Inventiones math. 17, 67–78 (1972)
© by Springer-Verlag 1972

The Global Homological Dimension of Some Algebras of Differential Operators

Jan-Erik Björk (Stockholm)

Introduction

In this paper we compute the global homological dimension of some filtered rings whose associated graded rings are commutative and regular noetherian rings. Here follow some "classical" examples.

When K is a field we let $A_n(K)=K[X_1,\ldots,X_n,\partial/\partial X_1,\ldots,\partial/\partial X_n]$ be the Weyl Algebra. It is the ring of differential operators in n variables. A filtration on $A_n(K)$ arises if we introduce the usual degree of a differential operator and the associated graded ring is the polynomial ring in $2n$ variables over K.

Another example is $F_n(K)=K[\![X_1\ldots X_n]\!][\partial/\partial X_1\ldots\partial/\partial X_n]$, the ring of differential operators with coefficients in the ring of formal power series in n variables over K.

A closely related example is $H_n(\Delta^n)=\mathcal{O}(\Delta^n)[\partial/\partial z_1\ldots\partial/\partial z_n]$, the ring of holomorphic differential operators with coefficients in the ring of germs of holomorphic functions on the closed polydisc Δ^n in C^n.

In [10] Roos proves that the left global dimension of $A_n(K)$ is n, provided that K is a field of characteristic 0. We remark that if $\operatorname{char}(K)\neq 0$ then the global dimension of $A_n(K)$ is $2n$ (see [8]).

The *Main Theorem*, proved in Section 5, only computes global dimensions when algebras of differential operators arise from *abelian* Lie algebras of derivations on a regular commutative noetherian ring. To what extent our present results can be generalised to non-abelian cases remains open. A typical example arises if we consider the n-sphere S^n in R^{n+1} and let $D(S^n)$ be the algebra of differential operators on S^n which is generated by the real-analytic functions and the vector fields on S^n which are invariant under the usual action of $SO(n+1)$ on S^n. It is easily verified that $n\leq \text{l. gl. dim}(D(S^n))\leq 2n$ and we believe that the correct value is n.

Because of these possible extensions we have introduced a rather general class of rings of differential operators in Section 1 and in Section 3 we have proved some results which may have applications to non-abelian cases. We will use some standard results, collected in Section 2, dealing with flatness, localisations and filtrations etc.

The core of this paper occurs in the *Main Lemma* from Section 3. With its aid we compute the global dimension of $F_n(K)$ and then we have only continued with several technical steps in order to arrive at the *Main Theorem.*

1. Constructing Rings of Differential Operators

The material in this section is wellknown and we refer to [9] for more details. Consider a commutative field K and a commutative K-algebra R. There arises the Lie algebra $\mathrm{Der}_K(R,R)$ consisting of all K-derivations on R. Let L be a finite dimensional K-subalgebra of $\mathrm{Der}_K(R,R)$.

Given a K-basis $\{u_1 \dots u_n\}$ for L a ring $B=R_L[X_1 \dots X_n]$ is constructed as follows:

Let $R\{Y_1 \dots Y_n\}$ be the free left R-module whose basis consists of all non-commuting monomials $M_\alpha(Y)$ in the variables $Y_1 \dots Y_n$. Then $R\{Y_1 \dots Y_n\}$ becomes an associative ring via the multiplication rules: $Y_i x = x Y_i + u_i(x)$ for all x in R and each $i=1 \dots n$.

Finally B is the ring $R\{Y_1 \dots Y_n\}/J$, where J is the two-sided ideal generated by the elements $Y_i Y_k - Y_k Y_i - \sum c_{ikv} Y_v$. Here $\{c_{ikv}\} \in K$ are the structure constants of L, i.e. $[u_i, u_k] = \sum c_{ikv} u_v$ and finally the canonical map from $R\{Y_1 \dots Y_n\}$ onto B is R-linear and maps Y_i into X_i.

The Poincare-Birkhoff-Witt Theorem holds for B. So if F_k is the two-sided R-submodule of B generated by all monomials $M_\alpha(X)$ of total degree $|\alpha| = \alpha_1 + \dots + \alpha_n \leqq k$, then $\{F_k\}$ is a filtration on B. The associated graded ring is the polynomial ring in n commuting variables over R.

It follows that B is noetherian (i.e. left and right noetherian) if and only if R is so. From a general result in [12] we have the following estimates for the left global dimensions:

$$\mathrm{l.\,gl.\,dim}(R) \leqq \mathrm{l.\,gl.\,dim}(B) \leqq \mathrm{l.\,gl.\,dim}(R) + n.$$

Suppose next that R is an integral domain and let S be a multiplicative subset of $R \setminus 0$. We have the localisation R_S and each $\delta \in \mathrm{Der}_K(R,R)$ is extended to a K-linear derivation on R_S by the rule: $\delta(x/s) = (\delta(x)s - x\delta(s))/s^2$. In particular the previous Lie algebra L gives rise to a Lie algebra $L_S \subset \mathrm{Der}_K(R_S, R_S)$, so we may construct the ring $B_S = (R_S)_{L_S}[X_1 \dots X_n]$. It is easily verified that B_S is the classical two-sided quotient ring of B constructed from the multiplicative subset S of B. In particular $B_S \otimes_B B_S \cong B_S$ and B_S is a flat extension of B, i.e. B_S contains B as a subring and it becomes flat both as a left and as a right B-module.

