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Relations between K_2 and Galois Cohomology

John Tate (Cambridge, Mass.)

To Jean-Pierre Serre

§ 1. Introduction

In this paper we establish a natural isomorphism, for a global field F, between K_2F and the quotient of the Galois cohomology group $H^1(F, (\mathbf{Q}/\mathbf{Z})(2))$ by its maximal divisible subgroup. This isomorphism, first conjectured by Lichtenbaum, is not new; indeed it and some of its consequences have already been used by several authors [4–6, 9, 10, 14, 15] in studying K_2 of global fields. But so far only a sketch of a proof has been published, in [25] together with [24]. Here we give the details of that proof, in a slightly simpler arrangement in which no use is made of Iwasawa's theory of $\mathbf{Z}_l$-extensions.

The organization of the paper is as follows. In § 2, we review some general facts about the continuous cochain cohomology of groups, especially with l-adic coefficient modules. The facts are quite well known and elementary but are basic for the sequel. Therefore we include a summary of them for the convenience of the reader.

In § 3 we construct, for any field F, and any prime $l \neq \operatorname{char} F$, a homomorphism h from K_2F to $H^2(F, \mathbf{Z}_l(2))$. The construction of h depends on the description of K_2F by symbols, i.e., on Matsumoto's theorem. We show that if a certain auxiliary homomorphism h_1 is injective, then $\operatorname{Ker} h$ is l-divisible and $\operatorname{Coker} h$ has no l-torsion. It follows that if the field F satisfies the two conditions (a) that K_2F is a torsion group with no non-zero divisible subgroup, and (b) that h_1 is injective for F, then h induces an isomorphism from the l-primary part of K_2F to the torsion subgroup of $H^2(F, \mathbf{Z}_l(2))$. Condition (a) is known to hold for a global field F; the aim of §§ 4 and 5 is to show that condition (b) does also.

In § 4 we give a criterion for h_1 to be injective for a field F. To show injectivity one can assume F contains the l-th roots of unity, in which case $h_1: K_2F/lK_2F \to \operatorname{Br}_l F$ is a map given by the theory of cyclic algebras. ($\operatorname{Br}_l F$ is the group of elements of order dividing l in the Brauer group of F.) It is an open question whether h_1 is injective for every field. We show that the injectivity of h_1 is equivalent to the kernel of $F^\cdot \otimes F^\cdot \to \operatorname{Br}_l F$ being generated by the elements of the form $a \otimes b$ contained in it. (Incidentally, the question whether h_1 is surjective is the classical one of whether an element of order l in the Brauer group is a product of cyclic algebras.)

In § 5 we show, using class field theory, that our criterion for h_1 to be injective is satisfied for a global field F, and the basic isomorphism between K_2F and Galois cohomology (Theorem (5.4)) follows as indicated above.

In § 6 we derive some consequences. For a global F containing a primitive l-th root of unity z we show that every element of order l in K_2F is of the form $\{z, a\}$ for some $a \in F^{\cdot}$, and that if A denotes the group of $a \in F^{\cdot}$ such that $\{z, a\} = 1$, then $A/(F^{\cdot})^l$ is of order l^{r_2+1} where r_2 is the number of complex places of F. We also establish the isomorphism mentioned at the beginning of this introduction and in addition show that the maximal divisible subgroup of $H^1(F, (\mathbf{Q}/\mathbf{Z})(2))$ is isomorphic to $((\mathbf{Q}/\mathbf{Z})')^{r_2}$ (where the $'$ means omit the p-primary component if F is a function field of characteristic p).

This last result, whose statement has nothing to do with K_2, is equivalent to the fact (Theorem (6.5)) that $H^1(F, \mathbf{Z}_l(2))$ is a $\mathbf{Z}_l$-module of rank r_2. In the function field case this can easily be proved directly using the fact that q^2 is not an eigenvalue for Frobenius acting on the Jacobian. (By Weil's theorem those eigenvalues have absolute value $q^{1/2}$.) In the number field case this result is equivalent (as indicated in [25]) to the fact that $\mathbf{Z}_l(2)$ does not occur as a submodule of Iwasawa's module $X = \mathrm{Gal}(M/K)$ (cf. [13]). In the number field case I know of no proof of this which does not go back via theorems of Matsumoto and Bass to Garland's proof that $H^2(SL_n(O_F), \mathbf{R}) = 0$ for large n, a proof which involves Riemannian geometry and harmonic forms!

These results are presumably part of a broader picture. Lichtenbaum and Quillen conjecture (cf. [15]) that for $i = 1, 2$ and for $n \geqq 1$ the Galois cohomology group $H^i(F, \mathbf{Z}_l(n))$ is related to $\mathbf{Z}_l \otimes K_{2n-i}F$. On the other hand, one form of Leopoldt's conjecture is that $H^1(F, \mathbf{Z}_l)$ be isomorphic to $\mathbf{Z}_l^{1+r_2}$ (cf. [13], § 2).

§ 2. Continuous Cochain Cohomology

In this section we review the basic definitions and properties of the cohomology theory which is used in the sequel. Let G be a topological group and M a topological G-module. For each integer $n \geqq 0$ we denote by $C^n(G, M)$ the group of *continuous* maps of the n-fold product G^n into M. One defines homomorphisms

$$d_n \colon C^n(G, M) \to C^{n+1}(G, M)$$

as usual, by

$$(d_0 f)(s) = sf(\cdot) - f(\cdot) \quad \text{(where } \cdot \text{ is the unique element of } G^0\text{)},$$

$$(d_1 f)(s, t) = sf(t) - f(st) + f(s),$$

$$(d_2 f)(s, t, u) = sf(t, u) - f(st, u) + f(s, tu) - f(s, t), \quad \text{etc.}$$

In this way one gets a complex $C^{\cdot}(G, M)$ whose cohomology groups are denoted by $H^n(G, M)$. The group $H^0(G, M)$ can be identified with M^G via the map $f \mapsto f(\cdot)$.

Exact Cohomology Sequence. Suppose

$$0 \to L \to M \to N \to 0$$

is an exact sequence of topological G-modules such that the topology of L is induced by that of M, and such that the map $M \to N$ has a continuous section, this section being just a map, not necessarily a homomorphism. Then the resulting sequence of complexes

$$0 \to C^{\cdot}(G, L) \to C^{\cdot}(G, M) \to C^{\cdot}(G, N) \to 0$$

is exact, and we obtain an exact cohomology sequence

$$\cdots \to H^r(G, M) \to H^r(G, N) \xrightarrow{\delta} H^{r+1}(G, L) \to H^{r+1}(G, M) \to \cdots.$$

Note that this sequence is at our disposal in particular in case L is an *open* submodule of M and $N = M/L$ is the quotient module, with the quotient topology, which is discrete.

Cup Products. Suppose

$$B\colon M_1 \times M_2 \to M$$

is a continuous G-pairing, i.e., a continuous biadditive map such that $(sx_1) \cdot (sx_2) = s(x_1 \cdot x_2)$, for $x_i \in M_i$, $s \in G$, where $x_1 \cdot x_2$ denotes $B(x_1, x_2)$. Then B induces biadditive maps

$$C^m(G, M_1) \times C^n(G, M_2) \to C^{m+n}(G, M)$$

via the well-known formulas

$$(f \cup g)(s_1, \ldots, s_{m+n}) = f(s_1, \ldots, s_m) \cdot s_1 \cdots s_m g(s_{m+1}, \ldots, s_{m+n}).$$

This "cup-product" of cochains satisfies the identity

$$\delta(f \cup g) = (\delta f) \cup g + (-1)^m f \cup (\delta g)$$

and consequently induces pairings

$$H^m(G, M_1) \times H^n(G, M_2) \to H^{m+n}(G, M).$$

Restriction and Transfer. Let H be a subgroup of G and M a topological G-module. Restriction of cochains from G to H gives a homomorphism of complexes $C^{\cdot}(G, M) \to C^{\cdot}(H, M)$ which induces "restriction" homomorphisms

$$\text{res}\colon H^n(G, M) \to H^n(H, M).$$

Restriction commutes with the maps in exact cohomology sequences, and with the cup products. In dimension 0, it is the inclusion $M^G \hookrightarrow M^H$.

