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Semi-Regularity and de Rham Cohomology

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§0. Introduction

Let X be a smooth, projective variety over $\mathbf{C}$. (We will work over $\mathbf{C}$ to fix ideas. By the Lefschetz principle, all our results are valid over any ground field of characteristic 0.) Let $Z \subset X$ be a subscheme of codimension p which is a *local complete intersection*. After some preliminaries on de Rham cohomology and deformation theory, we define the *semi-regularity* map

$$\pi\colon H^1(Z, N_{Z/X}) \to H^{p+1}(X, \Omega_X^{p-1}),$$

where $N_{Z/X}$ is the normal bundle of Z in X. Z is said to be semi-regular if π is injective.

Our principal results are:

Theorem. *Suppose Z is semi-regular in X. Then the Hilbert scheme* $\mathrm{Hilb}(X/\mathbf{C})$ *is smooth at the point corresponding to Z.*

Theorem. *Let $f\colon X \to S$ be a smooth, projective morphism, with S smooth, connected, and of finite type over $\mathbf{C}$. Let $o \in S$ and let $z \in \Gamma(S, \mathbf{R}^{2p} f_*(\Omega_{X/S}^{\bullet}))$ be a horizontal section of the de Rham cohomology. Suppose that $z_0 = z \mid X_0 \in H_{DR}^{2p}(X_0 \mid \mathbf{C})$ is algebraic, representing a local complete intersection $Z_0 \subset X_0$ which is semi-regular in X_0. Then $z_s = z \mid X_s$ is algebraic for all $s \in S$.*

This theorem is related to a conjecture of Grothendieck ([8], footnote 13). More precisely, it reduces Grothendieck's conjecture to the problem of finding semi-regular representatives for a given algebraic cycle class.

The central point in the argument is the following compatibility. Consider a diagram with cartesian square

$$\begin{array}{ccccc} Z_0 & \longrightarrow & X_0 & \hookrightarrow & X \\ & \overset{\text{flat}}{\searrow} & \downarrow & & \downarrow \text{smooth} \\ & & S_0 & \hookrightarrow & S, \end{array}$$

where S is affine, S_0 is defined by a square-zero ideal I, and Z_0 is a local complete intersection in X_0. The obstruction to lifting Z_0 to $Z \hookrightarrow X$ is given by $\alpha \in H^1(Z_0, N_{Z_0/X_0}) \otimes I$; we can view $\pi(\alpha)$ as a class in $H^{p+1}(X, I\Omega^{p-1}_{X/S})$.

On the other hand, the cohomology class $[Z_0] \in F^p H^{2p}_{DR}(X_0/S_0)$ lifts, at least under suitable hypotheses, to a horizontal class $V \in F^{p-1} H^{2p}_{DR}(X/S)$ (here $F^\bullet$ is the Hodge filtration). Then (again under suitable hypotheses), the class $\bar{V} \in H^{p+1}(X, \Omega^{p-1}_{X/S})$ induced by V is given by $\pi(\alpha)$. In particular, $V \in F^p H^{2p}_{DR}(X/S)$ if and only if $\pi(\alpha) = 0$.

§ 1. Definition of Semi-Regularity Map: Examples

Let X be smooth and projective of dimension n over $\mathbf{C}$, and let $Z \subset X$ be a local complete intersection of codimension p. If $I \subset \mathcal{O}_X$ is the defining ideal, the normal bundle is given by

$$N_{Z/X} = \mathcal{H}om_{\mathcal{O}_Z}(I/I^2, \mathcal{O}_Z)$$

and is locally free. Write $\omega_X = \Omega^n_{X/\mathbf{C}}$, $\omega_{Z/X} = \overset{p}{\wedge} N_{Z/X}$, $\omega_Z = \omega_{Z/X} \otimes \omega_X$. We remark that

$$\Omega^{p-1}_X \cong (\Omega^{n-p+1}_X)^* \otimes \omega_X, \qquad \overset{p-1}{\wedge} N_{Z/X} \cong N^*_{Z/X} \otimes \omega_{Z/X}.$$

(Let me ignore the problem of signs in defining these isomorphisms. For our purposes, it is sufficient that maps be defined up to sign.)

The natural map

$$\varepsilon\colon\ N^*_{Z/X} \to \Omega^1_X \otimes \mathcal{O}_Z$$

gives rise to an element

$$\begin{aligned}\overset{p-1}{\wedge} \varepsilon \in \mathcal{H}om_{\mathcal{O}_Z}\big(\overset{p-1}{\wedge} N^*, \Omega^{p-1}_X \otimes \mathcal{O}_Z\big) &= \Gamma\big((\Omega^{n-p+1}_X)^* \otimes \omega_X \otimes \omega_{Z/X} \otimes N^*\big)\\ &= \mathcal{H}om_{\mathcal{O}_X}(\Omega^{n-p+1}_X, \omega_Z \otimes N^*).\end{aligned}$$

The induced map on cohomology

$$\overset{p-1}{\wedge} \varepsilon\colon\ H^{n-p-1}(X, \Omega^{n-p+1}_X) \to H^{n-p-1}(Z, \omega_Z \otimes N^*)$$

dualizes (Grothendieck duality, cf. Kleiman [14] or Hartshorne [13]) to give the *semi-regularity* map

$$\pi\colon\ H^1(Z, N_{Z/X}) \to H^{p+1}(X, \Omega^{p-1}_X).$$

(1.1) **Proposition.** *Suppose Z is a divisor on X. Then the semi-regularity map $\pi\colon H^1(Z, N) \to H^2(X, \mathcal{O}_X)$ arises as the boundary map in the cohomology sequence associated to*

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(Z) \to N \to 0.$$

The notion of semi-regularity for divisors (with the definition of π given in (1.1)) is due to Kodaira-Spencer [17]. They prove:

(1.2) **Theorem.** (i) *Let $Z \subset X$ be a divisor which is semi-regular in X. Then the corresponding point $Z \in \mathrm{Hilb}(X/\mathbf{C})$ is smooth.*

(ii) *Suppose Z is smooth of codimension $\geqq 1$ and $H^1(Z, N_{Z/X}) = (0)$. Then $Z \in \mathrm{Hilb}(X/\mathbf{C})$ is smooth.*

I have two examples of semi-regularity in codimension >1 with $H^1(Z, N_{Z/X}) \neq (0)$.

