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A Characterization of Orders of Finite Lattice Type

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In this paper we give a characterization of orders of finite lattice type via the global dimension and a property of relative injective and projective lattices. Since an order is of finite lattice type if and only if this is true at all completions (cf. [4]), it is enough to treat the local problem here.

§ 1. Notation and Statement of the Results

We shall use the following notation throughout this paper.

K = $\mathfrak{p}$-adic completion of an algebraic number field at the finite prime $\mathfrak{p}$

R = ring of integers in K

πR = Jacobsonradical of R

A, B = finite dimensional semi-simple K-algebras

Λ, Γ = R-orders in A and B resp.

(We recall that an R-order Λ in A is a subring of A with the same identity as A, containing a K-basis of A and consisting entirely of integral elements over R.)

${}_\Lambda\mathfrak{M}^f\,(\mathfrak{M}^f_\Lambda)$ = category of finitely generated left (right) Λ-modules

${}_\Lambda\mathfrak{M}^0\,(\mathfrak{M}^0_\Lambda)$ = category of left (right) Λ-lattices

(A left Λ-lattice M is a finitely generated left Λ-module, which is R-free.)

${}_\Lambda\mathfrak{P}\,(\mathfrak{P}_\Lambda)$ = category of projective left (right) Λ-lattices

${}_\Lambda\mathfrak{I}\,(\mathfrak{I}_\Lambda)$ = full subcategory of injective objects in ${}_\Lambda\mathfrak{M}^0\,(\mathfrak{M}^0_\Lambda)$

${}_\Lambda\mathfrak{B} = {}_\Lambda\mathfrak{P} \cap {}_\Lambda\mathfrak{I}$ = *bijective* objects in ${}_\Lambda\mathfrak{M}^0$.

For $M \in {}_\Lambda\mathfrak{M}^0$ we denote by

${}_M\mathfrak{C}$ the full subcategory of ${}_\Lambda\mathfrak{M}^0$, whose objects are $N \in {}_\Lambda\mathfrak{M}^0$, $N | M^{(s)}$ for some non-negative integer s ($X|Y$ indicates that X is isomorphic to a direct summand of Y, $Z^{(s)}$ is the direct sum of s copies of Z)

$G \in {}_\Lambda\mathfrak{M}^0$ is called an *additive generator* for Λ if ${}_G\mathfrak{C} = {}_\Lambda\mathfrak{M}^0$

$n(\Lambda)$ = number of non-isomorphic indecomposable objects in ${}_\Lambda\mathfrak{M}^0$, or infinity, if this number is not finite

${}_\Lambda\mathfrak{N}$ = $\{M_1, \ldots, M_t\}$ = a set of all non-isomorphic indecomposable objects in ${}_\Lambda\mathfrak{M}^0$, if $n(\Lambda) < \infty$.

Since lattices are R-projective, we have a *duality*

$$\mathfrak{D}\colon {}_{\Lambda}\mathfrak{M}^0 \to \mathfrak{M}^0_{\Lambda},$$
$$M \mapsto M^* = \mathrm{Hom}_R(M, R),$$

with $\mathfrak{D}^2 \sim 1_{{}_{\Lambda}\mathfrak{M}^0}$, which preserves isomorphism and decomposition etc.

Hence there are only finitely many non-isomorphic indecomposable left Λ-lattices if and only if the same is true for the right lattices.

The above notation is changed accordingly for other orders under consideration.

(1.1) *Remark.* If $n(\Lambda) < \infty$, ${}_{\Lambda}\mathfrak{N} = \{M_1, \ldots, M_t\}$, then $G = \bigoplus_{i=1}^{t} M_i$ is an additive generator for Λ. Conversely, if Λ has an additive generator, then $n(\Lambda) < \infty$.

(1.2) **Theorem.** I) *Assume that Λ has an additive generator G and put $\Gamma = \mathrm{End}_{\Lambda}(G)$. Then Γ is an R-order in the semi-simple K-algebra $B = \mathrm{End}_{\Lambda}(K \otimes_R G)$, satisfying:*

(i) *the left and right global dimension of Γ is $\leqq 2$,*

(ii) *there exists a projective resolution of left Γ-lattices*

$$0 \to P_1 \to P_0 \to \Gamma^* \to 0$$

with $P_0 \in {}_{\Gamma}\mathfrak{J}$,

(iii) *Λ is Morita-equivalent to* $\mathrm{End}_{\Gamma}(P_0)$.