If H is open and of finite index in G, as we now suppose, then there is a "transfer" homomorphism going in the direction opposite to restriction. Let R be a set of representatives of the right cosets of H in G, so that $G = \bigcup_{r \in R} Hr$, disjoint union. With the aid of R we define a homomorphism of complexes $t_R\colon C^{\cdot}(H, M) \to C^{\cdot}(G, M)$ as follows:

$$(t_R f)(s_1, \ldots, s_n) = \sum_{r \in R} r_0^{-1} f(r_0 s_1 r_1^{-1}, r_1 s_2 r_2^{-1}, \ldots, r_{n-1} s_n r_n^{-1}),$$

where the $r_i \in R$ are determined inductively as functions of r and the given n-tuple $(s_1, \ldots, s_n) \in G^n$ by

$$r_0 = r, \quad \text{and} \quad r_i \in H r_{i-1} s_i, \qquad 1 \leqq i \leqq r.$$

The homotopy class of t_R is independent of R, and so also are the resulting "transfer" homomorphisms

$$\mathrm{tr}: H^n(H, M) \to H^n(G, M).$$

In dimension 0 the transfer is the trace: $M^H \to M^G$ which carries $x \in M^H$ into $\sum_{r \in R} r^{-1} x$. The transfer maps commute with the maps in exact cohomology sequences. In relation to the cup-product they satisfy

$$\mathrm{tr}(x \cup \mathrm{res}\, y) = (\mathrm{tr}\, x) \cup y, \qquad \text{for } x \in H^m(H, M_1), \qquad y \in H^n(G, M_2).$$

This can be verified by very tedious direct computation. (For our application the cases $m, n \leqq 1$ will suffice.)

l-Adic Cohomology. Let l be a prime number, $\mathbf{Z}_l$ the ring of l-adic integers and T a topological G-module which, as topological group, is a finitely generated $\mathbf{Z}_l$-module with the natural topology, and on which G operates $\mathbf{Z}_l$-linearly.

(2.1) **Proposition.** *Let Y be a finitely generated $\mathbf{Z}_l$-submodule of $H^n(G, T)$. The quotient group $H^n(G, T)/Y$ contains no non-zero subgroup which is l-divisible.* (A group Z is said to be *l-divisible* if $Z = lZ$.)

Suppose $x_i \in H^n(G, T)$, $0 \leqq i < \infty$, such that $x_i \equiv l x_{i+1} \pmod{Y}$ for all i. We must show $x_0 \in Y$. Let y_j, $1 \leqq j \leqq m$, be a finite set of generators for Y. For each i, let f_i be an n-cocycle representing x_i, and for each j, let g_j be an n-cocycle representing y_j. Then there are $(n-1)$-cochains h_i and elements $a_{ij} \in \mathbf{Z}_l$ such that

$$f_i = l f_{i+1} + \sum_{j=1}^{m} a_{ij} g_j + d h_i$$

for each $i \geqq 0$. Multiplying this equation by l^i and summing over i gives

$$\sum_{i \geqq 0} l^i f_i = \sum_{i \geqq 1} l^i f_i + \sum_{i \geqq 0} \sum_{j=1}^{m} l^i a_{ij} + \sum_{i \geqq 0} l^i d h_i$$

or

$$f_0 = \sum_{j=1}^{m} a_j g_j + d h$$

with $a_j = \sum_{i \geqq 0} l^i a_{ij}$ and $h = \sum_{i \geqq 0} l^i h_i$. The use of infinite sums here is formally justified by the fact that T is the inverse limit of its quotient $T/l^i T$ and consequently

$$C^n(G, T) = \varprojlim_i C^n(G, T/l^i T)$$

is a projective limit of modules, each of which is killed by a fixed power of l.

Corollary. *The $\mathbf{Z}_l$-module $H^n(G, T)$ is finitely generated if and only if $H^n(G, T)/l H^n(G, T)$ is finite.*

Indeed, if the quotient H^n/lH^n is finite there is a finitely generated submodule Y such that $H^n = lH^n + Y$, i.e., such that H^n/Y is l-divisible. Then $H^n = Y$ by the proposition just proved.

(2.2) **Proposition.** *For each $n>0$ there is an exact sequence*

$$0 \to \varprojlim_i{}^1 H^{n-1}(G, T/l^i T) \to H^n(G,T) \to \varprojlim_i H^n(G, T/l^i T) \to 0.$$

Corollary. *If the groups $H^{n-1}(G, T/l^i T)$ are finite for each i, or if the maps $H^{n-1}(G, T/l^{i+1} T) \to H^{n-1}(G, T/l^i T)$ are surjective for each i, then we have isomorphisms $H^n(G,T) \xrightarrow{\sim} \varprojlim_i H^n(G, T/l^i T)$.*

Let $u_i \colon C^n(G, T/l^{i+1} T) \to C^n(G, T/l^i T)$ be the canonical map. Since $T/l^i T$ is discrete, this map is surjective for each i. It follows easily that the sequence

$$(*) \quad 0 \to C^n(G,T) \to \prod_{i \geqq 1} C^n(G, T/l^i T) \xrightarrow{1-u} \prod_{i \geqq 1} C^n(G, T/l^i T) \to 0$$

is exact, where the map $1-u$ is defined by

$$((1-u)f)_i = f_i - u_i f_{i+1}$$

for $f = (f_i) \in \prod_{i \geqq 1} C^n(G, T/l^i T)$. The proposition follows immediately from the long exact cohomology sequence associated to the short exact sequence of complexes $(*)$.

Suppose now that T is torsion free. Tensoring it, over $\mathbf{Z}_l$, with the exact sequence $0 \to \mathbf{Z}_l \to \mathbf{Q}_l \to \mathbf{Q}_l/\mathbf{Z}_l \to 0$ gives an exact sequence

$$(**) \quad 0 \to T \to V \to W \to 0$$

in which V is a finite-dimensional vector space over $\mathbf{Q}_l$, T is an open compact subgroup, and W a discrete divisible l-primary torsion group.

(2.3) **Proposition.** *Suppose G is compact. Then, in the exact cohomology sequence associated with $(**)$ the kernel of the connecting homomorphism*

$$\delta \colon H^{n-1}(G, W) \to H^n(G, T)$$

is the maximal divisible subgroup of $H^{n-1}(G, W)$, and its image is the torsion subgroup of $H^n(G, T)$.

Since V is a vector space over $\mathbf{Q}_l$, so is $H^{n-1}(G, V)$, and its image, $\operatorname{Ker}\delta$, is therefore divisible. On the other hand, by Proposition (2.1), each divisible subgroup of $H^{n-1}(G, W)$ must be in $\operatorname{Ker}\delta$. Since G is compact and W discrete, a cochain $f \colon G^{n-1} \to W$ takes on only a finite set of values, and since W is a torsion group, it follows that $H^{n-1}(G, W)$ is a torsion group. Thus the image of δ is a torsion subgroup of $H^n(G, T)$. On the other hand, the image of δ is the kernel of a map to the $\mathbf{Q}_l$-vector space $H^n(G, V)$ which is torsion free, and consequently all the torsion in $H^n(G, T)$ must be in $\operatorname{Im}\delta$.

§ 3. The l-Adic Symbol

Let F be a field, F_s a separable algebraic closure of F, and $G_F = \operatorname{Gal}(F_s/F)$ the Galois group of F_s over F. For each integer $m \geqq 1$ prime to the characteristic of F, let μ_m denote the group of m-th roots of unity in F_s. For any field E, let $E^{\bullet}$ denote the multiplicative group of E.

