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Triple Collision in the Planar Isosceles Three Body Problem

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Abstract. We employ a device due to McGehee to discuss the qualitative behavior of orbits which reach or come close to triple collision in a special case of the planar three body problem. We show that there exist infinitely many orbits which both begin and end in triple collision. Nearby orbits behave in different ways depending on whether they pass close to the collinear or equilateral triangle central configuration. Finally, we discuss a new type of orbit in the three body problem which we call "billiard shots".

Introduction

The goal of this paper is to investigate the qualitative behavior of solutions of the planar isosceles three body problem which begin of end in triple collision or which pass close to a triple collision. The isosceles problem is a special case of the planar three body problem which reduces to a Hamiltonian system with two degrees of freedom. Briefly, one takes two equal masses whose initial position and velocity are symmetric with respect to the y-axis in the plane, and a third mass whose initial position and velocity lie along this axis. So the masses form a (possibly degenerate) isosceles triangle in the plane. Because of the symmetry of the problem, the masses will remain in such a configuration for all time. Hence, after fixing the center of mass at the origin, we have a system with only two degrees of freedom.

The main tool in the study of triple collision is a device due to McGehee by which the singularity due to triple collision is replaced by an invariant two-dimensional manifold over which the flow extends smoothly. If one understands the flow on this collision manifold completely, then one can "read off" the behavior of orbits which reach or pass close to triple collision.

This is precisely the procedure McGehee followed in studying triple collision in the collinear three body problem [3]. There he showed that, for most choices

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of the masses, a near-collision orbit tends to leave a neighborhood of triple collision with the masses receiving an arbitrarily large velocity. Two of the masses travel in one direction in a tight binary system, while the third mass separates from them in the opposite direction.

A natural question is how do these results extend to the planar or three dimensional problem. Various authors have discussed the triple collision manifold in the planar case [10, 13]. This manifold turns out to be four-dimensional and the flow on it is not at present well understood. We hope that studying the simpler flow on the triple collision manifold in the isosceles case will give some insight into the structure of this flow in the planar problem. Certainly most of the phenomena we describe below will occur in some form in the full three-body problem.

One of the major differences between the isosceles and the collinear problems is the possibility of both collinear and equilateral triangle central configurations in the former. We recover easily the classical result that any orbit which begins or ends in triple collision must assume one of these configurations as it approaches the singularity. Furthermore, we show that the natural collinear collision orbit is the only one which approaches the collinear central configuration, while there are two smooth one parameter families of orbits which tend to the triangular configurations.

When the third mass is relatively small, the structure of these orbits is extremely complicated. We exhibit infinitely many orbits which both begin and end in triple collision (the so called ejection-collision orbits) and which may be characterized as follows. The binary system simply makes one cycle near the x-axis, beginning and ending at the origin and with exactly one point of zero velocity for each mass. Meanwhile the third mass oscillates rapidly near the origin up and down the y-axis. In fact, we show that for any sufficiently large integer k, there is an ejection-collision orbit of this type in which the third mass passes through the center of mass of the binary system exactly k times. This proves a result determined numerically by Broucke [1].

We remark that these ejection-collision orbits are all close to the natural collinear ejection-collision orbit in which the binary system remains on the x-axis and the third mass is at rest at the origin. Thus this orbit is unstable in the sense that nearby orbits follow it for a long time before jumping off into a triangular configuration and colliding.

As in the collinear three body problem, the flow on the triple collision manifold also allows us to determine the behavior of orbits which pass close to triple collision. There are basically two possibilities. Either the binary system escapes along the x-axis with finite velocity, leaving the third mass oscillating near their center of mass, or else the third mass leaves either up or down the y-axis with arbitrarily large velocity. In this case, the binary system travels in the opposite direction, undergoing a sequence of binary collisions in the process.

The flow on the triple collision manifold also allows us to prove the existence of "billiard shot" orbits in the isosceles problem. These are orbits which feature a rapid binary oscillation and the third mass traveling up the y-axis before triple collision. After passing close to collision, the third mass (or billiard ball) is left oscillating rapidly through the center of mass while the binary system separates in opposite directions near the x-axis.

1. The Triple Collision Manifold

Our goal in this section is to exploit a device introduced by McGehee [3] to blow up the singularity in the system at triple collision. Most of the material here is exactly the same as in McGehee's paper. We refer to that paper or to [2] for motivation of some of the changes of variables below. However, for completeness, we include all of the technical details here.

Recall the setting of the planar isosceles three body problem. One is given three mass points in the plane with masses $m_1 = m_2$ and m_3. The third mass is initially positioned on the y-axis, with velocity parallel to the y-axis. The two equal masses have initial position and velocity symmetric with respect to the y-axis. So the particles initially lie at the vertices of an isosceles triangle. The masses are assumed to attract each other under the Newtonian law of attraction. Because of the symmetry of the problem, the particles will then always lie at the vertices of some isosceles triangle.

Note that binary collisions are only possible between the two equal masses· if m_3 is involved in a collision, it must necessarily collide with both m_1 and m_2 at the same instant. It is this particular occurrence we shall be mainly concerned with below.

If we fix the center of mass of the system at the origin, then the system has only two degrees of freedom. One can use either heliocentric or Jacobi coordinates to describe the resulting motion; we shall use the latter. Let x_1 denote the distance from m_2 to m_1, so $x_1 \geqq 0$. See Fig. 1. Also let x_2 denote the directed distance along the y-axis from the center of mass of m_1 and m_2 to m_3. So x_2 is positive when m_3 lies above the binary system, and negative when m_3 lies below. We have a binary collision precisely when $x_1 = 0$, and we have total collapse or triple collision whenever $x_1 = x_2 = 0$.

