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Trigonometric Sums, Green Functions of Finite Groups and Representations of Weyl Groups

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Introduction

Let G be a connected reductive algebraic group over the finite field $k \simeq \mathbb{F}_q$, and let $\mathfrak{g}$ be its Lie algebra. Assume, for simplicity, that there is a "Killing form" B on $\mathfrak{g}$, i.e. a G-invariant non-degenerate symmetric bilinear form which is defined over k. Denote by F the Frobenius morphism. Let ψ be a nontrivial character of the additive group of k, with values in a field of characteristic 0. In this paper we study trigonometric sums of the form

$$S^G(A, A') = \sum_{X \in O(A')^F} \psi(B(A, X)),$$

where A and A' are in the finite Lie algebra $\mathfrak{g}^F$ of fixed points of F and $O(A')$ is the G-orbit of A' under adjoint action. We are interested, in particular, in the case that A is nilpotent and A' strongly regular (i.e. its centralizer in G is a maximal torus). The corresponding trigonometric sums were introduced in [25] and it was shown there that if $G = GL_n$ they are very closely related to the Green polynomials of GL_n, which govern the character theory of $GL_n(k)$. In fact, if $G = GL_n$ or if G is semisimple and if the characteristic p of k is sufficiently good, there is a G-equivariant bijection of the unipotent set of G onto the nilpotent set of $\mathfrak{g}$, which commutes with F [24]. This bijection allows one to pass from nilpotents of $\mathfrak{g}^F$ to unipotents of G^F. The $S^G(A, A')$ then lead to functions on the unipotent set of G^F which are likely to be related to values of certain irreducible characters of G^F on the unipotent set.

Using the results of this paper, D.A. Kazhdan [15] has proved that such a relation does indeed exist, and that the $S^G(A, A')$ are related to the irreducible characters of the finite group G^F which are constructed by Deligne and Lusztig in [10], if p and q are sufficiently large.

The results of this paper give, as a complement to those of [10], formulas for the values of the irreducible characters of [10] on the unipotents of G^F (if p and q are large).

We shall now give a brief review of the contents of the paper. The first section contains some results (essentially well-known) about the description of trigonometric sums like $S^G(A, A')$ via l-adic cohomology. We also compute, as an example and for later use, the $S^G(A, A')$ for $G = GL_2$. No. 2 deals with algebraic questions. Since a "Killing form" does not always exist, we have preferred to avoid an identification of $\mathfrak{g}$ with its dual vector space $\mathfrak{g}'$ via such a form, working instead directly with $\mathfrak{g}'$. We then require a number of results about $\mathfrak{g}'$, in particular about the existence of strongly regular elements in $\mathfrak{g}'$ and $(\mathfrak{g}')^F$. These are established in no. 2.

In no. 3 we take up the trigonometric sums $S^G(A, A')$ in the setting of the previous paragraph. We take $A' \in (\mathfrak{g}')^F$ strongly regular and $A \in \mathfrak{g}^F$ arbitrary. The main result 3.15 of this section is a reduction theorem (proved by cohomological methods). Let $A = A_s + A_n$ be the Jordan decomposition of A and H the connected centralizer of A_s. Then 3.15 reduces the determination of $S^G(A, A')$ to that of the $S^H(A_n, A')$. This result is similar to [10, Th. 4.2].

No. 4 deals with the $S^G(A, A')$ for nilpotent A. The centralizer T of A' is a maximal torus of G which is defined over k. Let W be the Weyl group of T. If B is a Borel subgroup of G containing T there is $w \in W$ such that $FB = w \cdot B$. Let $\mathscr{B}_A$ be the variety of Borel subgroups of G whose Lie algebra contains A. The main result 4.4 of no. 4 is that there exists a graded representation r_B^* of W on the l-adic cohomology of $\mathscr{B}_A$ with constant coefficients, such that $S^G(A, A')$ is given (up to a constant) by the alternating sum of the traces of the $F^* r_B^i(w)$ on the cohomology groups $H^i(\mathscr{B}_A)$. It follows, in particular, that $S^G(A, A')$ depends only on the centralizer T of A'.

Using the last fact, we define in no. 5 Green functions $Q_{T,G}$, which are functions on the nilpotent set of $\mathfrak{g}^F$ indexed by (G^F-conjugacy classes of) maximal tori of G defined over k. Under some restrictions on p and q, we prove that they satisfy orthogonality relations (see 5.6). By Kazhdan's results these Green functions can be connected with the Green functions of [10, no. 4].

As a consequence of the results of no. 4 and no. 5 we obtain in no. 6 a realization of the irreducible representations of the Weyl group W of G. To do this we study the behaviour of the $Q_{T,G}$ for large q. The results are as follows. Assume p sufficiently large. Fix a nilpotent $A \in \mathfrak{g}$, let $Z_G(A)$ be its centralizer and $C(A)$ the quotient of $Z_G(A)$ by its identity component. We have $e(A) = \dim \mathscr{B}_A = \frac{1}{2}(\dim Z_G(A)$-rank $G)$.

There is a representation s of $C(A)$ in $H^{2e(A)}(\mathscr{B}_A)$: it is the permutation representation defined by the action of $C(A)$ on the irreducible components of $\mathscr{B}_A$ of dimension $e(A)$. The results of no. 5 give a representation of W in $H^{2e(A)}(\mathscr{B}_A)$, which turns out to commute with s. We prove that $C(A) \times W$ acts irreducibly in the non-zero isotypic subspaces of s and if A runs through a set of representatives of the nilpotent G-orbits in $\mathfrak{g}$, then each irreducible representation of W is obtained in a unique manner (see 6.10). A reduction argument then gives a similar statement in characteristic 0 (6.14). It would be interesting to have a more direct proof of such results over $\mathbb{C}$.

In the last section we discuss a number of examples and special cases.

It will be clear from the preceding that l-adic cohomology provides the key to the nontrivial results of this paper. I am greatly indebted to P. Deligne, who showed me (in a letter of November, 1973) how to use this key, by determining the $S^G(A, A')$ for A and A' strongly regular. The method used in no. 3 and no. 4 is due to Deligne. It is a pleasure to thank him for his help and interest.

Part of the work of this paper was done during visits to the Institut des Hautes Etudes Scientifiques in 1973 and 1975. I want to thank the I.H.E.S. for its hospitality.

Some Notations. If S is a finite set, then $|S|$ denotes its cardinal number. If G is a group then $Z_G(\ )$ and $N_G(\ )$ denote centralizers and normalizers.

Algebraic varieties and algebraic groups are taken in the sense of [3]. An algebraic variety V over the field k is identified with its set $V(\bar{k})$ of $\bar{k}$-rational points, $\bar{k}$ an algebraic closure of k. If k is finite, and F the corresponding Frobenius morphism of V then the finite set of fixed points of F on V(i.e. the set of k-rational points $V(k)$) is written V^F.

The Lie algebra of an algebraic group $G, \ldots$ is written with the corresponding gothic letter $\mathfrak{g}, \ldots$

A graded module $\oplus M^i$ is written M^* (since these usually come from cohomology, we use superscripts); similarly for graded sheaves. A homomorphism of graded objects is denoted $f^* = (f^i)$. However, F^* *always* denotes a cohomology homomorphism defined by a Frobenius morphism.

1. Trigonometric Sums

1.1. Let k be a finite field with q elements, of characteristic p. Denote by $\bar{k}$ an algebraic closure of k and by F the Frobenius automorphism $x \mapsto x^q$ of $\bar{k}$.

Let $l \neq p$ be a prime number. Denote by E an algebraic extension of the field $\mathbb{Q}_l$ of l-adic numbers, which contains the p-th roots of unity. We fix a nontrivial character $\psi: k \to E^*$ of the additive group of k.

1.2. Let $\mathbb{A}^n$ be n-dimensional affine space over k. Let V be a k-subvariety of $\mathbb{A}^n$ and P a polynomial function on $\mathbb{A}^n$, defined over k. Denote by Y_P the subvariety of $\mathbb{A}^1 \times V$ formed by the (x, v) with $x^q - x = P(v)$.

If $a \in k$, the k-rational points of Y_{P+a} are the (x, v) with $x \in k$, $v \in V^F$ (the fixed point set of F in V) and $P(v) = -a$. The additive group k acts on Y_P, the action being given by

$$\tau(a)(x, v) = (x + a, v),$$

for $a \in k$. Fix $a \in k$ and let $b \in \bar{k}$ be such that $b^q - b = a$. Then $f: (x, v) \mapsto (x + b, v)$ defines an isomorphism (over $\bar{k}$) of Y_P onto Y_{P+a} and we have

$$F \cdot f = f \cdot \tau(a) \cdot F. \tag{1}$$

1.3. We have

$$\sum_{v \in V^F} \psi(P(v)) = q^{-1} \sum_{a \in k} \psi(-a) \, |Y_{P+a}^F|$$

and by a result of Grothendieck [11] the number of k-rational points $|Y_{P+a}^F|$ is given by

$$|Y_{P+a}^F| = \sum_{i \geqq 0} (-1)^i \operatorname{Tr}(F^*, H_c^i(Y_{P+a}, E)),$$

where F^* is the homomorphism induced by F in the l-adic cohomology with proper support and values in the constant sheaf E. It follows from (1) that ($\tau(a)^*$ denoting the induced homomorphism)

$$\mathrm{Tr}(F^*, H_c^i(Y_{P+a}, E)) = \mathrm{Tr}(F^* \tau(a)^i, H_c^i(Y_P, E)),$$

whence

$$\sum_{v \in V^F} \psi(P(v)) = q^{-1} \sum_{i \geq 0} (-1)^i \sum_{a \in k} \psi(-a) \,\mathrm{Tr}(F^* \tau(a)^i, H_c^i(Y_P, E)). \tag{2}$$

Let $H_c^i(Y_P, E)_\psi$ be the subspace of $H_c^i(Y_P, E)$ formed by the elements x such that

$$\tau(a)^i x = \psi(a) x,$$

for all $a \in k$. It is an F^*-stable subspace. Then (2) can be rewritten as

$$\sum_{v \in V^F} \psi(P(v)) = \sum_{i \geq 0} (-1)^i \,\mathrm{Tr}(F^*, H_c^i(Y_P, E)_\psi). \tag{3}$$

The cohomological interpretation of trigonometric sums given by (3) is equivalent to the one given in [9, no. 8]. Let $\pi\colon Y_P \to V$ be the projection. The direct image sheaf $\pi_! E$ decomposes into a direct sum of locally constant sheaves $\mathscr{S}_\phi$ on V, ϕ running through $\mathrm{Hom}(k, E^*)$ and we have

$$H_c^i(Y_P, E)_\psi \cong H_c^i(V, \mathscr{S}_\psi).$$

The cohomology groups on the right-hand side are the ones used in [loc.cit.].

1.4. In this paper we are interested in the trigonometric sums figuring in (3) in a particular situation.

Let G be a connected reductive linear algebraic group over k. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}'$ its vector space dual. Both $\mathfrak{g}$ and $\mathfrak{g}'$ are viewed as affine spaces defined over k. Let $\langle , \rangle$ denote the canonical pairing of $\mathfrak{g}$ and $\mathfrak{g}'$.

Denote by Ad and Ad' the adjoint representations of G in $\mathfrak{g}$ and its contragredient, respectively. Both are defined over k. If A is an element of $\mathfrak{g}$ or $\mathfrak{g}'$ we denote by $O(A)$ its G-orbit (for Ad or Ad').

Fix elements $A \in \mathfrak{g}^F$ and $A' \in (\mathfrak{g}')^F$. The particular situation of the paper is that where $\mathbb{A}^n = \mathfrak{g}'$, $V = O(A')$ and $P(X') = \langle A, X' \rangle$. In that situation we denote the Y_P of 1.2 by $Y_{A,A'}$ or $Y_{A,A'}^G$. In this case the trigonometric sum of (3) is

$$\sum_{X' \in O(A')^F} \psi(\langle A, X' \rangle).$$

We denote it by $S(A, A')$ or $S^G(A, A')$.

1.5. *Example.* If $G = GL_2$ these sums can easily be computed. Then $\mathfrak{g} = \mathfrak{gl}_2$, the Lie algebra of all 2×2-matrices and $\mathfrak{g}'$ can be identified with $\mathfrak{g}$ via the pairing $(X, X') \mapsto \mathrm{Tr}(XX')$. Let $A' \in (\mathfrak{g}')^F$ be a noncentral semisimple element. Its centralizer is a maximal k-torus T in G. Let $\mathfrak{t}$ be its Lie algebra and $r (= 1, 2)$ its k-rank.

1.6. **Lemma.** (i) *If A is a noncentral semisimple element of $\mathfrak{g}^F$ then*

$$S(A, A') = (-1)^r q \sum_B \psi(\langle B, A' \rangle),$$

the sum being taken over the elements of $\mathfrak{t}^F$ which are G^F-conjugate to A;

(ii) *If* $A \neq 0$ *is nilpotent then*

$$S(A, A') = (-1)^r q.$$

There are $\sigma, \delta \in k$ such that

$$O(A') = \left\{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} \mid x + t = \sigma, xt - yz = \delta \right\}.$$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$S(A, A') = \sum_{e \in k} n_e \psi(e),$$

where n_e is the number of solutions in k^4 of

$$\begin{aligned} xt - yz &= \delta \\ x + t &= \sigma \\ ax + cy + bz + dt &= e \end{aligned}$$

These numbers can easily be computed, and the statement of 1.6 emerges.

If $p \neq 2$ a similar result is true for SL_2, as follows from the fact that then $\mathfrak{gl}_2$ is the direct sum of $\mathfrak{sl}_2$ and its center.

2. Strongly Regular Elements in the Dual of a Lie Algebra

2.1. The notations are as in 1.4. However, in most of this section k may be any field. Let T be a maximal torus of G (not necessarily defined over k). Let Φ be the root system of (G, T). For each $\alpha \in \Phi$ there is a 1-parameter subgroup $U_\alpha \subset G$, let $X_\alpha \in \mathfrak{g}$ be a nonzero tangent vector in its Lie algebra.

$W = W(T) = N_G(T)/T$ denotes the corresponding Weyl group. If $w \in W$, then $n_w \in N_G(T)$ denotes a representative.

We denote by $\mathfrak{t}'$ the subspace of $\mathfrak{g}'$ orthogonal to all X_α ($\alpha \in \Phi$). From the decomposition

$$\mathfrak{g} = \mathfrak{t} + \sum_{\alpha \in \Phi} \bar{k} X_\alpha$$

it is clear that $\mathfrak{t}'$ can be viewed as the dual of $\mathfrak{t}$. We call $\mathfrak{t}'$ a *toral subspace* of $\mathfrak{g}'$. Two such subspaces are conjugate by an element of $\mathrm{Ad}'(G)$. An element of $\mathfrak{g}'$ is *semisimple* if it lies in a toral subspace. (It should be remarked at this point that there is a Jordan decomposition theory for $\mathfrak{g}'$, due to Kac and Weisfeiler [14]. We shall not need it here.)

If $X' \in \mathfrak{g}'$, its centralizer $Z_G(X')$ in G is

$$Z_G(X') = \{x \in G \mid \mathrm{Ad}'(x) X' = X'\}$$

and its connected centralizer is the identity component of that group. A semisimple element of $\mathfrak{g}'$ is *regular* if its connected centralizer is a maximal torus and *strongly regular* if its centralizer is a maximal torus. As we shall make use of strongly regular elements in the sequel, a discussion of their properties is needed.

2.2. First some formulas. For $\alpha\in\Phi$ let x_α be an isomorphism of the additive group onto U_α. Its differential dx_α is a nonzero linear map of $\bar{k}$ into $\mathfrak{g}$ and we assume that $dx_\alpha(1)=X_\alpha$. We also assume the x_α to be such that for $t\neq 0$

$$w_\alpha(t)=x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$$

normalizes T. Put $H_\alpha=[X_\alpha, X_{-\alpha}]$. Define elements $X'_\alpha\in\mathfrak{g}'$ by

$$\langle X_\alpha, X'_\beta\rangle=\delta_{-\alpha,\beta},$$
$$\langle \mathfrak{t}, X'_\alpha\rangle=0,$$

and define $H'_\alpha\in\mathfrak{t}'$ by $\langle H, H'_\alpha\rangle=d\alpha(H)$, if $H\in\mathfrak{t}$. Then $\langle H_\alpha, H'_\alpha\rangle=2$.