Let l be a prime different from the characteristic of F. For each integer $i \geqq 0$ we have a commutative diagram of discrete G_F-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mu_{l^i} & \longrightarrow & F_s^{\bullet} & \xrightarrow{l^i} & F_s^{\bullet} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle ''l''} & & \uparrow{\scriptstyle l} & & \uparrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mu_{l^{i+1}} & \longrightarrow & F_s^{\bullet} & \xrightarrow{l^{i+1}} & F_s^{\bullet} & \longrightarrow & 0
\end{array}$$

in which the rows are exact. Passing to the inverse limit over i we obtain a sequence

$$0 \to \mathbf{Z}_l(1) \to \varprojlim F_s^{\bullet} \to F_s^{\bullet} \to 0.$$

Since the maps $\mu_{l^{i+1}} \to \mu_{l^i}$ are surjective this last sequence is exact, and $\mathbf{Z}_l(1) = \varprojlim (\mu_{l^i})$ is a free $\mathbf{Z}_l$-module of rank 1. For integers $m \in \mathbf{Z}$ we define G_F-modules $\mathbf{Z}_l(m)$ inductively by

$$\mathbf{Z}_l(0) = \mathbf{Z}_l, \quad \mathbf{Z}_l(m+1) = \mathbf{Z}_l(m) \underset{\mathbf{Z}_l}{\otimes} \mathbf{Z}_l(1) \quad \text{for } m \geqq 0,$$

and

$$\mathbf{Z}_l(m-1) = \mathrm{Hom}(\mathbf{Z}_l(1), \mathbf{Z}_l(m)), \quad \text{for } m \leqq 0.$$

For any $(\mathbf{Z}_l, G_F)$-module M and any integer $m \in \mathbf{Z}$ we put $M(m) = M \underset{\mathbf{Z}_l}{\otimes} \mathbf{Z}_l(m)$. There are canonical isomorphisms $M(m)(n) \simeq M(m+n)$.

As is often done in writing about Galois cohomology, we will write $H^r(F, M)$ instead of $H^r(G_F, M)$ to denote the continuous cochain cohomology groups of G_F with coefficients in M. Since $F_s^{\bullet}$ is discrete, the exact sequence above gives rise to an associated cohomology sequence and in particular to a homomorphism

$$d_F: F^{\bullet} = H^0(F, F_s^{\bullet}) \to H^1(F, \mathbf{Z}_l(1)).$$

(3.1) **Theorem.** *There exists a unique homomorphism*

$$h = h_F: K_2 F \to H^2(F, \mathbf{Z}_l(2))$$

such that

$$h(\{a, b\}) = d_F a \cup d_F b$$

for each pair of elements a and b in $F^{\bullet}$.

Let $(a, b)_F = d_F a \cup d_F b$. To prove the theorem we have to show that $(,)_F$ is a "symbol", i.e., is bilinear and satisfies

$$(*) \quad (a, 1-a)_F = 0, \quad \text{if } a \in F^{\bullet}, \quad a \neq 1.$$

The bilinearity is obvious from the definition. To prove $(*)$ we let D_F denote the subgroup of $H^2(F, \mathbf{Z}_l(2))$ which is generated by the elements of the form $\mathrm{Tr}_{E/F}(a, 1-a)_E$, where E is any finite extension of F in F_s, and $a \in E^{\bullet}$, $a \neq 1$, and where $\mathrm{tr}_{E/F}$ denotes the cohomological transfer from G_E to G_F. To prove the theorem it suffices to show $D_F = 0$, and by Proposition 2.1 (with the Y in that proposition equal to 0) it is enough for this to show D_F is l-divisible.

Let $a \in E^\cdot$, $a \neq 1$, where E/F is a finite subextension of F_S/F. Let $X^l - a = \prod_i f_i(X)$ with $f_i(X)$ monic and irreducible in $E[X]$. For each i, let a_i be a root of $f_i(X)$ in F_S and let $E_i = E(a_i)$. Then

$$1 - a = \prod_i f_i(1) = \prod_i N_{E_i/E}(1 - a_i).$$

Hence

$$(a, 1-a)_E = (a, \prod_i N_{E_i/E}(1-a_i))_E = \sum_i (a, N_{E_i/E}(1-a_i))_E.$$

By Lemma 3.2 below we can write this as

$$\sum_i \mathrm{tr}_{E_i/E}(a, 1-a_i)_{E_i} = \sum_i \mathrm{tr}_{E_i/E}(a_i^l, 1-a_i)_{E_i} = \sum_i l\, \mathrm{tr}_{E_i/E}(a_i, 1-a)_{E_i}.$$

Finally, applying $\mathrm{tr}_{E/F}$ and using the transitivity of the transfer we find that an arbitrary generator $\mathrm{tr}_{E/F}(a, 1-a)_E$ of D_F is in lD_F. Thus D_F is l-divisible as claimed.

In the proof just finished we used for the extensions E_i/E the following lemma:

(3.2) **Lemma.** *Let E/F be a finite subextension of F_S/F. Let $a \in F^\cdot$ and $b \in E^\cdot$. Then* $\mathrm{tr}_{E/F}(a, b)_E = (a, N_{E/F}\, b)_F$.

We must show

$$\mathrm{tr}_{E/F}(d_E a \cup d_E b) = d_F a \cup d_F N_{E/F} b.$$

This follows from the identity

$$\mathrm{tr}(\mathrm{res}\, \alpha \cup \beta) = \alpha \cup \mathrm{tr}\, \beta,$$

together with the fact that d commutes with restriction and transfer.

Consider now the following diagram, in which $E = F(\mu_l)$, $\Delta = \mathrm{Gal}(E/F)$, and the maps are as explained below:

$$(3.3)\qquad \begin{array}{ccccccccc}
(\mu_l \otimes E^\cdot)^\Delta & \xrightarrow{\gamma} & K_2 F & \xrightarrow{l} & K_2 F & \longrightarrow & K_2 F/lK_2 F & \longrightarrow & 0 \\
\wr\downarrow i & & \downarrow h & & \downarrow h & & \downarrow h_1 & & \\
H^1(F, \mu_l \otimes \mu_l) & \xrightarrow{\delta} & H^2(F, \mathbf{Z}_l(2)) & \xrightarrow{l} & H^2(F, \mathbf{Z}_l(2)) & \longrightarrow & H^2(F, \mu_l \otimes \mu_l) & &
\end{array}$$

The bottom row is part of the exact cohomology sequence associated with the exact sequence

$$0 \to \mathbf{Z}_l(2) \xrightarrow{l} \mathbf{Z}_l(2) \to \mu_l \otimes \mu_l \to 0.$$

The top row is not necessarily exact at the left-hand $K_2 F$, but is exact everywhere else. We define the homomorphism γ and the isomorphism i first in the case $E = F$, $\Delta = (1)$, i.e., in the case in which the l-th roots of 1 are in F. In that case, γ is the homomorphism defined by $\gamma(z \otimes a) = \{z, a\}$ for $z \in \mu_l$, $a \in F^\cdot$, and i is defined by $i(z \otimes a) = z \cup d_1 a$, where $d_1 : F^\cdot \to H^1(F, \mu_l)$ is the connecting homomorphism in the cohomology sequence associated with $0 \to \mu_l \to F_s^\cdot \xrightarrow{l} F_s^\cdot \to 0$. Since $H^1(F, F_s^\cdot) = 0$ (Hilbert Theorem 90), the map d_1 induces an isomorphism

$(\mathbf{Z}/l\mathbf{Z})\otimes F^{\cdot}=F^{\cdot}/lF^{\cdot}\xrightarrow{\sim}H^1(F,\mu_l)$, whether or not $\mu_2\subset F$. When $\mu_l\subset F$ it follows that the map i is an isomorphism

$$\mu_l\otimes F^{\cdot}\xrightarrow{\sim}\mu_l\otimes H^1(F,\mu_l)=H^1(F,\mu_l\otimes\mu_l).$$

In the general case γ and i are defined by the commutativity of the diagram

$$\begin{array}{ccccc} H^1(F,\mu_l\otimes\mu_l) & \xleftarrow{\ i\ } & (\mu_l\otimes E^{\cdot})^{\Delta} & \xrightarrow{\ \gamma\ } & (K_2F)_l \\ \downarrow{\scriptstyle\text{res}} & & \downarrow{\scriptstyle\text{incl}} & & \downarrow{\scriptstyle f} \\ H^1(E,\mu_l\otimes\mu_l) & \xleftarrow[\sim]{\ i_E\ } & (\mu_l\otimes E^{\cdot}) & \xrightarrow{\ \gamma_E\ } & (K_2E)_l, \end{array}$$

where γ_E and i_E are as described above ($\mu_l\subset E$). On the right we are using the notation X_l for the kernel of $l\colon X\to X$. This diagram does define maps γ and i, and i is an isomorphism, because the outside vertical arrows give isomorphisms

$$H^1(F,\mu_l\otimes\mu_l)\xrightarrow{\sim}H^1(E,\mu_l\otimes\mu_l)^{\Delta}\quad\text{and}\quad(K_2F)_l\xrightarrow{\sim}(K_2E)_l^{\Delta}.$$

Indeed l is prime to the degree $[E:F]$, and hence the following lemma implies that those maps have kernel and cokernel equal to 0.