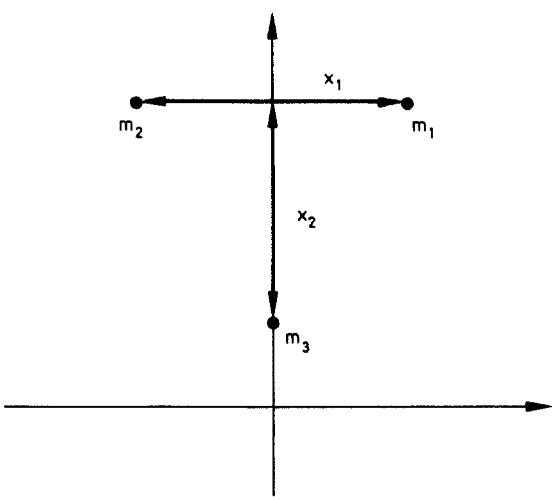

Fig. 1. Jacobi coordinates in the plane

In these coordinates, the equations of motion may be written

$$\ddot{x}_1 = \frac{-2m_1}{x_1^2} - \frac{8m_3 x_1}{(x_1^2+4x_2^2)^{3/2}}$$
$$\ddot{x}_2 = \frac{-8(2m_1+m_3)x_2}{(x_1^2+4x_2^2)^{3/2}} \tag{1.1}$$

See Pollard [4] for the derivation of these equations.

Let p_1 and p_2 denote the momenta conjugate to x_1 and x_2 defined by

$$p_1 = m_1 \dot{x}_1/2$$
$$p_2 = 2m_1 m_3 \dot{x}_2/(2m_1+m_3)$$

Then the equations of motion may be written as a first order system of differential equations in Hamiltonian form

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} \tag{1.2}$$

with the Hamiltonian or total energy given by

$$H(x_1, x_2, p_1, p_2) = \frac{p_1^2}{m_1} + \frac{(2m_1+m_3)p_2^2}{4m_1 m_3} - \frac{m_1^2}{x_1} - \frac{4m_1 m_3}{(x_1^2+4x_2^2)^{1/2}} \tag{1.3}$$

It is well known that H is a constant of the motion for (1.2).

We may write this system more compactly by introducing the following notation. Let $x=(x_1, x_2)$ and $p=(p_1, p_2)$. Let $M=2m_1+m_3$ be the total mass of the system. Then we may write

$$H(x,p) = (1/2)p^t A^{-1} p + V(x) \tag{1.4}$$

where A^{-1} is the 2×2 matrix

$$A^{-1} = \begin{pmatrix} 2/m_1 & 0 \\ 0 & M/2m_1 m_3 \end{pmatrix} \tag{1.5}$$

and where the potential energy $V(x)$ is given by

$$V(x) = -\frac{m_1^2}{x_1} - \frac{4m_1 m_3}{(x_1^2+4x_2^2)^{1/2}} \tag{1.6}$$

Note that $V(x)<0$ and that V is homogeneous of degree -1.

Thus we have an analytic Hamiltonian system defined for $x_1>0$, i.e., on the space $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$. Our initial goal will be to extend this system to the boundary $x_1=0$ where the collisions occur. This will be accomplished using a sequence of changes of variables due to McGehee.

We first introduce "polar" coordinates into configuration space, together with a scaling of the radial and tangential components of momentum. That is, we introduce

$$\begin{aligned} r &= (x^t A x)^{1/2} \\ s &= x/r \\ v &= r^{1/2}(s \cdot p) \\ u &= r^{1/2}(A^{-1} p - (s \cdot p) s) \end{aligned} \tag{1.7}$$

Note that $s^t A s = 1$ and $s^t A u = 0$. Also, r is the moment of inertia of the system. We also scale the time variable of the system by

$$dt = r^{3/2}\, d\tau$$

In these variables, the system (1.2) becomes

$$\begin{aligned} \frac{dr}{d\tau} &= r\,v \\ \frac{dv}{d\tau} &= u^t A u + (1/2)\, v^2 + V(s) \\ \frac{ds}{d\tau} &= u \\ \frac{du}{d\tau} &= -(1/2)\, v\, u - (u^t A u)\, s - \operatorname{grad} V(s) \end{aligned} \tag{1.8}$$

Here $V(s)$ is the restriction of V to the sphere $s^t A s = 1$, and grad $V(s)$ represents its gradient in the metric induced by A.

Now introduce

$$\begin{aligned} s &= (A^{-1})^{1/2} (\cos\theta, \sin\theta) \\ u &= u(A^{-1})^{1/2} (-\sin\theta, \cos\theta) \end{aligned}$$

with $-\pi/2 < \theta < \pi/2$. The equations become

$$\begin{aligned} \frac{dr}{d\tau} &= r\,v \\ \frac{dv}{d\tau} &= u^2 + (1/2)\, v^2 + V(\theta) \\ \frac{d\theta}{d\tau} &= u \\ \frac{du}{d\tau} &= -(1/2)\, v\, u - V'(\theta) \end{aligned} \tag{1.9}$$

The energy relation (1.4) goes over to

$$r\,e = (1/2)\, v^2 + (1/2)\, u^2 + V(\theta) \tag{1.10}$$

where e is the constant value of the Hamiltonian and where

$$V(\theta) = -\frac{m_1^2}{\sqrt{2/m_1}\cos\theta} - \frac{4m_1 m_3}{\sqrt{(2/m_1)\cos^2\theta + \frac{2M}{m_1 m_3}\sin^2\theta}} \tag{1.11}$$

Note that binary collisions at $x_1=0$ and $x_2>0$ correspond to $\theta=\pi/2$, while $x_1=0, x_2<0$ corresponds to $\theta=-\pi/2$. Finally, triple collision $x_1=x_2=0$ corresponds to $r=0$.

