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Cycles for the Dynamical Study of Foliated Manifolds and Complex Manifolds

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To Jean-Pierre Serre

A classical cycle on a manifold is determined in a neighborhood of a point by giving a finite number of weightes branches passing through the neighborhood. It is natural to generalize this picture by considering a kind of cycle defined locally by an infinite family of branches weighted by a non-negative measure on the family.

For example one could take cycles locally defined by measured families of leaves in a foliated manifold, or measured families of (local) irreducible complex subvarierties passing near a point in a complex manifold.

Such cycles determine currents (linear functionals on C^∞-forms) in the sense of de Rham [De] and Schwartz [Sc]. These cycles have proved useful in the study of foliations [P], [RS], [Sch]. Also one obtains a single geometric notion interpolating between the two extremes-classical cycles and closed differential forms. Our original goal here was to characterize such cycles for foliations among all the currents.

The idea is to consider currents which are "directed" by an a-priori given field of cones in the spaces of tangent p-vectors. Such a positivity condition leads to a compact convex cone of currents with a compact convex subcone of cycles (closed currents).

One has (Part I) transversal intersection theory for these "directed cycles" and approximation by diffuse cycles (those given by closed differential forms). Moreover, because of the compactness one can apply the basic tools of linear analysis such as the theorems of Hahn-Banach and Choquet.

The former allows one to construct closed C^∞-forms satisfying positivity conditions (on the cone field) because of the duality between forms and currents [Sc]. The latter allows one to decompose globally the directed cycles into a mass distribution of irreducible cycles – those determined by extreme rays of the cone of cycles. (Work of Ruelle implies such decompositions are unique in the case of foliations. The decompositions are not unique in the case of cycles directed by "complex" $2p$-vectors in a complex manifold.)

In the case of foliated manifolds, the notion of "directed cycle" provides the desired characterization of cycles defined locally by measured families of leaves, (or transverse invariant measures [P] and [RS]). There are many elementary applications of this characterization to the dynamical study of foliations (Part II).

In the case of complex manifolds the complex-directed cycles (called complex cycles) include those defined locally by measured families of local complex analytic subvarieties. It is unknown to the author if complex cycles are more general.

In a Kaehler manifold the complex cycles yield natural compact convex cones $[\mathbb{C}_p] \subset H_{2p}(M,R)$ and these cones generate the Hodge spaces $\mathscr{H}_p \subset H_{2p}(M,R)$, where $\mathscr{H}_p$ = real points $H_{p,p}(M,\mathbb{C})$ (Theorem III.17). There are other "elementary" new results about compact complex manifolds in Part III.

Directed cycles are flat currents or flat chains in the sense of [F] and [W] – thus they are geometric in character. For example, such a non-trivial d-cycle cannot be supported on a set whose Hausdorf dimension is less than d. ([F] 4.1.20).

The latter works especially [F] contain a wealth of non-trivial local geometric information concerning these cycles – hopefully relevant to future study.

Before giving a detailed description of the contents I would record my debt to inspiring conversations with Rufus Bowen, Boris Moizshezon, Joe Plante, David Ruelle, Harold Rosenberg, Bill Thurston and Alan Weinstein as well as the stimulation of very interesting recent theorems of Dan Asimov [As] and David Fried [Fr] on flows.

Contents

Summary

Part I: The Basic Results

(§ 1) A *cone structure* (I.2) is a continuous field of cones of p-vectors on a manifold. One has *currents* (I.4) and *cycles* (I.6) *tangent* to the structure and differential forms *transverse* to (or positive on) directions of the structure (I.3). Theorem I.7 expresses a duality between tangential cycles and transversal closed forms. I.7 uses Hahn Banach, the duality between forms and currents, and a compactness phenomenon of these currents (I.5).

(§ 2) Currents tangent to cone structures are represented by measures (I.8). They can be approximated by diffuse currents (differential forms) tangent to slightly larger cone structures (I.9). This yields a picture of structure cycles in dimension one and in codimension one (I.10). It also implies transversal cycles behave homologically as expected (I.11).

(§ 3) In case the cone structure is given by an oriented foliation one can analyze the cycles—*foliation cycles.* Locally these are distributions of leaves (I.12). Thus they are in one to one correspondance with transversal invariant measures for foliations (I.13).

Part II: Applications to Foliations

(§ 4) i) One either has transversal invariant measures for a foliation or closed forms positive on the foliation (II.1), (II.2) gives a more precise statement. In the *non-degenerate case* where one has both foliation cycles and transversal closed forms, these objects define compact dual cones in homology and cohomology (II.3). Moreover, the cycles in one homology class form a compact convex set (II.4).

ii) A non-compact Riemann manifold is "closed at ∞" if the volume form is not the differential of a bounded form (II.5). These manifolds include those of subexponential growth (II.7) and define foliation cycles if they occur as leaves of foliations (II.8).

iii) The Poincaré recurrence set is the union of the supports of all foliation cycles II.9. It is closed and invariant (II.10) and has somewhat surprising stability properties (II.11) and (II.12) when one deforms the foliation.

iv) A vanishing cycle (II.14) yields a foliation cycle. (II.15)

(§ 5) The property in codimension one "all leaves have exponential growth" is stable under perturbation (II.17).

A totally recurrent foliation can be approximated by one defined by a closed one form (II.18). To be so approximated is characterized by homological properties of transversal one-cycles (II.19).

A codimension one foliation with all leaves non-compact is transversal to a volume preserving flow (II.20).

(§ 6) Novikov's theorem that every foliation of S^3 has a compact leaf is proved using "Haefliger's argument", (II.15) and (II.16).

(§ 7) In a foliation with all leaves compact the obvious necessary homological condition on the leaves to have a closed form positive on the foliation is also sufficient (II.23). A basic theorem [EMS] on bounds on volumes is reproven.

(§ 8) Invariant measures for flows correspond to foliation cycles (II.24) which can be approximated by circles nearly tangent to the flow (II.25) using the ergodic theorem.

A compact set either contains recurrence or there is a gradient function (II.26). We prove Schwartzman's homological criterion for a flow to have a global cross section (II.27).

A totally recurrent flow can be approximated by a volume preserving flow (II.28).

(II.29) gives the condition that there is a volume preserving flow transverse to a given $(n-1)$ plane field.

(II.30) characterizes flows which may be approximated by volume preserving flows.

Part III: Applications to Other Structures

(§ 9) (III.1) gives a dichotomy (for the dynamical structure determined by a field of light cones) between recurrence defined by one-cycles and global functions increasing along cone directions. (This is closely related to a theorem of Hawking.)

(§ 10) (III.2) describes a geometrical cone structure Poincaré dual to a sympletic structure. Examples: 4 manifolds and complex structures.

(§ 11) Compact complex manifolds have natural cone structures defined by the complex directions. (Example following III.2).

These are always complex cycles in $\mathbb{C}$-dimension one (III.10).

The complex cycles form *natural* compact convex cones (III.13).

A Kaehler metric determines a diffuse complex cycle of dimension $n-1$ (III.15). The cone in homology $[\mathbb{C}_p] \subset H_{2p}(M, R)$ determined by the complex cycles of a Kaehler manifold generates the Hodge space $\mathscr{H}_p$ = real points $H_{p,p}(M, \mathbb{C})$ (III.17).

III.18, III.19, III.20, III.21, III.22 are corollaries of the latter.

$$\{\text{algebraic cycles}\} = \{\text{complex cycles}\} \cap \{\text{rectifiable currents}\} \qquad \text{(III.23)}$$

Preliminary Remarks

We will always work in a compact region of a smooth n-manifold of class C^∞. We use C^∞ forms $\mathscr{D}_p$ and their dual functionals the currents $\mathscr{D}'_p$ [De] and [Sc]. We make use of the beautiful formal properties:

i) $\mathscr{D}_p$ and $\mathscr{D}'_p$ are strong duals of one another [Sc].

ii) $\mathscr{D}_p$ and $\mathscr{D}'_p$ are locally convex and Montel (bounded sets are precompact) [Sc].

iii) If $p+q=n$, there are natural inclusions $\mathscr{D}_q \to \mathscr{D}'_p$ defined by an orientation of M (assumed closed) and the wedge product of forms. The image of $\mathscr{D}_q$ in $\mathscr{D}'_p$ are called the diffuse p-currents. They are dense [De].

iv) There are natural intersection pairings $p+q=n+r$,

(diffuse p-currents) $\times$ (q-currents) $\to$ (r-currents).

We denote all the natural pairings involving currents, forms, p-vectors, and q-covectors by " $\wedge$ ".

The inclusion iii) is a precise form of Poincaré duality. At each point where a q-form ω is non-zero (the interior support of ω) there is a well-defined ray of p-vectors – the direction of ω. The current defined by ω has a closed support, (the closure of the interior support).

Although we work in the general setting above to use the compactness and Hahn-Banach theorem, the currents we construct are usually currents of degree 0. They define functionals on continuous forms and can be represented in terms of non-negative measures (measures are always non-negative here) and measurable functions into p-vectors (Part I).

The structures we consider such as foliations need not be C^∞. For most of the work on foliations it is sufficient to have smoothness of class $C^{1,0}$ (leaves are C^1 and tangent planes vary continuously.) Everything works for foliations with local charts of class C^1.

Part I: Cone Structures, Cycles, and Closed Forms

§ 1. Cone Structures on Manifolds

Definition I.1. A *compact convex cone* C in a (locally convex topological) vector space over R is a convex cone which for some (continuous) linear functional L satisfies $L(x)>0$ for $x\neq 0$ in C and $L^{-1}(1)\cap C$ is compact. The latter set is called a base for the cone. We will often identify a base with the set of rays in the cone, denoted $\mathbf{C}$. The kernel of L is called a strictly supporting hyperplane of the cone C.