(3.4) **Lemma.** *Let L/F be a Galois extension of finite degree n with group G. Then the kernel and cokernel of the functorial maps*

$$K_2F\xrightarrow{\ f\ }(K_2F)^G\quad\text{and}\quad H^i(F,M)\xrightarrow{\ f\ }H^i(L,M)^G$$

(for any integer i and any topological G-module M) are killed by n.

This lemma is an immediate consequence of the existence of a transfer map in the opposite direction to f satisfying the identities

$$\operatorname{tr}(f(x))=nx\quad\text{and}\quad f(\operatorname{tr}(y))=\sum_{s\in G}sy.$$

The cohomological transfer is discussed in §2; for the K_2-transfer see for example [17], §14.

The left hand square in diagram (3.3) is commutative. To prove this we can and do assume $\mu_l\subset F$ since, by Lemma (3.4), the map $H^2(F,\mathbf{Z}_l(2))\to H^2(E,\mathbf{Z}_l(2))$ is injective. Let $z\in\mu_l$ and $a\in F^{\cdot}$. Let ζ and α denote inverse images of z and a in $\varprojlim F_s^{\cdot}$. Then $h(\gamma(z\otimes\alpha))$ is the cohomology class of the cocycle $d\zeta\cup d\alpha$. On the other hand, $l\zeta$ is an element of $\mathbf{Z}_l(1)$ mapping to z under the natural map $\mathbf{Z}_l(1)\to\mu_l$. Consequently $\delta(i(z\otimes\alpha))$ is the class of the cocycle $l^{-1}d(l\zeta\cup d\alpha)=d\zeta\cup d\alpha$, so $\delta i(z\otimes\alpha)=h\gamma(z\otimes\alpha)$ as claimed.

The middle square of (3.3) is obviously commutative, and h_1 is defined so that the right square is commutative. It is easy to check that h_1 is the map which carries the class of $\{a,b\}$ (mod lK_2F) into $d_1a\cup d_1b$, where the connecting homomorphism d_1 is as in the definition of the map i above.

In stating the next theorem we will use the following notation concerning an abelian group A.

$A_{l\text{-div}}=$ the maximal l-divisible subgroup of A.

$A_{l^n}=$ the group of elements $a\in A$ such that $l^na=0$.

$A\{l\}=\bigcup_{n=1}^{\infty}A_{l^n}=l$-primary part of A.

(3.5) **Theorem.** (a) *The kernel of the map h in diagram* (3.3) *contains* $(K_2F)_{l\text{-div}}$, *and h maps $(K_2F)_l$ onto $H^2(F, \mathbf{Z}_l(2))_l$.*

(b) *Suppose the map h_1 in diagram* (3.3) *is injective. Then the kernel of h is $(K_2F)_{l\text{-div}}$ and the cokernel of h is l-torsion free.*

Corollary. *If h_1 is injective then $K_2F\{l\}$ is the direct sum of its maximal divisible subgroup, which is killed by h, and a subgroup which is mapped isomorphically by h onto $H^2(F, \mathbf{Z}_l(2))\{l\}$.*

Part (a) of the theorem results from Proposition (2.1) and easy diagram chasing with (3.3), using the surjectivity of i and $\operatorname{Im}\gamma \subset (K_2F)_l$. Part (b) results on further chasing of (3.3), again using the surjectivity of i.

To derive the corollary from the theorem, note that since Coker h is l-torsion free, the image of h contains $H^2(F, \mathbf{Z}_l(2))\{l\}$. Let A be the inverse image of that group under h. Then h induces an isomorphism $A/A_{l\text{-div}} \xrightarrow{\sim} H^2(F, \mathbf{Z}_l(2))\{l\}$. Since an extension of an l-primary torsion group by an l-divisible group splits, $A_{l\text{-div}}$ is a direct summand of A, and from that the corollary follows immediately.

§ 4. Criteria for the Injectivity of h_1

For which fields F is the map $h_1 = h_1^F$ in diagram (3.3) injective? I know of no field for which h_1 has been shown to be non-injective, so it is possible that the answer is "for all fields". In this section we reduce the question to one on cyclic algebras and give criteria which will enable us to prove injectivity for arithmetic fields.

(4.1) **Lemma.** *Let $E = F(\mu_l)$. If h_1^E is injective (resp. bijective) then h_1^F is injective (resp. bijective).*

Let $\Delta = \operatorname{Gal}(E/F)$. The diagram

$$\begin{array}{ccc} K_2F/lK_2F & \longrightarrow & (K_2E/lK_2E)^{\Delta} \\ \Big\downarrow{\scriptstyle h_1^F} & & \Big\downarrow{\scriptstyle h_1^E} \\ H^1(G_F, \mu_l \otimes \mu_l) & \longrightarrow & H^1(G_E, \mu_l \otimes \mu_l) \end{array}$$

is obviously commutative. The horizontal arrows can be proved bijective by using the transfer maps in the opposite direction, as in the proof of Lemma (3.4), and Lemma (4.1) follows immediately.

For the rest of this section we suppose $\mu_l \subset F$. Then there are canonical isomorphisms

$$H^2(G_F, \mu_l \otimes \mu_l) \approx \mu_l \otimes H^2(G_F, \mu_l) \approx \mu_l \otimes \operatorname{Br}_l F,$$

where $\operatorname{Br}_l F$ denotes the group of elements of order dividing l in the Brauer group $\operatorname{Br} F \approx H^2(G_F, F_s^{\cdot})$. Viewing h_1 as a map into $\mu_l \otimes \operatorname{Br}_l F$, we can describe it in terms of "cyclic algebras". Let z be a primitive l-th root of 1 in F. For $a, b \in F^{\cdot}$ let (a, b) denote the element of $\operatorname{Br}_l F$ represented by the central simple "cyclic" algebra $A_z(a,b)$ over F defined by

$$A_z(a,b)=F[\alpha,\beta]; \quad \alpha^l=a, \quad \beta^l=b, \quad \beta\alpha=z\alpha\beta.$$

Then (with the appropriate sign convention) we have

(4.2) $h_1(\{a,b\})=z\otimes(a,b)$.

This follows for example from a cohomology computation carried out by Serre in [21, Ch. XIV, § 12]; see also Weil [26, Ch. IX, §§ 3, 4, 5], and Milnor [17, § 15]. Our notation $A_z(a,b)$ is that of Milnor.

(4.3) **Proposition.** *For $a,b\in F^{\cdot}$ the following statements are equivalent:*
 (i) $\{a,b\}\in lK_2F$,
 (ii) $h_1(\{a,b\})=0$, (ii)′ $(a,b)=0$,
 (iii) *b is a norm from the extension $F(a^{1/l})$.*

It is trivial that (i) implies (ii). The equivalence of (ii) and (ii)′ follows from (4.2). The equivalence of (ii)′ and (iii) is well known (see Milnor (loc. cit.) Theorem 15.7 or Serre (loc. cit.) Proposition 4 (iii)). To prove (iii) ⇒ (i), let tr (resp. N) denote the K_2-transfer (resp. the field-theoretic norm) from $F(a^{1/l})$ to F. Suppose $\beta\in F(a^{1/l})$ and $N\beta=b$. Then

$$\{a,b\}=\{a,N\beta\}=\mathrm{tr}\{a,\beta\}=l\,\mathrm{tr}\{a^{1/l},\beta\}\in lK_2F$$

as was to be shown.