Now the system (1.9) is analytic at $r=0$. In fact, when $r=0$, we have $dr/d\tau=0$. Hence $r=0$ is an invariant manifold for the flow. Thus we have removed the singularity which corresponds to triple collision. In its place we have pasted a smooth manifold which is called the triple collision manifold. Orbits which previously began or ended at triple collision in finite time are now slowed down so that they tend to the triple collision manifold as $t\to-\infty$ or $t\to\infty$. And orbits which pass close to triple collision now behave very much like orbits on the triple collision manifold itself. Hence understanding the flow on this manifold gives a good deal of information about these near-collision orbits. It is this flow which we now study.

The system (1.9) defines an analytic vector field for $0\leq r<\infty$, $v\in\mathbb{R}$, $-\pi/2<\theta<\pi/2$, $u\in\mathbb{R}$. We first extend this system over the boundary $\theta=\pm\pi/2$. This is accomplished by using a regularization due to Sundman [11, 12]. Our treatment here differs slightly from McGehee's.

Let $W(\theta)=-(\cos\theta)V(\theta)$. Note that W is a positive analytic function on $[-\pi/2, \pi/2]$. We introduce a new variable

$$w=\frac{\cos\theta}{\sqrt{W(\theta)}}u$$

as well as another change of time scale

$$\frac{d\tau}{dt}=\frac{\cos\theta}{\sqrt{W(\theta)}}.$$

We remark that this new time variable t is different from the original time variable.

The system (1.9) now goes over to

$$\begin{aligned}
\frac{dr}{dt} &= \frac{\cos\theta}{\sqrt{W(\theta)}} r v \\
\frac{dv}{dt} &= \sqrt{W(\theta)}\left(1-\frac{\cos\theta}{2W(\theta)}(v^2-4re)\right) \\
\frac{d\theta}{dt} &= w \\
\frac{dw}{dt} &= (\sin\theta)\left(-1+\frac{\cos\theta}{W(\theta)}(v^2-2re)\right) \\
&\quad -\frac{v w\cos\theta}{2\sqrt{W(\theta)}}+\frac{W'(\theta)}{W(\theta)}\cos\theta - w^2/2)
\end{aligned} \tag{1.12}$$

with the energy relation

$$\frac{w^2}{2\cos\theta}-1=\frac{\cos\theta}{W(\theta)}(r\,e-v^2/2) \tag{1.13}$$

This system is an analytic vector field on $[0,\infty)\times\mathbb{R}\times[-\pi/2,\pi/2]\times\mathbb{R}$, so the singularities due to binary collisions have now been removed. Actually, since $dv/dt>0$ when $\theta=\pm\pi/2$, it follows that solutions have been extended through binary collision by an elastic bounce.

The triple collision manifold remains intact under this change of variables and is given by the energy relation in $r=0$:

$$w^2/2+\frac{v^2\cos^2\theta}{W(\theta)}=\cos\theta \tag{1.14}$$

This surface is sketched in Fig. 2. Note that it is topologically a sphere with four points removed. This is exactly the same as in the collinear three body problem. However, the flow induced by (1.12) on this surface is dramatically different from the collinear case.

The flow on this manifold is determined by the differential equation

$$\begin{aligned}
\frac{dv}{dt}&=\sqrt{W(\theta)}\left(1-\frac{\cos\theta}{2W(\theta)}v^2\right)\\
\frac{d\theta}{dt}&=w\\
\frac{dw}{dt}&=(\sin\theta)\left(-1+\frac{v^2\cos\theta}{W(\theta)}\right)-\frac{v\,w\cos\theta}{2\sqrt{W(\theta)}}\\
&\quad+\frac{W'(\theta)}{W(\theta)}(\cos\theta-w^2/2)
\end{aligned} \tag{1.15}$$

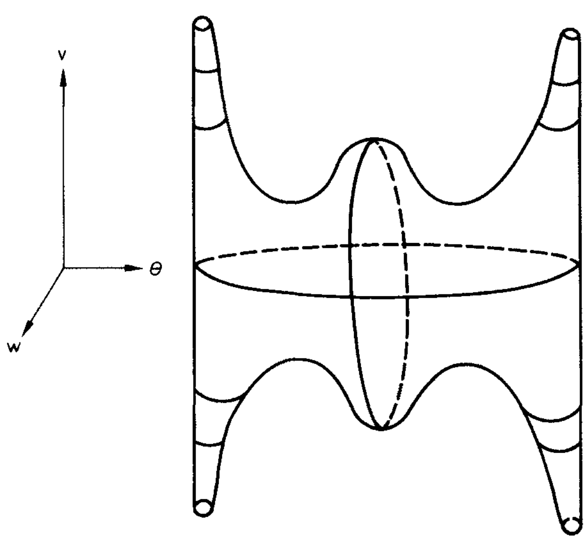

Fig. 2. The triple collision manifold in the isosceles three body problem

We shall discuss the main properties of this flow in the next section.

Remark 1. $W(\theta)$, and hence the collision manifold, depend on the choice of masses m_1 and m_3. One can remove this dependence by introducing $\eta = v/\sqrt{W(\theta)}$. The differential equation becomes

$$\begin{aligned}\frac{d\eta}{dt} &= 1-(1/2)\eta^2\cos\theta-(1/2)\frac{W'(\theta)}{W(\theta)}\eta w\\ \frac{d\theta}{dt} &= w\\ \frac{dw}{dt} &= (\sin\theta)(-1+\eta^2\cos\theta)-(1/2)\eta w\cos\theta\\ &\quad+\frac{W'(\theta)}{W(\theta)}(\cos\theta-w^2/2)\end{aligned} \tag{1.16}$$

with energy relation

$$w^2+\eta^2\cos^2\theta=2\cos\theta \tag{1.17}$$

which is independent of the masses.