We call an R-order satisfying (i) *and* (ii) *an $\mathfrak{A}$-order.*

II) *Conversely, assume that Γ is an $\mathfrak{A}$-order in B, and let P_0 be as in* (ii). *Then $\Lambda = \mathrm{End}_{\Gamma}(P_0)$ is an R-order in $A = \mathrm{End}_B(K \otimes_R P_0)$ satisfying*

(i) *$n(\Lambda) < \infty$, ${}_{\Lambda}\mathfrak{N} = \{\mathrm{Hom}_{\Gamma}(P_0, I_i),\ 1 \leqq i \leqq s\}$, if I_i, $1 \leqq i \leqq s$, are the non-isomorphic indecomposable objects in ${}_{\Gamma}\mathfrak{J}$,*

(ii) *Γ is Morita-equivalent to $\mathrm{End}_{\Lambda}(G)$, where $G = \bigoplus_{i=1}^{s} \mathrm{Hom}_{\Gamma}(P_0, I_i)$.*

(1.3) **Lemma.** *The left global dimension of Λ* (gl. dim(Λ))$\leqq n+1$ *if and only if the left global dimension of ${}_{\Lambda}\mathfrak{M}^0$,* (gl. dim$^0\,\Lambda$)$\leqq n$.

The proof consists of a straightforward generalization of [1, 2.2].

(1.4) *Remarks.* 1) By means of (1.2) the orders of finite lattice type are known, if one knows all $\mathfrak{A}$-orders. Moreover, this connection is a very close one, since as a bonus one also gets the non-isomorphic indecomposable lattices.

2) To test whether an order Γ is an $\mathfrak{A}$-order one has to do the following two things:

(i) Find a minimal projective resolution for the Jacobson-radical $J(\Gamma)$ of Γ and check whether this resolution has length $\leqq 1$,

(ii) compute Γ^* and check, whether the projective cover of Γ^* is bijective.

Given a particular order, (i) and (ii) are easily checked.

3) (1.2) can also be formulated globally.

§ 2. Proof of (1.2), I)

Let G be an additive generator for Λ. Let $\Gamma = \operatorname{End}_\Lambda(G)$.

To avoid too complicated notation we write $(X, Y)_\Lambda$ for $\operatorname{Hom}_\Lambda(X, Y)$ etc.

With G we associate the functor

(2.1)
$$\mathfrak{F}: {}_\Lambda\mathfrak{M}^0 \to {}_\Gamma\mathfrak{M}^0, \quad M \mapsto (G, M)_\Lambda,$$

which is faithful, since G is a generator in ${}_\Lambda\mathfrak{M}^0$. (A generator in ${}_\Lambda\mathfrak{M}^0$ is a left Λ-lattice M such that $\Lambda | M^{(n)}$ for some n. This should not be confused with additive generator.) Moreover, it induces an equivalence between

(2.2)
$${}_\Lambda\mathfrak{M}^0 \text{ and } {}_\Gamma\mathfrak{P} \quad \text{(cf. [2])}.$$

(2.3) **Lemma.** *The left and right global dimension of Γ is $\leqq 2$.*

Proof. Let $T \in {}_\Gamma\mathfrak{M}^f$, and choose a resolution

$$0 \to \operatorname{Ker}\varphi \to Q_1 \xrightarrow{\varphi} Q_0 \to T \to 0$$

with $Q_i \in {}_\Gamma\mathfrak{P}$, $i = 1, 2$.

Because of (2.2), $Q_i \cong (G, M_i)_\Lambda$ with $M_i \in {}_\Lambda\mathfrak{M}^0$, $i = 1, 2$. Then

$$(Q_1, Q_0)_\Gamma \cong ((G, M_1)_\Lambda, (G, M_0)_\Lambda)_\Gamma \cong (M_1, M_0)_\Lambda;$$

i.e., there exists $\psi \in (M_1, M_0)_\Lambda$ such that $\varphi = \operatorname{Hom}(1_G, \psi)$. But $\mathfrak{F}$ is left exact and so $\operatorname{Ker}\varphi \cong (G, \operatorname{Ker}\psi)_\Lambda$. By (2.2) we conclude $\operatorname{Ker}\varphi \in {}_\Gamma\mathfrak{P}$ and the left global dimension of Γ is $\leqq 2$.

As for the right global dimension we observe that $\Gamma = (G^*, G^*)_\Lambda$ and so we have a functor

(2.4)
$$\mathfrak{F}^*: \mathfrak{M}_\Lambda^0 \to \mathfrak{M}_\Gamma^0, \quad M \mapsto (G^*, M)_\Lambda,$$

which is faithful, since G^* is an additive generator for $\mathfrak{M}_\Lambda^0$. Now one shows as above, that the right global dimension of Γ is $\leqq 2$.