Suppose now that R_0 is a K-subalgebra of R and assume that each $\delta \in L$ maps R_0 into itself. The restricted action of L to R_0 gives a Lie algebra $L_0 \subset \mathrm{Der}_K(R_0, R_0)$ and we get the ring $B_0 = (R_0)_{L_0}[X_1 \dots X_n]$.

B_0 is a subring of B and using the filtrations on B and B_0 the result below follows from the material in [12].

Lemma 1.1. *Let* $R_0 \subset R$ *and* L *be as above. If* R *is flat* (resp. faithfully flat) *over* R_0, *then* B *is so over* B_0.

2. Preliminaries

2.a. *A Remark on Global Dimensions.* For a ring A which is noetherian, i.e. left and right noetherian, its global dimension can be computed via flat resolutions. We say that a (non-zero) left A-module M has *weak dimension n* if and only if

$$\mathrm{Tor}_{n+1}^A(N, M)=0 \quad \text{for all right } A\text{-modules } N,$$

while $\mathrm{Tor}_n^A(N^*, M)\neq 0$ for some N^*.

If A is noetherian and if $\mathrm{w.dim}_A(M)\leqq n$ for all left A-modules M, while equality holds for some M, then the left and the right global dimension of A are both equal to n. Recall here that when A is noetherian and if $\mathrm{gl.dim}(A)=n\geqq 1$, then A contains a left ideal L such that $\mathrm{w.dim}_A(L)=n-1$.

When A is noetherian and $\{M_\alpha\}$ is a family of left A-modules, then $\mathrm{w.dim}_A(M)=n$, where $n=\sup\{\mathrm{w.dim}_A(M_\alpha)\}$ and $M=\prod M_\alpha$.

2.b. *Flat Extensions.* Let A be a subring of the ring B. We say that B is a flat extension of A if B is flat as a left and as a right A-module. When M is a left A-module we get the extended left B-module ${}_{(B)}M = B\otimes_A M$.

The following wellknown result holds.

Lemma 2.b.1. *Let* $A\subset B$ *and suppose that* B *is flat as a right* A*-module. Then* $\mathrm{w.dim}_B({}_{(B)}M)\leqq \mathrm{w.dim}_A(M)$ *for all left* A*-modules* M.

Next we consider a left B-module N and via scalar restriction we get the left A-module ${}_AN$. Under the condition that $B\otimes_A B\cong B$ it follows that $N\cong {}_{(B)}({}_AN)$ and the result below holds.

Lemma 2.b.2. *Let* B *be a flat extension of* A *satisfying* $B\otimes_A B\cong B$. *Then* $\mathrm{w.dim}_B(N)\leqq \mathrm{w.dim}_A({}_AN)$ *for all left* B*-modules* N. *As a consequence* $\mathrm{gl.dim}(B)\leqq \mathrm{gl.dim}(A)$ *provided that* A *and* B *are noetherian.*

Here follows another wellknown result.

Lemma 2.b.3. *Let* B *be a flat extension of* A *and let* M *be a left* A*-module. If* $M_0={}_A({}_{(B)}M)$, *then* $\mathrm{w.dim}_A(M_0)\leqq \mathrm{w.dim}_B({}_{(B)}M)$.

Recall that B is a *faithful flat* extension of A when B is flat over A and when proper one-sided ideals of A remain proper in B after an extension, i.e. $BL \neq L$ for all left ideals $L \neq A$ and similarly for right ideals.

When B is faithfully flat over A it follows that $L = BL \cap A$ for all left ideals L of A. As a consequence the A-linear map from M into ${}_A({}_{(B)}M)$ is injective for all left A-modules M.

Next follows a very useful result discovered by McConnell.

Lemma 2.b.4. *Let* $A \subset B$ *be a pair of noetherian rings and suppose that* B *is a faithful flat extension of* A. *If we also assume that* gl. dim(A) *and* gl. dim(B) *are both finite, then* gl. dim$(A) \leqq$ gl. dim(B).

Proof. Suppose that $s =$ gl. dim$(A) >$ gl. dim(B). Choose a left A-module M such that w. $\dim_A(M) = s$ and put $\tilde{M} = B \otimes_A M$. Then Lemma 2.b.3 shows that w. $\dim_A({}_A\tilde{M}) \leqq$ gl. dim$(B) < s$.

If $Q = \tilde{M}/M$ is considered as a left A-module we get the exact sequence $0 \to M \to \tilde{M} \to Q \to 0$, which implies that w. $\dim_A(Q) = s+1$, a contradiction.

We finish this section with another useful result.

Lemma 2.b.5. *Let* B *be a noetherian ring and let* J *be a two-sided ideal of* B. *Suppose that* w. $\dim_B(J) = s$, *where* J *has been considered as a left* B*-module. If now* gl. dim$(B/J) = t > 0$, *then* w. $\dim_B(L) \leqq s+t$ *for each left ideal* L *of* B *containing* J.

Proof. When B/J is considered as a left B-module we see that w. $\dim_B(B/J) \leqq s+1$ and hence w. $\dim_B(K) \leqq s+1$, where K is a projective left B/J-module which has been considered as a left B-module.

