq 12k - 2$,
 $m(f) = 4k - 3$, $c(\mathbf{UQ}_k) = 8k$.

94_k. $j_{y^{3k}}^* f = \varphi + x^2y^k + ax y^k z$, $a^2 \neq a \Rightarrow f \in \mathbf{UR}_k$, $\mu(f) \geq 12k - 2$,
 $m(f) = 4k - 3$, $c(\mathbf{UR}_k) = 8k$.

95_k. $j_{y^{3k}}^* f = \varphi + x^2y^k \Rightarrow f \in \mathbf{US}_k$, $\mu(f) \geq 12k - 1$, $m(f) = 4k - 3$,
 $c(\mathbf{US}_k) = 8k + 1$.

96_k. $j_{y^{3k}}^* f = \varphi + x y^k z \Rightarrow f \in \mathbf{UT}_k$, $\mu(f) \geq 12k - 1$, $m(f) = 4k - 3$,
 $c(\mathbf{UT}_k) = 8k + 1$.

Class V

97. $j_3 f(x, y, z) = x^2 y \Rightarrow f \sim x^2 y + \alpha(y, z) + x\beta(z) \Rightarrow 98$.

In theorems 98 and 102 φ is one of the 10 polynomials:

$$z^4 + z^3 y, z^3 y + z^2 y^2, z^2 y^2 + z y^3, z^4 + z^2 y^2, z^4, z^3 y, z^2 y^2, z y^3, y^4, 0.$$

98. $f = x^2 y + \alpha(y, z) + x\beta(z)$, $j_3 f = x^2 y \Rightarrow$ one of the four possibilities holds:

$$\begin{aligned} j_{x^2 y, y^4, z^4} f &\approx x^2 y + z^4 + a z^3 y + b z^2 y^2 + z y^3, \quad \Delta \neq 0, \quad ab \neq 9 \Rightarrow 99; \\ &\approx x^2 y + z^4 + b z^3 y + z^2 y^2, \quad b^2 \neq 4 \quad \Rightarrow 100; \\ &\approx x^2 y + z^3 y + a z^2 y^2 + y^4, \quad 4a^3 + 27 \neq 0 \quad \Rightarrow 101; \\ &\approx x^2 y + \varphi \quad \Rightarrow 102. \end{aligned}$$

99. $j_{x^2 y, y^4, z^4} f = x^2 y + z^4 + a z^3 y + b z^2 y^2 + z y^3$, $\Delta \neq 0 \Rightarrow f \in V_{1,0}$.

100. $j_{x^2 y, y^4, z^4} f = x^2 y + z^4 + b z^3 y + z^2 y^2$, $b^2 \neq 4 \Rightarrow f \in V_{1,p}$, $p > 0$.

101. $j_{x^2 y, y^4, z^4} f = x^2 y + z^3 y + a z^2 y^2 + y^4$, $4a^3 + 27 \neq 0 \Rightarrow f \in V_{1,p}^*$, $p > 0$.

102. $j_{x^2 y, y^4, z^4} f = x^2 y + \varphi \Rightarrow \mu(f) \geq 17$, $m(f) \geq 3$, $c(f) \geq 13$.

103. $j_3 f(x, y, z) = x^3 \Rightarrow \mu(f) \geq 18$, $m(f) \geq 4$, $c(f) \geq 13$.

104. $j_3 f(x, y, z) = 0 \Rightarrow \mu(f) \geq 27$, $m(f) \geq 10$, $c(f) \geq 16$.

105. Corank $f > 3 \Rightarrow \mu(f) \geq 16$, $m(f) \geq 5$, $c(f) \geq 10$.

§ 3. The Proofs

Theorems 1, 17 and 25 are obvious. Theorems 12₁, 15, 16 are proved in [8]. The proofs of theorems 12_k ($k > 1$), 24, 31, 32, 38–46, 65, 72, 73, 79, 80, 81, 87, 92–96, 100, 101 are based on new techniques as compared with [8] (some spectral sequence) and are not given here.