We now come to the proof of (1.2), (ii):

Since G is a generator, we have $\Lambda | G^{(n)}$ for some n and so

$$G \cong \mathrm{Hom}_\Lambda(\Lambda, G)_\Gamma | \mathrm{Hom}_\Lambda(G^{(n)}, G) \cong \Gamma^{(n)}.$$

Hence $G \in \mathfrak{P}_\Gamma$. Similarly $G^* \in \mathfrak{M}_\Lambda^0$ implies $G^* \in {}_\Gamma\mathfrak{P}$. Therefore $G^* \in {}_\Gamma\mathfrak{B}$ and $G \in \mathfrak{B}_\Gamma$.

Since ${}_\Lambda\mathfrak{J} = \{P^* : P \in \mathfrak{P}_\Lambda\}$, every Λ-lattice M has an injective envelope in ${}_\Lambda\mathfrak{J}$. Let $I \in \mathfrak{J}_\Lambda$ be an injective envelope for $G^* \in \mathfrak{M}_\Lambda^0$. Then we have an exact sequence

$$(2.5) \qquad 0 \to G^* \to I \to C \to 0$$

with $C \in \mathfrak{M}_\Lambda^0$. We apply the functor $\mathfrak{F}^*$ to this sequence and obtain the exact sequence of right Γ-lattices

$$0 \to \Gamma \to (G^*, I)_\Lambda \to C' \to 0,$$

where C' is a Γ-lattice, since it is a submodule of $(G^*, C)_\Lambda$. Applying the duality $\mathfrak{D}$ to this sequence we get the exact sequence of left Γ-lattices

$$(2.6) \qquad 0 \to C'^* \to (G^*, I)_\Lambda^* \to \Gamma^* \to 0.$$

Then $(G^*, I)_\Lambda^* \in {}_\Gamma\mathfrak{J}$, since it is the dual of a projective lattice. We have to show that $(G^*, I)_\Lambda^* \in {}_\Gamma\mathfrak{P}$. But this is the case if and only if $(G^*, I)_\Lambda \in \mathfrak{J}_\Gamma$.

But $I | \Lambda^{*(n)}$ for some n; hence

$$(G^*, I)_\Lambda | (G^*, \Lambda^*)_\Lambda^{(n)} \cong (\Lambda, G)_\Lambda^{(n)} \cong G^{(n)}.$$

But $G \in \mathfrak{J}_\Gamma$, and so $(G^*, I)_\Lambda \in \mathfrak{J}_\Gamma$. By (1.3) and (2.3) $C'^* \in {}_\Gamma\mathfrak{P}$. This shows that Γ satisfies (ii) of (1.2). Observe that in (1.2), (ii) we have $P_0 = (G^*, I)_\Lambda^*$.

Finally we prove (1.2), (iii). Put $\Lambda' = \mathrm{End}_\Gamma(P_0)$. Then

$$\Lambda' = \big((G^*, I)_\Lambda^*, (G^*, I)_\Lambda^*\big)_\Gamma \cong \big((G^*, I)_\Lambda, (G^*, I)_\Lambda\big)_\Gamma.$$

But $\mathfrak{F}^*$ is faithful, and so we have

$$\Lambda' \cong (I, I)_\Lambda \cong (I^*, I^*)_\Lambda.$$

Observe that all the above isomorphisms are natural. It therefore remains to show that $(I^*, I^*)_\Lambda$ is Morita-equivalent to Λ; for this it is necessary and sufficient that $I^* \in {}_\Lambda\mathfrak{M}^0$ is a progenerator. Firstly $I^* \in {}_\Lambda\mathfrak{P}$ since $I \in \mathfrak{J}_\Lambda$. Moreover, because of (2.5) we have an epimorphism of left Λ-modules.

$$I^* \to G \to 0$$

Therefore I^* is a generator, G being one. This completes the proof of (1.2), I).

§ 3. Proof of (1.2), II)

Let now Γ be an $\mathfrak{A}$-order and let P_0 be the projective cover of Γ^*. Then P_0 is bijective, and if we put $\Lambda=(P_0, P_0)_\Gamma$, then the functor

(3.1) $$\mathfrak{G}: {}_\Gamma\mathfrak{M}^0 \to {}_\Lambda\mathfrak{M}^0, \quad N \mapsto (P_0, N)_\Gamma$$

is dense, P_0 being projective.