Using induction over w. $\dim_{B/J}(K)$ it follows that w. $\dim_B(K) \leqq s+1+$w. $\dim_{B/J}(K)$ for every finitely generated left B/J-module K. In particular $K = L/J$ is a finitely generated left ideal of B/J and hence we get w. $\dim_{B/J}(K) \leqq$ gl. dim$(B/J)-1$ which gives w. $\dim_B(K) \leqq s+t$.

Finally the exact sequence $0 \to J \to L \to L/J \to 0$, implies that w. $\dim_B(L) \leqq s+t$.

2.c. *Krull Dimensions.* We employ Krull dimensions in the sense of [7]. Let us consider the following situation: R is a local noetherian ring whose Krull dimension is n. Let B be a flat extension of R and suppose also that B is noetherian and that the maximal ideal $\mathfrak{m}$ of R is such that $B\mathfrak{m}$ is a proper two-sided ideal of B. Let k be the left Krull dimension of the ring $B/B\mathfrak{m}$.

Using the fact that $L = BL \cap R$ holds for all ideals L of R the result below easily follows, using an induction over the deviations of decreasing sequences of ideals in R.

Lemma 2.c.1. *In the situation above it follows that the left Krull dimension of* B *is at least* $n+k$.

Here follows a case where the situation above occurs. Let R be a local noetherian ring containing a field K such that $R=K+\mathfrak{m}$ and let n be its Krull dimension. Let L be an s-dimensional K-subalgebra of $\mathrm{Der}_K(R,R)$ and put $B=R_L[X_1\dots X_s]$.

If $\delta(\mathfrak{m})\subset\mathfrak{m}$ for all $\delta\in L$, then $B\mathfrak{m}$ is a two-sided ideal of B and $B/B\mathfrak{m}\cong U(L)$, where $U(L)$ is the envelopping algebra of the Lie algebra L over K. Using Lemma 2.c.1 we conclude that the left (and the right) Krull dimension of B is at least $n+\mathrm{Kr.dim}(U(L))$.

Since B also admits a filtration whose associated graded ring is the polynomial ring in s variables over R we also have $\mathrm{Kr.dim}(B)\leqq n+s$. As a consequence $\mathrm{Kr.dim}(B)=n+s$, provided that L is abelian.

3. The Main Lemma

In this section R is a commutative, noetherian and regular K-algebra. L is an s-dimensional K-subalgebra of $\mathrm{Der}_K(R,R)$ and we put $B=R_L[X_1\dots X_s]$. To each multiplicative subset S of $R\setminus 0$ we get the ring B_S as in Section 1. We write $B_{\mathfrak{p}}$ instead of B_S, when $S=R\setminus\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of R.

With these notations we have the result below.

Main Lemma. *Let $\mathcal{M}$ be a family of multiplicative subsets of R. If* $\mathrm{gl.dim}(B_S)<\mathrm{gl.dim}(B)=k\geqq 2$, *for each S in $\mathcal{M}$, then B contains a left ideal L_0 such that* $\mathrm{w.dim}_B(L_0)=k-1$, *while $L_0\cap R$ is a prime ideal of R which intersects each S in $\mathcal{M}$.*

Proof. Let L be a left ideal of B such that $\mathrm{w.dim}_B(L)=k-1$. Since B is left noetherian we may assume that L is maximal among all left ideals with this property, i.e. that $\mathrm{w.dim}_B(L_1)<k-1$ if L_1 is a left ideal strictly containing L.

Put $M=B/L$ and $M_S=B_S\otimes_B M$ for each S in $\mathcal{M}$, while $\tilde{M}=\prod M_S$. There is a B-linear map ϕ which sends $m\in M$ into $\prod 1\otimes m$ in $\tilde{M}$. We know that

$$\mathrm{w.dim}_B(M_S)\leqq \mathrm{w.dim}_{B_S}(M_S)\leqq \mathrm{gl.dim}(B_S)\leqq k-1 \quad \text{for each } S.$$

It follows that $\mathrm{w.dim}_B(\tilde{M})\leqq k-1$ and then the sequence $0\to M\to\tilde{M}\to Q\to 0$, with $Q=\tilde{M}/M$, cannot be exact because it would follow that $\mathrm{w.dim}_B(Q)=k+1$.

Hence there is some $m\neq 0$ in M such that $\phi(m)=0$ which means that the left R-ideal $\mathrm{Ann}_R(m)$ intersects each S in $\mathcal{M}$. Because R is noetherian we can find $m_1\in M$ such that $\mathrm{Ann}_R(m_1)$ is a prime ideal $\mathfrak{p}$ of R containing $\mathrm{Ann}_R(m)$. Let $x_1\in B\setminus L$ be mapped into m_1 and put

$$L_1=Bx_1+L \quad\text{and}\quad W=\{b\in B\colon bx_1\in L\}.$$

Then $L_1/L \cong B/W$ and $W \cap R = \not p$. It remains to show that $\mathrm{w.dim}_B(W) = k-1$. The maximal property of L shows that $\mathrm{w.dim}_B(L_1) < k-1$ which implies that $\mathrm{w.dim}_B(L_1/L) = k$ and hence $\mathrm{w.dim}_B(B/W) = k$ which gives $\mathrm{w.dim}_B(W) = k-1$.