Definition I.2. A *cone structure* on a closed subset X of a smooth manifold M is a continuous field of compact convex cones $\{C_x\}$ in the vector spaces $\Lambda_p(x)$ of tangent p-vectors on M, $x\in X$. Continuity of cones is defined by the Hausdorf metric on the compact subsets of the rays in Λ_p. Namely the bases of the cones move continuously relative to the metric $h(\mathbf{C},\mathbf{C}')=\max(\sup_{c\in\mathbf{C}}\rho(c,\mathbf{C}'), \sup_{c'\in\mathbf{C}'}\rho(c',\mathbf{C}))$ where ρ is a convenient metric on rays defined in some local trivialization of Λ_p.

Definition I.3. A differential p-form ω (of class C^∞) on M is *transversal* to the cone structure C if $\omega(v)>0$ for each $v\neq 0$ in $C_x\subset\Lambda_p(x)$, $x\in X$.

Proposition I.4. *A cone structure C admits transversal p-forms.*

Proof. A strictly supporting hyperplane for C_x in $\Lambda_p(x)$ can be extended easily to a p-form satisfying $\omega\geqq 0$ on C with strict inequality holding for non zero p-vectors on a neighborhood of x. Now one takes convex combinations of these forms using a partition of unity to find a transversal form.

Remark. If X is compact then clearly any transversal p-form has a positive lower bound on the p-vectors of unit length in the cones C_x (relative to any convenient Riemann metric).

Definition I.4. A *Dirac current* is one determined by the evaluation of p-forms on a single p-vector at one point. The *cone of structure currents* $\mathscr{C}$ associated to the cone structure C is the closed convex cone of currents generated by the Dirac currents associated to elements of C_x, $x\in X$.

Proposition I.5. *If X is compact the cone of structure currents $\mathscr{C}$ associated to a cone structure C on X is a compact convex cone.*

Proof. Let ω be a p-form positive on C and let λ be a positive lower bound for the values of ω on the unit p-vectors of C_x (relative to a convenient metric). Let $\bar{\mathscr{C}}=\mathscr{C}\cap\omega^{-1}(1)$ where ω denotes the linear functional $c\to\int_c\omega$. $\bar{\mathscr{C}}$ is closed by definition. To show that $\bar{\mathscr{C}}$ is compact we need only show that the values $(\bar{\mathscr{C}},\eta)$ are bounded for any fixed form η [Sc, p. 74]. If m denotes the maximum value of η on any unit p-vector along X then clearly $(\bar{\mathscr{C}},\eta)\subset[-a,a]$ where $a=m/\lambda$. For $|\int_c\eta|\leqq|m/\lambda\int_c\omega|\leqq m/\lambda$, when $c\in\bar{\mathscr{C}}$. Note by density it is enough to take c to be a finite sum of Dirac currents in the above calculations. This proves the proposition.

Remarks. Structure currents are currents of integration or currents of degree zero. Namely, the values are small on forms whose coefficients are small. This implies they can be described in terms of measures (Proposition I.8 below). Then the proposition above follows from the well known fact that probability measures on a compact space comprise a compact convex set in the dual space of continuous functions [Ph].

Definition I.6. If C is a cone structure, the *structure cycles* of C are the structure currents which are closed as currents.

Remark. We will try to develope the idea that structure cycles with compact support are like global recurrent solutions of the dynamical structure associated to C.

Now we give one of the main technical results of the paper which follows readily from the Hahn-Banach theorem, the duality between forms and currents, and the compactness.

First we note that in the space of p-currents there are two natural subspaces, the boundaries $\mathscr{B}$ and the cycles $\mathscr{Z}$. Thus cone structures C are partitioned into

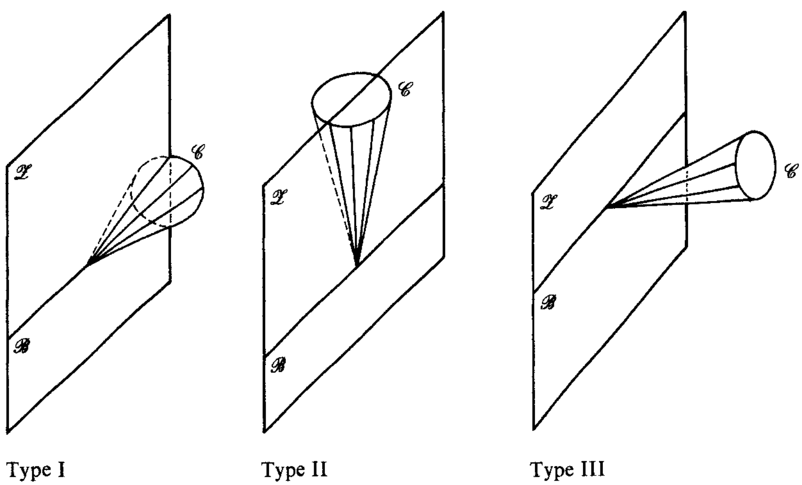

Fig. 1

three groups according to the position of $\mathscr{C}$ the cone of currents determined by C relative to $\mathscr{B}$ and $\mathscr{Z}$.

Suppose C is a cone structure of p-vectors defined on a compact subspace X in the interior of M which is also compact (with or without boundary).

Theorem I.7. i) *There are always non-trivial structure cycles in X or closed p-forms on M transversal to the cone structure (i.e. positive on the cones C_x).*

ii) *If no closed transverse form exists some non-trivial structure cycle in X is homologous to zero in M (type I).*

iii) *If no non-trivial structure cycle exists some transversal closed form is cohomologous to zero (Type III).*

iv) *If there are both structures cycles and transversal closed forms (type II) then*

a) *the natural map*

(structure cycles on $X \rightarrow$ homology classes in M)

is proper and the image is a compact cone $\mathbb{C} \subset H_p(M, R)$

b) *the interior of the dual cone $\mathbb{C}' \subset H^p(M, R)$*[1] *consists precisely of the classes of closed forms transverse to C.*

Proof. First we note that the closed currents $\mathscr{Z}$ forms a closed subspace of the p-currents on M (∂ is continuous). Second the boundaries form a closed subspace of the cycles $\mathscr{Z}$ (being defined by the vanishing of periods $\int \omega_i = 0$ where ω_i ranges over of $H^p(M, R)$). Third by the Hahn Banach theorem any closed subspace of currents which does not intersect a compact set[2] can be extended to a closed subspace of codimension one not intersecting that set. Fourth, a closed hyperplane of currents determines (up to a non-zero constant) a C^∞ form on M which vanishes only on that hyperplane because currents and forms are dual spaces [Sc].

To prove the statements of the theorem let $\bar{\mathscr{C}}$ be any compact base of the cone (I.5) and we obtain a compact set not intersecting zero to which the above remarks may be applied.

To prove iii) extend the cycles to a closed hyperplane not intersecting $\bar{\mathscr{C}}$. Choosing the direction determined by $\bar{\mathscr{C}}$ gives an exact form (up to a positive constant) which is positive on $\bar{\mathscr{C}}$.

To prove ii) extend the boundaries to a hyperplane not intersecting $\bar{\mathscr{C}}$ if no structure cycle bounds. Then one has a closed p-form transversal to the structure. This is a contradiction so ii) must be true.

Note the extension just made could have begun with any hyperplane in $\mathscr{Z}$ containing $\mathscr{B}$ which does not intersect $\bar{\mathscr{C}}$. This proves iv) b).

Now iv) a) is clear because the fibres (up to scaling) are the intersection of $\bar{\mathscr{C}}$ with translates of $\mathscr{B}$.

Statement i) is a consequence of ii) and iii) and the theorem is proved.

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[1] $\mathbb{C}'$ is defined by $(\mathbb{C}', \mathbb{C}) \geqq 0$

[2] Of course convex

§ 2. *Local Study of Structure Currents*

We begin the local analysis of the structure currents of C. First note if μ is a probability measure on M and v is a continuous field of p-vectors satisfying $v(x)\in C_x$ for each x then the pair (v,μ) defines a structure current for C denoted $\int v\,d\mu$. Namely $(\int v\,d\mu)(\omega)=\int(\omega\wedge v)\,d\mu$ any continuous p-form ω. Actually, v need only be μ-integrable in order that $\int v\,d\mu$ defines a structure current, and we have

Proposition I.8. *Any structure current c may be represented $c=\int_X v\,d\mu$ where μ is a non-negative measure on X (assumed compact) and v is a μ-integrable function into p-vectors satisfying $v(x)\in C_x$.*

Proof. The question is local and we can work on a small neighborhood U where the bundle Λ_p is trivialized $U\times\Lambda_p(x_0)$ and the cones C_x vary only a little. By the Riesz representation theorem c may be represented by a measure ν with values in $\Lambda_p(x_0)$. On a Borel set A we will have $\nu(A)\in C_A=\bigoplus_{a\in A} C_a$.

It is easy to construct a non-negative measure μ on X (from the components of ν) so that ν is absolutely continuous with respect to μ. The Radon-Nikodym derivative $\frac{d\nu}{d\mu}=\lim_{A\to a}\frac{\nu(A)}{\mu(A)}$ is defined μ almost everywhere and we have the representation $c=\int v\,d\mu$ where $v=\frac{d\nu}{d\mu}$. Note that $v(a)\in C_a$ follows from $\nu(A)\in C_A$.

Remark. i) One may think of $c=\int v\,d\mu$ as a linear combination of dirac currents $v(x)$ (in the structure) with the weighting provided by μ. For example $\partial c=\int(\partial v(x))\,d\mu$.

ii) The representation is not unique because we can take any positive continuous function ρ on X and replace (v,μ) by $(v/\rho,\rho\mu)$. The representation in terms of the Λ_p-valued measure ν (of the proof) is unique.

iii) Note the special case $X=M$, μ is a smooth measure determined by an n-form Ω (and an orientation of M), and v is a smooth field of p-vectors. Then $\int v\,d\mu$ is the *diffuse current* determined by the q-form $\Omega\wedge v$ (via the same orientation of M).

Now we consider approximating currents by diffuse currents. These considerations are well known except possibly the geometric property iii) below.

The diffusion of currents can be accomplished by the linear operation

$$z\to\pi_*((M\times z)\wedge U_\varepsilon)=D_\varepsilon z \qquad \text{(Fig. 2)}$$

where π is the projection on the first factor in $M\times M$ and U_ε is a closed n-form Poincaré dual to the diagonal in $M\times M$.