Let ${}_\Gamma\mathfrak{L}$ be the full subcategory of ${}_\Gamma\mathfrak{M}^0$, whose objects are those lattices $N \in {}_\Gamma\mathfrak{M}^0$ such that there exists an epimorphism

(3.2) $$P_0^{(n)} \to N \to 0 \quad \text{for some } n.$$

If $\tau_N: P_0 \otimes_\Lambda (P_0, N)_\Gamma \to N$, $p\otimes\varphi \mapsto p\varphi$ is the trace map, then $N\in{}_\Gamma\mathfrak{L}$ if and only if τ_N is an epimorphism.

(3.3) **Lemma.** *$\mathfrak{G}$ induces an equivalence between ${}_\Gamma\mathfrak{L}$ and ${}_\Lambda\mathfrak{M}^0$.*

Proof. (i) $\mathfrak{G}|_{{}_\Gamma\mathfrak{L}}$ is faithful; in fact, we have for $N_1, N_2 \in {}_\Gamma\mathfrak{L}$:

$$(N_1, N_2)_\Gamma \cong (P_0 \otimes_\Lambda (P_0, N_1)_\Gamma, N_2)_\Gamma$$
$$\cong ((P_0, N_1)_\Gamma, (P_0, N_2)_\Gamma)_\Lambda.$$

For the first isomorphism observe that $\operatorname{Ker}\tau_N$ is the torsionpart of $P_0 \otimes_\Lambda (P_0, N)_\Gamma$.

(ii) $\mathfrak{G}|_{{}_\Gamma\mathfrak{L}}$ is dense, in fact let $M\in{}_\Lambda\mathfrak{M}^0$ be given. Then $M\cong(P_0, N)_\Gamma$, since $\mathfrak{G}$ is dense. Put $N_1=\operatorname{Im}\tau_N$. Then $(P_0, N)_\Gamma \cong (P_0, N_1)_\Gamma$ and $\mathfrak{G}|_{{}_\Gamma\mathfrak{L}}$ is dense.

(3.4) **Lemma.** *$N\in{}_\Gamma\mathfrak{L}$ if and only if $N\in{}_\Gamma\mathfrak{J}$.*

Proof. (i) If $N\in{}_\Gamma\mathfrak{L}$, then we have an exact sequence

$$0 \to C \to P_0^{(n)} \to N \to 0$$

of Γ-lattices. Applying the duality $\mathfrak{D}$, we get the exact sequence of right Γ-lattices

$$0 \to N^* \to P_0^{*(n)} \to C^* \to 0.$$

Since P_0^* is Γ-projective, and since C^* has homological dimension $\leqq 1$, $N^*\in\mathfrak{P}_\Gamma$ (cf. (1.3)). But then $N\in{}_\Gamma\mathfrak{J}$.

(ii) Let now $N\in{}_\Gamma\mathfrak{J}$. Then $N|\Gamma^{*(n)}$ for some n and we surely have an epimorphism $P_0^{(n)}\to N\to 0$. Hence $N\in{}_\Gamma\mathfrak{L}$.

We now are in the position to prove (1.2), II).

Let $I_1, \dots, I_s$ be the non-isomorphic indecomposable objects in ${}_\Gamma\mathfrak{J}$. Then because of (3.3) and (3.4),

$${}_\Lambda\mathfrak{N} = \{(P_0, I_1)_\Gamma, \dots, (P_0, I_s)_\Gamma\}.$$

Let $I = \bigoplus_{i=1}^{s} I_i$; we have to show that Γ and $\mathrm{End}_\Lambda((P_0, I)_\Gamma)$ are Morita-equivalent. However

$$\Gamma' = \mathrm{End}_\Lambda((P_0, I)_\Gamma) \cong \mathrm{End}_\Gamma(I) \cong \mathrm{End}_\Gamma(I^*),$$

and it remains to show that I^* is a progenerator for $\mathfrak{M}_\Gamma^0$. In fact, this says that the categories $\mathfrak{M}_{\Gamma'}^0$ and $\mathfrak{M}_\Gamma^0$ are Morita-equivalent. But then also the categories ${}_{\Gamma'}\mathfrak{M}^0$ and ${}_\Gamma\mathfrak{M}^0$ are Morita-equivalent via the duality $\mathfrak{D}$. I^* surely is projective. Moreover, $\Gamma^* | I^{(n)}$ for some n and therefore $\Gamma | I^{*(n)}$ and hence $I^* \in \mathfrak{M}_\Gamma^0$ is a generator. This completes the proof of (1.2), II).

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