(1.3) *Example.* Let Z be a non-singular non-hyperelliptic curve of genus 3, and let X be the Jacobian. Then Z is semi-regular in X.

Proof. It suffices to show the map

$$\varphi\colon \Gamma(Z, \Omega_X^2 \otimes \mathcal{O}_Z) \to \Gamma(\omega_Z \otimes N^*)$$

dual to π is surjective. The exact sequence

$$0 \to \wedge^2 N^* \to \Omega_X^2 \otimes \mathcal{O}_Z \to N^* \otimes \omega_Z \to 0$$

gives rise to a sequence of cohomology

$$\begin{aligned} 0 \to \Gamma(\wedge^2 N^*) \to \Gamma(\Omega_X^2 \otimes \mathcal{O}_Z) &\xrightarrow{\varphi} \Gamma(\omega_Z \otimes N^*) \\ \to H^1(\wedge^2 N^*) \to H^1(\Omega_X^2 \otimes \mathcal{O}_Z) &\to H^1(\omega_Z \otimes N^*) \to 0, \end{aligned}$$

and we must show

(1.3.1) $$h^1(\Omega_X^2 \otimes \mathcal{O}_Z) = h^1(\wedge^2 N^*) + h^1(\omega_Z \otimes N^*).$$

Note

(1.3.2) $$h^1(\Omega_X^2 \otimes \mathcal{O}_Z) = \frac{3(3-1)}{2} \cdot 3 = 9$$

and by Riemann Roch

(1.3.3) $$h^1(\wedge^2 N) = h^1(\omega_Z^{-1}) = h^0(\omega_Z^{\otimes 2}) = 3 \cdot 3 - 3 = 6.$$

I claim the map

$$r\colon \Gamma(\Omega_X^1 \otimes \omega_Z) \to \Gamma(\omega_Z^{\otimes 2})$$

induced from the exact sequence

(1.3.4) $$0 \to N^* \otimes \omega_Z \to \Omega_X^1 \otimes \omega_Z \to \omega_Z^{\otimes 2} \to 0$$

is surjective. To see this, factor r

$$\begin{array}{ccc} \Gamma(\Omega_X^1 \otimes \omega_Z) & \xrightarrow{r} & \Gamma(\omega_Z^{\otimes 2}) \\ \wr\| & & \uparrow \mu \\ \Gamma(\Omega_X^1) \otimes \Gamma(\omega_Z) & \cong & \Gamma(\omega_Z) \otimes \Gamma(\omega_Z). \end{array}$$

By a classical theorem of Noether, μ is surjective in the non-hyperelliptic case, proving the claim

Now the cohomology sequence of (1.3.4) gives

$$(1.3.5)\qquad \begin{aligned} h^1(N^*\otimes\omega_Z) &= h^1(\Omega^1_X\otimes\omega_Z) - h^1(\omega_Z^{\otimes 2}) \\ &= h^1(\Omega^1_X\otimes\omega_Z) = 3. \end{aligned}$$

Combining (1.3.2), (1.3.3), and (1.3.5) proves (1.3.1).

(1.4) *Example.* Let W be smooth and projective of dimension $2m+1$ over $\mathbf{C}$. Let $Z\subset W$ be a smooth subvariety of dimension m. One can show that there exist smooth hypersurface sections $X\subset W$ of arbitrarily large degree with $Z\subset X$. Moreover, for X of sufficiently large degree, Z is semi-regular in X; although it may well happen that $H^1(Z, N_{Z/X})\neq(0)$ for any such X.

§ 2. Deformation Theory

A standard reference for this section is [10], 221.

Let A be an artinian $\mathbf{C}$-algebra, $I\subset A$ a square-zero ideal, $A_0=A/I$. Write $S=\operatorname{Spec}(A)$, $S_0=\operatorname{Spec}(A_0)$, and consider a diagram

$$\begin{array}{ccc} Z_0 & & \\ \downarrow & & \\ X_0 & \hookrightarrow & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ S_0 & \hookrightarrow & S \end{array}$$

with cartesian square. Assume f is smooth and of finite type, and Z_0 is a local complete intersection of codimension p in X_0.

Let $J_0\subset\mathcal{O}_{X_0}$ (resp. $J_0'\subset\mathcal{O}_X$) be the ideal of Z_0 in X_0 (resp. Z_0 in X). We have exact (locally split) sequences

$$\begin{array}{lccccccccc} (2.1) & 0 \longrightarrow & I\otimes\mathcal{O}_{Z_0} & \longrightarrow & J_0'/J_0'^2 & \longrightarrow & J_0/J_0^2 & \longrightarrow 0 \\ & & \downarrow{\scriptstyle d\otimes 1} & & \downarrow & & \downarrow{\scriptstyle u} & \\ (2.2) & 0 \longrightarrow & \Omega^1_{S/\mathbf{C}}\otimes\mathcal{O}_{Z_0} & \longrightarrow & \Omega^1_{X/\mathbf{C}}\otimes\mathcal{O}_{Z_0} & \longrightarrow & \Omega^1_{X_0/S_0}\otimes\mathcal{O}_{Z_0} & \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow & \\ (2.3) & 0 \longrightarrow & \Omega^1_{S/\mathbf{C}}\otimes\mathcal{O}_{X_0} & \longrightarrow & \Omega^1_{X/\mathbf{C}}\otimes\mathcal{O}_{X_0} & \longrightarrow & \Omega^1_{X_0/S_0} & \longrightarrow 0. \end{array}$$