We will choose U_ε so that as $\varepsilon\to 0$ U_ε approaches the diagonal current in the following senses

i) in the topology of n-currents on $M\times M$,

ii) the support of U_ε is contained in the ε neighborhood of the diagonal,

iii) the rays of n-vectors determined by the non-zero values of U_ε approach those determined by the oriented tangent planes of the diagonal.

Given such a system of n-forms U_ε we have the

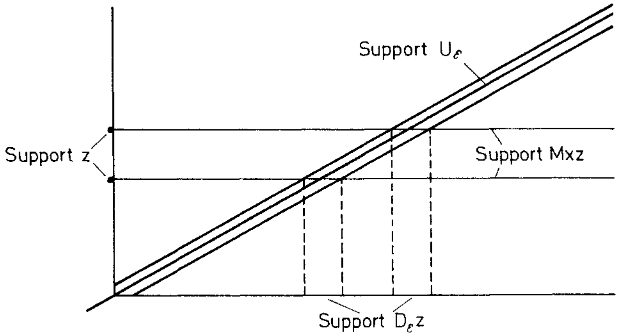

Fig. 2

Proposition I.9. *D_ε defines a linear operation on currents which commutes with ∂ and preserves homology classes. For structure currents z relative to C,$D_\varepsilon z$ is a diffuse current (a q-form, $p+q=n$) relative to a slightly larger cone structure $\tilde{C}$ ($C_x \subset$ interior $\tilde{C}_x$). Finally interior support ($D_\varepsilon z$) contains support (z).*

Proof. The first statement follows directly from the definition. For the second represent z by $\int v\,d\mu$ (I.8). By linearity $D_\varepsilon z = \int D_\varepsilon v(x)\,d\mu$, and it suffices to analyze $D_\varepsilon v$ where v is a dirac current at x.

Now $D_\varepsilon v$ is by direct calculation a q-form. Properties ii) and iii) insure the existence of $\tilde{C}$ so that $D_\varepsilon v$ is a structure current for $\tilde{C}$. The result follows for z by linearity (we integrate the function $D_\varepsilon v(x)$ with values in q-forms against the measure μ, $D_\varepsilon z = \int D_\varepsilon(v(x))\,d\mu$).

Because the local currents produced are structure currents for $\tilde{C}$ there is no cancellation of support when we take linear combinations and the last statement holds. (We are reduced to the corresponding statement about the interior support of a diffused non-negative measure.)

Now we will construct the form U_ε. We will do this for the integration current defined by any compact submanifold V^v contained in W. Then we restrict to the special case of the diagonal in $M \times M$.

Proposition [De]. *If* (V) *is the current of integration defined by a compact submanifold V of W then there is a system of closed Poincaré dual forms v_ε satisfying as $\varepsilon \to 0$,*

i) *v_ε converges to (V) in the space of currents on W,*

ii) *support v_ε is contained in the ε-neighborhood of V,*

iii) *the rays of v-vectors determined by the non-zero values of v_ε approach those determined by the oriented tangent planes of V.*

Proof. For the proof we first diffuse the diagonal in $W \times W$ (over V) by a stepwise operation. Cover a neighborhood of V by a finite number of parallelized regions of $W_1, W_2, \ldots, W_r$. Over W_i in $W \times W$ consider the d-current (d = dimension W) made out of parallel copies of the diagonal (Figure 2) (defined by the parallelization and an a-priori identification of a tubular neighborhood of the diagonal and the tangent bundle). These translates are forced to become tangent to the diagonal at ∂W_i and are weighted with a smooth transverse measure which becomes the dirac measure at ∂W_i. (We present the current U_i as a measured family (B_i, μ_i) of smooth manifolds which are each small partial displacements of the diagonal. Where the submanifolds all agree the measure on the family becomes the Dirac transversal measure in the manifold.)

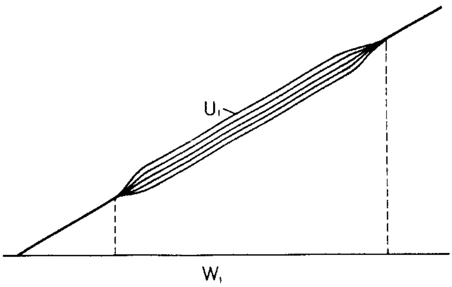

Fig. 3

This construction yields a closed d-current U_i which is a (decomposable) form over W_i and agrees with the diagonal current outside W_i. Now one can apply the diffusion operators $D_i z = \pi_*((W \times z) \wedge U_i)$ successively to the current (V), namely $D_1(V)$, $D_2 D_1(V), \ldots, D_r \cdots D_1(V)$ to construct the desired diffusion v_ε of V.

One actually defines $D_1 V$, $D_2 D_1 V$, etc. by intersecting families of transversal manifolds (and projecting). Thus $D_1 V$ is presented as a (B_1, μ_1) family, $D_2 D_1 V$ as a $(B_1 \times B_2, \mu_1 \times \mu_2)$ family, etc.

One knows [De] that these currents obtained by intersection agree with $\pi_*((W \times V) \wedge U_1)$, $\pi_*((W \times D_1 V) \wedge U_2)$, etc. when either factor is diffuse (equal to a form). This proves the proposition. (An analogous procedure is described analytically in [De].)

Now we give an approximation corollary.

Corollary I.10. i) *Any structure cycle in dimension one is approximated by a volume preserving flow.*

ii) *Any structure cycle in codimension one is approximated by a volume preserving codimension one foliation.*

Proof. i) A structure cycle z in dimension one can be diffused to a closed $(n-1)$) form ω. Where ω is non-zero we have the flow lines determined by $\ker \omega$ which nearly point in the cone directions of the cone structure associated to z.

ii) A structure cycle z in dimension $(n-1)$ can be diffused to a closed one form η. Where η is η non-zero we have a foliation of codimension one defined by $\ker \eta$ whose leaves (with orientation induced by η and the orientation of M) nearly point in cone directions associated to z.

Note these submanifolds (the flow lines and the leaves) are measured by the corresponding forms (in the transverse direction) to give currents approximating the original cylces (see § 3).

Now we prove the plausible theorem that two (positively transverse) structure cycles in complementary dimensions $(p+q=n)$ which intersect have a positive self-intersection number.

Assume C and C' are two cone structures of dimension p and q respectively on X and X', two compact subsets of M. At each point x of the intersection $X \cap X'$ suppose $(\mathbf{C}_x) \wedge (\mathbf{C}'_x) > 0$ in $\Lambda_n(x)$.

Let z and z' be two structure cycles for C and C' so that support $z \cap$ support z' $\neq \emptyset$. Then

Theorem I.11. *The intersection number* (z, z') *is positive.*

Proof. Diffuse z to a closed q-form Z using the D_ε of (I.9). For ε sufficiently small Z is positive on $\mathbf{C}'_x$ for x in support Z. Since interior support Z contains support z (1.9) it must intersect support z'. Consequently, $(z, z') = \int_{z'} Z > 0$.

Note. i) It would have been easy in the Proposition (above) to achieve such a v_ε if the normal bundle to V were trivial. We average parallel copies of V.

ii) More generally if the normal bundle admitted a flat orthogonal connection, v_ε could be represented as a diffuse foliation cycle (§ 3) made out of leaves of integral solutions of the connection.

iii) In general, the intersection of the submanifolds of the diffusion v_ε could provide a picture of the real characteristic classes of the normal bundle of V.

Patodi's work shows an analogous statement is true analytically when one considers the diagonal in $M \times M$ and heat diffusion. The characteristic classes appear as derivatives of the diffusion near $t=0$.

iv) In a similar vein the invariant of Godbillon-Vey is the second derivative (at $t=0$) of an intersection constructed by deforming a foliation transverse to itself. If ω defines a foliation $\mathscr{F}$ of codimension 1 on M^3 and X is a vector field so that $\omega(X)=1$, then

$$gv(\mathscr{F}) = \lim_{t \to 0} 1/t^2 \int_M \varphi_t \omega \wedge d\omega, \qquad \varphi_t = \exp tX.$$

§ 3. Local Study of Foliation Cycles

In case the cone structure is determined by an oriented foliation there is a precise local description of the structure cycles.

We assume the foliation is of class C^1 and choose a closed neighborhood B foliated by closed disks L_y on leaves. Let D denote the quotient of B by the leaves L_y, $D = \{y\}$.

Let (L_y) denote the current of integration on L_y (oriented by $\mathscr{F}$). Suppose Z is any foliation cycle whose support intersects B.

Theorem I.12. *There is a non-negative measure μ on D so that on interior B the current Z may be represented $\int_D (L_y)\, d\mu$. μ is unique in the interior of D.*

Proof. Step 1: Let v denote a continuous field of pure p-vectors providing the leaves of $\mathscr{F}$ with volumes (generating Λ_p (tangent spaces to $\mathscr{F}$)). Then a direct application of Riez representation yields a non-negative measure v on M so that Z (or any other foliation current) is represented on M by $Z(\omega) = \int_M \omega(v)\, d\mu$. (Compare I.8)

Step 2: Let $\bar{\nu}$ denote the measure ν restricted to B and denote by π the natural projection $B \to D$. We disintegrate the measure $\bar{\nu}$ over π. Namely if $\mu' = \pi_* \bar{\nu}$ there is family of probability measures η_y on B satisfying

i) for each continuous function h on B, the function $y \to \int_B h\, d\eta_y$ is Borel measurable,

ii) the support of η_y is contained in L_y,

iii) for each continuous function h on B,

$$\int_B h\, d\bar{\nu} = \int_D \left(\int_{L_y} h\, d\eta_y \right) d\mu' \qquad \text{[Bo, p. 58].}$$

We write the last equation $\bar{\nu} = \int_D \eta_y\, d\mu'$. Let $[L_y]$ denote the current $\omega \to \int_{L_y} \omega(v)\, d\eta_y$. Then iii) becomes $Z = \int_D [L_y]\, d\mu'$.

Step 3: The support of $\partial[L_y]$ does not intersect interior B for μ' almost all y in $\mathring{D}$ = interior D.