Remark. The idea to use the map ϕ is due to Roos who employed such a map in [10, Proposition 1].

Here follow some useful consequences of the Main Lemma.

Lemma 3.1. *Let R and B be as before. Then*

$$\mathrm{gl.dim}(B) = \sup\{\mathrm{gl.dim}(B_{\mathfrak{m}})\},$$

as $\mathfrak{m}$ runs over all maximal ideals of R.

Lemma 3.2. *Let R, L and B be as before and recall that* $\dim_K(L) = s$. *If* $\mathrm{gl.dim}(B) > s$, *then B contains a left ideal L such that $L \cap R$ is a non-zero prime ideal of R while* $\mathrm{w.dim}_B(L) \geqq s$.

Proof. We can apply the Main Lemma where $\mathcal{M}$ consists of the set $S = R \setminus 0$. To see this we notice that $B_S = D_{L_S}[X_1 \dots X_s]$, where D is the quotient field of R and then a result in [12] shows that $\mathrm{gl.dim}(B_S) \leqq s$.

4. Computation of gl. dim $(F_n(K))$

From now on we will only consider rings of differential operators arising from abelian Lie algebras of derivations. The rings obtained in the abelian case are called Ore's polynomial rings and here we describe how these are constructed.

Let δ be a derivation on a ring R, where R need not be commutative, then Ore's polynomial ring $B = R_\delta[X]$ arises as follows:

B is a free left R-module, with a basis given by $\{1, X, X^2, \dots\}$ and it becomes a ring via the multiplication rule $rX = Xr + \delta(r)$ for all r in R.

More generally we let $\Delta = \{\delta_i\}_1^s$ be an s-tuple of commuting derivations on R and get the ring $B = R_\Delta[X] = R_{\delta_1 \dots \delta_s}[X_1 \dots X_s]$.

Here $\{X_i\}_1^s$ commute in B while $rX_i = X_i r + \delta_i(r)$ for all r in R. It is important to notice that B also arises as follows:

Put $B_1 = R_{\delta_1}[X_1]$ and extend δ_2 to a derivation on B_1 by the rule $\delta_2(X_1) = 0$. This gives the ring $B_2 = B_{1\,\delta_2}[X_2]$ and finally $B_S = B$.

When R and $B = R_\Delta[X]$ are as above a result in [12] gives the estimates:

$$\mathrm{l.gl.dim}(R) \leqq \mathrm{l.gl.dim}(B) \leqq \mathrm{l.gl.dim}(R) + s.$$

Theorem 4.1. *Let K be a field of characteristic* 0 *and let* $F_n(K) = K[\![X_1 \dots X_n]\!][\partial/\partial X_1 \dots \partial/\partial X_n]$. *Then* $\mathrm{gl.dim}(F_n(K)) = n$.

Proof. To each $1 \leqq s \leqq n$ we put $R_s = K[\![X_1 \dots X_s]\!]$ and recall that R is a regular local noetherian ring of dimension s. The group $GL(n, K)$ operates on $F_n(K)$ as follows: If $\xi = (k_{vi}) \in GL(n, K)$ we put

$$\tilde{X}_i = \sum k_{vi} X_v \quad \text{and} \quad \tilde{Y}_i = \sum \rho_{iv} Y_v, \qquad \text{where } (\rho_{iv}) = \xi^{-1} \text{ in } GL(n, K).$$

Then there exists an automorphism of the K-algebra $F_n(K)$ which sends X_i into $\tilde{X}_i$ and Y_i into $\tilde{Y}_i$.

Suppose now that $\operatorname{gl.dim}(F_n(K)) = n+s$, with $s \geqq 1$. It follows from Lemma 3.2 that $F_n(K)$ contains a left ideal L whose weak dimension is $n+s-1$, while $L \cap R_n \neq 0$.

Using the Preparation Theorem in R_n we may assume that $L \cap R_n$ contains a Distinguished Weierstrass Polynomial $f \in R_{n-1}[X_n]$. If necessary we perform an automorphism of $F_n(K)$ by some element in $GL(n, K)$ to obtain this.

The division in R_n shows that $R_n = R_n f + R_{n-1}[X_n]$ and since $F_n(K) = \sum M_\alpha R_n$, where M_α runs over all monomials in $\{\partial/\partial X_i\}$ we conclude that $F_n(K) = F_n(K) f + B_1$, where

$$B_1 = F_{n-1}(K)[X_n, \partial/\partial X_n].$$

In particular $L = F_n(K) L_0$ where $L_0 = L \cap B_1$. Since R_n is a flat extension of $R_{n-1}[X_n]$ it follows from Lemma 1.1 that $F_n(K)$ is so over B_1. It follows that $L \cong_{(F_n(K))} L_0$ and then Lemma 2.b.1 gives

$$\operatorname{w.dim}_{F_n(K)}(L) \leqq \operatorname{w.dim}_{B_1}(L_0).$$

The last inequality implies that $\operatorname{gl.dim}(B_1) \geqq n+s$.