We have the following formulas for the action of G on $\mathfrak{g}'$:

$$\begin{aligned}\mathrm{Ad}'x_\alpha(t)A' &= A'-t\langle H_\alpha, A'\rangle X'_\alpha \quad (A'\in\mathfrak{t}'),\\ \mathrm{Ad}'x_\alpha(t)X'_{-\alpha} &= X'_{-\alpha}+tH'_\alpha-t^2X'_\alpha,\\ \mathrm{Ad}'x_\alpha(t)X_\beta &= X'_\beta+\sum_{i\geq 1}c_i t^i X_{\beta+i\alpha} \quad (\beta\neq-\alpha),\end{aligned} \tag{1}$$

where the c_i of the last formula are in the prime field. These formulas are straightforward consequences of well-known ones for the adjoint representation. We find from (1)

$$\mathrm{Ad}'w_\alpha(t)A'=A'-\langle H_\alpha, A'\rangle H'_\alpha.$$

Also, T acts trivially on $\mathfrak{t}'$. If $w\in W$, $A'\in\mathfrak{t}'$ write $w\cdot A'=\mathrm{Ad}(n_w)A'$. Then the last formula reads

$$s_\alpha\cdot A'=A'-\langle H_\alpha, A'\rangle H'_\alpha, \tag{2}$$

where $s_\alpha\in W$ is the reflection defined by α.

2.3. **Lemma.** *Let $A'\in\mathfrak{t}'$. Then $Z_G(A')$ is generated by T, the U_α with $\langle H_\alpha, A'\rangle=0$, and the n_w with $w\cdot A'=A'$.*

Fix a Borel subgroup $B\supset T$. It defines a set of positive roots Φ^+, and the U_α with $\alpha>0$ generate the unipotent radical U of B. If $w\in W$, let $U_w\subset U$ be the subgroup generated by the U_α with $\alpha>0$, $w\alpha<0$.

Now let $g\in G$, $\mathrm{Ad}'(g)A'=A'$. By Bruhat's lemma we can write $g=b^{-1}n_w u$, with $b\in B$, $u\in U_w$. So

$$\mathrm{Ad}'(n_w u n_w^{-1})(w\cdot A')=\mathrm{Ad}'(b)A'.$$

The left-hand side is of the form $w\cdot A'+\sum_{\alpha<0}u_\alpha X_\alpha$ and the right-hand side is of the form $A'+\sum_{\alpha>0}v_\alpha X_\alpha$, as follows from (1). Consequently, we have $w\cdot A'=\mathrm{Ad}'(u)A'=\mathrm{Ad}'(b)A'$. Since T lies in $Z_G(A')$, we have that $B\cap Z_G(A')$ is generated by the U_α which it contains. The assertion now readily follows.

2.4. **Lemma.** *$A'\in\mathfrak{t}'$ is strongly regular if and only if no element $w\neq 1$ of W fixes A'.*

This is a consequence of 2.3, using (2).

2.5. **Proposition.** (i) *If $p\neq 2$ then $\mathfrak{g}'$ contains strongly regular elements;*

(ii) ($p=2$, *G quasi-simple*). *If* $\mathfrak{g}'$ *does not contain strongly regular elements then G is either of type* B_l ($l \geqq 2$), C_l, D_{2l}, E_7, E_8, F_4, G_2 (*i.e. one of the types whose Weyl group contains* -1) *or G is isomorphic to* $SO(4l+2)$ *for some* $l \geqq 1$.

If $\mathfrak{g}'$ does not contain strongly regular elements then by 2.4 some $w \neq 1$ acts trivially on $\mathfrak{g}'$. We now use the following lemma. For part (i) of it see e.g. [22, p. 210] and part (ii) is a direct consequence of (i).

2.6. **Lemma.** *Let* Γ *be a finite group of linear transformations of a real vector space V, fixing a lattice L. Let l be a prime number and let* $\bar{\gamma}$ *be the transformation of* L/lL *induced by* $\gamma \in \Gamma$.

(i) *If* $\gamma \neq 1$, $\bar{\gamma}=1$, *then* $l=2$ *and the order of* γ *is a power of* 2;

(ii) *If the kernel* Δ *of the reduction map* $\gamma \mapsto \bar{\gamma}$ *is nontrivial it is a normal 2-subgroup* Δ *of* Γ. *The isotypic subspaces of V for* Δ *are permuted by the elements of* Γ.

The character group X of T is a W-module and there is a W-equivariant isomorphism $\phi: X \otimes_{\mathbb{Z}} k \xrightarrow{\sim} \mathfrak{t}'$, with

$$\langle A, \phi(\lambda \otimes x)\rangle = x\, d\lambda(A).$$

So the action of W on $\mathfrak{t}'$ is obtained by extension of the base field from the action of W on X/pX. We can then apply 2.6, with $\Gamma = W$, $V = X \otimes_{\mathbb{Z}} \mathbb{R}$, $L = X \otimes 1$, $l = p$ and 2.5 (i) follows from 2.6 (i). It remaining to prove 2.5 (ii).

So let $p=2$, G quasi-simple. If $-1 \in W$ then no strongly regular elements exist in $\mathfrak{t}'$. Let $-1 \notin W$ and assume that (in the previous situation) the group Δ is nontrivial. Now G is of one of the types A_l, D_{2l+1}, E_6. A Weyl group of type A_l (i.e. $\mathfrak{S}_{l+1}$) does not contain a nontrivial normal 2-subgroup if $l > 1$, and neither does the Weyl group of type E_6 (which is an extension of $\mathbb{Z}/2\mathbb{Z}$ by a simple group of order 25920, see e.g. [5, p. 229]). A Weyl group of type D_{2l+1} is a semi-direct product $\mathfrak{S}_{2l+1} \cdot (\mathbb{Z}/2\mathbb{Z})^{2l}$. We then must have $\Delta \subset (\mathbb{Z}/2\mathbb{Z})^{2l}$. Using 2.6 (ii) it follows that we have equality. The assertion for this type can now be checked by looking at the 3 possible actions of W on X.

2.7. **Lemma.** *Let* $f: G \to G_1$ *be a separable morphism, whose kernel is central. If* $A' \in \mathfrak{g}_1'$ *is strongly regular then so is* $(df)' A' \in \mathfrak{g}'$.

$(df)'$ is the dual of $df: \mathfrak{g} \to \mathfrak{g}_1$. Since f is separable, df is surjective and $(df)'$ injective. Let $fT = T_1$. We identify the Weyl group of T_1 in G_1 with W. We may assume $A' \in \mathfrak{t}_1'$. Then $w \cdot (df)' A' = (df)' A'$ implies $w \cdot A' = A'$, by the injectivity of df'. This proves the assertion.

Let $\bar{G}$ be the adjoint group of G. It is characterized by the following properties: $\bar{G}$ is a semisimple adjoint group, there is a surjective morphism $\pi: G \to \bar{G}$ whose scheme-theoretic kernel is central and which defines an isomorphism of the unipotent radical of any Borel subgroup onto its image.

2.8. **Lemma.** *There is a strongly regular element* $A' \in \bar{\mathfrak{g}}'$ *such that* $(d\pi)' A'$ *is also strongly regular if and only if either p is odd and* Φ *has no components of type* A_2, *in case* $p=3$; *or if* $p=2$ *and G has only irreducible components of type* A_l ($l \neq 1, 3$) *and* E_6.

Passing to the quotient of G by its connected center and using 2.7, we may assume that G is semisimple. It suffices to deal with the case that G is moreover simply connected, and then we may also assume G to be quasi-simple.

The character group of T is isomorphic to the weight lattice P of the root system Φ. Let $Q \subset P$ be the root lattice. Putting $\pi T = \bar{T}$ we have $\mathfrak{t}' \simeq P \otimes_{\mathbb{Z}} \bar{k}$, $\bar{\mathfrak{t}}' \simeq Q \otimes_{\mathbb{Z}} \bar{k}$, and $(d\pi)'$ is defined by the injection $Q \to P$. It follows that an A' as required does not exist if and only if there is $w \in W - \{1\}$ such that the image I of $Q \otimes \bar{k}$ in $P \otimes \bar{k}$ is fixed elementwise by w. First let Φ be of type A_l. By [5, p. 251] we have that $P \otimes \bar{k} \simeq \bar{k}^{l+1}/\bar{k}$ ($\bar{k}$ being imbedded diagonally) and that I is the image of the subspace of $\bar{k}^{l+1}$ formed by the vectors with coordinate sum 0. The $w \in W$ fixing all of I then correspond to the $\sigma \in \mathfrak{S}_{l+1}$ such that

$$x_{\sigma(1)} - x_1 = \dots = x_{\sigma(l+1)} - x_{l+1},$$

for all $x_i \in \bar{k}$ with $x_1 + \dots + x_{l+1} = 0$. If σ has an orbit of length 1 and $l > 1$ this implies $\sigma = 1$. The same is true if σ has an orbit of length > 2 and $l > 2$ or if $l > 3$. It follows that we can have $\sigma \neq 1$ in only 3 cases: $p = 2$ and $l = 1, 3$ or $p = 3$ and $l = 2$. This proves the assertion for type A_l. By 2.7 we need only discuss the cases where π is not separable. If p is odd, there remains only the case E_6 ($p = 3$). The elements of W fixing I form a normal subgroup. If it is nontrivial, it is either W itself or has index 2 (by what we recalled in the proof of 2.6), so it contains all products of an even number of reflections of W. Since, as is easily seen, there is a product of 2 reflections of W which does not fix all of I, the assertion of 2.8 follows for odd p.

If $p = 2$, it follows from 2.5 (ii) that the required A' can only exist if G is of type A_l ($l \neq 1, 3$) or E_6. By what we saw above, A' does indeed exist in these cases. This concludes the proof.

We now assume again that $k \simeq \mathbb{F}_q$ is finite.

2.9. **Lemma.** *Assume that T is defined over k.*

(i) *If $\mathfrak{t}'$ contains strongly regular elements and $q \geqq |W|$ then $(\mathfrak{t}')^F$ contains strongly regular elements;*

(ii) *Let $\pi: G \to \bar{G}$ be the morphism onto the adjoint group, and let $\pi T = \bar{T}$. If $\bar{\mathfrak{t}}'$ contains an strongly regular element whose image in $\mathfrak{t}'$ is strongly regular and $q \geqq 2|W|$ then $(\bar{\mathfrak{t}}')^F$ contains such elements.*

By 2.4 the strongly regular elements of $\mathfrak{t}'$ are those lying outside $(|W| - 1)$ linear subspaces of $\mathfrak{t}'$. The number of non strongly regular elements of $(\mathfrak{t}')^F$ is thus $\leqq (|W| - 1) q^{\dim T - 1}$. If $q \geqq |W|$ this is less than $|(\mathfrak{t}')^F|$, which proves (i). The proof of (ii) is similar.

From the preceding results one deduces conditions for the existence of strongly regular elements in $(\mathfrak{g}')^F$. For example, such elements exist if $p \neq 2$ and $q \geqq |W|$ (by 2.5 (i) and 2.9 (i)).

Finally, some odds and ends about strongly regular elements. Let T be F-stable, let $A' \in (\mathfrak{t}')^F$ be strongly regular. Denote by $O_G(A')$ ($O_{G^F}(A')$) the orbit of A' under G (resp. G^F). Let $\phi(g) = \mathrm{Ad}'(g) A'$.

2.10. **Lemma.** (i) *ϕ induces a k-isomorphism of algebraic varieties $G/T \xrightarrow{\sim} O_G(A')$;*

(ii) $O_G(A')^F = O_{G^F}(A')$.

ϕ is an orbit map for the action of T by right translations [3, p. 174]. By [loc.cit., 6.7, p. 180] it suffices for the proof of (i) to show that $\operatorname{Ker} d\phi \subset \mathfrak{t}$. This follows by using that

$$(d\phi)X_\alpha = -\langle H_\alpha, A'\rangle X'_\alpha,$$

which is a consequence of (1). Since T is connected we have $(G/T)^F = G^F/T^F$, which implies (ii).

2.11. **Lemma.** *Let $p=2$. Assume that G has semisimple rank 1 and that $\mathfrak{g}'$ contains strongly regular elements. Then there is a separable k-homomorphism of G onto GL_2, whose kernel is central.*

The adjoint group $\bar{G}$ of G is k-isomorphic to PSL_2. It follows that there is an F-stable maximal torus T of G whose image in $\bar{G}$ is k-split. Let $X = X(T)$ be the character group of T, let $\pm\alpha \in X$ be the roots of T. Let $w \in G^F$ be a representative of the nontrivial element of the Weyl group of T. Then w acts on X and interchanges α and $-\alpha$, moreover $(w-1)X \subset \mathbb{Z}\alpha$. Since $\mathfrak{t}'$ contains strongly regular elements, w does not act trivially on $X/2X$ (see 2.4), whence $(w-1)X \not\subset 2X$ and $\alpha \notin 2X$. The sublattice $\mathbb{Z}\alpha + \operatorname{Ker}(w-1)$ of X cannot be X itself, otherwise G was isomorphic to the product of PSL_2 and a torus, and $\mathfrak{g}'$ could not contain strongly regular elements. It follows that the derived group G' of G is k-isomorphic to SL_2. Let C be the connected center of G and put $T' = T \cap G'$, this is a maximal torus of G'. The product morphism $G' \times C \to G$ leads to an injection i of X into $X(T') \times X(C)$, which is bijective on $i^{-1}(0, X(C))$.

There is $\omega \in X(T')$ such that $i\alpha \in \{2\omega\} \times X(C)$. Put $x = i^{-1}(2\omega, 0)$, then $w \cdot x - x = -\alpha$.

The Frobenius automorphism $F \in \operatorname{Gal}(\bar{k}/k)$ acts on all character groups and i commutes with F. It follows that $Fx = x$. Now x and wx define a sublattice X_1 of X of rank 2, which is stable for w and F, and contains α. The quotient G_1 of G by the central k-torus of G orthogonal to X_1 is a product of SL_2 and a 1-dimensional central torus. G_1 cannot be a direct product, because $\mathfrak{g}'$ contains strongly regular elements. Hence $G_1 \simeq GL_2$, and the assertion follows.

In the last result of this section the assumption that k is finite is unnecessary.

2.12. **Lemma.** *Let $A' \in \mathfrak{t}'$ be strongly regular. Assume that P is a parabolic subgroup of G such that A' is orthogonal to the Lie algebra of the unipotent radical of P. Then $T \subset P$.*

Fix a Borel subgroup $B \supset T$. There exist a parabolic subgroup $Q \supset B$ and $g \in G$ such that $P = \operatorname{ad}(g)\,Q$. Let V be the unipotent radical of Q then $\langle \mathfrak{v}, \operatorname{Ad}(g^{-1})\,A'\rangle = 0$. By Bruhat's lemma we can write $g^{-1} = b n_w u$, where $b \in B$ and with u in the subgroup U_w of B generated by the U_α with $\alpha > 0$, $w\alpha < 0$. We then have $\langle \mathfrak{v}, \operatorname{Ad}(n_w u)\,A'\rangle = 0$. By (1)

$$\operatorname{Ad}(u)\,A' = A' + \sum_{\substack{\alpha > 0 \\ w\alpha < 0}} \xi_\alpha X'_\alpha, \tag{3}$$

with $\xi_\alpha \in \bar{k}$. It follows that for each α such that $\xi_\alpha \neq 0$ in (3) we have $X_{-w\alpha} \notin \mathfrak{v}$. By familiar properties of the parabolic subsystems of Φ we conclude that for such α

we have $U_{w\alpha} \subset Q$. Now write $u = \prod_{\substack{\alpha>0 \\ w\alpha<0}} u_\alpha$ with $u_\alpha \in U_\alpha$. The order of the α occuring in this product is taken to be compatible with decreasing height. It then follows by induction on the height of α, using the strong regularity of A', that for all α such that $u_\alpha \neq 1$ we have $U_{w\alpha} \subset Q$. Consequently, $P = \mathrm{ad}(g)\, Q = \mathrm{ad}(n_w^{-1}) Q$ and $T \subset P$.