Remark 2. If we let ε denote the mass ratio m_3/m_1, then one computes easily that

$$W(\theta)=m_1^{5/2}\left(\frac{1}{\sqrt{2}}+\frac{4\varepsilon^{3/2}\cos\theta}{\sqrt{2\varepsilon+4\sin^2\theta}}\right)$$

$$\frac{W'(\theta)}{W(\theta)}=\frac{-8\sqrt{2}\,\varepsilon^{3/2}(2+\varepsilon)\sin\theta}{(2\varepsilon+4\sin^2\theta)(\sqrt{2\varepsilon+4\sin^2\theta}+4\sqrt{2}\,\varepsilon^{3/2}\cos\theta)}.$$

Since W'/W depends only on ε, it follows that (1.16) represents a one parameter family of vector fields depending on the mass ratio.

2. The Flow on the Triple Collision Manifold

The object of this section is to describe in detail the flow generated by (1.15) on the triple collision manifold. We will show that there are some striking differences between this flow and that studied by McGehee in the collinear three body problem. In the next sections, we will interpret what these results mean in terms of actual collision and near-collision orbits in the isosceles problem.

One of the most important features of the flow on the triple collision manifold is the equilibrium or rest points. These correspond to the central configurations of the isosceles problem, as we show below. First, however, we compute the equilibrium solutions on the triple collision manifold.

Proposition 1. *Suppose* (v_0, θ_0, w_0) *is a rest point for the flow on the triple collision manifold. Then*

1. $w_0 = 0$
2. $V'(\theta_0) = 0$
3. $v_0^2 = -2V(\theta_0)$.

Proof. From the energy relation (1.14), it follows that dv/dt and $d\theta/dt$ vanish iff $w_0 = 0$. If $w_0 = 0$, we also have

$$\frac{dw}{dt} = \sin\theta + \frac{W'(\theta)}{W(\theta)}\cos\theta = \frac{-\cos^2\theta\, V'(\theta)}{W(\theta)}$$

since $V(\theta) = -W(\theta)/\cos\theta$. Hence $\frac{dw}{dt} = 0$ iff $V(\theta_0) = 0$. Hence we have a rest point iff $w_0 = 0$ and $V'(\theta_0) = 0$. The energy relation then determines v_0. qed

So to each critical point θ_0 of $V(\theta)$, there correspond exactly two equilibrium solutions of the flow on the triple collision manifold, one with $v_0 < 0$ and the other with $v_0 > 0$. Such critical points are called *central configurations*. One of our results below is that any triple collision orbit approaches one of these equilibrium solutions. That is, we have proven the classical result that the masses approach a central configuration as they approach triple collision.

Before proving this, we calculate the number of central configurations in the isosceles problem.

Proposition 2. *There are three central configurations for the planar isosceles three body problem. These are a non-degenerate minimum for* V *at* $\theta = 0$ *and two non-degenerate maxima at* $\theta = \arctan(\pm\sqrt{3m_3/M})$.

Proof. This is a straightforward calculation. qed

Our next observation about the flow is that it is gradient-like. Recall that this means that there is a smooth real-valued function which increases along all non-stationary solutions. On the triple collision manifold, the projection onto the v-axis serves as this gradient function. Indeed, from (1.15), we have

$$\frac{dv}{dt} = W(\theta)^{1/2}\left(\frac{w^2}{2\cos\theta}\right) \geqq 0.$$

So v is non-decreasing along orbits. Only when $w = 0$ do we have $dv/dt = 0$. But, at such points,

$$\begin{aligned}\frac{dw}{dt} &= \sin\theta + \frac{W'(\theta)}{W(\theta)}\cos\theta \\ &= \frac{-V'(\theta)\cos^2\theta}{W(\theta)} \neq 0\end{aligned}$$

provided $V'(\theta) \neq 0$. Hence v in fact increases along all non-stationary orbits of the system. We record this fact as another proposition.

Proposition 3. *The coordinate v increases along all non-equilibrium orbits of the flow on the triple collision manifold. Hence the flow is gradient-like.*

Suppose now that a solution of the isosceles problem begins or ends at triple collision. Such a solution must be forward or backward asymptotic to the triple collision manifold. Since the flow there is gradient-like, it follows that triple collision orbits must be asymptotic to one of the equilibria. This fact has a classical interpretation in terms of central configurations.

Suppose first that a triple collision orbit approaches one of the two equilibria at $\theta=0$. Interpreting this in terms of the original coordinates, it follows that

$$x_2/x_1=(M/4m_3)\sin\theta/\cos\theta \to 0$$

so that the masses tend toward a straight line as they approach collision. This is the collinear central configuration discovered by Euler. Similarly, if the orbit approaches the equilibria at $\theta=\arctan(\pm 3m_3/M)^{1/2})$, then

$$x_2^2/x_1^2=M\sin^2\theta/4m_3\cos^2\theta \to 3/4.$$

This means that the masses approach an equilateral triangle configuration as they tend toward collision. Thus we have recovered the following classical theorem.

Theorem. (Euler, Lagrange). *Any triple collision orbit in the isosceles three body problem tends to either a collinear* ($\theta=0$) *or an equilateral triangle* ($\theta\neq 0$) *central configuration.*

In the sequel, we will identify the collinear central configuration by θ_0 and the two triangular central configurations by θ_+ and θ_-, with $\theta_-<0<\theta_+$.

Remark. At this point it might be helpful to recall the relationship between the new r, v, θ, w coordinates and the old Jacobi coordinates. The variables r and θ represent polar coordinates (in the A-metric) in configuration space, while v and w are scaled velocity coordinates, v in the radial direction and w in the θ-direction. For solutions which pass close to triple collision, r must pass close to zero, and the θ-coordinate then describes the configuration assumed by the three particles. For example, if $\theta=0$, the particles lie close to a straight line, while for $\theta=\pm\pi/2$, the equal masses are close to double collision with m_3 (relatively) far away and above or below them. See Fig. 3.