Now we consider the ring $M_1 = \mathcal{M}_{n-1}[X_n, \partial/\partial X_1 \dots \partial/\partial X_n]$, where $\mathcal{M}_{n-1}$ is the quotient field of R_{n-1}. Since $\mathcal{M}_{n-1}[X_n, \partial/\partial X_n]$ is the Weyl Algebra in one variable over the field $\mathcal{M}_{n-1}$ of characteristic 0, its global dimension is 1. We conclude that

$$\operatorname{gl.dim}(M_1) \leqq n,$$

since M_1 arises from this Weyl Algebra and $n-1$ derivations, namely $\partial/\partial X_1 \dots \partial/\partial X_{n-1}$.

Observe that M_1 also arises as the localisation of B_1 by the multiplicative set $R_{n-1} \setminus 0$. An application of the Main Lemma shows that if $\operatorname{gl.dim}(B_1) = n+k$, with $k \geqq 1$, then B_1 contains a left ideal L_1 such that $L_1 \cap R_{n-1} \neq 0$ while $\operatorname{w.dim}_{B_1}(L_1) = n+k-1$.

Again we may assume that $L_1 \cap R_{n-1}$ contains a Weierstrass polynomial from $R_{n-2}[X_{n-1}]$. So if we put

$$B_2 = F_{n-2}(K)[X_{n-1}, X_n, \partial/\partial X_{n-1}, \partial/\partial X_n],$$

then it follows that L_1 is generated by $L_1 \cap B_2$. The same argument as before implies that

$$\text{gl. dim}(B_2) \geqq \text{gl. dim}(B_1) \quad \text{and hence gl. dim}(B_2) > n.$$

Now we consider $M_2 = \mathcal{M}_{n-2}[X_{n-1}, X_n, \partial/\partial X_1 \dots \partial/\partial X_n]$ and observe that M_2 is Ore's Polynomial Ring over the Weyl Algebra $A_2(\mathcal{M}_{n-2})$ arising from $n-2$ derivations, namely $\partial/\partial X_1 \dots \partial/\partial X_{n-2}$.

Using Roos' result on the Weyl Algebra $A_2(\mathcal{M}_{n-2})$ it follows that gl. dim$(M_2) \leqq n$.

If we let $B_3 = F_{n-3}(K)[X_{n-2} \dots \partial/\partial X_n]$ together with the established estimate gl. dim$(B_2) > n$, we can prove that gl. dim$(B_3) \geqq$ gl. dim$(B_2) > n$ by the same methods as above.

Inductively we finally arrive at B_n and obtain gl. dim$(B_n) > n$. But here B_n is the Weyl Algebra in n variables over K, so this contradicts the result by Roos. Hence we conclude that gl. dim$(F_n(K)) \leqq n$ must hold.

Replacing the Scalar Field K by a Ring

Theorem 4.2. *Let R be a regular, noetherian and commutative ring of dimension k. Assume also that R contains the field of rational numbers. For $n \geqq 1$ we consider the ring*

$$F_n(R) = R[\![X_1 \dots X_n]\!]\,[\partial/\partial X_1 \dots \partial/\partial X_n].$$

Then gl. dim$(F_n(R)) = k + n$.

Proof. We use induction over dim(R) and observe that Theorem 4.1 gives the result when dim$(R) = 0$. Let us put $B = F_n(R)$ and observe that B is already free as a left $R[\![X_1 \dots X_n]\!]$-module, which implies that gl. dim$(B) \geqq k + n$ holds.

An obvious companion to Lemma 3.1 shows that

$$\text{gl. dim}(B) = \sup\{\text{gl. dim}(R_{\mathfrak{m}}[\![X_1 \dots X_n]\!]\,[\partial/\partial X_1 \dots \partial/\partial X_n])\},$$

as $\mathfrak{m}$ runs over all maximal ideals of R. Hence we may assume that R is local. The induction hypothesis implies that

$$\text{gl. dim}(R_{\mathfrak{p}}[\![X]\!]\,[\partial/\partial X] < n + \dim(R),$$

if $\mathfrak{p}$ is a prime ideal of the local ring R strictly contained in its maximal ideal $\mathfrak{m}$.

Since $s = \text{gl. dim}(B) \geqq n + \dim(R)$ an application of the Main Lemma shows that B contains a left ideal L such that

$$\text{w. dim}_B(L) = s - 1, \quad \text{while } B\mathfrak{m} \subset L.$$

Notice that $B\mathfrak{m}$ is a two-sided ideal of B and that $B/B\mathfrak{m} \cong F_n(K)$, where K is the field $R/\mathfrak{m}$. Since R is a Q-algebra it follows that $\mathrm{char}(K)=0$ and then Theorem 4.1 implies that gl. $\dim(B/B\mathfrak{m})=n$.

Next the regularity of R implies that $\mathfrak{m}$ is generated by k parameters an application of the Standard Resolution (see [2, pp. 150–153]) implies that $\mathrm{w.dim}_B(B\mathfrak{m}) \leqq k-1$.

At this stage an application of Lemma 2.b.5 shows that $\mathrm{w.dim}_B(L) \leqq n+k-1$, which gives the desired estimate $s \leqq n+k$.

5. The Main Theorem

In this section R is a commutive, noetherian and regular ring satisfying the additional property below.

Assumption (A). R contains a subfield K of characteristic 0 such that $R/\mathfrak{m}$ is an algebraic extension of K for each maximal ideal $\mathfrak{m}$ of R.