3. The Trigonometric Sums $S^G(A, A')$: A Reduction Theorem

3.1. We assume the situation of 1.4. Let T be an F-stable maximal torus of G. We use the notations of 2.1. Let $\mathfrak{t}'_0$ be the set of strongly regular elements of $\mathfrak{t}'$. Assume that $\mathfrak{t}'_0 \neq \phi$ (by 2.5 (i) this is so, for example, if $p \neq 2$). $\mathfrak{t}'_0$ is an irreducible k-subvariety of $\mathfrak{t}'$, stable for the action of the Weyl group W on $\mathfrak{t}'$.

Fix an element $A \in \mathfrak{g}^F$. Let $\mathscr{Y}^G_{A,T} = \mathscr{Y}$ be the k-subvariety of $\mathbb{A}^1 \times G/T \times \mathfrak{t}'_0$ consisting of the (x, gT, A') such that

$$x^q - x = \langle A, \mathrm{Ad}'(g) A' \rangle.$$

The projection $\pi\colon \mathscr{Y} \to \mathfrak{t}'_0$ is a morphism of k-varieties. By 2.10 (i) the fibre $\pi^{-1} A'$ is isomorphic to the variety $Y^G_{A,A'}$ of 1.4, if $A' \in (\mathfrak{g}')^F$. So we have put these varieties, with A fixed, in an algebraic family.

W acts on $\mathscr{Y}$ by

$$w \cdot (x, gT, A') = (X, g n_w^{-1} T, w \cdot A'),$$

and π commutes with the actions of W on $\mathscr{Y}$ and $\mathfrak{t}'$. Also, F acts on W and we have, on $\mathscr{Y}$,

$$F \cdot w = Fw \cdot F. \tag{1}$$

Finally, k acts on $\mathscr{Y}$, as in 1.2, and if we let it act trivially on $\mathfrak{t}'$ then π commutes with k.

3.2. Let $\mathscr{B}^G$ be the variety of Borel subgroups of G. It is a projective k-variety of dimension $d = d(G)$, where $|\Phi| = 2d$. Denote by $\mathscr{B}^G_A$ the subvariety of $\mathscr{B}^G$ formed by the Borel subgroups whose Lie algebra contains A.

Let $A = A_s + A_n$ be the Jordan decomposition of A, then $A_s, A_n \in \mathfrak{g}^F$. Denote by $H = Z_G(A_s)^0$ the connected centralizer of A_s. This is a reductive subgroup of G, containing a maximal torus of G.

Fix a Borel subgroup $B \supset T$.

3.3. **Proposition.** (i) *If $B_1 \in \mathscr{B}^G_{A_s}$ then $B_1 \cap H$ is a Borel subgroup of H, whose Lie algebra is the intersection $\mathfrak{b}_1 \cap \mathfrak{h}$;*

(ii) *Any $B_1 \in \mathscr{B}^G_{A_s}$ is of the form* $\mathrm{ad}(g) B$ *with* $\tau(B_1) = (\mathrm{Ad}\, g)^{-1} A_s \in \mathfrak{t}$;

(iii) *τ is a surjective map of $\mathscr{B}^G_{A_s}$ onto the set of conjugates of A_s in* $\mathfrak{t}$, *its fibers are the irreducible components of $\mathscr{B}^G_{A_s}$;*

(iv) *H acts transitively on each irreducible component Z of $\mathscr{B}^G_{A_s}$. The map $B_1 \mapsto B_1 \cap H$ defines an isomorphism of $\mathscr{B}^G_A \cap Z$ onto $\mathscr{B}^H_{A_n}$;*

(v) *The connected components of $\mathscr{B}^G_A$ are the intersections of $\mathscr{B}^G_A$ with the irreducible components of $\mathscr{B}^G_{A_s}$.*

This is similar to [10, Prop. 4.4], where analogous questions are treated for semisimple elements of G. The proof of (i) uses an argument like that used in the proof of 2.3. The proof of (ii), (iii) and the first part of (iv) is along the lines of that of [loc. cit.] and may be omitted. Let $Z = \mathrm{ad}(H) B_1$ be an irreducible component of $\mathscr{B}^G_{A_s}$. Then $\mathscr{B}^G_A \cap Z$ consists of the $\mathrm{ad}(h) B_1$ with $A \in \mathrm{Ad}(h)(\mathfrak{b}_1 \cap \mathfrak{h})$, and the second point of (iv) follows from (i).

Finally, (v) follows from the fact that $\mathscr{B}^H_{A_n}$ is connected [24, p. 379].

3.4 We shall now establish some cohomological results about the varieties $\mathscr{Y}^G_{A,T}$ and $Y^G_{A,A'}$. We first work over $\bar{k}$, and forget for the moment the action of F.

Let f_B the morphism $\mathscr{Y}^G_{A,T} \to \mathscr{B}^G \times \mathfrak{t}'_0$ with

$$f_B(x, gT, A') = (\mathrm{ad}\,(g)\, B, A').$$

If we let k act trivially on $\mathscr{B}^G \times \mathfrak{t}'_0$ then f_B commutes with k. Hence k acts on the higher direct image sheaves $\mathscr{S}^i = R^i f_{B,!} E$, where E stands for the constant l-adic sheaf defined by the field E of 1.1. If ψ is as in 1.1, let $\mathscr{S}^i_\psi$ be the ψ-part of $\mathscr{S}^i$. Let U be the unipotent radical of B, generated by the U_α with $\alpha > 0$ (for the order on Φ defined by B).

3.5. **Lemma.** (i) $\mathscr{S}^i = 0$ *if* $i \neq 2d$;

(ii) $\mathscr{S}^{2d}_\psi$ *is supported by* $\mathscr{B}^G_A \times \mathfrak{t}'_0$;

(iii) *In a point* $\xi \in \mathscr{B}^G_A \times \mathfrak{t}'_0$ *the stalk* $(\mathscr{S}^{2d}_\psi)_\xi$ *is* E.

Let $\xi = (\mathrm{ad}\,(g)\, B, A') \in \mathscr{B}^G \times \mathfrak{t}'_0$. The stalk $(\mathscr{S}^i)_\xi$ in the geometric point ξ is given by

$$(\mathscr{S}^i)_\xi = H^i_c(f_B^{-1}\xi, E)$$

(see [19, exp. XVII, Prop. 5.2.8] for the corresponding result in étale cohomology). Now $f_B^{-1}\xi$ consists of the (x, guT, A') with $u \in U$ and

$$x^q - x = \langle A, \mathrm{Ad}'\,(gu)\, A' \rangle.$$

Let the X'_α $(\alpha \in \Phi)$ be as in 2.2. By formula (1) of no. 2,

$$u \mapsto (\mathrm{Ad}'\,(u) - 1)\, A'$$

defines a morphism of U to the subspace $\mathfrak{b}^\perp$ of $\mathfrak{g}'$ orthogonal to $\mathfrak{b}$, which is spanned by the X'_α with $\alpha > 0$. Since A' is strongly regular, the morphism is bijective. Since $\mathrm{Ad}'\,(U)\, A'$ is closed, by a theorem of Kostant-Rosenlicht (for a proof see [2, p. 474–475]), we must have

$$\mathrm{Ad}'\,(U)\, A' = A' + \mathfrak{b}^\perp.$$

If $\xi \notin \mathscr{B}^G_A \times \mathfrak{t}'_0$, i.e. $\mathrm{Ad}\,(g)^{-1} A \notin \mathfrak{b}$, it follows that $f_B^{-1}\xi \simeq \mathbb{A}^d$, and that the action of an element of k on $f_B^{-1}\xi$ corresponds to a translation in some coordinate. Observe that translations act trivially on the cohomology of $\mathbb{A}^d$ (by [10, 6.4], for example). If $\xi \in \mathscr{B}^G_A \times \mathfrak{t}'_0$ then $f_B^{-1}\xi \simeq \mathbb{A}^d \times k$, where k acts by translations in the second factor. 3.5 now follows by using that $H^i_c(\mathbb{A}^d, E) = 0$ if $i \neq 2d$ and $H^{2d}_c(\mathbb{A}^d, E) = E$ (consequences of facts which are hidden in the depths of [19]).

3.6. Fix a set $V \subset G$ such that the $\mathrm{Ad}(v)^{-1} A_s (v \in V)$ are the distinct conjugates of A_s in $\mathfrak{t}$. By 3.3 the irreducible components of $\mathscr{B}^G_{A_s}$ are the varieties

$$Z_v = \mathrm{ad}(Hv)\, B \times \mathfrak{t}'_0.$$

If $v \in V$ there is a unique closed subgroup $U_{B,v}$ of U, normalized by T, such that

$$B = (B \cap \mathrm{ad}(v)^{-1} H) \cdot U_{B,v}$$

Then $U_{B,v}$ is generated by the U_α with $\alpha > 0$, $U_\alpha \not\subset \mathrm{ad}(v)^{-1} H$, whence $\dim U_{B,v} = d - d'$, where $d' = d(H)$.

We have $A \in \mathfrak{h}$, $\mathrm{ad}(v)\, T \subset H$. Let

$$\mathscr{Y}_v = \{(x, h\,\mathrm{ad}(v)\, T, \mathrm{Ad}'(v)\, A') \in \mathbb{A}^1 \times H/\mathrm{ad}(v)\, T \times \mathrm{Ad}'(v)\, \mathfrak{t}'_0 \mid$$
$$x^q - x = \langle A, \mathrm{Ad}'(hv)\, A' \rangle\}.$$

If $\mathrm{ad}(v)\, T$ is F-stable then $\mathscr{Y}_v$ is isomorphic to an open subvariety of $\mathscr{Y}^H_{A, \mathrm{ad}(v) T}$. Let $\Phi_{B,v}$ be the morphism

$$\mathscr{Y}_v \times U_{B,v} \to \mathscr{Y}^G_{A,T}$$

with

$$\Phi_{B,v}((x, h\,\mathrm{ad}(v)\, T, \mathrm{Ad}'(v)\, A'), u) = (x, hvuT, A').$$

The restriction of $\Phi_{B,v}$ to $\mathscr{Y}_v \times \{1\}$ is independent of B.

3.7. **Lemma.** *$\Phi_{B,v}$ is an isomorphism of $\mathscr{Y}_v \times U_{B,v}$ onto $f_B^{-1} Z_v$.*

$f_B^{-1} Z_v$ is the set of $(x, hvuT, A')$ with $h \in H$, $u \in U_{B,v}$ and

$$x^q - x = \langle \mathrm{Ad}(v^{-1} h^{-1})\, A, \mathrm{Ad}'(u)\, A' \rangle.$$

By formulas (1) of no. 2, $\mathrm{Ad}'(u) A' - A'$ is a linear combination of the X'_α with $\alpha > 0$, $U_\alpha \subset \mathrm{ad}(v)^{-1} H$. Since $\mathrm{Ad}(v^{-1} h^{-1}) A$ lies in the Lie algebra of $\mathrm{ad}(v)^{-1} H$, which is spanned by $\mathfrak{t}$ and the X_α with $U_\alpha \subset \mathrm{ad}(v)^{-1} H$, it follows that we may drop the $\mathrm{Ad}'(u)$ in the last formula. This implies the assertion.

It follows from 3.7 that there is an isomorphism

$$\Phi_B: \coprod_{v \in V} (\mathscr{Y}_v \times U_{B,v}) \tilde{\to} f_B^{-1} \mathscr{B}^G_{A_s}, \tag{2}$$

deduced from the $\Phi_{B,v}$.

Let $\pi_v: \mathscr{Y}_v \to \mathfrak{t}'_0$ be projection on the last factor, followed by $\mathrm{Ad}'(v)^{-1}$.

3.8. **Proposition.** *Φ_B determines an isomorphism*

$$\phi_B: (R^* \pi_! E)_\psi \tilde{\to} \bigoplus_{v \in V} (R^* \pi_{v,!} E)_\psi (-2d + 2d').$$

(the integer in brackets denoting a dimension shift).

If σ is the projection of $\mathscr{B}^G \times \mathfrak{t}'_0$ on its second factor we have $\pi = \sigma f_B$. Apply the Leray spectral sequence for a composite morphism [11]. It follows from 3.5 that the support of $R^* f_{B,!} E$ lies in $\mathscr{B}^G_{A_s} \times \mathfrak{t}'_0 = \bigcup_{v \in V} Z_v$. Let i be the inclusion $f_B^{-1} \mathscr{B}^G_{A_s} \to \mathscr{Y}^G_{A,T}$, then $\pi i = \pi' \Phi_B$, where π' is the morphism of the left-hand side of (2) deduced from the π_v. It also follows by using 3.5 that there is a restriction isomorphism $(R^*\pi_! E)_\psi \simeq (R^*(\pi i)_! E)_\psi$. Since $U_{B,v}$ is an affine space of dimension $d-d'$, we have

$$(R^*\pi_! E)_\psi \tilde{\to} \bigoplus_{v\in V} (R^*\pi_{v,!} E)_\psi(-2d+2d').$$

Putting all this together we obtain 3.8.

3.9. The isomorphism of 3.8 depends on the choice of the Borel group $B\supset T$. If B' is another one, then $\phi_{B'}\phi_B^{-1}$ is an automorphism, and it follows from the definitions that it comes from automorphisms $\eta_v(B,B')$ of the summands $(R^*\pi_{v,!}E)_\psi(-2d+2d')$. We shall prove, in fact, that $\eta_v(B,B')$ is a multiplication (by ± 1). Before doing so, we establish a result about the trigonometric sums $S^G(A,A')$. So we have to bring in the ground field k and the Frobenius morphism F. First observe that we may assume V chosen such that $FV=V$. In fact, F permutes the elements of V up to a factor in H. It follows from Lang's theorem, applied to H, that these factors may be taken to be 1.

3.10. **Corollary.** *Assume that all $\eta_v(B,B')$ are scalar multiplications. Let $A'\in(\mathfrak{t}')^F$ be strongly regular. Then*

$$S^G(A,A') = q^{d-d'} \sum_{v\in V^F} \eta_v(B,FB)\, S^H(A, \mathrm{Ad}'(v)\, A').$$

In this formula, we have identified (as we may) $\mathrm{Ad}'(v)\mathfrak{t}'$ with the toral subspace of $\mathfrak{h}'$ defined by the maximal torus $\mathrm{ad}(v)\, T$ of H (see 2.1).

The stalk of $(R^*\pi_! E)_\psi$ in A' is $H_c^*(Y^G_{A,A'}, E)_\psi$. The isomorphism ϕ_B of 3.8 leads to a stalk isomorphism

$$H_c^*(Y^G_{A,A'}, E)_\psi \tilde{\to} \bigoplus_{v\in V} H_c^*(Y^H_{A,\mathrm{Ad}'(v)A'}, E)(-2d+2d').$$

Using the fact that

$$F\Phi_{B,v} = \Phi_{FB,Fv} F$$

it follows that the action of F^* on the left-hand side of this formula corresponds to the endomorphism of the right-hand side which is, on the v-component

$$\eta_v(B,FB)\, q^{d-d'} F^*$$

(the q-power coming from the fact that F^* acts as multiplication by this power on $H_c^{2d-2d'}(\mathbb{A}^{d-d'}, E)$). The Corollary then follows by using what was established in 1.3.

3.11. We now discuss the $\eta_v(B,B')$. Let $B=B_0, B_1, \ldots, B_l = B'$ be a "minimal gallery", i.e. the B_i are Borel subgroups containing T such that B_i and B_{i+1} are adjacent (= their intersection has codimension 1 in each of them) and such that l is minimal. If $B' = w\cdot B = \mathrm{ad}(n_w)\, B$, then $l = l(w)$, the length of w with respect to the set of generators defined by B. Put

$$\varepsilon_G(B,B') = (-1)^l.$$

3.12. **Lemma.** *$\eta_v(B,B')$ is multiplication by $\varepsilon_G(B,B')\,\varepsilon_H(H\cap \mathrm{ad}(v)B, H\cap \mathrm{ad}(v)B')$.*

From the definitions we see that

$$\eta_v(B,B'') = \eta_v(B,B')\,\eta_v(B',B''),$$

from which it follows that it suffices to prove the assertion in the case that B and B' are adjacent. Then $B'=s_\alpha\cdot B$, where s_α is a reflection in a simple root $\alpha\in\Phi$. Now $B=(B\cap B')\cdot U_\alpha$, $B'=(B\cap B')\cdot U_{-\alpha}$. Moreover, $H\cap \mathrm{ad}(v)B$ and $H\cap \mathrm{ad}(v)B'$ are Borel subgroups of H which are either equal or adjacent. The second case prevails if and only if $U_{B,v}=U_{B',v}$. Now if this equality holds, the isomorphisms ϕ_B^{-1} and $\phi_{B'}^{-1}$ coincide on the v-component, whence $\eta_v(B,B')=1$. This proves the lemma in that case.