Let t (resp. u) be the homomorphism of $F^{\cdot}\otimes F^{\cdot}$ onto K_2F/lK_2F (resp. into $\mathrm{Br}_l F$) induced by the bilinear function $\{a,b\}$ mod lK_2F (resp. (a,b)). Clearly $\mathrm{Ker}\,t$ is generated by the elements of the form $a\otimes b$ with $a+b=1$, and those of the form $a^l\otimes b=l(a\otimes b)$. Hence $\mathrm{Ker}\,t$ is a subgroup of $F^{\cdot}\otimes F^{\cdot}$ which is generated by the *decomposable* elements in it. (By a decomposable element of a tensor product we mean one of the form $a\otimes b$.) By Proposition (4.3), the decomposable elements of $\mathrm{Ker}\,t$ are the same as the decomposable elements of $\mathrm{Ker}\,u$. Since h_1 carries $\{a,b\}$ (mod lK_2) to $z\otimes(a,b)$, the kernel of h_1 is isomorphic to $\mathrm{Ker}\,u/\mathrm{Ker}\,t$. Hence:

(4.4) **Theorem.** *If $\mu_l\subset F$ the kernel of the map h_1 in diagram* (3.3) *is isomorphic to* $\mathrm{Ker}\,u/(\mathrm{Ker}\,u)'$, *where $u\colon F^{\cdot}\otimes F^{\cdot}\to \mathrm{Br}_l F$ is the map given by the cyclic algebra symbol (a,b), and where* $(\mathrm{Ker}\,u)'$ *is the subgroup of* $\mathrm{Ker}\,u$ *generated by the decomposable elements of* $\mathrm{Ker}\,u$. *In particular, h_1 is injective if and only if* $\mathrm{Ker}\,u$ *is generated by its decomposable elements.*

Corollary. *If the cyclic algebra symbol (,) satisfies the following two conditions, then h_1 is injective:*

(i) *Given $a,b,c,d\in F^{\cdot}$ such that $(a,b)=(c,d)$, there exist elements $x,y\in F^{\cdot}$ such that*

$$(a,b)=(x,b)=(x,y)=(c,y)=(c,d).$$

(ii) *Given a_1, a_2, b_1, $b_2\in F^{\cdot}$ there exist elements c_1, c_2 and d in $F^{\cdot}$ such that $(a_1,b_1)=(c_1,d)$ and $(a_2,b_2)=(c_2,d)$.*

Indeed suppose these conditions are satisfied. By (i), if $(a,b)=(c,d)$ then

$$a\otimes b - c\otimes d = \left(\frac{a}{x}\otimes b\right) + \left(x\otimes \frac{b}{y}\right) + \left(\frac{x}{c}\otimes y\right) + \left(c\otimes \frac{y}{d}\right)$$

is in $(\operatorname{Ker} u)'$, because $(a,b)=(x,b)$ implies that the decomposable element $(a/x)\otimes b$ is in $\operatorname{Ker} u$, hence in $(\operatorname{Ker} u)'$, and similarly for $x\otimes\left(\frac{b}{y}\right)$, etc. Hence, given (i) and (ii), any sum $(a_1\otimes b_1)+(a_2\otimes b_2)$ of two decomposable elements in $F^\cdot\otimes F^\cdot$ is congruent $\operatorname{mod}(\operatorname{Ker} u)'$ to one decomposable element $(c_1 c_2\otimes d)$. By induction on n it follows that any element $x=\sum_{i=1}^{n}(a_i\otimes b_i)$ in $F^\cdot\otimes F^\cdot$ is congruent $\operatorname{mod}(\operatorname{Ker} u)'$ to a decomposable element x_0. If $x\in\operatorname{Ker} u$, then $x_0\in\operatorname{Ker} u$ and, being decomposable, is in $(\operatorname{Ker} u)'$. Hence $\operatorname{Ker} u=(\operatorname{Ker} u)'$ as was to be shown.

Remarks. 1) condition (ii) just means that any two l-cyclic algebras over F have a common cyclic splitting field $F(d^{1/l})$, because the algebras over F which are split by $F(d^{1/l})$ are, as is well known, exactly those in the class (c,d) for some $c\in F^\cdot$.

2) Condition (i) could obviously be replaced by (i_n): Given $a,b,c,d\in F^\cdot$ there exist $x_i, y_i\in F^\cdot$, $1\leq i\leq n$, such that

$$(a,b)=(x_1,b)=(x_1,y_1)=(x_2,y_1)=(x_2,y_2)=\cdots=(x_n,y_n)=(c,y_n)=(c,d).$$

3) For $l=2$, condition (i) is always satisfied, even with $y=d$. Indeed, let D be the quaternion algebra over F whose class is (a, b) = (c, d). Let D^0 be the three-dimensional subspace of D consisting of the elements with trace 0. This space D^0 carries a non-degenerate symmetric bilinear form $\langle\alpha,\beta\rangle=\frac{1}{2}(\alpha\beta+\beta\alpha)\in F$. To say that the class of D is (a,b) just means that there are elements $\alpha,\beta\in D^0$ such that $\alpha\perp\beta$ (i.e., $\langle\alpha,\beta\rangle=0$) and such that $\alpha^2=a$ and $\beta^2=b$. Let α and β be such elements, and let γ and δ be such that $\gamma\perp\delta$ and $\gamma^2=c$, $\delta^2=d$. Since D^0 is three-dimensional there exists $\xi\neq 0$ in D^0 such that $\xi\perp\beta$ and $\xi\perp\delta$. Put $x=\xi^2$. Then the class of D is $(a,b)=(x,b)=(x,d)=(c,d)$.

4) In [11], Elman and Lam establish, in case $l=2$, some remarkable results on the injectivity of our h_1^F (their g_F). For example, they show that h_1 is injective if every element of $K_2F/2K_2F$ is a sum of five generators $\{a_i,b_i\}$. The case $l=2$ seems somewhat exceptional. Milnor [16] has shown that in that case $K_2F/2K_2F \approx I_F^2/I_F^3$, where I_F is the augmentation ideal of the Witt ring of quadratic forms over F.

(4.5) **Proposition.** *If* $\operatorname{Br}_l F$ *is cyclic, then the conditions* (i) *and* (ii) *of the corollary to Theorem* (4.4) *are satisfied, and* h_1 *is injective.*

For condition (ii) this is obvious. For condition (i) there is no problem if $\operatorname{Br}_l F=0$, so we may suppose $\operatorname{Br}_l F\approx \mathbf{Z}/l\mathbf{Z}$ and view $(\,,\,)$ as a bilinear form on the vector space $F^\cdot/(F^\cdot)^l$ over $\mathbf{Z}/l\mathbf{Z}$. We are given $(a,b)=(c,d)\in\operatorname{Br}_l F$ and must find $x,y\in F^\cdot$ such that $(a,b)=(x,b)=(x,y)=(c,y)=(c,d)$. Call $(a,b)=\alpha=(c,d)$.

If $\alpha=0$ we can take $x=y=1$. Suppose $\alpha\neq 0$. Then the forms $x\mapsto(x,b)$ and $x\mapsto(x,d)$ are non-zero. If these forms are *either* linearly independent of each other *or* equal, then we can take $y=d$, and find an x such that $(x,b)=\alpha$ and $(x,y)=(x,d)=\alpha$. Suppose now that those two non-zero linear forms are dependent but unequal. Then $l\neq 2$ (in harmony with Remark 3) above), so the bilinear form $(\,,)$ is alternating (any e in $F^{\cdot}$ is either an l-th power or is the norm of $e^{1/l}$ from the extension $F(e^{1/l})$; hence $(e,e)=0$). The forms $x\mapsto(x,c)$ and $x\mapsto(x,d)$ are independent of each other because $(c,d)\neq 0$. Take $y=cd$. Then $(c,y)=(c,d)=\alpha$, and the linear forms $x\mapsto(x,b)$ and $x\mapsto(x,y)=(x,cd)$ are linearly independent, so we can solve $(x,b)=\alpha=(x,y)$ as required.