Corresponding to these sequences, there are elements

(2.4)
$$\alpha \in \operatorname{Ext}^1_{\mathcal{O}_{Z_0}}(J_0/J_0^2, I \otimes \mathcal{O}_{Z_0})$$
$$\beta \in \operatorname{Ext}^1_{\mathcal{O}_{X_0}}(\Omega^1_{X_0/S_0}, \Omega^1_{S/\mathbf{C}} \otimes \mathcal{O}_{X_0})$$

and we have

(2.5)
$$(1 \otimes d)_* \alpha = u^*(\beta \otimes \mathcal{O}_{Z_0}).$$

(2.6) **Proposition.** *The obstruction to lifting Z_0 to a local complete intersection $Z \subset X$ is given by α.*

Proof. We must show that Z_0 lifts if and only if (2.1) splits. Suppose first that $Z \subset X$ lifts Z_0, and let $J \subset \mathcal{O}_X$ be the ideal of Z. Then $J \subset J_0'$ and the induced map

$$J/J \cap J_0'^2 \to J_0/J_0^2$$

is an isomorphism. Indeed, this is straightforward because Z is a local complete intersection in X.

Conversely, let K: $J_0/J_0^2 \to J_0'/J_0'^2$ be a splitting of (2.1) and let $J = K(J_0/J_0^2) + J_0'^2 \subset J_0'$. I claim $\mathcal{O}_{X/J}$ is flat over $\mathcal{O}_S$. Indeed, since Z_0 is flat over S_0, it suffices (E.G.A.0.6.6.9.1) to show

$$\mathcal{O}_{X/J} \otimes I \xrightarrow{\sim} I \cdot (\mathcal{O}_{X/J})$$

or in other words

$$I\mathcal{O}_X \cap J = IJ.$$

But a section v of $I\mathcal{O}_X \cap J$ vanishes in J_0/J_0^2 and so necessarily lies in $J_0'^2 \cap I\mathcal{O}_X$. Since $I \otimes \mathcal{O}_{Z_0} \hookrightarrow J_0'/J_0'^2$ we get

$$J_0'^2 \cap I\mathcal{O}_X = IJ_0' = IJ.$$

Finally, $\mathcal{O}_Z$ flat over $\mathcal{O}_S$ implies that $Z \subset X$ is a local complete intersection [20].

(2.7) *Remark.* The Kodaira-Spencer class

$$K_{X/S/\mathbf{C}} \in \operatorname{Ext}^1(\Omega^1_{X/S}, \Omega^1_{S/\mathbf{C}} \otimes \mathcal{O}_X)$$

is the class of the extension

$$0 \to \Omega^1_{S/\mathbf{C}} \otimes \mathcal{O}_X \to \Omega^1_{X/\mathbf{C}} \to \Omega^1_{X/S} \to 0.$$

With reference to (2.4), we have $\beta = K_{X/S/\mathbf{C}} \otimes \mathcal{O}_{S_0}$.

§3. de Rham Cohomology

In this section are listed some properties of de Rham cohomology to be used in the sequel.

(3.1) *Notation.* Given a morphism of schemes f: $X \to S$, we write $\mathcal{H}^q_{DR}(X/S) = \mathbf{R}^q f_*(\Omega^\bullet_{X/S})$, where $\Omega^\bullet_{X/S}$ is the de Rham complex and $\mathbf{R}^q$

denotes the q-th hyperderived functor. When S is affine, we set $H^q_{DR} = \Gamma \mathscr{H}^q_{DR}$.

(3.2) **Theorem** (Deligne, [2]). *Let S be a scheme over* Spec(**Q**) *and let $f: X \to S$ be a proper, smooth morphism. Then:*

(i) *The sheaves $R^q f_*(\Omega^p_{X/S})$ are locally free of finite type and commute with base change.*

(ii) *The spectral sequence*

$$E_1^{p,q} = R^q f_*(\Omega^p_{X/S}) \Rightarrow \mathscr{H}^{p+q}_{DR}(X/S)$$

degenerates at E_1.

(iii) *The sheaves $\mathscr{H}^*_{DR}$ are locally free of finite type and commute with base change.*

Let S be a T-scheme, $M = \mathscr{H}^q_{DR}(X/S)$. There is a canonical integrable connection, the *Gauss-Manin connection,*

(3.3) $$\tilde{\nabla}: M \to M \underset{\mathcal{O}_S}{\otimes} \Omega^1_{S/T}.$$

The spectral sequence (3.2)(ii) induces a filtration

$$M = M^{(0)} \supset M^{(1)} \supset \cdots \supset M^{(q)} \supset (0)$$

by locally free, locally direct summands on which the connection acts by

(3.4) $$\tilde{\nabla}(M^{(p)}) \subset M^{(p-1)} \otimes \Omega^1_{S/T} \qquad \text{(Griffith's Transversality)}.$$

Recall (2.7) there is a canonical Kodaira-Spencer class

(3.5) $$K_{X/S/T} \in \operatorname{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \Omega^1_{S/T} \otimes \mathcal{O}_X).$$

(3.6) **Proposition.** *The Gauss-Manin connection is related to the Kodaira-Spencer class* (3.5) *by the commutative diagram*

$$\begin{array}{ccc} M^{(p)}/M^{(p+1)} & \xrightarrow{\nabla} & M^{(p-1)}/M^{(p)} \otimes \Omega^1_{S/T} \\ \wr\| & & \wr\| \\ R^{q-p} f_*(\Omega^p_{X/S}) & \xrightarrow[(3.5)]{\cup K_{X/S/T}} & R^{q-p+1} f_*(\Omega^{p-1}_{X/S}) \otimes \Omega^1_{S/T}. \end{array}$$

Proof. See Griffiths [5].