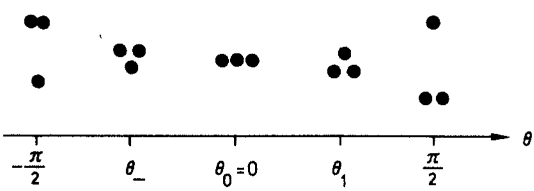

Fig. 3

We now compute the characteristic exponents of the rest points on the triple collision manifold. There will be an additional characteristic exponent in the direction transverse to the triple collision manifold which we will calculate later. All of these exponents occur as the eigenvalues of the linearization of (1.12) at a rest point $(0, v, \theta, 0)$. The resulting matrix is

$$\begin{pmatrix} \dfrac{v\cos\theta}{W(\theta)^{1/2}} & 0 & & * \\ \dfrac{2e\cos\theta}{W(\theta)^{1/2}} & -\dfrac{v\cos\theta}{W(\theta)^{1/2}} & & \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{-\cos^2\theta V''(\theta)}{W(\theta)} & -(\cos\theta/2)^{1/2}\operatorname{sgn} v \end{pmatrix} \tag{2.1}$$

This is easily computed from (1.12) using the energy relation (1.13), Proposition 1, and the fact that

$$\begin{aligned} \frac{d}{d\theta}\left(\frac{dw}{dt}\right) &= \frac{d}{d\theta}\left(\sin\theta + \frac{W'(\theta)\cos\theta}{W(\theta)}\right) \\ &= \frac{d}{d\theta}\left(\frac{-\cos^2\theta V'(\theta)}{W(\theta)}\right) \\ &= \frac{-\cos^2\theta V''(\theta)}{W(\theta)} \end{aligned}$$

at a rest point.

One checks easily that the characteristic exponents of the restriction of the flow to the triple collision manifold are then given by the eigenvalues of the submatrix

$$\begin{pmatrix} 0 & 1 \\ \dfrac{-\cos^2\theta V''(\theta)}{W(\theta)} & -((\cos\theta)/2)^{1/2}\operatorname{sgn} v \end{pmatrix} \tag{2.2}$$

These eigenvalues are easily computed and are found to be

$$\zeta^{\pm} = -(1/2)((\cos\theta)/2)^{1/2}\operatorname{sgn} v \pm (1/2)\sqrt{\frac{\cos\theta}{2} - \frac{4\cos^2\theta V''(\theta)}{W(\theta)}}$$

At the Lagrange (triangular) equilibria, Proposition 2 gives that $V''(\theta) < 0$, so the expression inside the radical is positive and greater than $((\cos\theta)/2)^{1/2}/2$. Hence one of $\zeta^{\pm}$ is positive and the other is negative. Consequently the Lagrange equilibria are saddles for all values of the masses.

At the Euler central configuration θ_0, the situation is quite different. Proposition 2 gives that $V''(\theta) > 0$, so both $\zeta^{\pm}$ have real parts with the same sign as $-(1/2)((\cos\theta)/2)^{1/2}\operatorname{sgn} v$. In particular, the equilibrium point is a sink when $v > 0$ and a source when $v < 0$. Moreover, the expression inside the radical is negative for certain values of the masses.

Proposition 4. *The sink and source at θ_0 have complex eigenvalues provided $m_3<(55/4)m_1$.*

Proof. Since $\theta_0=0$, the expression inside the radical for $\zeta^{\pm}$ is negative when

$$1+\frac{8V''(0)}{V(0)}<0$$

One computes easily that

$$V(0)=-\frac{m_1^{5/2}}{\sqrt{2}}-\frac{4m_1^{3/2}m_3}{\sqrt{2}}$$
$$V''(0)=7m_1^{5/2}/\sqrt{2}$$

The results then follows immediately. qed

In particular, when m_3 is relatively small, orbits on the triple collision manifold tend to spiral into the sink and away from the source. This will have a dramatic effect on the orbits which approach triple collision near the collinear central configuration.

We now turn our attention to the remaining qualitative feature of the flow on the triple collision manifold – the ultimate behavior of the stable and unstable manifolds of the four equilibria associated to the triangular central configurations $\theta_{\pm}$. Where the stable manifolds originate and the unstable manifolds die is of crucial importance for understanding near-collision orbits. Note that there are three major possibilities for the unstable manifolds. Since v must increase along these orbits, a typical branch of the unstable manifold may run up the left or right "arm" of the triple collision manifold, or else it may die in the sink. A degenerate possibility is that the unstable manifold might exactly match up with the stable manifold of one of the saddles.

Our first observation is that, by symmetry, one need only discuss a few of the invariant manifolds. Indeed, the system (1.15) is invariant under the reflection $(v,\theta,w)\to(v,-\theta,-w)$ and reversed by the reflection $(v,\theta,w)\to(-v,-\theta,w)$. Consequently, the system is also reversed by the composition of these reflections $(v,\theta,w)\to(-v,\theta,-w)$. Hence these symmetries preserve the invariant manifold structure of the system.

Our next observation is that some of the invariant manifolds of the saddles are easy to describe. For example, at the rest point $(+v_0,\theta_+,0)$ with $v_0>0$, one branch of the stable manifold must emanate from the source, while the other must run up the lower arm with $\theta=\pi/2$. This follows immediately from the gradient-like property of the flow together with the fact that $d\theta/dt>0$ when $w>0$ and $d\theta/dt<0$ when $w<0$. By the symmetries above, one can then determine the stable manifold at $(-v_0,\theta_-,0)$ as well as the unstable manifolds at $(+v_0,0)$ and $(+v_0,\theta_-,0)$. These are depicted in Fig. 4.