Now assume that $U_{B,v}\neq U_{B',v}$. We then have to prove $\eta_v(B,B')=-1$. Let P be the parabolic subgroup generated by B and B'. Its unipotent radical is $U\cap s_\alpha\cdot U$, let $P=L(U\cap s_\alpha\cdot U)$ be the Levi decomposition with $L\supset T$. There is a factorization of π

$$\mathcal{Y}^G_{A,T}\xrightarrow{\rho} G/P\times \mathfrak{t}'_0\to \mathfrak{t}'_0$$

where $\rho(x,gT,A')=(gP,A')$, and there are similar factorizations of the π_v, with morphisms $\rho_v\colon \mathcal{Y}_v\to G/P\times\mathfrak{t}'_0$. We have a result like 3.8 for ρ and ρ_v and it suffices to prove the analogue of the lemma in that situation. It also suffices to prove this for the stalk homomorphisms. So we may work in a fiber of ρ and those of interest are the fibers of the (hvP,A') with $h\in H$. Now

$$\rho^{-1}(hvP,A')\simeq Y^L_{\bar A,\bar A'}\times \mathbb{A}^{d-1},$$

where $\bar A$ and $\bar A'$ are the elements of $\mathfrak{l}$ and $\mathfrak{l}'$ defined by $\mathrm{Ad}(v^{-1}h^{-1})A$ and A'. The semisimple rank of L is 1. If $p=2$, it follows from 2.11 that $Y^L_{\bar A,\bar A'}$ is isomorphic to a similar variety with L replaced by GL_2. If $p\neq 2$ this is also true (and easy to see). It follows that the assertion of the lemma will hold if it holds for $G=GL_2$. We thus have reduced the proof to the case $G=GL_2$, and A regular semisimple. Taking for T a non-split maximal k-torus of GL_2 we see, by comparing 1.6 (i) and 3.10 that we shall indeed have $\eta_v(B,B')=-1$ for two opposite Borel subgroups of $G=GL_2$, if we know that $\eta_v(B,B')$ is a scalar multiplication.

In this case, the group H is a maximal torus. Let $\sigma\colon M\to\mathfrak{t}'_0$ be the Galois covering of $\mathfrak{t}'_0$ defined by the equation $x^q-x=\langle \mathrm{Ad}(v)^{-1}A,A'\rangle$, then $(R^*\pi_{v,!}E)_\psi=(R^*\sigma_!E)_\psi$ $(=\mathcal{S}$, say).

The sheaf $\sigma^*\mathcal{S}$ is constant, and hence the endomorphism of $\sigma^*\mathcal{S}$ defined by $\eta_v(B,B')$ is constant. Since the stalks are E, it follows that $\eta_v(B,B')$ is indeed a scalar multiplication. This finishes the proof.

As before, let T be a maximal F-stable torus of G and B a Borel subgroup containing T. Assume that $FB=w\cdot B$. Let $r(G)$ be the k-rank of G. The next result gives a description of $\varepsilon_G(B,FB)$.

3.13. **Lemma.** $\varepsilon_G(B,FB)=(-1)^{r(G)-r(T)}$.

Let X be the character group of T and $V=X\otimes_{\mathbb{Z}}\mathbb{R}$, so $\Phi\subset V$. There is a linear transformation σ of V, inducing a permuation of Φ such that $FU_\alpha=U_{w\sigma\alpha}$. Then σ keeps positive roots positive (the order being that defined by B) and hence induces a permutation of the corresponding basis S of Φ. Let V_u be the fixed point space of the linear transformation u of V. Then we have $r(G)=\dim V_\sigma$, $r(T)=\dim V_{w\sigma}$ (all this can be extracted from the results in [4, part E, Ch. II]). 3.13 now follows from the following result, which is purely a result about root systems, and in which σ may be any linear transformation with the above properties.

3.14. **Lemma.** $\det(w\sigma)=(-1)^{\dim V-\dim V_{w\sigma}}$.

(a) $w=1$. Let $n_e(n_o)$ be the number of orbits of even (odd) length of σ in S. Since the $\alpha\in S$ form a basis of V, we have that $\det(\sigma)$ is the sign of the permutation of S defined by σ, which equals $(-1)^{n_o}$. On the other hand, $\dim V_\sigma=n_e+n_o$ and $(-1)^{\dim V}=(-1)^{n_o}$. The assertion follows from these observations.

(b) $w\sigma$ has an eigenvalue 1. Let $(,)$ be a Euclidian metric on V which is W- and σ-invariant. σ has a eigenvector with eigenvalue 1 which is regular (i.e. not orthogonal to any root), e.g. $\sum_{\alpha>0}\alpha$ and it then follows from [26, 6.2 and 6.4] that there is $w'\in W$ with $w'\cdot V_{w\sigma}\subset V_\sigma$. Replacing $w\sigma$ by $w'(w\sigma)(w')^{-1}$ we get the situation that $V_{w\sigma}\subset V_\sigma$. If $V_{w\sigma}$ contains a regular element, then $V_{w\sigma}=V_\sigma$, $w=1$ and we are in case (a). Otherwise, there is a root orthogonal to $V_{w\sigma}$. Let Φ' be the set of these roots, it is a closed subsystem. w lies in the Weyl subgroup defined by Φ' [29, p. 10–11]. σ stabilizes Φ' and $\sigma|\Phi'$ has the same properties as σ. The assertion now follows by an induction on $|\Phi|$.

(c) $w\sigma$ has no eigenvalue 1. If $\alpha\in\Phi$ there is $x\in V$ with $(w\sigma-1)x=\alpha$. Then $(x,x)=(w\sigma x,w\sigma x)=(x+\alpha,x+\alpha)$, whence $2(x,\alpha)=-(\alpha,\alpha)$. It follows that $s_\alpha w\sigma x=x$. By case (b)

$$\det(w\sigma)=-\det(s_\alpha w\sigma)=(-1)^{\dim V-\dim V_{s_\alpha w\sigma}+1}.$$

Since $\dim V_{s_\alpha w\sigma}=1=\dim V_{w\sigma}+1$, the assertion follows.

We now come to the main result of this section.

3.15. **Theorem.** *Let $A'\in(\mathfrak{t}')^F$ be strongly regular. Then*

$$S^G(A,A')=(-1)^{r(G)-r(H)}q^{d(G)-d(H)}|H^F|^{-1}\sum_{\substack{g\in G^F\\ \mathrm{ad}(g)T\subset H}}S^H(A_n,\mathrm{Ad}'(g)A')\psi(\langle A_s,\mathrm{Ad}'(g)A'\rangle).$$

This follows from 3.10, 3.12 and 3.13, taking into account that

$$S^H(A,\mathrm{Ad}'(g)A')=S^H(A_n,\mathrm{Ad}'(g)A')\,\psi(\langle A_s,\mathrm{Ad}'(g)A'\rangle).$$

3.16. In the particular case that A is a regular semisimple element of $\mathfrak{g}^F$ we have that H is also a maximal torus. 3.15 then shows that $S^G(A,A')=0$ if H is not G^F-conjugate to T and

$$S^G(A,A')=(-1)^{r(H)-r(T)}q^{d(G)}\sum_{B'}\psi(\langle A,B'\rangle),$$

if $H=T$. The summation is over the distinct G^F-conjugates of A' in $\mathfrak{t}$. Such a formula was first proved by Deligne (unpublished). It is the analogue for finite Lie algebras of Chevalley's type, of a formula of Harish-Chandra for the Lie algebra of a compact Lie group [13, Th. 2, p. 104].

4. The Trigonometric Sums $S^G(A,A')$: A Nilpotent

3.15 reduces the determination of $S^G(A,A')$ to the case that A is nilpotent, to be discussed now. The notations are as before, and A is nilpotent. We begin with a further discussion of the sheaf $\mathcal{S}_\psi^{2d}$ of 3.5.

4.1. **Lemma.** *The restriction of* $\mathcal{S}_\psi^{2d}$ *to* $\mathcal{B}_A^G \times \mathfrak{t}'_0$ *is the constant sheaf* E.

Put $Z = f_B^{-1}(\mathcal{B}_A^G \times \mathfrak{t}'_0)$. By the base change theorem for cohomology with proper support (see [19, exp. XVII, Th. 2.5.6] for the étale cohomology version), applied to the cartesian diagram

$$\begin{array}{ccc} Z & \hookrightarrow & \mathcal{Y}_{A,T}^G \\ {\scriptstyle f_B|Z}\downarrow & & \downarrow{\scriptstyle f_B} \\ \mathcal{B}_A^G \times \mathfrak{t}'_0 & \hookrightarrow & \mathcal{B}^G \times \mathfrak{t}'_0 \end{array}$$

it suffices to prove that the sheaf $(R^{2d}(f_B|Z)_! E)_\psi$ is the constant sheaf E. Now from the proof of 3.5 we see that Z is the set of $(x, gT, A') \in \mathcal{Y}_{A,T}^G$ with $\mathrm{Ad}(g)^{-1} A \in \mathfrak{b}$ and

$$x^q - x = \langle \mathrm{Ad}(g)^{-1} A, A' \rangle.$$

Since $\mathrm{Ad}(g)^{-1} A$ is a nilpotent element of $\mathfrak{b}$ the right-hand side is 0. Consequently

$$Z \simeq k \times \rho^{-1} \mathcal{B}_A^G \times \mathfrak{t}'_0$$

where $\rho(gT) = \mathrm{ad}(g) B$. k acts by translations on the first factor, and $f_B|Z$ is projection on the last two factors, followed by (ρ, id).

Applying the base change theorem to

$$\begin{array}{ccc} \rho^{-1}\mathcal{B}_A^G \times \mathfrak{t}'_0 & \longrightarrow & G/T \\ \downarrow & & \downarrow{\scriptstyle \rho} \\ \mathcal{B}_A^G \times \mathfrak{t}'_0 & \longrightarrow & \mathcal{B}^G \end{array}$$

we see that it suffices to prove that $R^{2d}\rho_! E$ is a constant sheaf on $\mathcal{B}^G$. Now ρ is a locally trivial fibration for the Zariski topology (if O is a "big cell" in $\mathcal{B}^G$ then $\rho^{-1} O \simeq O \times \mathbb{A}^d$ and the big cells cover $\mathcal{B}^G$), so $R^{2d}\rho_! E$ is at any rate a locally constant l-adic sheaf. Since $\mathcal{B}^G$ is connected and simply connected [17, p. 285] it follows that we must have a constant sheaf [11, p. 06]. That it is the constant sheaf E follows from 3.5 (iii).

Let σ be the projection $\mathcal{B}^G \times \mathfrak{t}'_0 \to \mathfrak{t}'_0$, so $\pi = \sigma f_B$. Let $(R^* \pi_! E)_\psi$ be the ψ-part of the direct image $R^* \pi_! E$ (this is defined because of what was said in the last line of 3.1).

4.2. **Proposition.** (i) *$(R^* \pi_! E)_\psi$ is a constant sheaf of finite-dimensional vector spaces over E. The action of W on $\mathcal{Y}_{A,T}^G$ (see 3.1) defines a representation ρ^{i-2d} of W in $(R^i \pi_! E)_\psi$;*

(ii) *The factorization $\pi = \sigma f_B$ defines an isomorphism of constant sheaves*

$$\alpha_B^* : (R^* \pi_! E)_\psi \xrightarrow{\sim} H^*(\mathcal{B}_A^G, E)(-2d),$$

and we have

$$\alpha_{w \cdot B}^* = \alpha_B^* \rho^*(w)^{-1}.$$

There is a Leray spectral sequence

$$R^i\sigma_!(\mathcal{S}_\psi^j) \Rightarrow (R^*\pi_! E)_\psi$$

[11, p. 05]. By 3.5 the spectral sequence collapses, leading to an isomorphism

$$(R^*\pi_! E)_\psi \xrightarrow{\sim} R^*\sigma_!(\mathcal{S}_\psi^{2d})(-2d) \tag{1}$$

Let $\mathcal{T}$ be the sheaf on $\mathcal{B}^G$ supported by $\mathcal{B}_A^G$ whose restriction to $\mathcal{B}_A^G$ is the constant sheaf E. There is a cartesian diagram

$$\begin{array}{ccc} \mathcal{B}^G \times \mathfrak{t}_0' & \xrightarrow{\pi_1} & \mathcal{B}^G \\ \sigma \downarrow & & \downarrow \\ \mathfrak{t}_0' & \longrightarrow & \operatorname{Spec}(\bar{k}) \end{array}$$

and 4.1 implies that $\mathcal{S}_\psi^{2d} = \pi_1^* \mathcal{T}$. By the base change theorem it follows that $R^*\sigma_! \mathcal{S}_\psi^{2d}$ is a constant sheaf on $\mathfrak{t}_0'$. This is the first assertion of (i). Since π is W-equivariant the last assertion of (i) is also clear.

The fiber of the sheaf $R^*\sigma_! \mathcal{S}_\psi^{2d}$ in A' is $H^*(\sigma^{-1}A', E) = H^*(\mathcal{B}_A^G, E)$. The remaining assertion of (i) follows from the finite-dimensionality of the cohomology of the projective variety $\mathcal{B}_A^G$.

The isomorphism α_B^* is the one coming from (1). The last assertion of (ii) follows from the formula

$$f_{wB} \cdot w = (\mathrm{id}, w) \cdot f_B.$$

We now let the ground field k come into play. Since π is defined over k, the Frobenius morphism F defines an endomorphism (written, as usually, F^*) of the constant sheaf $(R^*\pi_! E)_\psi$.

Let $w \in W$ be such that $FB = w \cdot B$.

4.3. **Lemma.** *With the notations of* 4.2 *we have*

$$q^d F^* \alpha_B^i = \alpha_B^i F^* \rho^i(w).$$

From the diagram

$$\begin{array}{ccc} \mathcal{Y}_{A,T}^G & \xrightarrow{F} & \mathcal{Y}_{A,T}^G \\ f_B \downarrow & & \downarrow f_B \\ \mathcal{B}^G \times \mathfrak{t}_0' & \longrightarrow & \mathcal{B}^G \times \mathfrak{t}_0' \end{array}$$

we obtain

$$\tilde{F}(R^{2d} f_{FB,!}\, E)_\psi = R^{2d}(f_{B,!}\, \tilde{F} E)_\psi, \tag{2}$$

where $\tilde{F}$ denotes the inverse image morphism defined by F. There is a commutative diagram

$$\begin{array}{ccccc} (R^*\pi_! E)_\psi & \xrightarrow{\sim} & R^*\sigma_!(R^{2d} f_{FB,!}\, E)_\psi(-2d) & \xrightarrow{\sim} & H^*(\mathcal{B}_A^G, E)(-2d) \\ F^* \downarrow & & \downarrow & & \downarrow q^d F^* \\ (R^*\pi_! E)_\psi & \xrightarrow{\sim} & R^*\sigma_!(R^{2d} f_{B,!}\, \tilde{F} E)_\psi(-2d) & \xrightarrow{\sim} & H^*(\mathcal{B}_A^G, E)(-2d) \end{array}$$

where the composite of the horizontal arrows in the first (second) line is $\alpha^*_{FB}(\alpha^*_B)$. The middle vertical arrow is induced by the sheaf morphism $\tilde{F}$, using (2). The $\tilde{F}E$ in the second line is responsible for the vertical homomorphism $q^d F^*$. We find that

$$q^d F^* \alpha^i_{w \cdot B} = \alpha^i_B F^*,$$

whence 4.3, by 4.2 (ii).