The proofs of the classification theorems for unimodular singularities do not depend on these theorems.

The classification of unimodular singularities is obtained from the following theorems:

1–5, 6_{1,2}–9_{1,2}, 10₂, 11₂, 13–17, 18₁–21₁, 22₂, 23₂, 25, 26₁–30₁, 36₁, 37₁, 47, 48, 50–58, 59₁–62₁, 63₂, 64₂, 66, 67₁–71₁, 82, 83₁–86₁, 97, 98, 105.

Here one needs only the simplest cases (the first one) of theorems 10₂, 22₂, 29₁, 36, 47, 63₂, 70₁, 85₁, 98.

To classify all bimodular singularities one needs in addition theorems:

6₃–11₃, 12₂, 18₂–21₂, 22₃, 23₃, 24₂, 31₁–35₁, 36₂, 37₂, 59₂–62₂, 63₃, 64₃, 65₂, 72₁–76₁, 77₂, 78₂, 87₁–89₁, 90₂, 91₂, 99.

(In fact, only the simplest cases of theorems 10₃, 22₃, 36₂, 63₃, 77₂, 90₂ are needed.)

Theorems 6, 18, 26, 33, 59, 67, 74, 83, 88 are proved by the Newton method [9] of a moving ruler (line, plane). This method reduces the proof to the counting of the integer points in triangles resp. polyhedrons on the exponent plane (resp. in the space).

The proofs of theorems 3, 10, 13, 22, 29, 36, 47, 50, 52, 54, 56, 58, 63, 66, 70, 77, 82, 85, 90, 97, 98 can be reduced to the classifications of orbits of the actions of some quasihomogenous diffeomorphism groups on the spaces of quasihomogenous polynomials.

Some quasihomogenous Lie algebra "roots" technique reduces all these classification problems to geometrical problems as shown in the table below.

Theorem	Series	Geometrical problem
3	D	Linear classification of the 3-forms in \mathbb{C}^2 .
10, 22	J, Z	Affine classifications of the triples of points in \mathbb{C}^1 .
13	X	Linear classification of the 4-forms in \mathbb{C}^2 .
29	W	Linear classification of the couples of points in \mathbb{C}^1 .
36	X	Affine classification of the quadruples of points in \mathbb{C}^1 .
47	N	Linear classification of the 5-forms in \mathbb{C}^2 .
50	P	Linear classification of the 3-forms in \mathbb{C}^3 .
63	Q	Affine classification of the cubics with a finite cusp point in \mathbb{C}^2 .
70	S	Affine classification of the cubics with at least 2 finite nodes in \mathbb{C}^2 .
77	S*	Affine classification of the cubics in \mathbb{C}^2 , having the infinity line a simple tangent.
85	U	Affine classification of the cubics in \mathbb{C}^2 , admitting a centre of symmetry and having exactly 3 points at infinity.
90	U*	Affine classification of the cubics in \mathbb{C}^2 , having exactly 3 points at infinity.
98	V	Affine classification of the polynomials of degree ≤ 4 in 1 variable in \mathbb{C}^1 .

Theorems on the normal forms of the semiquasihomogenous functions (theorems 2, 4, 5, 7, 8, 9, 11, 14, 19, 20, 21, 23, 27, 28, 30, 34, 35, 37, 48, 51, 60, 61, 62, 64, 68, 69, 71, 75, 78, 84, 86, 89, 91, 99) and theorems 52, 54, 56, 58, 66, 82, 97 follow from section 7.3 in [8]. Theorems 49 and 102 are corollaries of 48 and 100, 101.

To prove theorems 53, 55, 57 we use the following lemmata (the terminology is that of [8]).