Proof. Since $Z = \int [L_y]\, d\mu'$ on B and Z is a cycle we have in $\mathring{B}$ = interior B

$$\partial \int_D [L_y]\, d\mu' = \int_D \partial[L_y]\, d\mu' = 0.$$

Consider a $C^\infty(p-1)$ form ω whose support lies in $\mathring{B}$ and a C^1 function on D, $h(y)$ whose support lies in $\mathring{D}$. Let $\varphi(y)$ denote the Borel measurable function of y, $y \to \partial[L_y](\omega)$.

The form $h(y) \cdot \omega$ can be C^1 approximated by C^∞ forms on $\mathring{B}$, so that $\partial Z(h(y) \cdot \omega)$ must be zero by continuity (∂Z is defined and continuous on C^1 forms and by hypothesis vanishes on C^∞ forms inside B.)[3]

Since $\partial[L_y](h(y) \cdot \omega) = h(y) \cdot \varphi(y)$ we conclude $\int h(y)\, \varphi(y)\, d\mu' = 0$. Since this is true for all such h it follows that $\varphi(y) = 0$ for μ' almost all points in $\mathring{D}$ (the measure $\varphi(y) \cdot \mu'$ annihilates all C^1 functions supported in $\mathring{D}$ so must be zero there).

Thus we remove a set of μ' measure zero from $\mathring{D}$ and conclude (in $\mathring{B}$) $\partial[L_y](\omega) = 0$ for the rest. Doing this for an appropriate countable set of ω's we conclude support $\partial[L_y]$ does not intersect interior B for μ' almost all y in $\mathring{D}$.

Step 4: By step 3 we may write $[L_y] = d(y)\, (L_y)$ on interior L_y for μ' almost all y in interior D. (The only closed p-currents on a connected p-manifold are constant multiples of the integration current.) But then if $\mu = d(y)\, \mu'$ we have $Z = \int [L_y]\, d\mu' = \int d(y)(L_y)\, d\mu' = \int (L_y)\, d\mu$ on $\mathring{B}$ and we're done with the first part.

Step 5: The uniqueness of μ on $\mathring{D}$ is similar to step iii) only simpler. Q.E.D.

Now we can easily relate foliation cycles and transversal invariant measures for a foliation ([P], [RS]). Let us first define a presentation of the foliation and the transversal measure.

Denote by T a finite union of closed disks transverse to the foliation whose interiors meet every leaf. If a path on one leaf connects two points x and y in T with y in the interior the foliation determines a homeomorphism germ, embedding a neighborhood of x in D into one of y in D. The foliation is essentially determined by this data.

---

[3] This is the point where the C^1-hypotheses is used. Otherwise the computation works for locally rectifiable leaves with continuously varying plane field

Definition. A *transversal invariant measure* for the foliation is a non-negative measure of finite mass on T which is compatible with all the germs of homeomorphisms determined by the foliation.

Remark. i) Any other transversal disk inherits a measure by sliding it piecemeal into T. In this way different presentations of the measure can be compared and we can pass to the equivalence class of presentations which we also call a transversal invariant measure.

ii) This intrinsic transversal measure of a foliation of dimension p together with an orientation determine a closed p-current [RS]. One merely integrates local forms over the leaves and averages the result in the transverse direction with the measure. The invariance property of the measure makes the procedure independant of the way a global form is broken into local forms by a partition of unity [RS], [EMS]. The current just described is clearly a foliation cycle.

Theorem I.13. *If $\mathcal{F}$ is an oriented p-dimensional foliation of a compact manifold, invariant transversal measures and foliation cycles are in a canonical one to one correspondance.*

Proof. Theorem I.12 tells us how to construct an invariant measure from a foliation cycle Z. Cover the manifold by flow boxes (whose interiors cover) $B_1, B_2, \ldots, B_n$. Express Z locally in B_i as $\int_{D_i} (L_y)\, d\mu_i$. The μ_i determine the measure on the transversal $T = \cup D^i$.

The invariance of $\{\mu_i\}$ can be reduced to the question of compatibility for the case of one small flow box contained inside a larger one. This case is clear from the statement of I.12.

We have described above (and in [RS]) the procedure in the other direction. The two compositions are each the identity.

Examples. i) (Flows). For one-dimensional foliations there are many examples of foliation cycles. The basic reason is that a long piece of one orbit normalized by dividing by its length is essentially a cycle [Sch].

a) The irrational flow on the torus provides an interesting example. Here the current is equal to that given by the one form whose kernel defines the foliation and whose transverse measure comes from the natural metric on the torus (Fig. 4).

b) If we split open the torus along one irrational flow line and insert a doubly tapering strip – the Denjoy example – the foliation cycle from a) concentrates on a closed invariant set locally homeomorphic to a line product the Cantor set (Fig. 5). This cycle is strictly between classical cycles and differential forms.

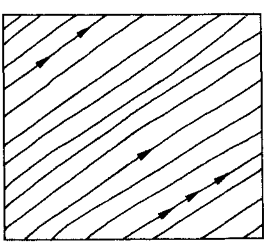

Fig. 4

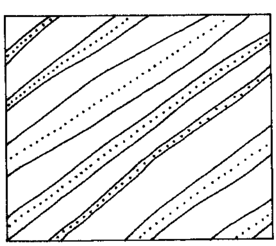

Fig. 5

ii) (2-dimensional leaves). Now foliation cycles may or may not exist. The irrational plane foliation of $T^3 = S^1 \times S^1 \times S^1$ is like a) above and one leaf may be split as in *b*). The Anosov foliations of the unit tangent bundle of a negative curved surface do not have transverse measures. (Note Theorem II.2 is clear in this case. One pulls up the volume form on the base to get the positive form. This form is exact because the Euler characteristic is non-zero.)

iii) (Discrete groups). If Γ is a group of measure preserving diffeomorphisms of a compact manifold F used to construct a bundle over a manifold B, there is a natural foliation of the total space transverse to the fibres. The invariant measure on F determines a foliation cycle. There are numerous recent and classical theorems related to this cycle.

iv) (Geometry of leaves). Plante [P] generalized i) by observing a leaf of subexponential growth (of volume of balls of radius R) leads to asymptotic homology classes. We consider a more general geometric condition "closed at ∞" in part II that leads to foliation cycles. The most general intrinsic geometric condition on a abstract leaf so that its closure in a foliation always carries a cycle is not known (see Part II).

Part II: Applications to Foliations (§ 4–§ 8)

§ 4. Foliations of Arbitrary Dimension and Codimension

We consider now the case when the cone structure (Part I) is determined by an oriented foliation. For this discussion it is enough to assume that the ambient manifold M is of class C^∞, and foliation is of class C^1. (Many of our statements are true under weaker hypothesis.) We say one foliation approximates another if their tangent planes are close. We say a differential form is bounded (on a non-compact Riemann manifold) if the coefficients in an orthonormal base are bounded. Forms are of class C^∞ unless otherwise stated.

In § 3 we have shown how foliation cycles correspond to transversal invariant measures and we refer to the two notions interchangeably (fixing the orientation of $\mathscr{F}$ (Theorem I.13).

Then we have

Theorem II.1. *Suppose M is compact and $\mathscr{F}$ is an orientable p-dimensional foliation of M. Then one either has*

i) *a non-trivial foliation cycle (equivalently a non-trivial transversal invariant* measure*)* or

ii) *a closed p-form transverse to the foliation (namely, a closed p-form positive on the foliation).*

In fact we have the more precise statement,

Theorem II.2. i) *If there is no transversal invariant measure for $\mathscr{F}$ there is an exact p-form positive on the leaves of $\mathscr{F}$.*

ii) *If there is no closed form positive on the leaves of $\mathscr{F}$ then some transversal invariant measure determines a foliation cycle in the trivial homology class.*

Suppose there are both transversal invariant measures for $\mathscr{F}$ and closed p-forms positive on the leaves of $\mathscr{F}$. We say $\mathscr{F}$ is *non-degenerate* in this case. Consider the closed cone $\mathbb{C}$ in $H_p(M, R)$ determined by the foliation cycles and the open cone $\mathscr{C}$ in $H^p(M, R)$ determined by closed forms positive on $\mathscr{F}$. Let $\mathbb{C}' \subset H^p(M, R)$ denote the dual cone of $\mathbb{C}$.

Theorem II.3. *If $\mathscr{F}$ is a non-degenerate foliation the cone $\mathbb{C}$ is a compact convex cone. The cone $\mathscr{C}$ is the interior of the dual cone $\mathbb{C}'$.*

Theorem II.4. *If $\mathscr{F}$ is a non-degenerate foliation, the natural map from the cone of invariant transversal measures to $\mathbb{C} \subset H_p(M, R)$ is proper – the convex set of measures determining one homology class is compact.*

Theorems II.1, II.2, II.3, II.4 are reinterpretations of (I.7) in the context of foliations possible because of (I.13). They are also true in the relative case. Namely if K is a compact subset of M one considers foliation cycles with support in K and closed forms positive on the tangent directions of $\mathscr{F}$ along K. Then the theorems are true on and near K (Theorem I.7).

Geometry of Leaves and Foliation Cycles

Here we generalize the theorem of Plante [P] that a leaf of subex-ponential growth determines a transversal measure.

Definition II.5. A non-compact complete Riemannian manifold L is "not closed at ∞" if it is possible to solve the equation $d\eta = \omega$ where ω is a bounded volume form ($\omega = \lambda(x) \cdot$ (unit volume form) where $0 < c \leqq \lambda(x) \leqq C < \infty$) and η is a bounded form (see [SW]). Otherwise we say L is "closed at ∞".

Note that this notion is independent of the quasi-isometry (diffeomorphisms with bounded distortion) class of the metric. Also note that leaves in foliations of compact manifolds inherit a natural quasi-isometry class of metrics (see [PS]).

The standard line and the Euclidean plane are "closed at ∞". The hyperbolic plane (constant negative curvature) is "not closed at ∞"[4].