Suppose next that A is a local ring containing a subfield K such that $A=K+\mathfrak{m}$, where $\mathfrak{m}$ is its maximal ideal. Let $\Delta=\{\delta_i\}_1^s$ be an s-tuple of commuting derivations on the K-algebra A. Ro each i we get a K-linear map T_i from the K-space $V=\mathfrak{m}/\mathfrak{m}^2$ into K as follows

$$T_i(\bar{x})-\delta_i(x)\in\mathfrak{m}, \qquad \text{for all } x \text{ in } \mathfrak{m}.$$

The K-dimension of the family $\{T_i\}_1^s$ in the dual space of V gives an integer denoted by $\mathrm{rank}_\Delta(\mathfrak{m})$.

Suppose now that R is a ring satisfying (A) and let $\Delta=\{\delta_i\}_1^s$ be an s-tuple of commuting K-derivations on R. Let us then define an integer $\mathrm{rank}_\Delta(\mathfrak{m})$, when $\mathfrak{m}$ is a maximal ideal of R.

Firstly we take the local ring $R_\mathfrak{m}$ and extend each δ_i to a K-derivation on this ring. Let $A=\hat{R}_\mathfrak{m}$ be its completition in the $\mathfrak{m}$-adic topology. If K_1 is the integral closure of K in A, then K_1 is a field and

$$A \cong K_1[\![X_1 \dots X_n]\!], \qquad \text{where } n=\dim(R_\mathfrak{m}).$$

Each δ_i admits a unique extension to a derivation $\tilde{\delta}_i$ on A. Since $\mathrm{char}(K)=0$ and each $\delta_i(K)=0$, it follows that $\tilde{\delta}_i(K_1)=0$.

Now we define $\mathrm{rank}_\Delta(\mathfrak{m})$ to be $\mathrm{rank}_{\Delta_e}(\hat{\mathfrak{m}})$, where $\Delta_e=\{\tilde{\delta}_i\}_1^s$.

Main Theorem. *Let R be a commutative, noetherian and regular ring satisfying the condition* (A) *with respect to a subfield K. Let $\Delta=\{\delta_i\}_1^s$ be an s-tuple of commuting K-derivations on R and put $B=R_\Delta[X]$.*

Then gl. $\dim(B)=\sup\{\dim(R_\mathfrak{m})+s-\mathrm{rank}_\Delta(\mathfrak{m})\}$, *as $\mathfrak{m}$ runs over the maximal ideals of R.*

The proof requires some technical results dealing with a commuting family of K-derivations on the local ring $S_n(K)=K[\![X_1 \dots X_n]\!]$ whose maximal ideal is denoted by $\mathfrak{m}$. Our aim is to establish the result below.

Proposition 5.1. *Let $\Delta = \{\delta_i\}_1^s$ be an s-tuple of commuting K-derivations on $S_n(K)$ and let $t = \operatorname{rank}_\Delta(\mathfrak{m})$. Then there exists $(k_{\nu i}) \in GL(n, K)$ such that if we put $\gamma_i = \sum k_{\nu i}\delta_\nu$, then $\mathfrak{m}$ has generators $\xi_1, \dots, \xi_n$ such that if $S_n(K)$ is identified with $K[\![\xi_1 \dots \xi_n]\!]$, then $\gamma_i = \partial/\partial\xi_i$ for $1 \leqq i \leqq t$ while $\gamma_j(\mathfrak{m}) \subset \mathfrak{m}$ for $t < j \leqq s$.*

The proof follows from a series of preliminary results.

Lemma 5.1. *Let δ be a K-derivation on $S_n(K)$ such that $\delta(\mathfrak{m})$ is not contained in $\mathfrak{m}$. Then $\mathfrak{m}$ has generators $\xi_1, \dots, \xi_n$, such that $\delta(\xi_1) = 1$ while $\delta(\xi_i) = 0$ for $2 \leqq i \leqq n$. So if $S_n(K)$ is identified with $K[\![\xi_1 \dots \xi_n]\!]$, then δ is the usual derivation $\partial/\partial\xi_1$.*

Proof. Since δ is K-linear we can choose $(k_{\nu i}) \in GL(n, K)$ such that if $y_i = \sum k_{\nu i} x_\nu$, then $\delta(y_1) - 1 \in m$ and $\delta(y_i) \in m$ for $2 \leqq i \leqq n$.

Then we can find $t_1 \dots t_n$ in $\mathfrak{m}^2$ such that $\delta(y_1 + t_1) - 1 \in \mathfrak{m}^2$ and $\delta(y_i + t_i) \in \mathfrak{m}^2$ for $2 \leqq i \leqq n$. For example $\delta(y_1) = 1 + k_1 y_1 + \dots + k_n y_n + u$, where $k_i \in K$ and $u \in \mathfrak{m}^2$. Then we can put $t_1 = -(k_1 y_1^2/2 + k_2 y_1 y_2 + \dots + k_n y_1 y_n)$.

In general we see that if $x \in \mathfrak{m}^\nu$ and $\delta(x) \in \mathfrak{m}^\nu$ for some $\nu \geqq 1$, then $\delta(x) = y_1^\nu f_0 + \dots + f_\nu + u$, where $u \in \mathfrak{m}^{\nu+1}$ and f_i is a homogenous polynomial of degree i in $K[y_2 \dots y_n]$.