Now consider the branches of the unstable manifold at $(+v_0,\theta_{\pm},0)$ which satisfy: the w-coordinate along the local unstable manifold is positive. Let γ_+ denote the branch at $(-v_0,\theta_+,0)$ and γ_- the branch at the other equilibrium. As above, symmetry arguments determine the remaining invariant manifolds in

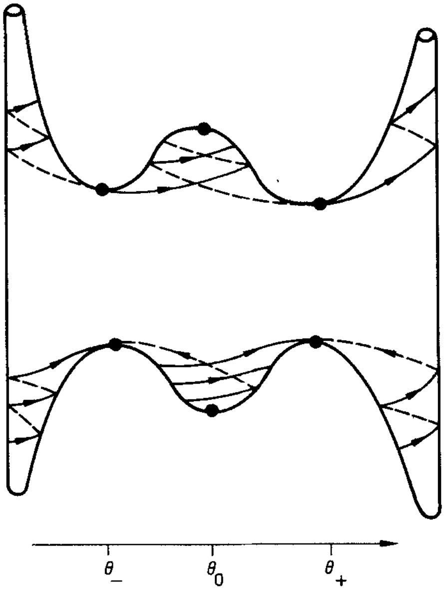

Fig. 4. Some of the invariant manifolds of the saddles

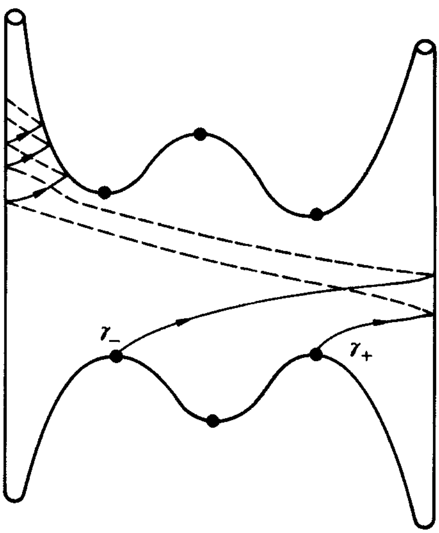

Fig. 5. The behavior of γ_+ and γ_- in the allowable case

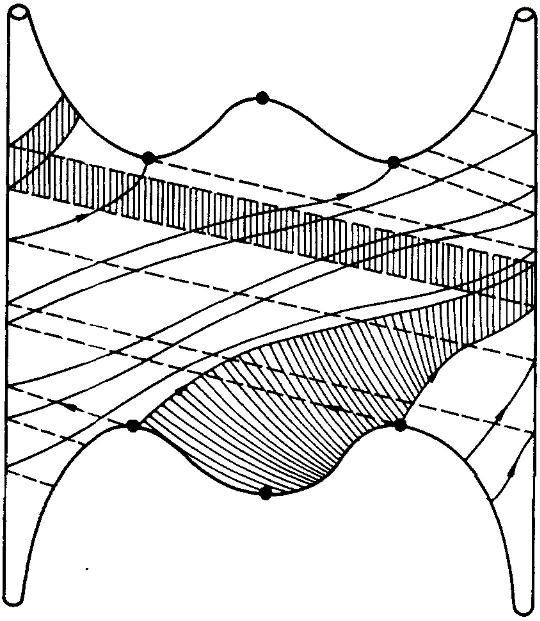

Fig. 6. The phase portrait of the flow on the collision manifold when the set of masses are allowable

terms of γ_+ and γ_-. For example, the first symmetry maps γ_+ and γ_- to the remaining branches of the unstable manifolds at θ_- and θ_+ respectively. Hence the ultimative behavior of all of the remaining invariant manifolds depend on the behavior of γ_+ and γ_-. These must be determined numerically. R. Moeckel [15] has some results along these lines for both this problem and the full planar problem.

For all values of the masses we have computed thus far, γ_+ tends to run up the upper arm of the triple collision manifold with $\theta=-\pi/2$ as depicted in Fig. 5. We conjecture that this is in fact true for all masses. The situation for γ_- is somewhat more subtle. Again, for all values of the masses that have been checked, γ_- runs up the same arm as γ_+. However, in doing so, γ_- passes close to the saddle point $(v_0, \theta_-, 0)$. It is conceivable that, for some values of the masses, γ_- meets the stable manifold of this saddle, or even falls into the sink. One checks easily that if γ_+ runs up the arm with $\theta=-\pi/2$, then γ_- cannot run up the arm with $\theta=\pi/2$.

Most of our results below depend only on the behavior of γ_+. We will call a set of masses allowable if both γ_+ and γ_- run up the arm with $\theta=-\pi/2$. Figure 6 gives a complete picture of the flow in case the masses are allowable. Whether this picture holds for all masses is an interesting question which awaits further numerical study.

3. Collision Orbits

In this section we use the results of the previous section to describe the set of orbits which begin or end in triple collision. Orbits which begin at collision are called *ejection* orbits; orbits which end at triple collision are called *collision* orbits; and orbits which do both are known as *ejection-collision* orbits. As we showed in the previous section, any such orbit must be asymptotic to one of the equilibria associated to a central configuration. That is, they lie on the stable or unstable manifolds of these equilibria. We denote the set of ejection and collision orbits at θ_0 by $E(\theta_0)$ and $C(\theta_0)$ respectively. Similarly, $E(\theta_\pm)$ and $C(\theta_\pm)$ denote the set of ejection and collision orbits at the triangular central configurations.

The linearization (2.1) can be used to show that all of the equilibria on the collision manifold are hyperbolic and thus that the ejection and collision orbits form immersed submanifolds of phase space. To see this, observe that $v\cos\theta/W(\theta)^{1/2}$ is an eigenvalue of (2.1) with associated eigenvector $(v, e, 0, 0)$. This eigenvector is tangent to the energy level corresponding to e, as is seen by differentiating (1.13). Hence we have in addition to the dimensions in the triple collision manifold, one additional unstable eigenvalue whenever $v_0>0$ and one stable eigenvalue whenever $v_0<0$.

Hence we have the following:

Proposition 5. *In any energy level $H=e$, both $E(\theta_\pm)$ and $C(\theta_\pm)$ are two-dimensional manifolds, while $E(\theta_0)$ and $C(\theta_0)$ are one-dimensional. All ejection orbits emanate from the equilibria with $v_0>0$, whereas all collision orbits are asymptotic to the equilibria with $v_0<0$.*

This reproves the classical result due to Siegel that the set of collision and ejection orbits forms a union of lower dimensional submanifolds of each energy level. It also reproves a result of Saari [6] which in the isosceles case states that triple collision is improbable in the sense of Lebesgue measure. (Note that binary collisions occur for a large open set of initial conditions.)