Examples of fields F with $\mathrm{Br}_l F$ cyclic are locally compact non-discrete fields. Thus h_1 is injective for these. But it is well known to be surjective as well, hence it is bijective. By Lemma (4.1) the same is true if we drop the assumption $\mu_l\subset F$. Therefore

Corollary. *The map h_1 of diagram* (3.3) *is bijective for any locally compact non-discrete field F.*

Remark. This corollary also follows from results of Moore; see [18] or Milnor [17], appendix.

§ 5. The Main Theorem for Global Fields

In this section we suppose F is a global field, i.e., an **A**-field in the sense of [26]. For each place v of F we let F_v denote the completion of F at v. For $\alpha\in\mathrm{Br}\,F$ let α_v denote the image of α in $\mathrm{Br}\,F_v$.

(5.1) **Theorem.** *For a global field F the map h_1 in diagram* (3.3) *is bijective.*

By Lemma (4.1) we may assume F contains a primitive l-th root of unity z, and we can then view h_1 as a mapping to $\mathrm{Br}_l F$ as in § 4. The surjectivity of h_1 is well known; it follows from the fact that any element $\alpha\in\mathrm{Br}_l F$ has a cyclic splitting field of degree l. In fact more is true – for any finite set of elements $\alpha_i\in\mathrm{Br}_l F$, $1\leq i\leq n$, there is a common splitting field $F(d^{1/l})$. (One has only to choose d so that $F_v(d^{1/l})$ splits $(\alpha_i)_v$ for all i and all places v of F, and for this it suffices to arrange that d is not an l-th power in F_v for each of the (finite) set of places v such that $(\alpha_i)_v\neq 0$ for some i.) Then each α_i is of the form (c_i,d). In particular, condition (ii) of the corollary to Theorem (4.4) is satisfied. To show h_1 injective by that corollary and thereby complete the proof of Theorem (5.1) we will now show condition (i) is satisfied. We are given $(a,b)=(c,d)\in\mathrm{Br}_l F$ and must find $x,y\in F$ such that $(a,b)=(x,b)=(x,y)=(c,y)=(c,d)$. Let $\alpha=(a,b)=(c,d)$, and let S be the set of places v of F where $\alpha_v\neq 0$. For each $v\in S$, choose, by Proposition (4.5), elements $x_v, y_v\in F_v$ such that $(a,b)_v=(x_v,b)_v=(x_v,y_v)_v=(c,y_v)_v=(c,d)_v$. From the last of these equalities, y_v/d is a norm from $F_v(c^{1/l})$ to F_v; say $y_v/d=N_v t_v$, with $t_v\in F_v(c^{1/l})$. Choose $t\in F(c^{1/l})$ such that t/t_v is a local l-th power at v for $v\in S$. Put $y=dNt$, where N denotes norm from $F(c^{1/l})$ to F. Then $(c,y)=(c,d)$, and $(c,y)_v=(c,y_v)_v$ for $v\in S$. We must now find x in $F^{\cdot}$ such that $(x,b)=\alpha$ and $(x,y)=\alpha$. By the construction of y, these two equations have a local solution x_v at the places

$v \in S$; and for $v \notin S$ they have the trivial solution $x_v = 1$. Thus our proof of Theorem (5.1) is finished by the following more-or-less well-known lemma, a proof of which we include for the convenience of the reader. A different proof in a special case can be found in Serre [23], Chapter III, § 2, Theorem 4; I was first made aware of the result through an early mimeographed version of that book.

(5.2) **Lemma.** *Let $\alpha_1, \ldots, \alpha_r$ be a finite family of elements of* $\mathrm{Br}_l F$. *Let $a_1, \ldots, a_r \in F^\cdot$. If there exists for each place v of F an element $x_v \in F_v^\cdot$ such that $(a_i, x_v) = (\alpha_i)_v$ for each i, then there exists an element $x \in F^\cdot$ such that $(a_i, x) = \alpha_i$ for each i.*

For each place v, let A_v denote the subspace of $F_v^\cdot / F_v^{\cdot l}$ generated by the images $a_{i,v}$ of the a_i. The existence of x_v satisfying $(a_i, x_v) = (\alpha_i)_v$ implies the existence of a character χ_v of A_v such that $\chi_v(a_{i,v}) = \mathrm{inv}_v(\alpha_i)_v$ for each i, where $\mathrm{inv}_v \colon \mathrm{Br}\, F_v \to \mathbf{Q}/\mathbf{Z}$ is the "invariant" map. Let χ_v be such a character; we have $\chi_v = 0$ for almost all v, so that $\chi = \sum_v \chi_v$ is a character of the product $A = \prod_v A_v$. Now view A as a compact subgroup of J/J^l, where J is the idele group of F. The interrelationships of global and local class field theory and Kummer theory are expressed by the facts that J/J^l is its own Pontrjagin dual with respect to the pairing $\langle \xi, \eta \rangle = \sum_v \mathrm{inv}_v(\xi_v, \eta_v)_v$, and that $F^\cdot / F^{\cdot l}$ is a discrete subgroup of J/J^l which is its own exact orthogonal with respect to that pairing. Hence, by duality, to find $x \in F^\cdot$ such that $(a_i, x) = \alpha_i$ for each i, is the same as to find a character ψ of J/J^l which is trivial on $F^\cdot / F^{\cdot l}$ and whose restriction to A_v is χ_v for each v, i.e., whose restriction to A is χ. Such a character ψ exists if and only if χ is trivial on $A \cap (F^\cdot / F^{\cdot l})$. Now by construction we have $\chi(a_i) = 0$ for each i (because $\sum_v \mathrm{inv}_v(\alpha_i)_v = 0$). Thus the crux of the matter is to show that $A \cap (F^\cdot / F^{\cdot l})$ is spanned by the a_i. Let $a \in F^\cdot$ be such that $a_v \in A_v$ for each v. Then in the field $F(\ldots, a_i^{1/l}, \ldots)$, a is an l-th power locally everywhere, and hence is an l-th power globally. By Kummer theory it follows that a is dependent on the $a_i \bmod (F^\cdot)^l$. This completes the proof of the lemma.

Let S be a finite non-empty set of place of F including the archimedean ones. Let O_S denote the ring of S-integers in F, i.e., the ring of all $a \in F$ such that $v(a) \geqq 0$ for each place $v \notin S$. For each non-archimedean place v of F let $k(v)$ denote the residue field of v, and let $d_v \colon K_2 F \to k(v)^\cdot$ be the homomorphism given by the "tame symbol" at v (cf. e.g. [17], p. 98). The maps d_v for $v \notin S$ taken together give the map d^S in the following exact sequence

(5.3) $$0 \to K_2 O_S \to K_2 F \xrightarrow{d^S} \coprod_{v \notin S} k(v)^\cdot \to 0,$$

where denotes direct sum. The map d^S is surjective by a theorem of Moore [18] (see also [17], and [8]). Bass [1] showed that $\mathrm{Ker}\, d^S$ is the image of $K_2 O_S$, enabling Garland [12] to prove the finiteness of $\mathrm{Ker}\, d^S$ in the number field case. In the function field case $\mathrm{Ker}\, d^S$ was shown finite of order prime to $p = \mathrm{char}\, F$ by Bass-Tate [3]. Quillen [19] obtained (5.3) as a part of an infinite exact sequence of "localization" which showed (because $K_2(k(v)) = 0$) that $K_2 O_S$ is isomorphic to $\mathrm{Ker}\, d^S$. In what follows we don't make any use of Quillen's theorem other than to interpret results about $\mathrm{Ker}\, d^S$ as results about $K_2 O_S$; the reader who is unfamiliar with it can for the purposes of the present discussion interpret K_2O_S as a notation for $\operatorname{Ker} d^S$.