If we put $x' = -(y_1^{\nu+1}/\nu + 1 + y_1^\nu f_1/\nu + \dots + y_1 f_\nu)$, then we see that $x' \in \mathfrak{m}^{\nu+1}$ and $\delta(x + x') \in \mathfrak{m}^{\nu+1}$.

In this way we can inductively find $t_\nu^{(i)} \in \mathfrak{m}^\nu$ for all $\nu > 1$, such that $\delta(y_i + t_2^{(i)} + \dots + t_\nu^{(i)}) \in \mathfrak{m}^\nu$ for all $\nu \geqq 2$ and $2 \leqq i \leqq n$, while $\delta(y_1 + t_2^{(1)} + \dots + t_\nu^{(1)}) - 1 \in \mathfrak{m}^\nu$.

Finally we put $\xi_i = y_i + \sum t_\nu^{(i)}$ for each i. Then Nakayama's Lemma shows that $\xi_1, \dots, \xi_n$ generate $\mathfrak{m}$ and Krull's Intersection Theorem shows that $\delta(\xi_1) = 1$ while $\delta(\xi_i) = 0$ for $2 \leqq i \leqq n$.

Lemma 5.2. *Let δ be a derivation on $S_n(K) = K[\![x_1 \dots x_n]\!]$ which commutes with $\partial/\partial x_1$. Then $\delta(x_i) \in K[\![x_2 \dots x_n]\!] = S_{n-1}(K)$ for $1 \leqq i \leqq n$.*

Proof. We can write $\delta(x_i) = s_0 + x_1 s_1 + x_1^2 s_2 + \cdots$, with $s_i \in S_{n-1}(K)$. Then $(\partial/\partial x_1)(\delta(x_i)) = \delta(\partial x_i/\partial x_1) = 0$ and hence $0 = s_1 + 2x_1 s_2 + 3x_1^2 s_3 + \cdots$, which gives $s_1 = s_2 = \cdots = 0$, as required.

Lemma 5.3. *Let δ_2 be a K-derivation on $S_n(K) = K[\![x_1 \dots x_n]\!]$ which commutes with $\delta_1 = \partial/\partial x_1$ and satisfies $\delta_2(x_2) - 1 \in m$ and $\delta_2(x_1) \in \mathfrak{m}$. Then $\mathfrak{m}$ has generators $\xi_1, \dots, \xi_n$ such that when $S_n(K)$ is identified with $K[\![\xi_1 \dots \xi_n]\!]$, then $\delta_1 = \partial/\partial\xi_1$ and $\delta_2 = \partial/\partial\xi_2$.*

Proof. Firstly Lemma 5.2 shows that δ_2 induces a derivation on the subring $S_{n-1}(K) = K[\![x_2 \dots x_n]\!]$. Using Lemma 5.1 we can find $y_2 \dots y_n$ which generate the maximal ideal $\mathfrak{m}_0$ of $S_{n-1}(K)$ while $\delta_2 | S_{n-1}(K) = \partial/\partial y_2$, as $S_{n-1}(K)$ is identified with $K[\![y_2 \dots y_n]\!]$.

Now $\delta_2(x_1)\in \mathfrak{m}$ and it follows from Lemma 5.2 that $\delta_2(x_1)\in \mathfrak{m}_0$. The explicit calculations in Lemma 5.1 show that there exists a sequence $\{t_v\}$, with $t_v \in \mathfrak{m}_0^v$ for $v \geqq 2$, such that $\delta_2(x_1+t_2+\cdots+t_v)\in \mathfrak{m}^v$ for all $v\geqq 2$.

At this stage the desired result follows if we put $\xi_1 = x_1+\sum t_v$ and $\xi_i = y_i$ for $2\leqq i\leqq n$.

The next result follows by iterating the previous computations.

Lemma 5.4. *Let $\Delta=\{\delta_i\}_1^t$ be a commuting family of K-derivations on $S_n(K)$ such that $\operatorname{rank}_\Delta(\mathfrak{m})=t$. Then $\mathfrak{m}$ has generators $\xi_1\ldots\xi_n$ such that if $S_n(K)$ is identified with $K[\![\xi_1\ldots\xi_n]\!]$, then $\delta_i=\partial/\partial\xi_i$ for $i=1\ldots t$.*

We finally remark that Proposition 5.1 is an easy consequence of Lemma 5.4. Next follows the first step in the proof of the Main Theorem.

Proposition 5.2. *Let $\Delta=\{\delta_i\}_1^s$ be an s-tuple of commuting K-derivations on $S_n(K)$ with $t=\operatorname{rank}_\Delta(\mathfrak{m})$. If $B=S_n(K)_\Delta[X_1\ldots X_s]$, then* $\operatorname{gl.dim}(B)\leqq n+s-t$.

Proof. We may assume that the conclusion of Proposition 5.1 is already satisfied with $\gamma_i=\delta_i$ and $\xi_i=x_i$.