Proposition II.6. *If L is "not closed at ∞" there is a constant γ so that for all compact regions R on L there is an isoperimetric inequality volume $R \leqq \gamma$ (area ∂R).*

Proof. Apply Stokes and the relation $d\eta = \omega$.

Corollary II.7. [SW]. *If L is "not closed at ∞" then L has exponential growth of volume.*

Proof. Apply the above proposition to the regions on L

$$B(x, r) = \{y \in L_x\colon \text{distance}\,(x, y) \leqq r)\}$$

to see that vol $(B(x, r)) = V(x, r)$ satisfies $\limsup \frac{1}{r} \log V(x, r) > 0$ (the definition of "L has exponential growth of volume").

Note. If L is "not closed at ∞", then L satisfies a rather uniform exponential growth condition (e.g. Proposition II.6.)

[4] See Note Added in Proof. Solvable Lie groups (in natural metric) are "closed at ∞" if and only if they are unimodular. Covering spaces of compact manifolds with amerable covering group are "closed at ∞" (in lifted metric).

Theorem II.8. *If M is compact and $\mathscr{F}$ does not have an invariant transversal measure then every leaf is "not closed at ∞" (and satisfies the uniform exponential growth described above). Furthermore, if some leaf L is "closed at ∞" there must be an invariant transversal measure whose support is contained in the closure of L.*

Proof. The first part follows immediately from II.2 and the definition. The second part is the relative form.

Conclusion. One may ask conversely whether the isoperimetric inequality of Proposition II.6 implies M is "not closed at ∞". If this is true then effectively the second part of Theorem II.8 is equivalent to Plante's discussion [P] and not more general. For in fact Plante considers a sequence of regions R_i on leaves so that the ratio area ∂R_i/volume $R_i \to 0$ to define homology classes.

Such a "Plante sequence" may be viewed in our terms as determining a specific sequence of foliation currents $C_i = (1/\text{volume } R_i) \cdot (\int_{R_i} (\cdot))$ which

i) have mass 1 and thus form a precompact family of currents $\{C_i\}$,

ii) can only accumulate to foliation cycles because the (volume-area) condition insures mass $(\partial C_i) \to 0$.

The accumulation points of a "Plante sequence" determine non-trivial foliation cycles and thus non trivial transversal invariant measures.

Leaves which are "closed at ∞" provide a method at least as general. In fact the adjective "closed at ∞" could really be reserved for that geometric condition on a quasi-isometry class of non-compact Riemannian manifolds W so that any immersion $W \xrightarrow{f} M$ determines non-trivial closed currents (M compact, f of bounded distortion) carried by the closure of the image of f.

Recurrence and Foliation Cycles

Definition II.9. The union of the supports of all foliation cycles (or invariant transverse measures) of a foliation $\mathscr{F}$ is called the *Poincaré recurrence set of* $\mathscr{F}$, and denoted $P(\mathscr{F})$.

For one-dimensional foliations the Poincaré recurrence theorem establishes a recurrence property for this set.

For general foliations $P(\mathscr{F})$ contains all compact leaves and intersects the closure of those non-compact leaves which are "closed at ∞" (II.8).

Proposition II.10. *$P(\mathscr{F})$ is a closed invariant set when M is compact.*

Proof. If $C_1, C_2, \ldots$ is a sequence of foliations cycles of uniformly bounded mass, $S = \sum_i \frac{1}{2^i} C_i$ is also a foliation cycle. If $\{C_i\}$ is dense in the cycles of mass 1 then the support of S will be all of $P(\mathscr{F})$.

Now we prove a stability property of $P(\mathscr{F})$, again M is compact.

Theorem II.11. *Suppose $\mathscr{F}$ has no invariant transversal measure. Then any foliation $\mathscr{F}'$ whose tangent planes are sufficiently near to $\mathscr{F}$ also has this property. (Namely, $P(\mathscr{F})$ vacuous is a stable property of $\mathscr{F}$.)*

Proof. By Theorem II.2 there is an exact form ω positive on $\mathscr{F}$. If $\mathscr{F}'$ is near to $\mathscr{F}$ ω is positive on $\mathscr{F}'$. Then $\mathscr{F}'$ has no foliation cycle z because $\int_z \omega$ would be both zero and positive.

More generally let $\mathscr{U}(\mathscr{F})$ denote the open complement of the Poincaré recurrence set $P(\mathscr{F})$ and $K \subset \mathscr{U}(\mathscr{F})$ be the complement of a small open neighborhood η of $P(\mathscr{F})$. Then there is an $\varepsilon > 0$ depending on η and $\mathscr{F}$ with the following property.

Theorem II.12. *If $\mathscr{F}'$ is any foliation whose tangent planes are ε-close to those of $\mathscr{F}$, the support of any foliation cycle of $\mathscr{F}'$ intersects the neighborhood η of $P(\mathscr{F})$. In particular the compact leaves of $\mathscr{F}'$ and even the non-compact leaves of $\mathscr{F}'$ which are "closed at ∞" pass through η.*

Proof. By Theorem II.2 (relative case) there is an exact form on M positive on the foliation $\mathscr{F}$ near K. This form determines the ε and the above proof (II.11) applies.

Vanishing Cycles and Foliation Cycles

In this paragraph we work with finite chains and cycles which can be realized by mapping complexes into the manifold.

Definition II.13. A cycle C in a leaf L of the foliation $\mathscr{F}$ is *$\mathscr{F}$-homologous to zero* if C bounds a simply connected homology mapping to L.

Note if $C = \partial W$ in this sense the induced germ of foliation around W is isomorphic to $W \times$ (small transverse disk).

Definition II.14. A *vanishing cycle* C is one which may be embedded in a one parameter family C_t (homotopy of cycles) so that

i) each C_t lies on one leaf L_t, $0 \leqq t \leqq 1$, and $C_0 = C$.
ii) C_0 is not $\mathscr{F}$-homologous to zero.
iii) C_1 is $\mathscr{F}$-homologous to zero.

The following is motivated by Novikov's theorem that any codimension one foliation of S^3 has a compact leaf.

Theorem II.15. If a foliation $\mathscr{F}$ has a vanishing cycle of dimension one less than the leaf dimension, then $\mathscr{F}$ has a non-trivial foliation cycle.

Proof. First note an elementary property of k-chains W on an oriented k-manifold. One may write $W = P + N$ where the oriented k-simplices of P agree with the orientation and those of N do not. Then observe that support $\partial W =$ support $\partial P \cup$ support ∂N, i.e. there is no cancellation between ∂P and ∂N.

Thus in the argument to follow we may assume $C_1 = \partial W$ where W is a simply connected singular homology and W is positive.

Now by hypothesis C_1 may be moved through cycles on leaves back to C_0. We cannot extend this motion to one of a neighborhood of the homology W because C_0 is not $\mathscr{F}$-homologous to zero. On the other hand because there is no holonomy near W we can extend the isotopy of ∂W part of the way to W_t, $t > t_0$.

The amount we can push a given W_t further depends on the number of flow boxes it takes to cover W_t (taken with repetitions from a given finite collection covering M). As we approach the critical value of t the number of flow boxes required must become unbounded (or we could go further). See Figure 6.

We find then a sequence of chains W_i on leaves whose volumes become unbounded but whose boundaries have bounded mass. Then $(1/\mathrm{vol}\, W_i)\, (\int_{W_i})$ accumulates to a foliation cycle.

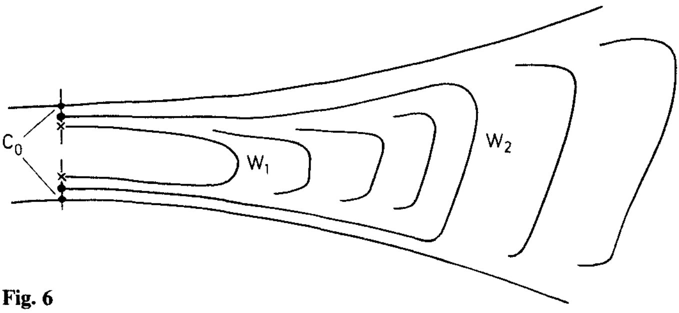

Fig. 6

§5. Codimension one Foliations

In codimension one special results are possible because of the abundance of closed transversal curves. For example, a leaf which cuts at least two times through a flow box is intersected by a closed transversal curve (Fig. 7).

Proposition II.16. *A foliation cycle in codimension one is either non-zero in homology or supported on compact leaves.*

Proof. A non-compact leaf in the support of the foliation cycle cuts through some flow box more than once (since M is compact) and we can draw a closed transversal curve (classical argument),

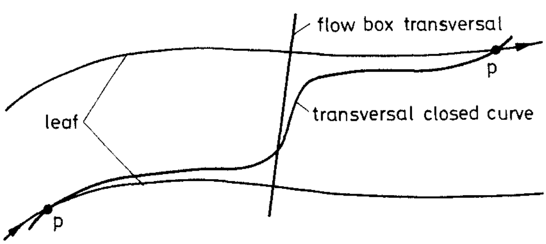

Fig. 7

The intersection number of the foliation cycle and this closed curve is positive – being the amount of mass deposited on the transversal – and each is essential in homology. (See Theorem I.11, or Theorem I.1 [P], or Proposition 2 [RS].)

Remark. In a beautiful study Plante [P] has shown the Poincaré set (union of supports of cycles) in codimension one is made out of compact leaves and leaves of polynomial growth. Moreover, the non-compact leaves correspond to truly irrational homology classes with arbitrarily small intersection numbers with transversal closed curves.

We can then reinterpret II.11 in codimension one as follows,

Theorem II.17. *If each leaf of a codimension one foliation $\mathscr{F}$ has exponential growth, then the same is true for any foliation $\mathscr{F}'$ whose tangent planes are sufficiently close to those of $\mathscr{F}$.*

Proof. If all leaves have exponential growth in codimension one then by Plante [P] there are no foliation cycles. The latter condition is stable under perturbation by II.11. But then the perturbation has leaves which are all "not closed at ∞" (II.8), in particular they all have exponential growth.