We note a general fact about modules with an integrable connection:

(3.7) **Proposition.** *Let k be a field of characteristic* 0, *M a finite $k[[t_1, \ldots, t_r]]$-module with integrable connection ∇. Let $M^\nabla = \operatorname{Ker}(\nabla)$. Then $M \cong M^\nabla \underset{k}{\otimes} k[[t_1, \ldots, t_r]]$.*

Proof. See Katz [15].

There is the following analogous result for the de Rham cohomology.

(3.8) **Proposition.** *Let A be a complete, local, augmented* **C**-*algebra* (e.g. A artinian), $S = \mathrm{Spec}(A)$, $f\colon X \to S$ *a smooth, proper morphism. Let $X_0 \subset X$ be the closed fiber. Then $H^*_{DR}(X/S) \cong H^*(X_0, \mathbf{C}) \underset{\mathbf{C}}{\otimes} A$.*

Proof. Using [12], (III.4.1.7) (the fundamental theorem on proper morphisms), one easily reduces to the case A artinian (this case will be the only one used in the sequel).

Let X^{an} be the associated analytic space, $\Omega^\bullet_{X^{an}/S}$ the complex of analytic differentials. The map

$$A_{X^{an}} \to \Omega^\bullet_{X^{an}/S}$$

is a quasi-isomorphism (Deligne [2]), so

$$H^*(X_0, \mathbf{C}) \otimes A \cong H^*(X^{an}, A_{X^{an}}) = H^*(X^{an}, \Omega^\bullet_{X^{an}/S}) \underset{(GAGA)}{\cong} H^*_{DR}(X/S)$$

proving (3.8).

(3.9) *Remark.* In technical terms, (3.8) gives a *stratification* on $H^*_{DR}(X/S)$. Cohomology classes of the form $c \otimes 1$, $c \in H^*(X_0, \mathbf{C})$ are said to be *horizontal.*

§4. Deformation of de Rham Cohomology

Consider a Cartesian diagram

$$\begin{array}{ccc} X_0 & \longrightarrow & X \\ \downarrow & \square & \downarrow f \\ S_0 & \longrightarrow & S \end{array}$$

with f smooth and projective, $S = \mathrm{Spec}(A)$ for an artinian C-algebra A, and S_0 defined by an ideal I of square zero. Let $F^p H^q_{DR}(X/S)$ denote the p-th level of the Hodge filtration on H^q_{DR}, and write

(4.1) $$\nabla : H^q(X_0, \Omega^p_{X_0/S_0}) \to H^{q+1}(X_0, \Omega^{p-1}_{X_0/S_0}) \underset{\mathcal{O}_S}{\otimes} \Omega^1_{S/\mathbf{C}}$$

for the map induced by $\beta = K_{X/S/\mathbf{C}} \otimes \mathcal{O}_{S_0}$ (2.7).

(4.2) **Proposition.** *Let $v_0 \in F^p H^q_{DR}(X_0/S_0)$ be a horizontal section, and let $\bar{v}_0 \in H^{q-p}(X_0, \Omega^p_{X_0/S_0})$ denote the induced class. Let $v \in H^q_{DR}(X/S)$ be the unique horizontal class lifting v_0, and assume the natural map*

$$d\colon I \to \Omega^1_{S/\mathbf{C}} \otimes \mathcal{O}_{S_0}$$

is injective. Then $v \in F^p H^q(X/S)$ *if and only if*

$$\nabla(\bar{v}) = 0 \quad in \quad H^{q-p+1}(X_0, \Omega^{p-1}_{X_0/S_0}) \otimes \Omega^1_{S/\mathbf{C}}.$$

Proof. Recall (3.2) that $H^q_{DR}(X/S)$ is a free A-module, and the $F^p H^q_{DR}(X/S)$ are direct summands. Let $\omega_1, \ldots, \omega_M, \ldots, \omega_N, \ldots, \omega_P$ be a basis for $H^q_{DR}(X/S)$ such that $\omega_1, \ldots, \omega_M$ is a basis for F^p and $\omega_1, \ldots, \omega_N$ is a basis for F^{p-1}. Write $v = \sum a_i \omega_i$ with $a_i \in I$, $i > M$. Let $\tilde{\nabla}$ denote the Gauss-Manin connection on $H^q_{DR}(X/S)$ and note (3.4)

$$\tilde{\nabla}\left(\bigoplus_1^M A\,\omega_i\right) \subset \bigoplus_1^N \omega_i.$$

By assumption, v is horizontal so

$$0 = \tilde{\nabla}(v) = \sum \omega_i \otimes d a_i + \sum a_i \tilde{\nabla}(\omega_i).$$

It follows that

$$\sum_{i>N} \omega_i \otimes d a_i \in F^{p-1} \otimes \Omega^1_A + I(H^q_{DR} \otimes \Omega^1_A).$$

Hence, $d a_i \in I\,\Omega^1_A$, $i > N$. Because $d\colon I \to \Omega^1_A \otimes A/I$ is injective, $a_i = 0$, $i > N$. Moreover, $v \in F^p$ if and only if $a_i = 0$, $i > M$, and this is equivalent to having

$$\sum_{i>M} \omega_i \otimes d a_i \equiv 0 \ (\mathrm{mod}\ F^p \otimes \Omega^1_A + I H^q_{DR} \otimes \Omega^1_A).$$

Since v is horizontal,

$$\tilde{\nabla}\left(\sum_{i \leq M} a_i \omega_i\right) \equiv -\sum_{i>M} \omega_i \otimes d a_i \quad (\mathrm{mod}\ I).$$

By (3.6), the left-hand side taken $\mathrm{mod}\, F^p \otimes \Omega^1_A + I$ is equal to $\nabla(\bar{v}_0)$. This proves (4.2).