4.4. **Theorem.** *Let $A \in \mathfrak{g}^F$ be nilpotent and $A' \in (\mathfrak{t}')^F$ strongly regular. Assume that $B \supset T$ is a Borel subgroup such that $FB = w \cdot B$. There exists a representation r^*_B of W in $H^*(\mathscr{B}^G_A, E)$ such that*

$$S^G(A, A') = q^d \sum_{i \geq 0} (-1)^i \operatorname{Tr}(F^* r^i_B(w)^{-1}, H^i(\mathscr{B}^G_A, E)). \tag{3}$$

By formula (3) of no. 1 we have

$$S^G(A, A') = \sum_{i \geq 0} (-1)^i \operatorname{Tr}(F^*, H^i_c(Y^G_{A,A'}, E)_\psi).$$

Now $H^i_c(Y^G_{A,A'}, E)_\psi$ is the stalk in A' of the constant sheaf $(R^i \pi_! E)_\psi$. Identifying $H^i_c(Y^G_{A,A'}, E)_\psi$ with $H^{i-2d}_c(\mathscr{B}^G_A, E)$ via the isomorphism α^i_B of 4.2 and putting

$$r^i_B(w) = \alpha^i_B \rho^i(w)(\alpha^i_B)^{-1},$$

the assertion follows from 4.3.

4.5. **Corollary.** *If $A \in \mathfrak{g}^F$ is nilpotent and A', A'' are two strongly regular elements in $(\mathfrak{t}')^F$ then $S^G(A, A') = S^G(A, A'')$.*

4.6. *Independence of Choices.* In the definition of $S^G(A, A')$ there enters the character ψ which was fixed in 1.1. Let us write for the moment $S_\psi(A, A')$ to indicate the dependence on ψ. If ψ' is another nontrivial character of k with values in E, there is $a \in k^*$ such that $\psi'(x) = \psi(ax)$. It follows from the definitions that

$$S_{\psi'}(A, A') = S_\psi(A, aA').$$

If A is nilpotent and A' strongly regular then 4.5 implies that this is also equal to $S_\psi(A, A')$. So, in these circumstances, $S^G(A, A')$ is independent of the choice of ψ.

It is clear from the definitions that the equivalence class of the representation r^*_B of 4.4 is independent of the choice of the Borel subgroup B. Also, the right-hand side of (3) must be independent of B. In fact, let $B_1 = w_1 \cdot B$ be another Borel subgroup containing T. By 4.2 (ii) we have

$$r^*_{B_1}(w) = r^*_B(w_1)^{-1} r^*_B(w) r^*_B(w_1).$$

If B is replaced by B_1 the element w of 4.4 gets replaced by $(Fw_1) w w_1^{-1}$ and the independence of B of the right-hand side of (3) can then also be checked by using that

$$\rho^*(Fw) = (F^*)^{-1} \rho^*(w) F^*,$$

which is a consequence of formula (1) of no. 3.

We now discuss some more results about the representation r^*_B. First, it is readily seen that its class is independent of the choice of ψ. Next we establish independence of k. In the definition of r_B^* the field k comes in, because k enters into the definition of $\mathscr{Y}_{A,T}^G$. Write $\mathscr{Y}_k$ to indicate this dependence.

Let σ be the projection of $\mathscr{Y}_k$ on $G/T \times \mathfrak{t}_0'$. Then $\mathscr{Y}_k$ is, via σ, a Galois covering of $G/T \times \mathfrak{t}_0'$, with group k. The character ψ of k defines an l-adic sheaf $\mathscr{T}_\psi$ on $G/T \times \mathfrak{t}_0'$ [9, p. 303]. If ρ is the projection of $G/T \times \mathfrak{t}_0'$ onto t_0', then

$$(R^* \pi_! E)_\psi = R^* \rho_! \mathscr{T}_\psi. \tag{4}$$

Let k' be a finite extension of k, of degree n. Denote by $\sigma', \ldots$ the previous objects for $\mathscr{Y}_{k'}, \ldots$. If τ is the morphism $\mathscr{Y}_{k'} \to \mathscr{Y}_k$ with

$$\tau(x, gT, A') = (x^{q^{n-1}} + \cdots + x^q + x, gT, A')$$

then $\sigma' = \sigma\tau$ and it follows that

$$\mathscr{T}_\psi = \mathscr{T}'_{\operatorname{Tr}\psi}, \tag{5}$$

where $\operatorname{Tr}\psi$ is defined by

$$\operatorname{Tr}\psi(x) = \psi(x^{q^{n-1}} + \cdots + x^q + x) \qquad (x \in k')$$

(4) and (5) yield an isomorphism

$$(R^* \pi_! E)_\psi \simeq (R^* \pi_!' E)_{\operatorname{Tr}\psi},$$

and it is easily checked that it commutes with W. This implies that ρ^*, and also r_B^*, is independent of k, up to equivalence. In other words: we have established that the representation of W on the cohomology of $\mathscr{B}_A^G$ has geometric significance.

It would be interesting to have such representations in characteristic 0. The method used above involves Artin-Schreier extensions and can thus be said to depend on the fact that the affine line is not simply connected in nonzero characteristics. It would also be interesting to establish independence of r_B^* of the characteristic p, if p is large enough. Such a result makes sense: if p is large than the classification of nilpotent classes of $\mathfrak{g}$ is independent of p [4, p. 247], and the $\mathscr{B}_A^G$ are obtained by reduction modulo p from similar objects in characteristic 0 (see 6.13).

5. Green Functions

5.1. Let T be an F-stable maximal torus of G such that $(\mathfrak{t}')^F$ contains a strongly regular element A'. We then define the *Green function* $Q_T = Q_{T,G}$ on the set of nilpotents of $\mathfrak{g}^F$ by

$$Q_{T,G}(A) = (-1)^{r(G)-r(T)} q^{-d(G)} S^G(A, A').$$

By 4.5 this is independent of the choice of A'. The independence of $S^G(A, A')$ of the character ψ (see 4.6) implies that $Q_{T,G}(A)$ is a rational number. It is clear that

$$Q_{T,G}(\operatorname{Ad}(g) A) = Q_{T,G}(A),$$

if $g \in G^F$.

We shall establish in this section a number of properties of the Green functions. We denote by $\mathscr{N}_G$ the set of nilpotent elements of $\mathfrak{g}$.

5.2. *Remarks.* (a) Q_T can already be defined if $\mathfrak{t}'$ contains strongly regular elements, namely by using formula (3) of no. 4. However, for the proofs of the results of this no. we need stronger assumptions.

(b) If there exists a G^F-equivariant bijection of the unipotent set of G^F onto the nilpotent set of $\mathfrak{g}^F$ (which is the case, for example if $G = GL_n$ or if G is semisimple and simply connected, and p is good, see [24]), then we can transfer the Green function to a function on the unipotent set of G^F.

It was shown by Kazhdan in [15], using the results of this paper, that if p and q are sufficiently large, one thus recovers the Green functions of [10, § 4]. 4.4 can then be interpreted as a description of values of the Green function $Q_{T,G}(u)$ of [10] in terms of the geometry of the variety $\mathscr{B}_u^G$ of Borel subgroups containing u. That the $Q_{T,G}(u)$ can be related to the trigonometric sums $S^G(A, A')$ is in accordance with a conjecture made in [25, p. 152].

First a simple lemma. Let $f\colon G \to G_1$ be a surjective k-morphism whose kernel is central. Put $fT = T_1$.

5.3. **Lemma.** *We have* $Q_{T,G} = Q_{T_1,G_1} \circ df$ *in the following cases:*

(a) *f is separable and* $(\mathfrak{t}')^F$ *contains a strongly regular element,*

(b) *f is the morphism onto the adjoint group and* $(\mathfrak{t}')^F$ *contains a strongly regular element A' such that $(df)'A'$ is also strongly regular.*

The proof is trivial, taking into account 2.7 for case (a). The condition of case (b) is satisfied if we are in the situation of 2.8 and q is sufficiently large.

5.4. Let T be as in 5.1. Assume that T is contained in an F-stable proper parabolic subgroup P. Let L be the (F-stable) Levi subgroup of P containing T and U the unipotent radical of P. Denote the projection $P \to L$ by π. We identify, as we may, the toral subspace $\mathfrak{t}'$ with the corresponding subspace of the dual $\mathfrak{l}'$ of the Lie algebra of L. In these circumstances $Q_{T,G}$ and $Q_{T,L}$ are defined.

5.5. **Proposition.**

$$Q_{T,G}(A) = |P^F|^{-1} \sum_{\substack{g \in G^F \\ \mathrm{Ad}(g) A \in \mathfrak{p}}} Q_{T,L}(\pi \,\mathrm{Ad}(g) A).$$

Let $A' \in (\mathfrak{t}')^F$ be strongly regular. Then

$$S^G(A, A') = |T^F|^{-1} \sum_{g \in G^F} \psi(\langle A, \mathrm{Ad}'(g) A' \rangle) = |T^F|^{-1}\, |U^F|^{-1} \sum_{\substack{g \in G^F \\ u \in U^F}} \psi(\langle A, \mathrm{Ad}'(gu) A' \rangle).$$

Now $\mathrm{Ad}'(U) A' = A' + \mathfrak{p}^{\perp}$ (see the proof of 3.5 for a similar fact). Hence the last sum equals

$$\sum_{\substack{y \in G^F \\ X' \in (\mathfrak{p}^{\perp})^F}} \psi(\langle A, \mathrm{Ad}'(g)(A' + X') \rangle) = |U^F| \sum_{\substack{g \in G^F \\ \mathrm{Ad}(g)^{-1} A \in \mathfrak{p}}} \psi(\langle A, \mathrm{Ad}'(g) A' \rangle)$$

$$= |U^F|\, |P^F|^{-1} \sum_{\substack{g \in G^F,\, y \in P^F \\ \mathrm{Ad}(g) A \in \mathfrak{p}}} \psi(\langle \mathrm{Ad}(g) A, \mathrm{Ad}'(y) A' \rangle).$$

The assertion now easily follows.

If T and T_1 are two maximal tori of G, put

$$N_G(T, T_1) = \{g \in G \mid \mathrm{ad}(g) T = T_1\}.$$

5.6. **Theorem** *(orthogonality relations of Green functions). Assume that $p \neq 2$ or that G is $\bar{k}$-isomorphic to GL_n, and that q is sufficiently large. Let T and T_1 be two F-stable maximal tori of G. Then*

$$|G^F|^{-1} \sum_{X \in \mathcal{N}_G^F} Q_{T,G}(X) Q_{T_1,G}(X) = |T^F|^{-1} |T_1^F|^{-1} |N_G(T, T_1)^F|.$$

We prove this by induction on the semisimple rank $s(G)$ of G. If $s(G)=0$ then G is a torus and the statement is trivial. Suppose now the theorem to be true for all G_1 with $s(G_1)<s(G)$. By 2.8 and 5.3 we may assume that G is either semisimple and adjoint or that $G \simeq GL_n$ (notice that the set of rational nilpotents of $\mathfrak{g}$ is not affected by passing to the adjoint group). In fact, if $p=3$ and if the root system Φ contains an irreducible component of type A_2 then 2.8 does not apply, and an extra argument is needed. Arguing as in the beginning of the proof of 2.8, we reduce to the case $G=SL_3$ (and $p=3$). Then $\mathfrak{g} \subset \mathfrak{gl}_3$ and it is easily seen that the orthogonality properties for G can be obtained from those for GL_3. Since the case $G=GL_n$ can be disposed of without assumptions on the characteristic (see below), this exceptional case does not cause trouble. Assume now that G is adjoint semisimple. Then the center of $\mathfrak{g}$ consists of 0 only. Let A', A'_1 be strongly regular elements in $(\mathfrak{t}')^F$, $(\mathfrak{t}'_1)^F$. By the orthogonality relations for the group characters of the finite group $\mathfrak{g}^F$ we have (the bar having the obvious meaning)

$$\sum_{X \in \mathfrak{g}^F} S^G(X, A') \overline{S^G(X, A'_1)} = |\mathfrak{g}^F| |G^F| |T^F|^{-1} \varepsilon_{A', A'_1}, \tag{1}$$

where $\varepsilon_{A', A'_1}=1$ if A' and A'_1 are G^F-conjugate, and 0 otherwise. On the other hand, this sum equals

$$\sum_{\substack{X \in \mathfrak{g}^F \\ X \text{ semisimple}}} \sum_{\substack{Y \in \mathfrak{g}^F \\ Y \text{ nilpotent} \\ [X, Y]=0}} S^G(X+Y, A') \overline{S^G(X+Y, A'_1)}.$$

Let H_X be the connected centralizer of $X \in \mathfrak{g}$. If X is semisimple and $[X, Y]=0$, then Y lies in the Lie algebra $\mathfrak{h}_X$ of H_X (see [3, p. 225]). Using 3.15 and the induction assumption, the last sum can be transformed into (up to a factor $q^{2d(G)}(-1)^{r(T)+r(T_1)}$)

$$\sum_{X \in \mathcal{N}_G^F} Q_{T,G}(X) Q_{T_1,G}(X) - |G^F| |T^F|^{-1} |T_1^F|^{-1} |N_G(T, T_1)^F|$$
$$+ |T^F|^{-1} |T_1^F|^{-1} \sum_{\substack{X \in \mathfrak{g}^F \\ X \text{ semisimple}}} |H_X^F|^{-1} \sum_{\substack{g, g_1 \in G^F \\ X \in \mathrm{ad}(g)\mathfrak{t} \cap \mathrm{ad}(g_1)\mathfrak{t}_1}}$$
$$\psi(\langle X, \mathrm{Ad}'(g) A' \rangle) \overline{\psi(\langle X, \mathrm{Ad}'(g_1) A'_1 \rangle)} |N_{H_X}(\mathrm{ad}(g) T, \mathrm{ad}(g_1) T_1)^F|.$$

The double sum equals

$$\sum_{\substack{X \in \mathfrak{g}^F \\ X \text{ semisimple}}} \sum_{\substack{g \in G^F \\ \mathrm{Ad}(g) X \in \mathfrak{t}^F \\ n \in N_G(T, T_1)^F}} \psi(\langle \mathrm{Ad}(g) X, A' \rangle) \overline{\psi(\langle \mathrm{Ad}(ng) X, A'_1 \rangle)}$$

$$= |G^F| \sum_{\substack{X \in \mathfrak{t}^F \\ n \in N_G(T, T_1)^F}} \psi(\langle X, A' \rangle) \overline{\psi(\langle \mathrm{Ad}(n) X, A'_1 \rangle)} = |G^F| |\mathfrak{t}^F| \varepsilon_{A', A'_1}$$

$$= |G^F| |\mathfrak{g}^F| q^{-2d(G)} \varepsilon_{A', A'_1}.$$

Comparison with (1) gives the result, for this case. If $G \simeq GL_n$, the center of $\mathfrak{g}$ is nonzero. However, in this case it is easily seen that $\mathfrak{t}'$ contains strongly regular elements orthogonal to the center of $\mathfrak{g}$. A similar argument then gives the assertion.

5.7. **Proposition.** *Under the assumption of* 5.6 *we have*

$$\sum_{X \in \mathcal{N}_G^F} Q_{T,G}(X) = |T^F|^{-1} |G^F|.$$

The method of proof is the same as that of 5.6. We now start with

$$\sum_{X \in \mathcal{N}_G^F} S^G(X, A') = 0,$$

if $A' \in (\mathfrak{t}')^F$ is strongly regular. Using 3.15 and induction we have, if G is semisimple and adjoint,

$$-\sum_{X \in \mathcal{N}_G^F} Q_{T,G}(X) + |T^F|^{-1} |G^F| = |T^F|^{-1} \sum_{\substack{X \in \mathfrak{g}^F \\ X \text{ semisimple}}} \sum_{\substack{g \in G^F \\ \mathrm{Ad}(g) X \in \mathfrak{t}^F}} \psi(\langle \mathrm{Ad}(g) X, A' \rangle)$$

$$= |T^F|^{-1} |G^F| \sum_{X \in \mathfrak{t}^F} \psi(\langle X, A' \rangle) = 0.$$

whence the assertion. If $G \simeq GL_n$ the argument is as in the proof of 5.6.