Note. i) A leaf of a foliation in a compact manifold has at most exponential growth. So the theorem may be paraphased: "the property – all leaves have maximal growth – is stable under perturbation in codimension one."

ii) The stability result of Theorem II.11 may also be sharpened in codimension one to "any leaf of $\mathscr{P}(\mathscr{F}')$ intersects the neighborhood η of $\mathscr{P}(\mathscr{F})$. For by Plante any leaf in the support of $\mathscr{P}(\mathscr{F}')$ is "closed at infinity".

Now consider the case when the Poincaré set $\mathscr{P}(\mathscr{F})$ is all of M. We say $\mathscr{F}$ is *totally recurrent* in this case.

Theorem II.18. *A totally recurrent codimension one foliation $\mathscr{F}$ can be (tangent plane) approximated by foliations defined by C^∞ closed one-forms.*

Proof. As in Proposition II.10 we can choose a foliation cycle Z whose support is all of M. Now apply the regularization procedure described in Proposition I.8, $Z \to \pi_*((M \times Z) \wedge U_\varepsilon)$. The result is a closed one-form ω_ε which is nowhere zero (the interior support is all of M (I.8)) and whose kernel approximates $\mathscr{F}$.

We can characterize such codimension one foliations.

Theorem II.19. *A transversely oriented codimension one foliation can be approximated by foliations defined by closed one-forms iff no transversal 1-cycle is homologous to zero.*

Proof. By transversal one cycle we mean a closed current in a transverse cone structure to $\mathscr{F}$. Let C_ε denote a cone field of 1-directions in the positive sense and such that C_ε approaches the positive half space as $\varepsilon \to 0$.

The hypothesis means we can have by I.7, a closed 1-form ω_ε positive on C_ε. The foliations defined by ω_ε approximate $\mathscr{F}$ as ε approaches zero.

Remark. The condition of II.19 can be rephased in terms of closed transversal curves and the classes they represent in $H_1(M, R)$.

The condition is – "*all closed transversal curves making a bounded angle with the foliation generate a compact convex cone in $H_1(M, R)$.*"

This condition is sufficient because we may approximate structure cycles in a cone of 1-directions by closed curves in a slight larger cone (the proof is the same as that of Proposition II.25).

The condition is also clearly necessary.

Finally we have the following

Theorem II.20. *Suppose each compact leaf in a transversely oriented codimension one foliation is cut by a closed transversal curve. Then there is a volune preserving flow transverse to the foliation.*

Proof. We may assume that each leaf is cut by a closed transversal curve (since this is already true for non-compact leaves). Then any foliation cycle is not homologous to zero by intersection theory (I.11). Thus there is a closed $(n-1)$ form ω positive on the foliation. The kernel of ω defines the direction field of the desired flow.

Remark. The necessity of the condition in Theorem II.20 is an easy consequence of the Poincaré recurrence theorem applied to the flow.

§6. A Homological Proof of Novikov's Theorem

We are in a position to give a rather understandable proof of Novikov's beautiful result. [H]

Theorem II.21 (Novikov). *A codimension one foliation of the three sphere has a compact leaf.*

Proof. Step 1: One produces (as usual) a vanishing cycle (II.14) using the simple connectivity of the sphere via the argument of Haefliger's thesis (see Remark below).

Step 2: A vanishing cycle leads to a foliation cycle (Theorem II.15, see Remark below).

Step 3: The foliation cycle must be supported on compact leaves because H_2 of the sphere is zero (Proposition II.16). Q.E.D.

Remarks. Step 1: Agrees with Novikov's first step [H]. One constructs a closed transversal γ (using any non-compact leaf), one spans γ by a two-disk in general

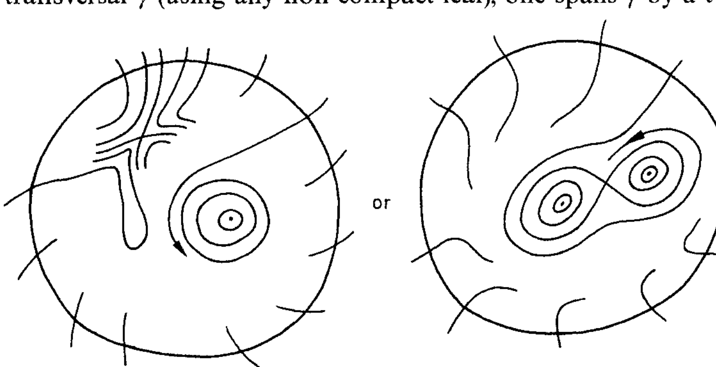

Fig. 8

position relative to $\mathscr{F}$, and one finds the vanishing cycle by considering the spiraling into centers of leaves of the induced foliation in the disk (Haefliger's use of Poincaré-Bendixson).

The vanishing cycle need not be embedded on its leaf to apply II.15.

Step 2: Is also Novikov's second step. The idea of the exploding region in Theorem II.15 is at the beginning of Novikov's geometric argument. (I am indebted to Harold Rosenberg for explaining this part of Novikov's argument to me.)

Step 3: Replaces Novikov's longer and more delicate geometric argument—which in fact gives the technically stronger result that the leaf containing the critical cycle is the boundary of a Reeb component (and doesn't use $H_2=0$). Note the foliation cycle also "finds the compact leaf".

§ 7. Compact Foliations

There has been recent progress in the study of *compact foliations*—compact manifolds foliated by compact leaves. There are examples where the volume of leaves is unbounded [S] and surprisingly delicate theorems about the non-existence of this phenomenon [M], [E], [EMS].

Here we recast one of the theorems of [EMS] in terms of foliation cycles. $(M, \mathscr{F})$ is heretofore a compact foliation.

Proposition II.22. *The extreme rays of the cone of foliation cycles in a compact foliation are the multiples of individual leaves.*

Proof. This is essentially done in [EMS] in different terms. There is a filtration of $M=\{X_\alpha\}$ by invariant sets where each leaf in $X_\alpha - X_{\alpha+1}$ has finite holonomy (and arbitrarily small invariant neighborhoods) in X_α. One merely looks at the smallest α so that X_α intersects the support of the cycle—to see that the transverse measure must reduce to a Dirac measure if the cycle is on an extreme ray in the cone of cycles.

In [EMS] a compact foliation was said to be "homologically positive" if the leaves determined classes in an open half-space of the real homology (for some orientation of the foliation). The definition may also be applied to any closed invariant subset X using Cech cohomology.

The following corollary was conjectured by Bob Edwards.

Corollary II.23. *If X is a "homologically positive" closed invariant set of a compact foliation, there is a closed p-form defined on a neighborhood of X which is positive on the tangent planes of $\mathscr{F}$ along X.*

Proof. The previous proposition applies just as well to X. Now any foliation cycle in X is by Choquet's theorem [Ph] a positive measure of extreme cycles (we suppose a normalization). If ω is a closed form defined on a neighborhood of X so that (leaf in X, ω)>0, then for any cycle $z=\int(L)\,dv$ (Choquet representation). $\int_z \omega = \int (L,\omega)\,dv > 0$. (This formula was also achieved in [EMS] by direct analysis.) Thus the foliation cycles on X are not boundaries in some closed neighborhood where ω is defined. By theorem II.2 there is a closed p-form positive on the tangent planes of $\mathscr{F}$ along X.

Now we recall the Moving leaf proposition of [EMS].

Proposition [EMS]. *If the volume of leaves is unbounded there is an isotopy of leaves L_t so that volume L_t is unbounded.*

Remark. The proof is a more delicate use of the filtration X_α and a theorem of Newman concerning periodic homeomorphisms.

Using the moving leaf and the positive form constructed above we can have a proof of the theorem of [EMS].

Theorem [EMS]. *If a compact foliation is "homologically positive" there is a bound on the volume of leaves.*

Proof. Consider the integral of the closed form of the corollary, η, over the moving leaf L_t: $\int_{L_t} \eta$. This integral is both constant (because η is closed and the L_t are homologous) and essentially the volume of L_t (because η is positive on $\mathscr{F}$ and M is compact). This contradiction proves the theorem.

Remark. A bound on volume allows one to describe the geometric structure of $\mathscr{F}$ rather completely as a generalized Seifert fibration. If there is no bound the possibilities are formidable – for example the filtration X_α referred to above *might* have any countable ordinal type.

§ 8. One Dimensional Foliations

A choice of a non-zero vector field X tangent to a one-dimensional foliation $\mathscr{F}$ determines a flow φ_t whose orbits are the leaves of $\mathscr{F}$. The vector field defines a map from measures to 1-currents, $\mu \to (X, \mu)$ defined by $(X, \mu)\ (\omega) = \int \omega(X)\, d\mu$.

Proposition II.24. *$\mu \to (X, \mu)$ defines continuous bijections*

i) *non negative mearues on $M \sim$ foliation currents along $\mathscr{F}$,*
ii) *measures invariant under flow $\sim$ foliation cycles along $\mathscr{F}$.*

Proof. Theorem I.12 (Step i) and Theorem I.13.

We could say that foliations cycles in dimension one correspond to that part of the ergodic theory of φ_t (the theory of invariant measures for the flow φ_t) which depends only on the orbit structure and not the parametrization.

If a φ_t-invariant measure μ is ergodic ($\varphi_t A = A$ implies $\mu A = 0$ or 1) one may reinterpret the ergodic theorem as follows ($\Gamma_t(x)$ denotes the orbit of x out to time t):

"for μ-almost all x the currents $1/t \cdot (\Gamma_t(x))$ converge as $t \to \infty$ to the foliation cycle (X, μ)" [Sch].

One corollary is the following

Proposition II.25. *Any foliation cycle in dimension 1 can be approximated by a sum of circles which are nearly tangent to the foliation.*

Proof. In fact if the cycle corresponds to an ergodic measure one waits until x reenters the front of a flow box about x and deflects the orbit slightly to close it. Then a single closed curve approximates $1/t \cdot \Gamma_t(x)$ and thus the current (X, μ), μ ergodic.

Any invariant measure is a approximated by finite convex sum of ergodic measures [Ph] and the result follows.