Lemma 1. Let $f_0 = ax_1x_2x_3 + x_1^{p_1} + x_2^{p_2} + x_3^{p_3}$, where $3 \leq p_1 \leq p_2 \leq p_3 > 3$, $a \neq 0$. Then

- 1) at O f_0 has an isolated critical point with $\mu = p_1 + p_2 + p_3 - 1$;
- 2) monomials $1, x_1x_2x_3, x_i^{s_i}$ ($0 < s_i < p_i$, $i = 1, 2, 3$) define a regular basis of the local ring;
- 3) condition A holds for the Newton filtration, defined by the 4 monomials of f_0 .

To prove this lemma we use the crossword technique (see [8], section 9.7). We find the following 2 geometrical facts:

Lemma 2. All the cycles formed by the admissible segments for f_0 , are trivial.

This follows from the linear independence of the 3 main segments; this independence holds when p_i satisfies the above stated restrictions.

Lemma 3. The maximal admissible chains are:

- 1) every point $x_i^{s_i}$ ($0 \leq s_i \leq p_i - 2$) is a trivial chain formed by this point;
- 2) there exist 4 finite maximal chains $x_1x_2 - x_3^{p_3-1}$, $x_2x_3 - x_1^{p_1-1}$, $x_3x_1 - x_2^{p_2-1}$, $x_1x_2x_3 - x_i^{p_i}$;
- 3) all other points have infinite admissible chains.

Lemma 1 is an easy corollary of lemmata 2 and 3.

The proofs of theorems 53, 55, 57.

These theorems can be stated as follows.

Theorem. *Let f have a critical point O of finite multiplicity and*

$$f = xyz + \alpha(x) + \beta(y) + \gamma(z), \quad j_2(\alpha, \beta) = 0, \quad j_3(\gamma) = 0.$$

Then in some neighbourhood of O

$$f \sim f_0 = axyz + x^p + y^q + z^r, \quad a \neq 0, \quad 3 \leq p \leq q \leq r < 3.$$

Proof. Let the first nonzero terms of the Taylor series for α, β, γ have exponents p, q, r . By changing if necessary the notations, we obtain $3 \leq p \leq q \leq r < 3$. By dilatations of coordinates we reduce f to the form $f_0 + f_1$, where all the exponents of the monomials of f_1 are above the Newton polyhedron of f_0 .

According to lemma 1, all the monomials of f_1 belong to the ideal $(\partial f_0 / \partial x_i)$, and f_0 satisfies the condition A of [8].

By theorem 9.5 of [8] we have $f \sim f_0$; theorems 53, 55, 57 are thus proved.

References

1. Arnold, V.I.: Critical points of smooth functions. In: The Vancouver International Congress of Mathematicians, 1974
2. Demazure, M.: Sousgroupes algébriques de rang maximum du groupe de Cremona. Annales Sc. de l'E.N.S., 4-e série, t.3, fasc 4, 507-588 (1970)
3. Gabrielov, A. M.: Bifurcations, Dynkin diagrams and the modules numbers of isolated singularities. Funct. Anal. and appl. **8**, N2 7-13 (1974)
4. Arnold, V.I.: Normal forms of functions near degenerate critical points, Weyl groups A_k, D_k, E_k and Lagrange singularities. Funct. Anal. and appl. **6**, N4 254-272 (1972)
5. Arnold, V.I.: Classification of unimodular critical points of functions. Funct. Anal. and appl. **7**, N3 75-76 (1973)
6. Dolgatchev, I. V.: Factor conical singularities of complex surfaces. Funct. Anal. and appl. **8**, N2 75-76 (1974)
7. Koulikov, V. S.: On some singularities of surfaces. Funct. Anal. and appl. **9**, N1 72-73 (1975)
8. Arnold, V.I.: Normal forms of functions in the neighbourhoods of degenerate critical points. Russian Math. surveys **29**, N2 11-49 (1974)
9. Newton, I.: La méthode des fluxions., trad. Buffon. Paris: Debure l'Ainé 1740
10. Klein, F.: Vorlesungen über die Entwicklung der Mathematik im 19 Jahrhundert. Berlin: Springer 1926
11. Poincaré, H.: Sur les formes cubiques ternaires et quaternaires. J. Ecole Polytechn. **50**, 199-253 (1881)

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