An application of Theorem 4.2 to the ring $R=K[\![x_{t+1}\ldots x_n]\!]$ proves that if

$$B_0=S_n(K)_{\delta_1\ldots\delta_t}[X_1\ldots X_t], \quad \text{then } \operatorname{gl.dim}(B_0)=n.$$

Since $B=B_{0,\delta_{t+1}\ldots\delta_s}[X_{t+1}\ldots X_s]$, we conclude that $\operatorname{gl.dim}(B)\leqq n+s-t$, as required.

Suppose next that R is a regular local noetherian ring satisfying condition (A) and let m be its maximal ideal. Its $\mathfrak{m}$-adic completition $\hat{R}$ is faithfully flat over R and it follows from Lemma 1.1 that $\hat{B}=\hat{R}_{\Delta_e}[X]$ is so over $B=R_\Delta[X]$, when $\Delta=\{\delta_i\}_1^s$ is an s-tuple of commuting K-derivations on R.

Using Lemma 2.b.4 and Proposition 5.2 it follows that $\operatorname{gl.dim}(B)\leqq \dim(R)+s-t$, where $t=\operatorname{rank}_\Delta(\mathfrak{m})$.

At this stage it follows from Lemma 3.1 that the estimate $\operatorname{gl.dim}(B)\leqq \sup\{\dim(R_{\mathfrak{m}})+s-\operatorname{rank}_\Delta(\mathfrak{m})\}$ holds in the Main Theorem.

It remains only to prove that equality holds. Using Lemma 2.b.2 it is sufficient to prove equality when R is local.

So let R be local with its maximal ideal $\mathfrak{m}$ and we let $\Delta=\{\delta_i\}_1^s$ be an s-tuple of commuting K-derivations on R with $t=\operatorname{rank}_\Delta(\mathfrak{m})$. Then we can choose $(k_{vi})\in GL(n,K)$ such that if $\gamma_i=\sum k_{vi}\delta_v$, then $\gamma_j(\mathfrak{m})\subset\mathfrak{m}$ for $t<j\leqq s$.

Let us put $B_1=R_{\gamma_{t+1}\ldots\gamma_s}[X_{t+1}\ldots X_s]$. Then Lemma 2.c.1 shows that the left Krull dimension of B_1 is at least $n+s-t$, where $n=\dim(R)$. Since B is Ore's Polynomial Ring $B_{1,\gamma_1\ldots\gamma_t}[X_1\ldots X_t]$ it follows that the left Krull dimension of B is at least $n+s-t$.

The proof is now finished by another recent result due to Roos. The result below is a special case of more general results dealing with Krull dimensions and global dimensions of filtered (non-commutative) Gorenstein rings (see [11]).

Theorem of Roos. *Let R be a filtered noetherian ring whose associated graded ring is a commutative regular noetherian ring. Then the left (and the right) Krull dimension of R is at most* gl. dim(R).

We finish by stating two consequences of the preceeding material.

Corollary 5.1. *For each ring arising in the Main Theorem the left and the right Krull dimension equals the global dimension.*

Corollary 5.2. *Let R and B be as in the Main Theorem. If B is a simple ring, i.e. if B has no proper two-sided ideals, then* gl. dim$(B) = \sup(\dim(R), s)$.

Acknowledgement. The ingenious proofs by J.-E. Roos in [10] have been the starting point for the present paper. His methods and some of his recent results will be used at several places.

References

1. Bourbaki, N.: Algebre commutatives. Ch. 1-3. Paris: Hermann 1964.
2. Cartan, H., Eilenberg, S.: Homological algebra. Princeton: University Press 1956.
3. Gopalakrishnan, N.S., Sridharan, R.: Homological dimension of Ore-extensions. Pac. J. of Math. **19**, 67–75 (1966).
4. Hochschild, G., Kostant, B., Rosenberg, A.: Differential forms on regular affine algebras. Trans. Amer. Math. Soc. **102**, 383–408 (1962).
5. Hart, R.: Krull dimension and global dimension of simple Ore-extensions. Math. Zeitschrift **121**, 341–346 (1971).
6. Nouaze, Y., Gabriel, P.: Journal of Algebra **6**, 77–99 (1967).
7. Rentschler, R., Gabriel, P.: Sur la dimension des anneaux et ensembles ordonnes. Comptes Rendus **265**, No. 2 (Ser. A), 712–715 (1967).
8. Rinehart, G.S.: Note on the global dimension of a certain ring. Proc. Amer. Math. Soc. **13**, 341–346 (1962).
9. Rinehart, G.S.: Differential forms on general commutative algebras. Trans. Amer. Math. Soc. **103**, 195–222 (1963).
10. Roos, J.-E.: Algebre homologique. Determination de la dimension homologique globale des algebres de Weyl. C. R. Acad. Sci. Paris **274**, Ser. A, 23–26 (1972).
11. Roos, J.-E.: The Weyl algebras are Gorenstein rings. Generalisations and applications. (To appear.)
12. Roy, A.: A note on filtered rings. Arc. der Math. **16**, 21–27 (1965).
13. McConnell, J.C., Robson, J.C.: Homomorphisms and extensions of modules over A_1 and related rings (Preprint), Leeds University (1971). (To appear in Journal of Algebra.)

Jan-Erik Björk
University of Stockholm
Department of Mathematics
Box 6701
S-113 85 Stockholm
Sweden

(Received March 4, 1972/April 4, 1972)