Remark. i) One example of this is a volume preserving flow – which now appears as approximately equal as a current to a number of long circles coiled neatly around so as to nearly fill up the manifold. This interpretation was suggested by Thurston.

ii) One may ask when the analogue of proposition II.25 holds for foliations of higher dimension.

Now we derive some corollaries of Theorem II.2 in the context of flows.

First suppose K is any compact set indide the manifold M on which we have a non-singular flow.

Theorem II.26. *The compact set K satisfies exactly one of the following*

i) *K contains a closed invariant set*

ii) *There is a gradient – like function f defined near K (that is, df is positive on the flow directions).*

Proof. Condition i) is equivalent to, there is a flow cycle whose support lies in K. In the absence of i) we must have ii) by the Theorem II.2 (relative).

Now take the case when K is all of M and suppose we are in the non-degenerate case (Theorem II.3).

Theorem II.27 (Schwartzman). *If each invariant measure for a flow determines a non-trivial homology class in $H_1(M, R)$ then the flow has a C^∞ global cross section, (A closed submanifold transverse to the flow and cutting every orbit.).*

Proof. By hypothesis we are in the non-degenerate case and there is a closed 1-form ω positive on the flow lines. If we assume the form ω has rational periods the path integrals $\int_p^x \omega$ define a submersion of M onto the circle (argument of Abel and Tischler). Any fibre provides the global cross section.

Remark. The condition is clearly necessary. Note that Theorem II.3 says any rational cohomology class supporting the cone $\mathbb{C}$ in homology can serve here.

Now we discuss *volume preserving flows*. First there is the analogue of Theorem II.18 (recall $P(\mathscr{F})$ is the union of the support of all transversal invariant measures, Definition II.9).

Theorem II.28. *Let $\mathscr{F}$ be a one dimensional (oriented) foliation which is totally recurrent – $P(\mathscr{F}) = M$. Then M may be (direction field) approximated by a volume preserving flow.*

Proof. We proceed as in II.18. By hypothesis there is a foliation cycle whose support is all of M. A diffuse approximation is a closed $(n-1)$ form defining the volume preserving flow (Proposition I.8).

Now suppose an $(n-1)$ plane field τ is given and we ask whether or not there are any volume preserving flows in the contractible space of flows transverse to τ.

Theorem II.29. *A transversely oriented plane field τ admits a transverse volume preserving flow iff no $(n-1)$-cycle tangent to τ is homologous to zero.*

Proof. By an $(n-1)$ cycle tangent to τ we mean a closed structure current for the cone structure determined by an orientation of τ. Then the theorem follows from II.2 which yields the desired closed $(n-1)$ form.

Remark. We recall a particular case – if τ defines a foliation with all leaves non-compact, there is a transversal volume preserving flow (Theorem II.20).

Although we haven't formulated the notion precisely – the existence of a tangent cycle means τ has a global solution. Thus a definite lack of integrability forces the transverse volume preserving flow. Of course complete non-integrability – τ defines a contact structure – implies there is a contact flow transverse to τ which is much more (If ω is a 1-form whose kernel is τ then $\omega \wedge (d\omega)^k$ is a volume form, $\dim M = 2k+1$, and $(d\omega)^k$ defines a volume preserving (in fact symplectic preserving) flow.).

We also have (analogous to Theorem II.19)

Theorem II.30. *A given flow may be approximated by a volume preserving flow iff no transverse $(n-1)$ cycle is homologous to zero. Using I.10 one sees the transverse $(n-1)$-cycle may be taken to be a closed submanifold. We assume the flow is transversely orientable.*

Proof. By transverse cycle we mean a structure cycle for a cone of directions transverse to the flow. Given this condition the closed $(n-1)$ forms positive on wider and wider cones (given by Theorem II.2) provide the approximation by volume preserving flows.

Remark. These theorems on volume preserving flows are in the spirit of recent work by Dan Asimov [As].

Part III: Applications to Other Structures

§9. Lorenz Geometry

A cone structure of 1-direction includes the case of a conformal class of Lorenz geometries – with the positive light cone determining the cone field.

Suppose M (not necessarily compact) is provided with a field of oriented light cones and let K be a compact region of M. Then we have (analogous to Theorem II.26),

Theorem III.1. *Exactly one of the following holds:*

i) *in K there is a non-trivial one-cycle "going in the light cone directions",*

ii) *there is a function defined in a neighborhood of K whose gradient is positive on the light cone directions.*

Proof. Either we have a cycle or an exact one form df by Theorem I.7.

Remark. The one cycle of proposition i) can be closely approximated by sums of closed curves whose directions are arbitrarily close to the light cones. (One applies diffusion Proposition I.10 and then the deflection argument of the proof of Theorem II.25.)

Note the one-cycle in the limit defines a recurrence set in K for the dynamical structure determined by the cone field.

With the approximation remark the above theorem comes close to one proved by Hawking ([HE], p. 198).

Theorem (Hawking). *A (non-compact) space-time has a cosmic time function iff it is stably causal (there is no closed curve which is almost time-like).*

Hawking's theorem suggests the basic technique here can be extended to non-compact manifolds.

§ 10. Symplectic Structures

We can describe a geometrical cone structure which plays a Poincaré dual role to a symplectic structure (a closed 2-form ω so that ω^n is a volume form).

Say that a cone structure of 2-directions is *ample* if at each point x the cone C_x intersects the linear span of the Shubert variety S_τ of every 2-plane τ at x (S_τ is the set of 2-planes which intersect τ in at least a line).

Now suppose M is a closed orientable even dimensional manifold.

Theorem III.2. i) *An ample cone structure of 2-directions on M always has non-trivial cycles.*

ii) *If no structure cycle is homologous to zero then M admits a symplectic structure (transverse to the ample 2-direction structure).*

Proof. If a 2-form ω is positive on the cone C_x then ω cannot have less than the maximal rank – for if ω were induced from a $(2n-2)$ dimensional quotient of the tangent space at x (dimension $M=2n$) ω would annihilate the linear span of the Shubert variety of the kernel (a 2-plane).

Such a form cannot be exact for its n-th power is a volume form, $\int_M \omega^n \neq 0$ and so $\omega, \omega^2, \ldots, \omega^n$ are all cohomologically non-trivial. Thus by Theorem I.7 there must be non-trivial structure cycles.

If for some ample 2-direction structure no cycle is homologous to zero, then Theorem I.7 gives a closed 2-form ω positive on the cone structure. By the first remark ω has maximal rank and so provides a symplectic structure on M.

Example. i) Let J denote a (continuously varying) complex structure on each tangent space, $J^2=-Id$. Then the complex lines at x generate a cone $C(J)_x$ of 2-directions in $\Lambda_2(x)$ which is ample. (For any plane τ and any vector v in τ the plane (v, Jv) belongs to S_τ and to the structure $C(J)_x$.)

Also $C(J)_x$ is a compact cone because in terms of standard coordinates on $\mathbb{C}^n$, $z_j = x_j + i y_j$, $\omega = \sum_i dx_i \wedge dy_i$ gives a supporting hyperplane for $C(J)_x$. So the $C(J)_x$ define a cone structure of 2-directions which is ample.

ii) We can ask whether a given almost complex structure J has a transversal symplectic structure ω – a closed 2-form ω positive on the complex lines. Then Theorem III.2 asserts

a) there are always 2-cycles going in complex directions,

b) there is a transversal symplectic structure iff none of these cycles is homologous to zero.

iii) A symplectic structure ω determines preferred coordinates whose overlap diffeomorphisms preserve the basic form $\sum_i dx_i \wedge dy_i$. (Darboux's Lemma). In particular the structure group of the tangent bundle is reduced to $Sp(2n, R)$ – the group preserving the basic form.

Since $Gl(n, \mathbb{C}) \subset Gl(2n, R)$ and $Sp(2n, R) \subset Gl(2n, R)$ have the same maximal compact $U(n) \subset Gl(2n, R)$ there is a well defined contractible set of almost complex structures J determined by ω. They are in fact characterized by the transversality condition $\omega(x, Jx) > 0$. In particular, every symplectic structure arises in the manner proposed by the Theorem III.2 – given ω choose a transversal J and the homological situation of the theorem holds for the ample cone of 2-directions determined by J.

Now we give a geometric condition to be able to deform a closed 2-form η of rank 2 on a 4-manifold to a symplectic form. Suppose the foliation defined by η is homologically non-degenerate (II.3). For example, suppose there is an immersed surface which is transversal to the foliation and cuts every leaf.

Theorem III.3. *A closed 2-form η of constant rank 2 on a 4-manifold which defines a homologically non-degenerate (II.3) foliation can be perturbed to a symplectic two form.*

Proof. By Theorem II.2 there is a closed 2 form ν which is positive on the leaves of the foliation defined by η. Let $\omega = \eta + \varepsilon \nu$. Then $\omega^2 = (\eta + \varepsilon \nu)^2 = 2\varepsilon\eta \cdot \nu + \varepsilon^2 \nu^2 \sim 2\varepsilon\eta \cdot \nu$ (for ε very small) is a volume form.

Example. Suppose M^4 has two transversal foliations $\mathscr{F}_1$ and $\mathscr{F}_2$. Assume $\mathscr{F}_1$ has a smooth transversal measure whose interior support is all of M and $\mathscr{F}_2$ has any transversal measure whose support cuts every leaf of $\mathscr{F}_1$. Then III.3 applies.

Such examples are not infrequent. One case is a fibration over a 2-manifold with fibre a 2-manifold and structure group discrete and measure preserving (for some measure). The latter include the examples of Thurston where M^4 is symplectic but the first Betti number is odd.

Remark. One can view a symplectic structure geometrically as a $(2n-2)$ cycle Z for a cone structure transverse to an ample 2-direction structure – if the support of the cycle is all of M. For one may diffuse Z into a symplectic form (if it is not already diffuse) by Proposition I.9.

For the canonical structure ω on complex projective space there are many ways to approximate the current ω by large numbers of complex hypersurfaces. Such descriptions restrict (by intersection) to approximate the symplectic form on any algebraic submanifold of projective space. This yields a geometric picture of any symplectic structure using Gromov's theorem that it may be induced by an immersion into complex projective space.