If T is a maximal torus, let $W(T)$ denote its Weyl group. We denote by $\sum_{(T)}$ summation over the G^F-conjugacy classes of F-stable maximal tori T. For the notion of a good prime, to be used in 5.8 see [4, p. 178, 185].

5.8. **Proposition.** *Assumptions of* 5.6, *assume moreover that p is good. Then*

$$\sum_{(T)} |W(T)^F|^{-1} Q_{T,G} = 1$$

If p is good the number of nilpotent elements of $\mathfrak{g}^F$ equals $q^{2d(G)}$ [24, p. 387] (where this is stated for semisimple and simply connected groups, the general result then easily follows). Using 5.6 and 5.7 we find that

$$\sum_{X \in \mathcal{N}_G^F} \left(\sum_{(T)} |W(T)^F|^{-1} Q_{T,G}(X) - 1\right)^2 = q^{2d(G)} - \sum_{(T)} |G^F| |T^F|^{-1} |W(T)^F|^{-1}.$$

The last sum equals the number of F-stable maximal tori of G, and this number also equals $q^{2d(G)}$, by a theorem of Steinberg [29, p. 96]. Hence the double sum is 0, which implies 5.8.

Remark. 5.6, 5.7 and 5.8 are counterparts of results proved in [10] for the Green functions of the group G^F, viz. Th. 6.9 and formulas (7.10.4), (7.13.1).

5.9. For later use we give a description of the $Q_{T,G}(A)$ for fixed A and variable T, which follows from 4.4. Fix an F-stable maximal torus T, let $W(T)$ be its Weyl group. Let T_1 be another such torus. If $T_1 = \mathrm{ad}(a) T$ then $n = a^{-1} F a$ normalizes T, let $w_1 \in W(T)$ be its image in the Weyl group. $\mathrm{ad}(a)$ defines an isomorphism $W(T) \xrightarrow{\sim} W(T_1)$.

There is an isomorphism $\phi\colon \mathcal{Y}_{A,T}^G \xrightarrow{\sim} \mathcal{Y}_{A,T_1}^G$, with

$$\phi(x, gT, A') = (x, g a^{-1} T_1, \mathrm{Ad}'(a) A').$$

If B is a Borel subgroup of G containing T, then $B_1 = \mathrm{ad}(a)B$ is one containing T_1 and, f_B being as in 3.4, we have $f_B = f_{B_1}\phi$. Moreover $FB_1 = \mathrm{ad}(a)(w_1 w) \cdot B_1$. The next proposition then follows from 4.4.

5.10. **Proposition.**

$$Q_{T_1,G}(A) = (-1)^{r(G)-r(T_1)} \sum_{i \geq 0} (-1)^i \mathrm{Tr}(F^* r_B^i (w_1 w)^{-1}, H^i(\mathscr{B}_A^G, E)).$$

Finally, another property of the Green functions. Let P be an F-stable parabolic subgroup of G, with unipotent radical U. Put $N(T, P) = \{g \in G \mid \mathrm{ad}(g) T \subset P\}$.

5.11. **Proposition.** *Assume that* $(\mathfrak{t}')^F \neq \phi$. *Then*

$$\sum_{X \in \mathfrak{u}^F} Q_{T,G}(X) = (-1)^{r(G)-r(T)} q^{-d(G)} |U^F| |T^F|^{-1} |N(T, P)^F|.$$

In particular, this sum is 0 *if* T *is not* G^F*-conjugate to a maximal torus of* P.

5.11 follows in a straightforward manner from the definition of $Q_{T,G}$, using 2.12.

6. A Realization of the Irreducible Representations of Weyl Groups

The realization of the title will be derived from the results of no. 4 and the orthogonality relations of 5.6. We begin with some preliminaries.

6.1. Let $\mathscr{Y} = \mathscr{Y}_{A,T}^G$ be as in 3.1, with A nilpotent. We use the notation of that section. The centralizer $Z = Z_G(A)$ of A in G operates on $\mathscr{Y}$ as follows

$$z \cdot (x, gT, A') = (x, zgT, A').$$

Z then also operates on all fibers $Y_{A,A'}^G$. The action of Z on $\mathscr{Y}$ commutes with that of W. It follows that Z operates on the sheaf $(R^* \pi_! E)_\psi$ of 4.2 (i) and that the representation of Z thus obtained commutes with the representation of W of 4.2. Since the identity component Z^0 of Z acts trivially on the stalks $H_c^*(Y_{A,A'}^G, E)_\psi$ of $(R^* \pi_! E)_\psi$ (by [10, 6.4]), it follows that Z^0 acts trivially on $(R^* \pi_! E)_\psi$. So the representation of Z comes in fact from a representation of the finite group $C(A) = Z/Z^0$.

If we let Z act on $\mathscr{B} \times \mathfrak{t}'_0$ via the obvious action on $\mathscr{B}$, the morphism f_B of 3.4 commutes with Z. Via the isomorphism α_B^* of 4.2 (ii) we then obtain a representation s_B^* of $C(A)$ in $H_c^*(\mathscr{B}_A^G, E)$, commuting with r_B^*.

6.2. Next some remarks involving the "fusion" in $\mathfrak{g}$ of nilpotent G^F-orbits of $\mathfrak{g}^F$. If M is an algebraic group over k, recall that the 1-cohomology set $H^1(k, M)$ is M modulo the equivalence relation: $m \sim m'$ if there exists $n \in M$ with $m' = (Fn)mn^{-1}$. Let $A_1 \in \mathfrak{g}^F$ be such that $A_1 = \mathrm{Ad}(x)A$, with $x \in G$. Then $z = (Fx)^{-1} x \in Z$. The image $h(A_1)$ of z in $H^1(k, C(A))$ depends only on the G^F-conjugacy class of A_1, and h defines a bijection of the set of G^F-conjugacy classes of elements of $\mathfrak{g}^F$ which are G-conjugate to A, onto $H^1(k, C(A))$. We have

$$|Z_G(A_1)^F| = \alpha |Z_G^0(A_1)|^F,$$

where, putting $c = zZ_0$, α is the number of $c' \in C(A)$ with $Fc' = cc'c^{-1}$. In particular, if F acts trivially on $C(A)$, we have $\alpha = |Z_{C(A)}(c)|$. All this is readily derived from the results of [4, part E, Ch. I].

Let $\phi: \mathscr{B}_A^G \to \mathscr{B}_{A_1}^G$ be the isomorphism $B' \mapsto \mathrm{ad}(x) B'$. Then $F \cdot \phi = \phi \cdot z^{-1} \cdot F$. If we identify $\mathscr{B}_{A_1}^G$ with $\mathscr{B}_A^G$ via ϕ, then the endomorphism F^* of $H^*(\mathscr{B}_{A_1}^G, E)$ corresponds to the endomorphism $F^* s_B^*(c)$ of $H^*(\mathscr{B}_A^G, E)$. With the notations of 4.4, we then have the following.

6.3. **Lemma.** (i) $S^G(A_1, A') = q^d \sum_{i \geq 0} (-1)^i \operatorname{Tr}(F^* s_B^i(c) r_B^i(w)^{-1}, H^i(\mathscr{B}_A^G, E))$;

(ii) $Q_{T,G}(A) = Q_{T,G}(A_1)$ *if* $s_B^*(c)$ *acts trivially on* $H^*(\mathscr{B}_A^G, E)$.

(i) follows from 4.4 and the preceding remarks, and (ii) is a direct consequence. (ii) explains why the Green functions $Q_{T,G}$ do not separate the nilpotent G^F-orbits of $\mathfrak{g}^F$.

6.4. We now recall some facts about maximal tori, also contained in [loc.cit.]. Let T be as before. It is known that there is a bijection of the set of G^F-conjugacy classes of maximal F-stable tori of G onto $H^1(k, W)$. Moreover, if T_1 is an F-stable torus in the class defined by $w_1 \in W$ then $|W(T_1)^F| = \alpha |T_1^F|$, where α is the number of elements $w' \in W$ with $F w' = w_1 w' w_1^{-1}$. In particular, if T is k-split, then F acts trivially on W and $H^1(k, W)$ is the set of conjugacy classes of W. Then the above α equals $|Z_W(w_1)|$.

To obtain the asymptotic result 6.6 about $Q_{T,G}$, we need the following lemma. Let X be a k-variety of dimension e. Assume, for simplicity, that its irreducible components of dimension e are defined over k, and let m be the number of them.

6.5. **Lemma.** (i) $H_c^{2e}(X, E) \simeq E^m$, *and* F^* *acts on* $H_c^{2e}(X, E)$ *as multiplication by* q^e;

(ii) *All complex conjugates of the eigenvalues of* F^* *on* $H_c^i(X, E)$ *with* $i < 2e$ *have absolute value* $\leq q^{e-\frac{1}{2}}$.

It should be borne in mind that $H_c^i(X, E) = 0$ unless $0 \leq i \leq 2e$.

Let S be the set of singular point of X, then $\dim S < e$. There is an exact sequence

$$\cdots \to H_c^i(X, E) \to H_c^i(S, E) \to H_c^{i+1}(X - S, E) \to \cdots,$$

from which we find, since $H_c^i(S, E) = 0$ for $i > 2e - 2$, that $H_c^{2e}(X, E) \simeq H_c^{2e}(X - S, E)$; the isomorphism being compatible with F^*. Now (i) follows from the results stated in [9, p. 281].

(ii) is a direct consequence of the (profound) results of Deligne stated in [20, Th. 2]. Perhaps there is a simpler proof of (ii).

Now let A and A_1 be as in 6.2. Put $e(A) = \dim \mathscr{B}_A^G$ and assume that all irreducible components of $\mathscr{B}_A^G$ of dimension $e(A)$ are defined over k. We use the notations of 6.2 and 6.3 and write $r_A = r_B^{2e(A)}$, $s_A = s_B^{2e(A)}$. The following result is then a consequence of 6.3 and 6.5.

6.6. **Lemma.**

$$Q_{T,G}(A_1) = (-1)^{r(G) - r(T)} q^{e(A)} \operatorname{Tr}(s_A(c) r_A(w)^{-1}, H^{2e(A)}(\mathscr{B}_A^G, E)) + O(q^{e(A) - \frac{1}{2}}).$$

There is an inequality for the dimension $e(A)$ which we shall need. For the moment, $\bar{k}$ may be any algebraically closed field. A is a nilpotent element of the Lie algebra $\mathfrak{g}$ of the reductive $\bar{k}$-group G.

6.7. **Proposition.** (i) $2e(A) \leqq \dim Z_G(A) - \operatorname{rank} G$.

(ii) *Equality holds in* (i) *if either* $p=0$ *or* p *is sufficiently large.*

The analogue of (i) for unipotent elements of G is proved in [30, p. 133]. For nilpotents of $\mathfrak{g}$ the proof is similar. It also follows as in [loc.cit.] that equality will hold in (i) if (T, W, B and U being as in no. 2) there is $w \in W$ such that the orbit $O_G(A)$ intersects $\mathfrak{u} \cap w \cdot \mathfrak{u}$ in a dense subset. It was pointed out to me by Steinberg that the results of Bala and Carter in [1] imply (ii). In fact, it is proved in [loc.cit.] that if p is either 0 or large enough, there exists a reductive subgroup H of G containing T and a parabolic subgroup Q of H such that $O_G(A)$ intersects the Lie algebra $\mathfrak{v}$ of the unipotent radical of Q in a dense subset. We may assume that Q contains the Borel subgroup $B \cap H$ of H and we then have to prove that there is $w \in W$ with $\mathfrak{v} = \mathfrak{u} \cap w \cdot \mathfrak{u}$.

Let Φ be the root system of (G, T) and Φ_1 that of (H, T), so $\Phi_1 \subset \Phi$. Let S_1 be the basis of Φ_1 defined by $B \cap H$. Let $S_2 \subset S_1$ span the root system Φ_2 of the Levi subgroup L of Q which contains T. Let $w_0, w_{1,0}, w_{2,0}$ be the elements of maximal lengths of the Weyl groups of Φ, Φ_1, Φ_2 (for the orders defined by $B, B \cap H, B \cap L$, respectively). Then $w = w_{2,0} w_{1,0} w_0$ is as required.

6.8. *Remarks.* (i) may also be proved by using the following consequence of 5.6

$$(Q_{T,G}(A))^2 \leqq |Z_G(A)^F| \, |T^F|^{-1} \, |W(T)^F|,$$

taking T to be k-split, using 6.5 and 6.6 and letting q tend to ∞. This requires some restrictions on p, in particular we must have $p>0$. The case $p=0$ can the be derived by a reduction argument.

(i), or rather its counterpart for unipotents of G, was apparently conjectured by Grothendieck in 1969. The author gave a proof using an elementary argument of the same nature as the one indicated in the previous paragraph (presented at the Oberwolfach meeting on algebraic groups in 1971). But Steinberg's proof, given in [30], is a better one.

6.9. We return to the case of a finite field k. We assume, for simplicity, that either G is quasi-simple not of type A_l and p is good, or $G \simeq GL_{l+1}$. Then the number of nilpotent G-orbits in $\mathfrak{g}$ is finite [4, p. 185]. We take k so large that (a) all such orbits are represented by elements $A \in \mathfrak{g}^F$ which are such that the irreducible components of $Z_G(A)$ and $\mathscr{B}_A^G$ are defined over k;

(b) the Green functions $Q_{T,G}$ are defined for all F-stable maximal tori T of G, and the orthogonality relations of 5.6 hold;

(c) G contains a maximal F-stable torus which is k-split.

Fix a nilpotent $A \in \mathfrak{g}^F$ with the properties of (a). Let $C(A)$ be as in 6.1. The representation s_A of $C(A)$ (see 6.6) is the permutation representation defined by the action of $C(A)$ on the set of irreducible components of $\mathscr{B}_A^G$ of maximal dimension $e(A)$.

If Γ is a finite group, denote by $\Gamma^\wedge$ the set of its irreducible E-valued characters. If $\phi \in C(A)^\wedge$, let $V_{A,\phi}$ be the ϕ-isotypic subspace of $H^{2e(A)}(\mathscr{B}_A^G, E)$. Since the representation r_A of W commutes with all $s_A(c)$, the $r_A(w)$ stabilize all $V_{A,\phi}$.

For each $\phi \in C(A)^\wedge$ such that $V_{A,\phi} \neq 0$ let $\chi_{A,\phi}$ be the character of W such that $\phi \otimes \chi_{A,\phi}$ is the character of the representation of $C(A) \times W$ in $V_{A,\phi}$. If k is replaced by an extension then $\chi_{A,\phi}$ does not change. So we may define $V_{A,\phi}$ and $\chi_{A,\phi}$ for all nilpotent $A\in\mathfrak{g}$.

Let Σ be the set of (A,ϕ), where $A\in\mathfrak{g}$ is nilpotent, $\phi\in C(A)^\wedge$, with $V_{A,\phi}\neq 0$ and $2e(A)=\dim Z_G(A)-\operatorname{rank}(G)$. Then G acts on Σ on the left, and $(A,\phi)\mapsto\chi_{A,\phi}$ defines a map ξ of $G\backslash\Sigma$ to characters of W.

We can now state the main result of this section.

6.10 **Theorem.** *ξ is a bijection of $G\backslash\Sigma$ onto $W^\wedge$.*

In particular, each irreducible character of W is of the form $\chi_{A,\phi}$, where the conjugacy class of A and, for fixed A, the character $\phi\in C(A)^\wedge$ are uniquely determined. Fix a maximal k-split F-stable torus T of G. Then F acts trivially on its Weyl group W, and if $B\supset T$ is a Borel subgroup then $FB=B$. So the element w of 4.4 equals 1. Let T_1 and T_2 be two F-stable maximal tori of G, defined by the conjugacy classes of the elements $w_i\in W$ (according to 6.4). Then 6.6 combined with 5.10 gives an asymptotic expression for large q for the $Q_{T_i,G}(A)$ $(i=1,2)$. Inserting these in the orthogonality relation of 5.6 for the tori T_1 and T_2, and using what was said in 6.2, we obtain the following formula

$$\sum{}'|Z_G(A)^F|^{-1}q^{2e(A)}\sum_{c\in C(A)}\tau_A(s_A(c)\,r_A(w_1)^{-1})\,\tau_A(s_A(c^{-1})\,r_A(w_2))$$
$$=\eta_{w_1,w_2}|T_1^F|^{-1}|Z_W(w_1)|+O(q^{-\operatorname{rank}(G)-\frac{1}{2}}). \tag{1}$$

The notations are as follows:

Σ' denotes summation over a set of representatives in $\mathfrak{g}^F$ of the nilpotent orbits of $\mathfrak{g}$, with the properties (a) of 6.9, $\tau_A(\ )=\operatorname{Tr}(\ ,H^{2e(A)}(\mathscr{B}_A^G,E))$, $\eta_{w_1,w_2}=1$ if w_1 and w_2 are conjugate in W and 0 otherwise.