§5. Local Cohomology. The Cycle Class

Let X be a topological space, $Z \subset X$ a subspace, and F an abelian sheaf on X. Then there are defined [11] *local cohomology groups* $H^i_Z(X, F)$ which fit into an exact sequence

(5.1) $$\to H^i_Z(X, F) \to H^i(X, F) \to H^i(X - Z, F) \to H^{i+1}_Z(X, F) \to.$$

When X is a scheme, $Z \subset X$ a closed subscheme, and F a coherent sheaf of $\mathcal{O}_X$-modules, there are canonical isomorphisms

(5.2) $$H^i_Z(X, F) \cong \varinjlim_k \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X/I^k, F)$$

where $I \subset \mathcal{O}_X$ is the ideal of Z.

The above definitions can be localized to give cohomology sheaves $\mathcal{H}_Z^i(X, F)$. In particular, when X and Z are as in (5.2) there are isomorphisms

(5.3) $$\mathcal{H}_Z^i(X, F) \cong \varinjlim_k \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_X/I^k, F),$$

(5.4) $$\mathcal{H}_Z^i(X, F) \cong R^{i-1}_{j_*}(F|X-Z), \qquad i>1 \text{ and } j\colon X-Z \hookrightarrow X.$$

The functors $\mathcal{H}_Z$ and H_Z are related by a spectral sequence

(5.5) $$E_2^{p,q} = H^p(X, \mathcal{H}_Z^q(X, F)) \Rightarrow H_Z^{p+q}(X, F).$$

Finally, these notions can be extended to define hypercohomology functors $\mathbf{H}_Z^*(X, F^\bullet)$ and $\mathbf{H}_Z^*(X, F^\bullet)$ when $F^\bullet$ is a complex of sheaves.

Let $f\colon X \to S$ be a smooth, projective morphism of schemes, where $S = \operatorname{Spec}(A)$ is the spectrum of an artinian, local $\mathbf{C}$-algebra. Let $Z \subset X$ be a local complete intersection of codimension p. We write $F^i\,\Omega^\bullet_{X/S}$ for the complex

$$(F^i\,\Omega_{X/S})^j = \begin{cases} 0 & j<i \\ \Omega^j_{X/S} & j \geqq i. \end{cases}$$

(5.6) **Proposition.** *There exists a canonical cycle class*

$$\{Z\} \in \mathbf{H}_Z^{2p}(X, F^p\,\Omega^\bullet_{X/S})$$

mapping to the de Rham class under the composition

$$\mathbf{H}_Z^{2p}(X, F^p\,\Omega^\bullet) \to \mathbf{H}^{2p}(X, F^p\,\Omega^\bullet) \to H^{2p}_{DR}(X/S).$$

Proof. We first describe a class $\{Z\}' \in H_Z^p(X, \Omega^p_{X/S})$. Since Z is a local complete intersection and Ω^p is locally free, $\mathcal{H}_Z^q(X, \Omega^p) = (0)$, $q \neq p$, so the spectral sequence (5.5) degenerates and

$$H_Z^p(X, \Omega^p_{X/S}) \cong \Gamma(X, \mathcal{H}_Z^p(X, \Omega^p)).$$

It therefore suffices to describe $\{Z\}'$ locally.

Let $\{U_i\}$ be an open affine cover of X such that $Z \cap U_i\colon f_i^{(1)} = \dots = f_i^{(p)} = 0$ for $f_i^{(j)} \in \Gamma(U_i, \mathcal{O}_X)$. The differential form

(5.7) $$\frac{df_i^{(1)} \wedge \dots \wedge df_i^{(p)}}{f_i^{(1)} \dots f_i^{(p)}}$$

gives rise to a cocycle in $\check{C}^{p-1}(V_i, \Omega^p)$, where V_i is the open affine cover of $U_i - Z$ by sets $V_i^{(j)} = U_i - \{f_i^{(j)} = 0\}$. Hence we get a class

$$\{Z\}'\,|\,U_i \in \Gamma(U_i, R^{p-1}_{j_*}(\Omega^p\,|\,U_i - Z)) \underset{(5.4)}{\cong} \Gamma(U_i, \mathcal{H}_Z^p(X, \Omega^p)).$$

These sections can be shown to patch to give $\{Z'\}\in\Gamma(X,\mathcal{H}_Z^p)\cong H_Z^p(X,\Omega^p)$ (cf. [10] Exposé 149 and [13], p. 176). We note also that the image of $\{Z\}'$ in $H^p(X,\Omega^p)$ is the Hodge class [10].

The exact sequence of complexes

$$0\to F^{p+1}\Omega^\bullet_{X/S}\to F^p\Omega^\bullet_{X/S}\to\Omega^p_{X/S}[-p]\to 0$$

gives

$$\begin{array}{ccccccc} \mathbf{H}_Z^{2p}(X,F^{p+1}\Omega) & \to & \mathbf{H}_Z^{2p}(X,F^p\Omega) & \to & H_Z^p(X,\Omega^p) & \xrightarrow{\partial} & \mathbf{H}_Z^{2p+1}(X,F^{p+1}\Omega). \\ \| & & & & \cup\!\!\!\shortmid & & \\ (0) & & & & \{Z\}' & & \end{array} \tag{5.8}$$

One sees that the left hand term of (5.8) vanishes by the degeneracy of (5.5). For the same reason

$$\mathbf{H}_Z^{2p+1}(X,F^{p+1}\Omega)\cong\Gamma(X,\mathcal{H}_Z^p(X,\Omega^{p+1}))$$

and $\partial\{Z\}'$ is represented by the cocycles

$$d\left(\frac{df_i^{(1)}}{f_i^{(1)}}\wedge\cdots\wedge\frac{df_i^{(p)}}{f_i^{(p)}}\right)\in\Gamma\big(U_i,R_{j_*}^{p-1}(\Omega^{p+1}\,|\,U_i-Z)\big). \tag{5.9}$$

The left hand side is clearly 0, so $\partial\{Z\}'=0$ and $\{Z\}'$ lifts to a unique $\{Z\}\in\mathbf{H}_Z^{2p}(X,F^p\Omega)$.