What is essential for us is the finiteness of $\operatorname{Ker} d^S$. It shows that K_2F is an extension (5.3) of a sum of finite cyclic groups by a finite group. Hence *K_2F is a torsion group with no non-zero divisible subgroup.* Putting this fact together with Theorem (5.1) and Theorem (3.5) (or its corollary) we find

(5.4) **Theorem.** *For a global field F the map h of Theorem* (3.1) *induces an isomorphism from the l-primary part of K_2F onto the torsion subgroup of $H^2(F, \mathbf{Z}_l(2))$.*

Since K_2F is the sum of its l-primary parts for $l \neq \operatorname{char} F$ this theorem gives a cohomological description of K_2F.

§ 6. Applications

Throughout this section F is a global field.

(6.1) **Theorem.** *The top row of diagram* (3.3) *is exact, i.e., the image of the map γ in diagram* (3.3) *is $(K_2F)_l$. In particular, if F contains a primitive l-th root of unity z, then every element of order l in K_2F is of the form $\{z, a\}$ for some $a \in F^{\bullet}$.*

This follows immediately from the injectivity of h, the surjectivity of i, and the exactness of the bottom row of the commutative diagram (3.3).

(6.2) **Theorem.** *Let S be a finite non-empty set of places of F containing the archimedean ones and the ones above l in the number field case. Let S_c denote the set of complex places of F. Suppose $\mu_l \subset F$. Then there is a natural exact sequence*

$$0 \to \mu_l \otimes \operatorname{Pic} O_S \to K_2O_S/lK_2O_S \xrightarrow{h_l^S} \Big(\coprod_{v \in S - S_c} \mu_l\Big)_0 \to 0,$$

where $(\coprod \mu_l)_0$ denotes the subgroup of the direct sum consisting of the elements $z = (z_v)$ such that $\sum z_v = 0$ (writing μ_l additively). The map h_l^S is that induced by the l-th power norm residue symbols for $v \in S - S_c$.

For $v \notin S$ the group $k(v)^{\bullet}$ is finite cyclic and we have canonical isomorphisms $k(v)^{\bullet}/(k(v)^{\bullet})^l \xrightarrow{\sim} (k(v)^{\bullet})_l \xrightarrow{\sim} \mu_l$, the first given by raising to the $(q-1)/l$ power, $q = |k(v)|$, and the second by choosing the root of unity in F representing a residue class of order l. Thus from (5.3) we derive an exact sequence which is the top row in the following commutative diagram

$$\begin{array}{ccccccccc}
(K_2F)_l & \xrightarrow{d^S} & \coprod_{v \notin S} \mu_l & \longrightarrow & K_2O_S/lK_2O_S & \longrightarrow & K_2F/lK_2F & \longrightarrow & \coprod_{v \notin S} \mu_l \longrightarrow 0 \\
\uparrow{\scriptstyle \gamma} & & \| & & & & {\scriptstyle h_l}\downarrow \wr & & \uparrow{\scriptstyle \mathrm{pr}} \\
\mu_l \otimes F^{\bullet} & \longrightarrow & \mu_l \otimes I_S & & & & \mu_l \otimes \operatorname{Br}_l F & \approx & \Big(\coprod_{v \notin S_c} \mu_l\Big)_0
\end{array}$$

In the left hand square, $I_S = \coprod_{v \notin S} \mathbf{Z}$ denotes the group of fractional ideals of O_S. The square commutes because, for $z \in \mu_l$ and $a \in F$, we have by definition of the tame symbol $(d^S\gamma(z \otimes a))_v = d_v\{z, a\} = z^{v(a)}$. Since γ is surjective (Theorem (5.1)), the cokernel of d^S is the same as that of the arrow below it. In view of the exact sequence $F^\cdot \to I_S \to \operatorname{Pic} O_S \to 0$, this cokernel is $\mu_l \otimes \operatorname{Pic} O_S$. On the other hand, using the isomorphism h_1 (Theorem (5.1)) and the theory of the Brauer group to replace K_2F/lK_2F by $(\coprod_{v \notin S_v} \mu_l)_0$, as indicated on the right of the above diagram, we see that the map of K_2O_S into K_2F/lK_2F gets replaced by the map h_1^S given by the l-th power norm residue symbols, and that its image is the kernel of the projection map pr in our diagram. The theorem follows, by the exactness of the horizontal row.

(6.3) **Theorem.** *Let r_2 be the number of complex places of F. Let $\varepsilon=1$ if $H^0(F, \mu_l \otimes \mu_l) \neq 0$, i.e., if $[F(\mu_l): F] \leq 2$, and let $\varepsilon=0$ otherwise. Then the kernel of the map γ in diagram* (3.3) *is an elementary abelian group of order $l^{r_2+\varepsilon}$. In particular, if F contains a primitive l-th root of unity z, and if A is the group of elements $a \in F^\cdot$ such that $\{z, a\}=0$, then $(A: (F^\cdot)^l) = l^{r_2+1}$.*

Suppose first that $\mu_l \subset F$. Let S be a finite set of primes as in Theorem (6.2), and large enough so that $\operatorname{Pic} O_S=0$ ($\operatorname{Pic} O_S$ of order prime to l would suffice). Extending the diagram used in proving Theorem (6.2) to the left we obtain an exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (K_2O_S)_l & \longrightarrow & (K_2F)_l & \longrightarrow & \coprod_{v \notin S} \mu_l & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \gamma^S} & & \uparrow{\scriptstyle \gamma} & & \| & & \\
0 & \longrightarrow & \mu_l \otimes O_S^\cdot & \longrightarrow & \mu_l \otimes F^\cdot & \longrightarrow & \mu_l \otimes I_S & \longrightarrow & 0
\end{array}$$

which shows that $\operatorname{Ker} \gamma = \operatorname{Ker} \gamma^S$, and that γ^S is surjective because γ is (Theorem (6.1)). Hence there is an exact sequence

(*) $0 \to \operatorname{Ker} \gamma \to \mu_l \otimes O_S^\cdot \to (K_2O_S)_l \to 0.$

The theorem for F containing μ_l follows readily, for by the S-unit theorem $\mu_l \otimes O_S^\cdot$ has order l^s, where s is the number of places in S, and on the other hand $(K_2O_S)_l$ has the same order as K_2O_S/lK_2O_S which is l^{s-r_2-1} by Theorem (6.2). However, to be able to treat later the case $\mu_l \not\subset F$ we must refine that argument, working with a Grothendieck group rather than just with group orders, to be able to make a Galois descent.

Suppose G is a group of automorphisms of F and suppose we have chosen an S which is stable under G. Then our diagrams are diagrams of G-modules and G-homomorphisms. Let $\mathfrak{M}_G$ denote the category of finite G-modules and for $M \in \mathfrak{M}_G$, let $[M]$ denote the corresponding element in the Grothendieck group of $\mathfrak{M}_G$.

(6.4) **Lemma.** *In the Grothendieck group of $\mathfrak{M}_G$ we have*

(a) $[\mu_l \otimes O_S^\cdot] = [\mu_l \otimes \mu_l] + [(\coprod_{v \in S} \mu_l)_0]$,

(b) $[(K_2O_S)_l] = [(\coprod_{v \in S - S_c} \mu_l)_0]$,

(c) $[\operatorname{Ker} \gamma] = [\mu_l \otimes \mu_l] + [\coprod_{v \in S_c} \mu_l]$,

where the subscript 0 *has the same notational significance as in Theorem* (6.2).