There are some special classical solutions of the three body problem associated with each central configuration. Assume $V'(\theta)=0$, so that θ is one of the central configurations. Assume moreover that $w=0$. At such a point, the system (1.12) becomes simply

$$\begin{aligned}\frac{dr}{dt}&=\frac{\cos\theta}{\sqrt{W(\theta)}}\,r\,v\\ \frac{dv}{dt}&=\frac{r\,e\cos\theta}{\sqrt{W(\theta)}}\\ \frac{d\theta}{dt}&=0=\frac{dw}{dt}\end{aligned}\tag{3.1}$$

Thus the two-dimensional r, v-plane is invariant whenever $V'(\theta)=0$ and $w=0$. We sketch the solutions of (3.1) in Fig. 7.

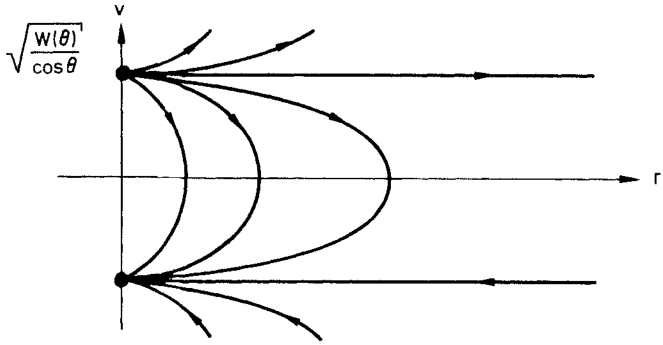

Fig. 7. The phase portrait in an invariant r, v-plane

Note that one can obtain these curves as the level sets of the restriction of the energy relation to the r, v-plane, which is given by

$$\frac{\cos\theta}{2W(\theta)}v^2 - 1 = \frac{\cos\theta}{W(\theta)}r\,e.$$

The two horizontal solutions shown in Fig. 6 occur when $e=0$. For $e<0$, there is a unique solution which begins at the equilibrium point with $v>0$ and ends at the equilibrium point with $v<0$. This solution is called a *homothetic* solution of the system. It is one in which the masses retain their configuration (up to a scale factor) for all time. See Wintner [14] for more details.

In particular, it follows from the above considerations that the one-dimensional sets $C(\theta_0)$ and $E(\theta_0)$ consist precisely of these special solutions of the three body problem. For negative energy, we therefore have that $C(\theta_0)=E(\theta_0)$, and thus we have a heteroclinic solution connecting the two equilibria associated to θ_0. This orbit is easy to describe in configuration space: the third mass is fixed at the origin while the other pair are ejected from either side of it, travel a finite distance along the x-axis, reach a point of zero velocity, and finally return to the origin and triple collision.

4. Near-Collision Orbits

In this final section, we show how the flow on the triple collision manifold can be used to give an almost complete description of the qualitative behavior of orbits which pass close to triple collision. For simplicity, we shall confine our attention to a single negative energy surface. Some of the results below can be extended to zero and positive energy levels with the obvious modifications.

We first consider a neighborhood of the collinear homothetic orbit described at the end of the last section. One branch of the stable manifolds of both of the Lagrangian equilibria with $v_0<0$ emanates from the source which corresponds to θ_0. For m_3 small enough, each of these branches spirals around the source infinitely often, as we see from Proposition 4. Consequently, $C(\theta_0)$ lies in the closure of both $C(\theta_+)$ and $C(\theta_-)$, and if we cut $C(\theta_0)$ by a local transversal section S, then $C(\theta_\pm)\cap S$ is again an infinite spiral converging down to $C(\theta_0)\cap S$ for m_3 small enough.

Similar statements hold for $E(\theta_0)$ and $E(\theta_\pm)$.

Now consider an initial condition p in the transversal S. If S is small enough, the orbit through p either meets $C(\theta_\pm)\cap S$ or $C(\theta_0)\cap S$, or else approaches but does not achieve triple collision. In the latter case, the orbit tends to follow orbits in the unstable manifold of the source which are not in the stable manifolds of the saddle points with $v_0<0$. Figure 5 shows that, in the allowable case, all such orbits run up the upper arms of the triple collision manifold. Hence the orbit through p behaves similarly if S is chosen small enough.

This has the following interpretation. An orbit close enough to the collinear homothetic solution either ends in triple collision or else passes close to triple collision but leaves in one of two ways: either m_3 exists up the y-axis and the binary system down the y-axis, or vice-versa. The closer the orbit approaches the triple collision manifold, the higher the v-value it achieves before leaving a neighborhood of triple collision. This implies that each mass leaves with very large velocity. The fact that orbits in the triple collision manifold tend to wind around the arms infinitely often also has an interpretation relative to the full flow: orbits which come close enough to triple collision feature an arbitrarily large number of binary collision between m_1 and m_2 while leaving a neighborhood of triple collision.

One has similar conclusions for the backward orbit through p. Such an orbit is either ejected from the origin, or else approaches and leaves a neighborhood of triple collision in a similar manner to those above. This leads to the possibility of orbits close to the collinear homothetic orbit which both begin and end in triple collision – i.e. additional ejection-collision orbits. These orbits of course must begin and end at the equilateral triangle central configurations, not at the collinear configurations. Their existence may be verified as follows.