Let $[Z]$ denote the image of $\{Z\}$ in $H_{DR}^{2p}(X/S)$. With notation as in (3.8), we have

$$\begin{array}{ccccc} [Z] & \longrightarrow & & & 0 \\ \Big\downarrow & & H_{DR}^{2p}(X/S) & \xrightarrow{r} & H_{DR}^{2p}(X-Z/S) \\ & & \downarrow & & \downarrow \\ [Z_0]_{d.R.}\otimes\mu\in & & H^{2p}(X_0,\mathbf{C})\underset{\mathbf{C}}{\otimes}A & \xrightarrow{r_{top}} & H^{2p}(X_0-Z_0,\mathbf{C})\otimes A. \end{array} \tag{5.10}$$

The kernel of r_{top} is isomorphic to A with generator $[Z_0]_{d.R.}\otimes 1$, so we are reduced to showing $\mu=1$ in (5.10).

Since X is projective over S, we can intersect Z with a linear space section of complementary dimension and reduce to the case $\dim Z=0$. By a trace argument we can reduce to the case Z is an S-point of X, then that Z is the intersection of $n=\dim X$ hyperplanes in general position, and then that Z is itself a hyperplane. Finally, this case can be checked by the exponential sequence

$$0\to\mathbf{Z}_X\to\mathcal{O}_X\to\mathcal{O}_X^*\to 0.$$

Details are omitted.

§6. Semi-Regularity

In this paragraph, the semi-regularity map π is reinterpreted via local cohomology, and the relation between π and deformation of de Rham cohomology is established.

Let notation be as in §5. Recall we defined a section of the local cohomology sheaf

$$\frac{df_i^{(1)}}{f_i^{(1)}}\wedge\cdots\wedge\frac{df_i^{(p)}}{f_i^{(p)}}\in\Gamma\big(U_i,\mathscr{H}_Z^p(X,\Omega^p)\big).$$

Note that this section is killed by multiplication by any function g vanishing on Z. In fact, $g=\sum_j a_j f_i^{(j)}$ so it suffices to check for $g=f_i^{(j)}$. But

$$\frac{df_i^{(1)}}{f_i^{(1)}}\wedge\cdots\wedge df_i^{(j)}\wedge\cdots\wedge\frac{df_i^{(p)}}{f_i^{(p)}}\in\check{C}^{p-1}(V_i,\Omega^p)$$

is a Čech coboundary, so the assertion follows.

Let $J\subset\mathcal{O}_X$ be the ideal of Z, and define $N_{Z/X}=\mathscr{H}om_{\mathcal{O}_Z}(J/J^2,\mathcal{O}_Z)$. It follows from the above that there is a morphism of sheaves

$$(6.1)\qquad \lrcorner\{Z\}'\colon\ N_{Z/X}\to\mathscr{H}_Z^p(X,\Omega^{p-1}).$$

Indeed, $df_i^{(1)}\wedge\cdots\wedge df_i^{(p)}$ defines a section $\sigma_i\in\Gamma(U_i,\overset{p}{\bigwedge}N^*_{Z/X})$, and interior multiplication $\alpha_i\lrcorner\sigma_i$ makes sense for $\alpha_i\in\Gamma(U_i,N)$. We can lift $\alpha_i\lrcorner\sigma_i$ to a section $\beta_i\in\Gamma(U_i,\Omega^{p-1}_{X/S})$ which is determined upto adding a section $g\cdot\tau$ with g vanishing along Z. Thus

$$\frac{\beta_i}{f_i^{(1)}\cdots f_i^{(p)}}=\alpha_i\lrcorner\{Z\}'$$

gives a well-determined class in $\mathscr{H}_Z^p(X,\Omega^{p-1})$.

(6.2) **Proposition.** *The diagram*

$$\begin{array}{ccc} H^1(Z,N_{Z/X}) & \xrightarrow{\lrcorner\{Z\}'} & H_Z^{p+1}(X,\Omega^{p-1}_{X/S}) \\ & {\scriptstyle\pi}\searrow \quad \swarrow & \\ & H^{p+1}(X,\Omega^{p-1}) & \end{array}$$

is commutative, where π is the semi-regularity map (§1). (*π was defined for X smooth over* **C**, *but the definition extends without difficulty.*)

Proof. Recall the fundamental local isomorphism ([13], p. 176) gives

$$(6.3)\qquad \mathscr{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_Z,F)\cong F\underset{\mathcal{O}_X}{\otimes}\omega_{Z/X}$$

where F is an $\mathcal{O}_X$-module and $\omega_{Z/X} = \bigwedge^{p} N_{Z/X}$. This gives mappings on the sheaf level

$$(6.4)\qquad \begin{array}{ccc}
N_{Z/X} \cong \bigwedge^{p-1} N^*_{Z/X} \otimes \bigwedge^{p} N_{Z/X} & \xrightarrow{(a)} & \Omega^{p-1}_{X/S} \otimes \bigwedge^{p} N_{Z/X} \\
 & \searrow^{\rho} & \downarrow{\scriptstyle (b)} \\
 & & \mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega^{p-1}) \\
 & & \downarrow{\scriptstyle (c)} \\
 & & \mathcal{H}^p_Z(X, \Omega^{p-1})
\end{array}$$

where

(a) is induced from $I/I^2 \hookrightarrow \Omega^1_{X/S} \otimes \mathcal{O}_Z$;

(b) is (6.3) with $F = \Omega^{p-1}_{X/S}$;

(c) is (5.3).