Let Y denote the quotient of $O_S^{\bullet}$ by its torsion subgroup $\mu(F)$. Since Y is $\mathbf{Z}$-free we have $[\mu_l \otimes O_S^{\bullet}] = [\mu_l \otimes \mu(F)] + [\mu_l \otimes Y]$. Clearly $\mu_l \otimes \mu(F) \approx \mu_l \otimes \mu_l$. By the non-vanishing of the regulator, the map $\eta \to (\log \|\eta\|_v)_{v \in S}$ maps Y to a lattice in $\coprod_{v \in S} \mathbf{R}$ spanning the hyperplane $(\coprod_{v \in S} \mathbf{R})_0$. Hence $Y \otimes \mathbf{R}$ is isomorphic as G-module to $(\coprod_{v \in S} \mathbf{R})_0$. Since a linear representation of G on a finite vector space over a field of characteristic zero is determined by its character, it follows that the isomorphism holds true if we replace the real field $\mathbf{R}$ by the rational field $\mathbf{Q}$. Hence Y contains a G-submodule X of finite index isomorphic to $(\coprod_{v \in S} \mathbf{Z})_0$. Since $Y/X \approx lY/lX$ it follows that $[Y/lY] = (X/lX]$. Tensoring with μ_l we conclude $[\mu_l \otimes Y] = [(\coprod_{v \in S} \mu_l)_0]$ and (a) follows. To prove (b) we note that $[(K_2 O_S)_l] = [K_2 O_S / l K_2 O_S]$ because $K_2 O_S$ is finite, and then use Theorem (6.2), recalling that we have chosen S large enough so that $\mu_l \otimes \operatorname{Pic} O_S = 0$. Formula (c) follows from the exact sequence (*) on subtracting (b) from (a), and this concludes the proof of the lemma.

The theorem follows easily from (c). Dropping now the assumption $\mu_l \subset F$, we apply the preceding to the field $F' = F(\mu_l)$ with $G = \operatorname{Gal}(F'/F)$. Since G is of order prime to l, the functor $M \mapsto M^G$ is exact on the category of G-modules of l-power order. Hence $\operatorname{Ker} \gamma_F = (\operatorname{Ker} \gamma_{F'})^G$. We have $(\mu_l \otimes \mu_l)^G = H^0(F, \mu_l \otimes \mu_l)$. Hence, by part (c) of the lemma, we will be done if we show that $(\coprod_{v \in S'_c} \mu_l)^G = \coprod_{v \in S_c} \mu_l$, where S_c (resp. S'_c) denotes the set of complex places of F (resp. of F'). If $l = 2$ we have $G = (1)$ and $F = F'$ and there is nothing to prove. If $l \neq 2$, then we have $\coprod_{v \in S_c} \mu_l \approx (\coprod_{v \in S_\infty} (F_v)^{\bullet})_l$, where S_∞ is the set of *all* archimedean primes of F, and the claim follows for example from the fact that $F \otimes \mathbf{R} = (F' \otimes \mathbf{R})^G$, or that the ideles of F are the ideles of F' fixed by G. This concludes the proof of Theorem (6.3).

(6.5) **Theorem.** *We have*

$$H^1(F, \mathbf{Z}_l(2)) \approx \mathbf{Z}_l^{r_2} \times (\mathbf{Z}/l^m \mathbf{Z}),$$

where m is the largest integer ≥ 0 *such that* $F(\mu_{l^m})$ *is contained in a composite of quadratic extensions of* F.

Let $X = H^1(F, \mathbf{Z}_l(2))$ for short. The quotient X/lX is isomorphic to the kernel of the map δ in diagram (3.3), and since h is injective $\operatorname{Ker} \delta$ is isomorphic to $\operatorname{Ker} \gamma$. By the Theorem (6.3) X/lX is finite, of order $l^{r_2 + \varepsilon}$, with $\varepsilon = 0$ or 1. By the corollary to Proposition (2.1), X is a finitely generated $\mathbf{Z}_l$-module. To conclude the proof we must show that the torsion part of X is 0 if $\varepsilon = 0$, i.e., if $H^0(F, \mu_l \otimes \mu_l) = 0$, and is cyclic of order l^m otherwise. But this is clear because, by Proposition (2.3), the torsion subgroup of X is the isomorphic image of $H^0(F, (\mathbf{Q}_l/\mathbf{Z}_l)(2))$.

Corollary. *$H^1(F, \mathbf{Q}_l(2))$ is a vector space of dimension r_2 over $\mathbf{Q}_l$ and the divisible part of $H^1(F, (\mathbf{Q}_l/\mathbf{Z}_l)(2))$ is isomorphic to $(\mathbf{Q}_l/\mathbf{Z}_l)^{r_2}$.*

Indeed that divisible part is a torsion group, and by Proposition (2.3), is the image of $H^1(F, \mathbf{Q}_l(2))$ by a map whose kernel is $X/X_{\text{tors}} \approx \mathbf{Z}_l^{r_2}$.

Let $(\mathbf{Q}/\mathbf{Z})(m)$ denote the direct sum, over all primes $l \neq \operatorname{char} F$, of the G_F-modules $(\mathbf{Q}_l/\mathbf{Z}_l)(m)$.

(6.6) **Theorem.** *There exists a unique homomorphism $g = g_F$ such that the following diagram is commutative for each prime $l \neq$ char F.*

$$\begin{array}{ccc} H^1(F, (\mathbf{Q}/\mathbf{Z})(2)) & \xrightarrow{g} & K_2F \\ \uparrow & & \downarrow h \\ H^1(F, (\mathbf{Q}_l/\mathbf{Z}_l)(2)) & \xrightarrow{\delta} & H^2(F, \mathbf{Z}_l(2)). \end{array}$$

The homomorphism g is surjective. Its kernel is the divisible part of $H^1(F, (\mathbf{Q}/\mathbf{Z})(2))$ and is isomorphic to $((\mathbf{Q}/\mathbf{Z})(0))^{r_2}$. In particular, g is an isomorphism if F is a function field or a totally real number field.

By Proposition (2.3), for each l, the image of δ is the torsion part of the H^2, which is isomorphic to the l-primary part of K_2F by h, and the kernel of δ is the divisible part of $H^1(F, (\mathbf{Q}_l/\mathbf{Z}_l)(2))$, which is isomorphic to $(\mathbf{Q}_l/\mathbf{Z}_l)^{r_2}$ by the corollary above. The theorem now follows because for discrete coefficient modules the functor $H^i(F,)$ commutes with direct sums (cf. e.g. [22], p. I-9).

Corollary. *Let L be a (possibly infinite) algebraic extension of a global field. There exists a unique homomorphism g such that the following diagram is commutative for each global subfield F of L, where the vertical arrows are the functorial maps*

$$\begin{array}{ccc} H^1(F, (\mathbf{Q}/\mathbf{Z})(2)) & \xrightarrow{g_F} & K_2F \\ \downarrow & & \downarrow \\ H^1(L, (\mathbf{Q}/\mathbf{Z})(2)) & \xrightarrow{g} & K_2L. \end{array}$$

The map g is surjective and its kernel is the union, over all global subfields F of L, of the images in $H^1(L, (\mathbf{Q}/\mathbf{Z})(2))$ of the divisible parts of $H^1(F, (\mathbf{Q}/\mathbf{Z})(2))$. In particular, g is an isomorphism if L is a function field or a totally real number field.

This follows immediately because both functors $F \mapsto K_2F$ and $F \mapsto H^1(F, (\mathbf{Q}/\mathbf{Z})(2))$ commute with direct limits.

Remark. It was Lichtenbaum who first conjectured the existence of the map g. It would be interesting to have a direct definition of it. When L contains the group μ of all roots of unity then g is easily described. We have then

$$\begin{aligned} H^1(F, (\mathbf{Q}/\mathbf{Z})(2)) &= \varinjlim_m H^1(F, \mu_m \otimes \mu_m) = \varinjlim_m (\mu_m \otimes H^1(F, \mu_m)) \\ &= \varinjlim_m (\mu_m \otimes F^{\cdot}) = \mu \otimes F^{\cdot}, \end{aligned}$$

and with that identification the map g takes $z \otimes a$ to $\{z, a\}$. Using this after adjoining the algebraic closure of the constant field, it is easy to get the description of K_2F for a global function field F which is given in [2], § 8.

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John Tate
Harvard University
Department of Mathematics
One Oxford Street
Cambridge, Mass. 02138
USA