Assume m_3 is small enough so that the source has complex eigenvalues. Choose a local transversal section S_0 which contains a smooth piece of the zero velocity curve in the energy surface, containing the point q at the intersection of the collinear homothetic orbit and the zero velocity curve. Our results above imply that $C(\theta_\pm)\cap S_0$ are both smooth spirals in S_0 converging to q. each of these spirals must therefore intersect the zero velocity curve infinitely often. By symmetry, any such point of intersection must then also lie in $E(\theta_\pm)$. Hence such initial conditions lead to ejection-collision orbits. See Fig. 8.

Observe that, as one of these orbits approaches triple collision, it crosses the hyperplane $\theta=0$ a large number of times. This implies that $x_2=0$ at each such crossing. Hence these ejection-collision orbits feature an oscillation of m_3 through the center of mass of the binary system.

We also observe that such an orbit looks almost like a collinear collision orbit until just before triple collision when the masses jump into a triangular configuration.

We close with a description of all possible behaviors for an orbit which passes close to triple collision with m_3 initially travelling up the y-axis. Let γ be

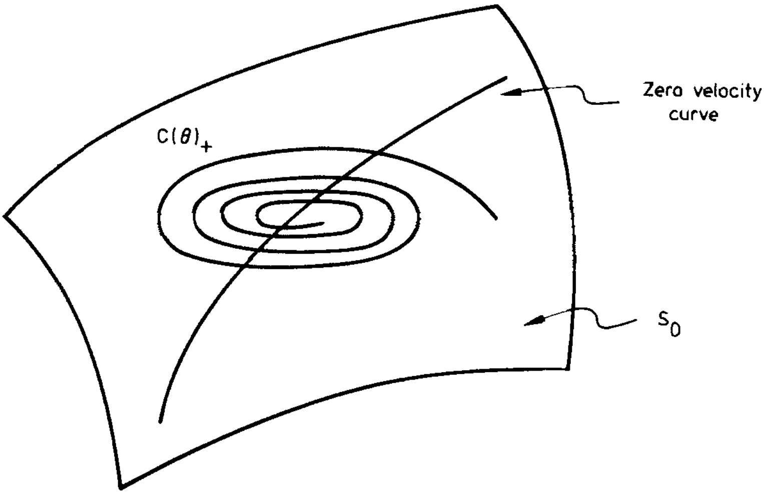

Fig. 8. The intersection of $C(\theta_+)$ and the zero velocity curve in S_0

an orbit on the triple collision manifold which runs up the lower arm with $\theta = -\pi/2$. We will describe non-collision orbits close to γ.

There are three main possibilities for γ in the allowable case:

i. γ runs up the arm with $v>0$, $\theta = -\pi/2$.
ii. γ runs up the arm with $v>0$, $\theta = +\pi/2$.
iii. γ dies in the sink.

In each case, nearby orbits behave in dramatically different ways. In case *i*, m_3 leaves a neighborhood of triple collision in the same direction in which it arrived, i.e. travelling down the y-axis. Meanwhile, m_1 and m_2 travel up the y-axis, oscillating in a tight binary system with a large number of double collisions.

In case ii, the situation is exactly opposite: m_3 exists up the y-axis and the binary system travles down the y-axis, again experiencing binary collisions.

Case iii is perhaps the most interesting; these are the orbits which we call "billiard shots". On the approach to triple collision the "cue-ball" m_3 races up the y-axis while the remaining masses m_1 and m_2 oscillate in a tight binary pair moving down the y-axis. After passing close to triple collision, m_3 is left oscillating rapidly near the center of mass of m_1 and m_2, while these masses fly apart in opposite directions along (roughly) the x-axis.

References

1. Broucke, R.: On the isosceles trinagle configuration in the planar general three body problem. Astron. Astrophys. **73** 303–313 (1979)
2. Devaney, R.: Morse-Smale singularities in simple mechanical systems. To appear in J. Diff. Geometry

3. McGehee, R.: Triple collision in the collinear three body problem. Inventiones Math. **27**, 191–227 (1974)
4. Pollard, H.: *Celestial Mechanics*, The Carus Mathematical Monographs, No. 18. Mathematical Association of America, 1976
5. Saari, D.: Singularities and collisions of Newtonian gravitational systems. Arch. Rational Mech. Anal. **49**, 311–320 (1973)
6. Saari, D.: Improbability of collisions in Newtonian gravitational systems, I and II, Trans. Amer. Math. Soc. **162**, 267–271 (1971) and **181**, 351–368 (1973)
7. Samarov, K.L.: Regularization of the isosceles three-body problem for a non-zero area vector. Soviet Math. Dokl. **18**, 345–248 (1977)
8. Siegel, C., Moser, J.: Lectures on Celestial Mechanics. Berlin: Springer-Verlag, 1971
9. Simo, C.: Regularization of triple collision in the general three body problem. In: Instabilities in Dynamical Systems, (V. Szebehely, ed.), pp. 301–307. Dordrecht: D. Reidel, 1979
10. Simo, C.: Masses for which triple collision is regularizable. To appear in the Proceedings of Mathematical Methods in Celestial Mechanics, Oberwolfach, 1978
11. Sundman, K.: Memoire sur le probleme des trois corps. Acta. Math. **36**, 105–179 (1912)
12. Sundman, K.: Nouvelles recherches sur le probleme des trois corps. Acta. Soc. Sci. Fenn. **35**, No. 9 (1909)
13. Waldvogel, J.: Stable and unstable manifolds in planar triple collision. In: Instabilities in Dynamical Systems, (V. Szebehely, ed.), pp. 263–271. Dordrecht: D. Reidel 1979
14. Wintner, A.: The Analytical Foundations of Celestial Mechanics. Princeton University Press, 1941
15. Moeckel, R.: Thesis, University of Wisconsin, Madison
16. Simo, C.: Analysis of Triple Collision in the Isosceles Problem. In: Classical Mechanics and Dynamical Systems. New York: Marcel Dekker 1980

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Note added in proof

Since this paper was written C. Simo has studied the behavior of the invariant manifolds on the triple collision manifold. He finds that there are, in fact, more possibilities than our allowable case [16].