The semi-regularity map π is dual to

$$\bigwedge^{p-1} \varepsilon\colon\ H^{n-p-1}(X, \Omega^{n-p+1}_{X/S}) \to H^{n-p+1}(Z, \omega_Z \otimes N^*_{Z/X}).$$

Tracing through the duality construction [13], one gets π via

$$\begin{array}{ccc}
H^1(N_{Z/X}) & \xrightarrow{\pi} & H^{p+1}(X, \Omega^{p-1}) \\
{\scriptstyle H^1(\rho)}\downarrow & & \uparrow \\
H^1(\mathcal{H}^p_Z(X, \Omega^{p-1})) & \cong & H^{p-1}_Z(X, \Omega^{p-1}).
\end{array}$$

The proposition now follows from

Claim. The maps ρ (6.4) and $\cdot \lrcorner \{Z\}'$ (6.1) coincide.

Proof of Claim. Most of this is straightforward checking and is left to the reader. We note only that the map

$$\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega^{p-1}) \xrightarrow{(c)} \mathcal{H}^p_Z(X, \Omega^{p-1})$$

can be computed locally as follows: let

$$E = \mathcal{O}_{U_i} f_i^{(1)} \oplus \cdots \oplus \mathcal{O}_{U_i} f_i^{(p)}$$

and consider the Koszul resolution

$$0 \to \bigwedge^{p} E \to \bigwedge^{p-1} E \to \cdots \to E \to \mathcal{O}_{U_i} \to \mathcal{O}_{Z \cap U_i} \to 0.$$

A section φ of $\mathscr{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega^{p-1})$ over U_i is represented by a homomorphism $\tilde{\varphi}\colon \overset{p}{\wedge} E \to \Omega^{p-1}$. Then (c) (φ) is represented by the cocycle

$$\frac{\tilde{\varphi}(f_i^{(1)} \wedge \cdots \wedge f_i^{(p)})}{f_i^{(1)} \ldots f_i^{(p)}}$$

in $\check{C}^{p-1}(V_i, \Omega^{p-1})$. This completes the proof of (6.2).

In what follows, we will consider a diagram

(6.5)
$$\begin{array}{ccccc} Z_0 & \hookrightarrow & Z_1 & & \\ \cap\downarrow & & \cap\downarrow & & \\ X_0 & \hookrightarrow & X_1 & \hookrightarrow & X \\ \downarrow & & \downarrow & & \downarrow f \\ \mathrm{Spec}(\mathbf{C}) & \hookrightarrow & \mathrm{Spec}(A/I) & \hookrightarrow & \mathrm{Spec}(A) \end{array}$$

with A an artinian local $\mathbf{C}$-algebra, $I \subset A$ an ideal such that $I \cdot m_A = (0)$, f a smooth map, and Z_1 a local complete intersection of codimension p. The obstruction to lifting Z_1 to a local complete intersection $Z \subset X$ is given (2.6) by an

(6.6) $$\alpha \in \mathrm{Ext}^1_{\mathcal{O}_{Z_1}}(J_1/J_1^2, I\,\mathcal{O}_{Z_1}) \cong H^1(Z_0, N_{Z_0/X_0}) \underset{\mathbf{C}}{\otimes} I$$

where $J_1 \subset \mathcal{O}_{X_1}$ is the ideal of Z_1.

Finally, we write

(6.7) $$\beta = K_{X/S,\mathbf{C}} \otimes \mathcal{O}_{S_1} \in \mathrm{Ext}^1_{\mathcal{O}_{X_1}}(\Omega^1_{X_1/S_1}, \mathcal{O}_{X_1} \otimes \Omega^1_{S/\mathbf{C}}).$$

For the restriction of the Kodaira-Spencer class. (2.7), and let $d\colon I \to \Omega^1_{S/\mathbf{C}} \otimes \mathcal{O}_{S_0}$ be the natural map.

(6.8) **Proposition.** *With the above notations,*

$$\beta \cup [Z_1] = (1 \otimes d) \cdot (\pi \otimes 1)(\alpha).$$

(6.9)
$$\begin{array}{ccc} [Z_1] \in H^p(X_1, \Omega^p_{X_1/S_1}) & \xrightarrow{\beta \cup \cdot} & H^{p+1}(X_1, \Omega^{p-1}_{X_1/S_1} \otimes \Omega^1_{S/\mathbf{C}}) \\ & & \uparrow 1 \otimes d \\ & & H^{p+1}(X_0, \Omega^{p-1}_{X_0}) \underset{\mathbf{C}}{\otimes} I \\ & & \uparrow \pi \otimes 1 \\ & & \alpha \in H^1(Z_0, N_{Z_0/X_0})_S \underset{\mathbf{C}}{\otimes} I. \end{array}$$

Proof. Note first that

$$H^{p+1}(X_0, \Omega^{p-1}_{X_0}) \underset{\mathbf{C}}{\otimes} I \cong H^{p+1}(X_1, \Omega^{p-1}_{X_1/S_1} \otimes I)$$

by base change, so the map $1 \otimes d$ in (6.9) makes sense. Replacing cohomology in (6.9) by local cohomology and using (6.2), it suffices to show

$$\beta \cup \{Z\}' = (1 \otimes d)(\alpha \lrcorner \{Z\}').$$

This equality follows easily from (2.5).