We have also used that we may in the right-hand side of the formula of 6.6, replace c by c^{-1} and w by w^{-1} (because the $Q_{T,G}$ take rational values, see 5.1).

Since the trace of $s_A(c)\,r_A(w)$ in $V_{A,\phi}$ equals $\phi(c)\,\chi_{A,\phi}(w)$ (or 0, if $V_{A,\phi}=0$), we have

$$\sum_{c\in C(A)}\tau_A(s_A(c)\,r_A(w_1)^{-1})\,\tau_A(s_A(c^{-1})\,r_A(w_2))$$
$$=\sum_{\substack{\phi_1,\phi_2\in C(A)^\wedge\\ V_{A,\phi_1}\neq 0,\,V_{A,\phi_2}\neq 0}}\phi_1(c)\,\phi_2(c^{-1})\,\chi_{A,\phi_1}(w_1^{-1})\,\chi_{A,\phi_2}(w_2)$$
$$=|C(A)|\sum_{\substack{\phi\in C(A)^\wedge\\ V_{A,\phi}\neq 0}}\chi_{A,\phi}(w_1^{-1})\,\chi_{A,\phi}(w_2),$$

by the orthogonality relations of the group characters of $C(A)$. Inserting this in (1), using 6.7 (i) and the well-known fact that if M is a connected k-group we have $|M^F|=q^{\dim M}+O(q^{\dim M-1})$, we find by letting q tend to ∞ from (1)

$$\sum{}''\Big(\sum_{\substack{\phi\in C(A)^\wedge\\ V_{A,\phi}\neq 0}}\chi_{A,\phi}(w_1^{-1})\,\chi_{A,\phi}(w_2)\Big)=\eta_{w_1,w_2}Z_W(w_1),$$

where Σ'' denotes summation over the G-orbits of nilpotents $A\in\mathfrak{g}$ with $2e(A)=\dim Z_A(G)-\operatorname{rank} G$. By the orthogonality relation for the group characters of W, the right-hand side equals

$$\sum_{\chi\in W^\wedge} \chi(w_1^{-1})\,\chi(w_2).$$

The assertion of 6.10 is now a consequence of the linear independence of the group characters $\chi\in W^\wedge$.

If $c\in C(A)$ let $n(A,c)$ denote the number of irreducible components of $\mathscr{B}_A^G$ of highest dimension $e(A)$ which are fixed by c. The following corollary is an immediate consequence of 6.10. The meaning of Σ'' is the same as above.

6.11. **Corollary.** $\Sigma''|C(A)|^{-1}\sum_{c\in C(A)} n(A,c)^2=|W|$.

6.12. *Remarks.* (a) 6.10 and 6.11 are, of course, true if G is any reductive group and p is good. [24, Th. 3.2] then implies similar results for the unipotents of G.

(b) It can be shown by an elementary argument that 6.11 is true as soon as the number of nilpotent G-orbits in $\mathfrak{g}$ is finite. A similar result holds for the unipotents G.

(c) Another proof of the analogue for unipotents of 6.11, along the lines of [30], was given by M. Cross (unpublished) and by R. Steinberg [31].

6.13. We shall now deduce from 6.10 a characteristic 0 result. Let G_0 be a group scheme over $\mathbb{Z}$ such that for each field k the scheme G_k obtained by base extension is a split semisimple algebraic group over k. We shall use a Dedekind ring R contained in a finite extension of $\mathbb{Q}$, with properties to be stated presently. We put $S=\operatorname{Spec} R$ and $G=G_0\times S$. The fibre of G in $s\in S$ is a split semisimple group G_s over a finite field if s is a closed point or over an algebraic number field, if s is the generic point ξ. We denote by $\mathfrak{g}_s$ the Lie algebra of G_s. The required properties of R are as follows: there is a finite set $\mathscr{A}$ of sections $S\to\operatorname{Lie}(G)$ such that the values $A_\xi\,(A\in\mathscr{A})$ in the generic point represent the nilpotent orbits of $\mathfrak{g}_\xi$, moreover all irreducible components of the centralizer $Z_{G_\xi}(A_\xi)$ and those of the varieties $\mathscr{B}_{A_\xi}^{G_\xi}$ are defined over the field $k(\xi)$ $(A\in\mathscr{A})$.

It follows from [4, p. 247] that there is a nonempty open set $U\subset S$ such that for all $s\in U$ the A_s $(A\in\mathscr{A})$ represent the nilpotent orbits of $\mathfrak{g}_s$. If $A\in\mathscr{A}$, let $Z_G(A)$ be its centralizer in G, this is an S-groupscheme. Its fibre in $s\in S$ is the centralizer $Z_{G_s}(A_s)$. There is a subgroup scheme $Z_G(A)^0$ whose fiber is s is the identity component of $Z_{G_s}(A_s)$ [18, exp. VIB, Th. 3.10]. By a theorem of Raynaud [16, p. 82], there exists a quotient $\tilde{C}(A)=Z_G(A)/Z_G(A)^0$, it is a finite group scheme over S. Its fiber in the generic point ξ is, by our choice of S, a trivial finite group scheme over $k(\xi)$. Hence we may assume, by shrinking U, that the restriction of $\tilde{C}(A)$ to U is a trivial group scheme, i.e. a product $C(A)\times U$, where $C(A)$ is a constant group scheme over S. We identify for each $s\in U$ the quotient $Z_{G_s}(A_s)/Z_{G_s}(A_s)^0$ with $C(A)$.

For each $A\in\mathscr{A}$ there is a projective S-scheme $\mathscr{B}_A^G$ such that its fiber $s\in S$ is the variety $\mathscr{B}_{A_s}^{G_s}$. Fix A and let $C_1,\dots,C_t$ be the irreducible components of $\mathscr{B}_A^G$. By [12, Prop. 15.5.9] we may assume (after shrinking) that the number of components of $\mathscr{B}_{A_s}^{G_s}$ is also t, for $s\in U$. We may also assume that $\dim(C_i)_s=\dim C_i$ and that the $(C_i)_s$ are the components of $\mathscr{B}_{A_s}^{G_s}$ $(s\in U)$. Now let $c\in C(A)$ and let i,j be such that $c(C_i)_\xi=(C_j)_\xi$. Then $c(C_i)_s=(C_j)_s$ on an open subset of S.

It follows that the action of the centralizers $Z_{G_s}(A_s)$ on the irreducible components of the $\mathscr{B}_{A_s}^{G_s}$ is "constant" on a non-empty open subset of S.

We can now formulate a result which is valid for $p=0$ (and for large p). Since we have not defined in characteristic 0 Weyl group representations on the $H^*(\mathscr{B}_A^G, E)$, the statement is somewhat less precise than that of 6.10. The notations are as in 6.9.

6.14. **Proposition.** *($p=0$ or p large). If $V_{A,\phi} \neq 0$ then $d_{A,\phi} = \phi(1)^{-1} \dim V_{A,\phi}$ is the degree of an irreducible representation of W. If A runs through a set of representations of the nilpotent G-orbits in $\mathfrak{g}$, then each such degree occurs exactly once as a $d_{A,\phi}$.*

This follows from the discussion of 6.13, using 6.7 (ii) and 6.10.

6.15. **Corollary.** *($p=0$ or p large). The number of nilpotent G-orbits in G is at most equal to the number of conjugacy classes of W.*

6.16. **Corollary.** *($p=0$ or p large). If $C(A)$ acts trivially on $\mathscr{B}_A^G$ then the number of irreducible components of $\mathscr{B}_A^G$ of maximal dimension equals the degree of an irreducible representation of W.*

7. Examples

In this section we first discuss the representations r_B^* of 4.4 in a few particular cases. Then we shall give some details about the Green functions of particular groups.

7.1. We use the notations of no. 4. We consider the case that $A=0$. The variety $\mathscr{Y}_{0,T}^G$ is isomorphic to $k \times G/T \times \mathfrak{t}'_0$, where k acts by translations in the first factor. The sheaf $(R^*\pi_! E)_\psi$ of 4.2 is now the constant sheaf $H_c^*(G/T, E)$. The Weyl group W acts on G/T by

$$w \cdot gT = g n_w^{-1} T$$

and it follows that the representation ρ^{i-2d} of 4.2 (i) is the representation of W in $H_c^i(G/T, E)$ defined by this action.

Let $\mathscr{B}$ be the variety of Borel subgroups of G. The α_B^* of 4.2 (ii) is now an isomorphism

$$H_c^*(G/T, E) \xrightarrow{\sim} H^*(\mathscr{B}, E)\ (-2d). \tag{1}$$

Assume, for simplicity, that G is k-split and semi-simple and that T is a k-split maximal torus. Then the Borel subgroup $B \supset T$ is defined over k, and (1) commutes with the Frobenius endomorphisms. One knows that $\mathscr{B}$ is the union of open Schubert cells X_w ($w \in W$), each of which is a locally closed subvariety, k-isomorphic to affine space of dimension $l(w)$ (where l is the length function on W defined by B). It follows readily by looking at the filtration of $\mathscr{B}$ defined by the closures of the X_w (which are unions of $X_{w'}$) that $H^{2i}(\mathscr{B}, E) \simeq E^{n_i}$, where n_i is the number of elements of W of length i, the other cohomology groups being 0. Moreover, F^* acts on $H^{2i}(\mathscr{B}, E)$ as multiplication by q^i. From (1) we deduce similar results for $H_c^*(G/T, E)$. Let F_w be the twisted Frobenius morphism on G/T and T, respectively, defined by $F_w(gT) = g n_w^{-1} T$ and $F_w t = n_w^{-1} t n_w$. By Grothendieck's formula for the number of rational points of a k-variety [11] we have

$$|(G/T)^{F_w}| = \sum_{i \geq 0} (-1)^i \operatorname{Tr}(F^* \rho^i(w)^{-1}, H_c^i(G/T, E)).$$

On the other hand, we have $(G/T)^{F_w} \simeq G^F/T^{F_w}$, and $|T^{F_w}| = \det(q-w)$ (see [4, p. 188]). Also, if $d_1, \dots, d_r$ are the degrees of the Weyl group W, the order of G^F equals

$$|G^F| = q^d(q^{d_1}-1)\dots(q^{d_r}-1).$$

Writing $r^*(w) = \alpha_B^* \rho(w)(\alpha_B^*)^{-1}$ (so that in the situation of 4.4 we have $r_B^* = r^*$) the preceding results and (1) imply

$$\prod_{i=1}^{r}(q^{d_i}-1)\det(q-w)^{-1} = \sum_{i\geq 0} q^i \operatorname{Tr}(r^i(w)^{-1}, H^i(\mathscr{B}, E)). \tag{2}$$

The Weyl group W acts in a vector space V over $\mathbb{Q}$, let S be the algebra of E-valued polynomial functions on V. Let I be the ideal of S generated by the nonconstant homogeneous W-invariants of S. Then S/I is a graded vectorspace on which W operates, W-isomorphic to the group algebra $E[W]$ (see [5, p. 107]). If χ is an irreducible E-valued character of W, let $(p_j(\chi))_{1\leq j\leq \chi(1)}$ be the set of degrees (with multiplicities) of S/I in which χ occurs. Putting

$$f_\chi(T) = \sum_{j=1}^{\chi(1)} T^{p_j(\chi)},$$

we have

$$|W|^{-1}\sum_{w\in W}\chi(w^{-1})\det(1-wT)^{-1} = f_\chi(T)\prod_{i=1}^{r}(1-T^{d_i})^{-1}$$

[26, p. 165], which implies, using the orthogonality relations for characters of W,

$$\prod_{i=1}^{r}(q^{d_i}-1)\det(g-w)^{-1} = \sum_{\chi\in W^\wedge}\chi(w)q^d f_\chi(q^{-1}).$$

Let ε be the sign character of W. Then the right-hand side equals (see [loc.cit.]).

$$\sum_\chi \varepsilon\chi(w)\sum_{j=1}^{\chi(1)} q^{p_j(\chi)}.$$

Comparing with (2) and using that $\chi(w) = \chi(w^{-1})$ for all characters χ of W (a consequence of [loc.cit., Th. 8.5]) we obtain the following result.

7.2. **Proposition.** *There exists a graded isomorphism $H^*(\mathscr{B}^G, E) \to S/I$, such that r^* corresponds to the natural action of W on S/I, multiplied by ε.*

This is related to familiar results in characteristic 0.

Let us remark, finally, that $Q_{T,G}(0) = (-1)^{r(G)-r(T)} q^{-d}|G^F|/|T^F|$ (notations of no. 5).

7.3. We next consider the case that the nilpotent A of no. 4 is contained in the Lie algebra of only one Borel subgroup. If B is this group and if the notations are as in 2.3, an A of the form $\sum_{\alpha>0} c_\alpha X_\alpha$ with $c_\alpha \neq 0$ for all simple α has this property (see [23, p. 137]). In good characteristics such elements are precisely the regular ones, whose centralizer has dimension equal to rank G [loc.cit., p. 138]. In bad characteristics this is not known.

Now $\mathscr{B}_A^G$ is reduced to a point. So, in the situation of 4.4, the representation $r_B^* = r_B^0$ is a 1-dimensional representation of W.

7.4. **Lemma.** *In this case, r_B^0 is the sign representation of W.*

The proof of this is rather similar to that of 3.12, so we shall be brief.

As in the proof of 3.12, we have to compare ϕ_B and $\phi_{B'}$ for two adjacent Borel subgroups B, B' containing T. Using the proper parabolic subgroup P containing B and B' one reduces to the rank 1 case. Then the result follows from 1.6 (ii) (using also 2.11, if necessary).

7.5. **Proposition.** *If A is contained in only one Borel subgroup then $Q_{T,G}(A)=1$.*

This follows from 7.4, using also 3.13. Of course, it must be assumed that $(\mathfrak{t}')^F$ (or at least $\mathfrak{t}'$, see 5.2 (a)) contains strongly regular elements.

The corresponding result for the Green functions of [10] is proved in [loc.cit., 9.16]. In that case the corresponding unipotent elements are indeed the regular ones, by a result of Steinberg [28, p. 59].

7.6. GL_n. Let $G=GL_n$. Then the $Q_{T,G}$ are defined in all characteristics, as soon as q is sufficiently large (since, as one readily sees, $\mathfrak{t}'$ always contains strongly regular elements). In fact, the $Q_{T,G}$ can be defined for any q. First, if the F-stable maximal torus T is minisotropic (i.e. if its image in SL_n is anisotropic) then one checks that $(\mathfrak{t}')^F$ always contains strongly regular elements, so then $Q_{T,G}$ is defined. From the classification of F-stable maximal tori of GL_n ([4, p. 126]) it follows that a non-minisotropic F-stable maximal torus T is contained in a proper F-stable parabolic subgroup P. Let P be minimal with this property and let L be the Levi subgroup of P containing T. Then L is F-stable and is a product of GL_m's. Moreover, T is minisotropic in L. We can then define $Q_{T,G}$ by the formula of 5.5, defining $Q_{T,L}$ in the obvious way. One can prove that the orthogonality relations of 5.6 remain valid.