§ 11. Complex Structures on Manifolds

A compact complex manifold M ($\mathbb{C}$-dimension$=n$) has $(n+1)$ natural cone structures $C_0, C_1, \ldots, C_n$ defined by the almost complex structure J.[5] At a point x, $C_p(x)$ is the compact convex cone in $\Lambda_{2p}(x)$ generated by the positive combinations of complex subspaces of $\mathbb{C}$-dimension p. (I.2 and Example following III.2.)

If we compare from a dynamical point of view a complex manifold and a foliated manifold – each has preferred local submanifolds which have unique maximal extensions – one can nope the study of these cone structures will be useful.

We will describe the situation as far as we understand it, vis a vis Part I. The statements are merely listed and proofs given in parentheses.

III.4. "*For each p the structure currents for C_p form a compact convex cone in the space of 2p-currents on M*".

((I.5) This cone is by definition (I.4) the convex closure of the Dirac currents obtained by evaluating forms on the single elements of $C_p(x)$.)

III.5. "*Each structure current c may be represented by a pair (μ, v), where μ is a non-negative measure on M and v is a μ-measurable function into 2p-vectors ($v(x) \in C_p(x)$), via an integral*

$$c = \int_M v(x)\, d\mu\text{"} \quad (I.8).$$

"*This representation is not unique although the related one as a 2p-vector valued measure is unique* (I.8).

III.6. "*The integration currents defined by compact complex submanifolds (with boundary) and their positive combinations are dense in the structure currents for C_p.*"

(Clearly the Dirac currents above can be so approximated.)

For this reason and some below we term the structure currents for C_p *complex currents* and the closed currents among them *omplex cycles.*

III.7. "*The cone of complex cycles (denoted $\mathbb{C}_p$) is a compact convex cone in the space of 2p-currents.*"

($\mathbb{C}_p$ is the intersection of the compact convex cone of complex currents with the closed subspace of cycles.)

III.8. Points on extreme rays of the compact cone $\mathbb{C}_p$ of complex cycles are called *irreducible complex cycles.*

"*Any complex cycle c can be represented by a measure on the space of irreducible cycles $c = \int c'\, d\mu$*". (Choquet's theorem [Ph].)

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[5] *Notation.* $\mathcal{J}$ denotes the circle action on forms and currents generated by the automorphism of tangent spaces J satisfying $J^2 = -\mathrm{Id}$. Example: When $n=4$, $C_2(x)$ is a 36-dimensional cone in a 70-dimensional space, $\Lambda^4 R^8$

Remark. The geometric analysis of irreducible complex cycles is completely open. These include classical irreducible subvarieties and (certain) foliation cycles in holomorphic foliations. An optimal conjecture for non-classical cycles would involve a representation by a non-compact analytic space V which is "closed at ∞" (II.5) relative to some metrical structure and which maps to M, $V \xrightarrow{f} M$ by a holomorphic map which is generically an immersion and has bounded distortion. Algebraic varieties in $\mathbb{C}^n$ project to such examples in torii.

III.9. *We do conjecture that "complex cycles can be locally represented near x in M as $c = \int c' \, dv$ where the c' are irreducible subvarieties passing near x and v is a non-negative measure on the space of such (regarded as a subspace of the currents on some neighborhood of x.)"*

Remark. The analogous local result holds for foliations (Theorem I.12).

III.10. "*A compact complex manifold M always has non-trivial complex cycles in $\mathbb{C}$-dimension one.*"

(III.2 and example following.)

III.11. "*If no complex cycle in dimension one is homologous to zero, the same is true for cycles in higher dimensions. In this case the manifold has a symplectic structure transverse to the complex structure*" (III.2).

III.12. "*For such complex-symplectic manifolds (which include Kaehler manifolds) the set of complex cycles in one homology class (the Chow space) forms a compact convex set*" (I.7 iv)a).

III.13. "*The Lie group of holomorphic automorphisms of a compact complex manifold acts on each cone of complex cycles. Each particular automorphism $M \xrightarrow{\sigma} M$ fixes rays in each cone. Namely for each automorphism σ and for each p there is a non-trivial complex cycle z and a positive number λ so that*

$\sigma z = \lambda z$ (as $2p$-currents)."

We are tacitly assuming in this argument that there are complex cycles in $\mathbb{C}$-dimension p.

(One applies the fixed point theorem for compact convex sets to the induced map on some base of the cone of cycles.)

III.14. Recall $\mathbb{C}_p$ denotes the cone of complex cycles of $\mathbb{C}$-dimension p. Let $D_p \subset \mathbb{C}_p$ denote the diffuse complex cycles. (Those currents in $\mathbb{C}_p$ which are given by (closed) $2n-2p$ forms).

"*If $p+q=n+r$, the natural intersection pairings (between diffuse currents and currents) satisfy*

$$D_p \cdot \mathbb{C}_q \subset \mathbb{C}_r. \qquad (*)$$

In particular if $p+q=n$, the intersection numbers $[D_p \cdot \mathbb{C}_q]$ are non-negative, and the diffuse complex cycles form a semi-ring under intersection."

(By (Remark iii) II.8) $(*)$ reduces to linear algebra at a point x. If Ω is a positive generator of $\Lambda_{2n}(x)$ and v and v' are in $C_p(x)$ and $C_q(x)$ one sees that $(\Omega \wedge v) \wedge v'$ is in $C_r(x)$ by considering the cases when v and v' are decomposable.)

III.15. "*A Kaehler metric on a compact complex manifold determines a diffuse complex cycle of* $\mathbb{C}$-*dimension* $n-1$."

(The metric can be written $\omega(x, Jy)+i\omega(x, y)$ where ω is a symplectic 2-form invariant under J, $\omega(x, y)=\omega(Jx, Jy)$. The $2n-2$ current determined by ω, the Kaehler form, is a diffuse complex cycle. For in a basis $\{z_i=x_i+iy_i\}$ of $\mathbb{C}^n$ the canonical 2-form $\Sigma\, dx_i \wedge dy_i$ on R^{2n} corresponds to the ray given by the sum of the coordinate complex hyperplanes of $\mathbb{C}^n=R^{2n}$ in Λ_{2n-2}. (Conversely a diffuse $(n-1)$ cycle whose interior support is all of M determines a Kaehler metric.))

III.16. "*A Kaehler manifold* M^n *has non-trivial complex cycles in every dimension* $0 \leqq d \leqq n$" (III.14, III.15).

Let $[\mathbb{C}_p] \subset H_{2p}(M, R)$ and $[D_p] \subset H_{2p}(M, R)$ denote the cones in homology determined by the complex cycles and the diffuse complex cycles. Let $\mathscr{H}_p \subset H_{2p}(M, R)$ denote the Hodge space, the real points of $H_{p,p}(M, \mathbb{C})$.

Theorem III.17. "*If* M *is a Kaehler manifold, the compact convex cone,* $[\mathbb{C}_p] \subset H_{2p}(M, R)$, *determined by the complex cycles is contained in and generates the Hodge space* $\mathscr{H}_p \subset H_{2p}(M, R)$."

Proof. Write $p+q=n$ and consider the three subspaces of $H_{2p}(M, R)$,

V_1 = closed $\mathscr{J}$-invariant $2q$ forms (mod exact).

V_2 = closed $\mathscr{J}$-invariant $2p$ currents (mod boundaries),

C = the linear span of $[\mathbb{C}_p]$.

The theorem follows from the relations

i) $V_1 \subset C \subset V_2$,

ii) $V_1 = V_2 = \mathscr{H}_p$.

The first inclusion of i) is the important one. First note that the q-th power η of a Kaehler form belongs to V_1 and also C (III.15). Moreover a small perturbation of η in the closed $\mathscr{J}$-invariant forms remains a complex cycle so stays in C. (Diffuse complex cycles ω contain those satisfying the following conditions: ω is a closed $2q$-form which is $\mathscr{J}$ invariant and at those points x where ω is not zero $\omega(x)$ is positive on the (non-zero part of the) dual cone in $\Lambda_{2q}(x)$ of $C_p(x) \subset \Lambda_{2p}(x)$, $p+q=n$. Since η is non-zero at every point and satisfies the condition, a C^0 perturbation will also.) By the open mapping principle for Frechet spaces such perturbations form an open set of V_1 and this implies all of V_1 is contained in C.

The second inclusion of i) is obvious from the definition. The relations of ii) are formal consequences of Hodge theory. Q.E.D.

As corollaries we have,

III.18. "*The cone* $[D_p]$ *has non void interior in the cone* $[\mathbb{C}_p]$."

III.19. "*If* M *is projective algebraic the rational points of the Hodge space intersect the interior of the cone* $[\mathbb{C}_p]$."

III.20. *"Under a small deformation of the complex structure of a Kaehler manifold the cones* $[\mathbb{C}_p] \subset H_{2p}(M, R)$ *maintain a constant dimension."*

III.21. *"The linear spaces generated by the cones* $[\mathbb{C}_p]$ *and* $[\mathbb{C}_q]$ *are dually paired under Poincaré duality* $p+q=n$."

III.22. *"The Hodge space* $\mathscr{H}_p \subset H_{2p}(M, R)$ *is generated by irreducible complex cycles."*

Proofs. III.18 follows directly from the proof of Theorem III.17. So does III.19 remembering that the natural η determines a rational point in the projective algebraic case. III.20 and III.21 are well known properties of the Hodge spaces. III.22 follows because linear combinations of extreme rays (the irreducible complex cycles) are dense in $\mathbb{C}_p$ (the complex cycles).

III.23. The cone $[\mathbb{C}_p] \subset \mathscr{H}_p \subset H_{2p}(M, R)$ adds precision to the questions about homology classes represented by algebraic subvarieties of a projective algebraic manifold. Besides III.19 one has,

"The algebraic classes lie in the cone $[\mathbb{C}_p]$. *In fact the positive algebraic cycles are precisely the rectifiable currents among the complex cycles* $\mathbb{C}_p$."

(The first statement is clear from the definitions. The second one follows easily from Theorem 5.2.1 [K].)

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