(6.10) **Corollary.** *With notations as above, assume that $d\colon I \to \Omega^1_{S/\mathbf{C}} \otimes \mathcal{O}_{S_0}$ is injective, and Z_0 is semi-regular in X_0. Then Z_1 lifts to $Z \subset X$ if and only if $[Z_1]_{d.\cdot R.} \in H^{2p}_{DR}(X_1/S_1)$ lifts to a horizontal class $z \in F^p H^{2p}_{DR}(X/S)$.*

Proof.

$$Z_1 \text{ lifts} \Leftrightarrow \alpha = 0 \Leftrightarrow (1 \otimes d)(\pi \otimes 1)(\alpha) = 0 \Leftrightarrow \beta \cup [Z_1] = 0 \underset{(4.2)}{\Leftrightarrow} [Z_1]_{d.\cdot R.}$$

lifts to a horizontal section of $F^p H^{2p}_{DR}(X/S)$.

§7. Applications

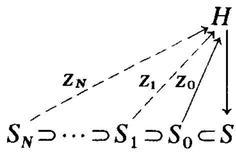
(7.1) **Theorem.** *Let $X \to S$ be a smooth, projective morphism with $S = \operatorname{Spec}(\mathbf{C}[[t_1, \dots, t_r]])$. Let $X_0 \subset X$ be the closed fiber, and let $Z_0 \subset X_0$ be a local complete intersection of codimension p. Suppose that the topological cycle class $[Z_0] \in H^{2p}$ lifts to a horizontal class $z \in F^p H^{2p}_{DR}(X/S)$, and that Z_0 is semi-regular in X_0. Then Z_0 lifts to a subscheme $Z \subset X$.*

Proof. Let $S_N = \operatorname{Spec}(\mathbf{C}[[t_1, \dots, t_r]]/(t)^{N+1})$ and note the maps

$$d\colon (t)^N/(t)^{N+1} \to \Omega^1_{S_N/\mathbf{C}} \otimes \mathcal{O}_{S_{N-1}}$$

are injective. Applying (6.10), we may lift step by step to find $Z_N \subset X_N = X \underset{S}{\times} S_N$ flat over S_N. Let H be the Hilbert scheme of X/S. The corresponding diagram

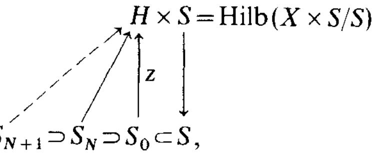
(7.2)



gives rise to an S-point $Z\colon S \to H$ lifting Z_0. The resulting scheme $Z \subset X$ is the desired one.

(7.3) **Theorem.** *Let X be smooth and projective over $\mathbf{C}$. Let $Z \subset X$ be a local complete intersection which is semi-regular in X. Then the corresponding point $Z \in \operatorname{Hilb}(X/\mathbf{C})$ is smooth.*

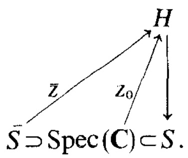
Proof. Let V be the cotangent space to $H = \operatorname{Hilb}(X/\mathbf{C})$ at Z, and write $S = \operatorname{Spec}(\widehat{\operatorname{Sym}}_{\mathbf{C}}(V))$, where $\widehat{\operatorname{Sym}}$ denotes the completed symmetric algebra. Applying (6.10) to $X \underset{\mathbf{C}}{\times} S$, we see that given a diagram



there exists a dotted arrow rendering the whole commutative. This implies that H is smooth at Z, as claimed.

(7.4) **Theorem.** *Let $X \xrightarrow{f} S \xrightarrow{g} \mathrm{Spec}(\mathbf{C})$ be morphisms, with f smooth and projective and g smooth, connected, and of finite type. Let $z \in \Gamma(S, \mathbf{R}^{2p} f_*(\Omega^\bullet_{X/S}))$ be a horizontal section and let $o \in S$. Suppose the restricted class $z_0 \in H^{2p}_{DR}(X_0/\mathbf{C})$ is algebraic, representing a local complete intersection, $Z_0 \subset X_0$ which is semi-regular in X_0. Then for all $s \in S$, $z_s \in H^{2p}_{DR}(X_s/\mathbf{C})$ is algebraic.*

Proof. Since $z_0 \in F^p H^{2p}_{DR}(X_0/\mathbf{C})$, it follows from the results of Deligne [3] that $z \in \Gamma(S, F^p \mathbf{R}^{2p} f_*(\Omega^\bullet))$. Let $\bar{S} = \mathrm{spec}(\hat{\mathcal{O}}_{S,0})$, $\bar{X} = X \underset{S}{\times} \bar{S}$, and let $\bar{z} \in F^p H^{2p}_{DR}(\bar{X}/\bar{S})$ be the pullback of z. Clearly, $\bar{z}$, Z_0, $\bar{X}$ satisfy the hypotheses of (7.1), so letting $H = \mathrm{Hilb}(X/S)$, we have a diagram



By a theorem of Artin [1], the existence of $\bar{Z}$ implies that Z_0 extends to an analytic map $Z: U \to H$, where $U \subset S$ is some complex neighborhood of o. Hence Z_0 lifts to an analytic family over U, so z_s is algebraic for $s \in U$. A simple argument using the Hilbert scheme shows

$$T = \{s \in S \mid z_s \text{ is algebraic}\}$$

is contained in a countable union of closed subvarieties of S. Since $U \subset T$, it follows that $T = S$, proving (7.4).

(7.5) *Remark.* Let $l \in \Gamma(S, \mathbf{R}^2 f_*(\Omega^\bullet))$ be the polarization class. The hypothesis on z_0 in (7.4) can be weakened to read: there exist integers $a, b, a \neq 0$, such that $a z_0 + b\, l_0^p$ is the class of a subscheme $Z_0 \subset X_0$ which is semi-regular and a local complete intersection. This gives a way of making the class z_0 "effective". The problem of constructing semi-regular representatives for algebraic cycle classes of codimension > 1 remains, however, wide open.

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