7.7 If $G=GL_n$, the centralizer of any element of G is connected [4, p. 233]. So all groups $C(A)$ of no. 6 are reduced to 1. The varieties $\mathscr{B}_A^G$ have been studied for $G=GL_n$ by N. Spaltenstein (see [21]). It follows from his results that the odd-dimensional cohomology of $\mathscr{B}_A$ vanishes, and that the eigenvalues of F^* on $H^{2i}(\mathscr{B}_A^G, E)$ all are q^i. Also, $\dim \mathscr{B}_A^G = \frac{1}{2}(\dim Z_G(A) - n)$ in all characteristics. Let T be a split maximal F-stable torus of GL_n, let T_w be an F-stable torus defined by twisting T with $w \in W$. The results just quoted and 5.10 then imply the following theorem, for $G=GL_n$.

7.8. **Theorem.** *There is a polynomial $\sum_{i\geq 0} a_i(w,A)T^i$ such that $Q_{T_w,G}(A) = \sum_{i\geq 0} a_i(w,A)q^i$. The function $w \mapsto a_i(w,A)$ is a group character of $W=\mathfrak{S}_n$.*

Using what was said in 7.6, it follows that this is true for all q. A priori, the $a_i(w,A)$ might depend on p, but from Green's results about these polynomials one knows that this is not so [4, p. 140].

The leading coefficient of the polynomial of 7.8 gives an irreducible character of $\mathfrak{S}_n$, and we obtain thus a parametrization of the irreducible characters of $\mathfrak{S}_n$ by the nilpotent orbits of $\mathfrak{gl}_n$, as follows from 6.10.

We next shall discuss the Green functions for groups of type B_2 and G_2. The general results to be discussed first will be useful for these special cases.

7.9. Let G be quasi-simple and k-split. Let now T be a split maximal F-stable torus of G and $B \supset T$ a Borel subgroup. B defines an ordening of the root system Φ of (G, T), let Φ^+ be the set of positive roots. We use the notations of 2.1. Assume that all X_α lie in $\mathfrak{g}^F$.

If $w \in W$, let U_w be the subgroup of the unipotent radical U of B generated by the U_α with $\alpha > 0$, $w^{-1}\alpha < 0$. By Bruhat's lemma, the variety of Borel subgroups $\mathscr{B}$ is the disjoint union of the open Schubert cells $X_w = \mathrm{ad}(U_w n_w)B$.

Φ contains a unique highest root λ. If there are different root lengths, there is also a highest short root μ. Then $\mu \neq \lambda$ (i.e. λ is long); the dual $\mu^\vee$ is the highest root of the dual root system $\Phi^\vee$.

7.10. **Lemma.** *If* $\alpha, \beta \in \Phi^+$ *and* $\mu+\alpha$, $\mu+\beta \in \Phi$ *then* $\mu+\alpha+\beta \notin \Phi$, $\mu + n\alpha \notin \Phi$ *for* $n \geqq 2$.

It suffices to prove this for the case that Φ is the smallest closed subsystem containing μ, α, β. This reduces the proof to the case where Φ has rank $\leqq 3$. The assertion is then easily checked directly.

7.11. **Proposition.** *If* $\alpha \in \Phi$ *then* $B^G_{X_\alpha}$ *is a union of the locally closed k-subvarieties* $X_w \cap \mathscr{B}^G_{X_\alpha} (w \in W)$, *each of which is k-isomorphic to an affine space.*

We may take $\alpha = \lambda$ or $\alpha = \mu$. If $X_\alpha \in \mathrm{Ad}(g)\mathfrak{b}$, we may take $g = u^{-1} n_w$, with $u \in U_w$. Then $\mathrm{Ad}(u) X_\alpha \in w \cdot \mathfrak{u}$. Since $(\mathrm{Ad}(u) - 1)X_\alpha$ is a linear combination of the X_β with $\beta \neq \alpha$, we must have $w^{-1} \cdot \alpha > 0$. Using standard formulas for the action of the groups U_β on X_α and 7.10, it follows that the above u are exactly those in the subgroup of U_w generately by the U_β with $\mathrm{Ad}(U_\beta) X_\alpha = X_\alpha$. This implies the assertion.

7.12. **Corollary.** $H^i(\mathscr{B}^G_{X_\alpha}, E) = 0$ *if i is odd. The eigenvalues of* F^* *on* $H^{2i}(\mathscr{B}^G_{X_\alpha}, E)$ *are* q^i.

This follows by looking at the filtration of $\mathscr{B}^G_{X_\alpha}$ defined by the closures of the subvarieties $X_w \cap \mathscr{B}^G_{X_\alpha}$, using 7.11.

7.13. **Corollary.** $\mathscr{B}^G_{X_\lambda}$ *is the union of the Schubert cells* X_w, *where* $w^{-1} \cdot \lambda > 0$.

This follows from the proof of 7.11.

7.14, There is a amusing application of 7.13 and 6.7 (ii). It follows from 7.13 that $\dim \mathscr{B}^G_{X_\lambda} = \max_{w \cdot \lambda > 0} l(w)$, where l is the length function on W defined by B. Let $\Phi' \subset \Phi$ be the closed subsystem formed by the roots orthogonal to λ, for a W-invariant Euclidean metric, and let W' be its Weyl group. It is then easy to see that the stabilizer in G of the line $\bar{k} X_\lambda$ in $\mathfrak{g}$ is the parabolic subgroup $BW'B$ of G. It follows that, r denoting the rank of G, we have

$$\dim Z_G(X) = \tfrac{1}{2}|\Phi| + \tfrac{1}{2}|\Phi'| + r - 1.$$

If p is sufficiently large it follows form 6.7 (ii) that

$$2 \max_{w \cdot \lambda > 0} l(w) = \tfrac{1}{2}|\Phi| + \tfrac{1}{2}|\Phi'| - 1.$$

Let D be the distinguished set of coset representatives of W/W' formed by the $d \in W$ with $l(dw') = l(d) + l(w')$, if $w' \in W'$ (see [5, p. 37]). It then follows that

$$2 \max_{\substack{d \cdot \lambda > 0 \\ d \in D}} l(d) = \tfrac{1}{2}|\Phi| - \tfrac{1}{2}|\Phi'| - 1. \tag{3}$$

Assume that all roots of Φ have the same length. Then by a result of Carter [6] we have, h denoting the Coxeter number of Φ and ht the height function on Φ,

$$l(d) = h - 1 - ht(d \cdot \lambda),$$

if $d \cdot \lambda > 0$. Hence $\max_{\substack{d \cdot \lambda > 0 \\ d \in D}} l(d) = h - 2$, and (3) implies that

$$|\Phi| - |\Phi'| = 4h - 6,$$

which is a known result, proved in [5, p. 170].

7.15. PSp_4. We now discuss the Green functions of a simple group of type B_2 ($p \neq 2$). We take $G = PSp_4$. We use the notations of 7.9. Now $\Phi^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta\}$, the long positive roots being $\beta, \alpha + 2\beta$. It follows from the results of [4, part E, Ch. IV, § 2] that there are 4 nilpotent classes in $\mathfrak{g}$. We shall represent them by elements X_i, with $\dim \mathscr{B}^G_{X_i} = i$. We have 3 classes represented by $X_4 = 0$, $X_0 = X_\alpha + X_\beta$ (the regular class), $X_2 = X_{2\alpha+\beta}$ (see 7.13). The remaining class is the subregular one, represented by $X_1 = X_{\alpha+\beta}$.

A discussion of subregular unipotents of G is given in [30, 3.10], and for nilpotents of $\mathfrak{g}$ the discussion is similar. One can also find the results for $\mathfrak{g}$ from those for G by using the Cayley map $x \mapsto (x-1)(x+1)^{-1}$. The result is that $\mathscr{B}^G_{X_1}$ is a union of 3 projective lines, which are defined over k (this last point following from 7.11). It follows from the results of [4, part E] that $Z_G(X_i)$ is connected unless $i = 1$ and that $C(X_1)$ has order 2. Using 6.11 it follows that $\dim V_{X_1, \phi} = 2$ if $\phi = 1$ and $= 1$ if $\phi \neq 1$ (notations of 6.9).

There are 5 nilpotent G^F-orbits in $\mathfrak{g}^F$, represented by X_0, X_1, X_2, X_4 and an element X_1' which is G-conjugate to X_1 (see 6.2). Denoting the centralizers by Z_i, Z_1', we have $|Z_0^F| = q^2$, $|Z_1^F| = 2q^3(q-1)$, $|Z_1'^F| = 2q^3(q+1)$, $|Z_2^F| = q^4(q^2-1)$, $|Z_4^F| = |G^F| = q^4(q^2-1)(q^4-1)$.

The Weyl group W has 5 conjugacy classes, represented by $1, s_\alpha, s_\beta, s_\alpha s_\beta$, $-1 = (s_\alpha s_\beta)^2$ (where s_γ is the reflection defined by $\gamma \in \Phi$). W has one irreducible character χ of degree 2, that of the standard representation, and 4 irreducible characters of degree 1, among which the sign character ε.

T being a k-split maximal torus, let T_w be the F-stable maximal torus obtained by twisting T with $w \in W$. Let $Q_w = Q_{T_w, G}$. One easily sees that the Lie algebras of the T_w contain F-stable strongly regular elements if $q > 3$, so the Q_w are defined for $q > 3$.

By 7.4 we have $Q_w(X_0) = 1$, moreover

$$Q_w(X_4) = \varepsilon(w)(q^2-1)(q^4-1)/(q^2 - \chi(w)q + \varepsilon(w)),$$

see the remark after 7.2.

It follows from 6.3 and 6.10 that

$$Q_w(X_1) = (\chi(w) + \tau(w))q + 1, \qquad Q_w(X_1') = (\chi(w) - \tau(w))q + 1,$$

where τ is a character of degree 1, different from $1, \varepsilon$, and that (using also 7.13)

$$Q_w(X_2)=\varepsilon\tau(w)q^2+\lambda(w)q+1,$$

where λ is a character of W. It remains to determine τ and λ.

Let $\mathfrak{v}$ be the subspace of $\mathfrak{g}$ spanned by $X_\alpha, X_{\alpha+\beta}, X_{\alpha+2\beta}$. It is the Lie algebra of the unipotent radical of an F-stable parabolic subgroup P. One verifies that $x_\alpha X_\alpha+x_{\alpha+\beta}X_{\alpha+\beta}+x_{\alpha+2\beta}X_{\alpha+2\beta}\in\mathfrak{v}^F$ is G^F-conjugate to X_1 if $x_\alpha\neq 0$ or $x_{\alpha+\beta}\neq 0$ and to X_2 if $x_\alpha=x_{\alpha+\beta}=0$, $x_{\alpha+2\beta}\neq 0$. Making use of the fact that T_{s_α} is not G^F-conjugate to a maximal torus of P, it follows from 5.11 that

$$(q^3-q)Q_{s_\alpha}(X_1)+(q-1)Q_{s_\alpha}(X_2)+Q_{s_\alpha}(X_4)=0,$$

and looking at coefficients of q^4 one sees that $\tau(s_\alpha)=1$, whence $\tau(s_\beta)=-1$.

To determine λ we use 5.7. A straightforward computation then gives that $\lambda=\chi$.

To sum up, we have the following determination of the Green functions of PSp_4 (for $p\neq 2$ and $q>3$):

$$Q_w(X_0)=1,\quad Q_w(X_1)=(\chi(w)+\tau(w))q+1,\quad Q_w(X_1')=(\chi(w)-\tau(w))q+1,$$
$$Q_w(X_2)=\varepsilon\tau(w)q^2+\chi(w)q+1,\ Q_w(X_4)=\varepsilon(w)(q^2-1)(q^4-1)/(q^2-\chi(w)q+\varepsilon(w)),$$

where χ is the character of the standard representation of W, ε its sign character and τ the character of degree 1 with $\tau(s_\alpha)=1$, $\tau(s_\beta)=-1$.

The reader can ascertain that the polynomials we have found are the ones occuring in the 5×5-matrix found by Mrs. B. Srinivasan when determining the irreducible characters of $Sp_4(k)$ [27, p. 506].

7.16. G_2. In type G_2 ($p\neq 2, 3$) one can determine the Green functions in a similar way. We briefly indicate how this can be done, leaving the details to the reader. The notations are as before.

We have now $\Phi^+=\{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$, the long positive roots being β, $2\alpha+\beta$, $3\alpha+2\beta$. There are 5 nilpotent classes in $\mathfrak{g}$, represented by $X_0=X_\alpha+X_\beta$, $X_1=X_\alpha+X_{2\alpha+\beta}$, $X_2=X_{2\alpha+\beta}$, $X_3=X_{3\alpha+2\beta}$, $X_6=0$, see e.g. [32]. We have again $\dim\mathscr{B}^G_{X_i}=i$; for $i=0,6$ this is easy, for $i=2,3$ the computation of 7.11 can be used, and then X_1 must be the subregular element, whence $\dim\mathscr{B}^G_{X_1}=1$ (see [30]). Moreover $Z_G(X_i)$ is connected unless $i=1$, and $C(X_1)\simeq\mathfrak{S}_3$ (this can be deduced, for example, from the results of [7] and [24]). Using the information on the structure of $\mathscr{B}^G_{X_i}$ contained in [30] it follows that $\dim V_{X_1,\phi}=2$ if $\phi=1$ and $=1$ if ϕ is the character of degree 2 of $\mathfrak{S}_3$, whereas $\dim V_{X_1,\phi}=0$ for the remaining character of $\mathfrak{S}_3$.

There are 7 nilpotent G^F-orbits in $\mathfrak{g}^F$, represented by the X_i and two extra elements X_1', X_1'' which are G-conjugate to X_1. We have $|Z_0^F|=q^2$, $|Z_1^F|=6q^4$, $|(Z_1')^F|=3q^4$, $|(Z_1'')^F|=2q^4$, $|Z_2^F|=q^4(q^2-1)$, $|Z_3^F|=q^6(q^2-1)$, $|Z_6^F|=|G^F|=q^6(q^2-1)(q^6-1)$ (see [7]).

The Weyl group W has 6 conjugacy classes, represented by $1, s_\alpha, s_\beta, s_\alpha s_\beta$, $(s_\alpha s_\beta)^2$, $-1=(s_\alpha s_\beta)^3$. Let χ be the character of the standard representation of W and ε the sign character. Now W has 4 characters of degree 1 and 2 irreducible characters of degree 2, viz. χ and $\varepsilon\chi$.

Let T_w and Q_w be as before. In this case, Q_w is defined for $q>5$. We have $Q_w(X_0)=1$ and

$$Q_w(X_6)=\varepsilon(w)(q^2-1)(q^6-1)/(q^2-\chi(w)\,q+\varepsilon(w)).$$

By 6.3 and 6.10 there are irreducible characters τ and λ_0 of degrees 1 and 2, with

$$Q_w(X_1)=(\lambda_0(w)+2\tau(w))q+1$$
$$Q_w(X_1')=(\lambda_0(w)-\tau(w))q+1$$
$$Q_w(X_1'')=\lambda_0(w)q+1.$$

Moreover we see from 7.13 that $\mathscr{B}_{X_3}^G$ can have only one irreducible component of highest dimension. Consequently, we have, by 6.3 and 6.10

$$Q_w(X_2)=\lambda_0\tau(w)q^2+\lambda_1(w)q+1$$
$$Q_w(X_3)=\varepsilon\tau(w)q^3+\lambda_2(w)q^2+\lambda_3(w)q+1,$$

where the λ_i are characters of W. It remains to determine τ and the λ_i.

Let $\mathfrak{v}$ be the subspace of $\mathfrak{g}$ spanned by X_α, $X_{\alpha+\beta}$, $X_{2\alpha+\beta}$, $X_{3\alpha+\beta}$, $X_{3\alpha+2\beta}$. It is the Lie algebra of the unipotent radical of a parabolic k-subgroup P. We have that $x_\alpha X_\alpha+x_{\alpha+\beta}X_{\alpha+\beta}+x_{2\alpha+\beta}X_{2\alpha+\beta}+x_{3\alpha+\beta}X_{3\alpha+\beta}+x_{3\alpha+2\beta}X_{3\alpha+2\beta}\in\mathfrak{g}^F$ is G^F-conjugate to X_1 if $x_\alpha x_{3\alpha+2\beta}$ is a nonzero square of k, to X_1 if $x_\alpha x_{3\alpha+2\beta}$ is a non-square, to X_2 if $x_\alpha\neq 0$, $x_{3\alpha+2\beta}=0$; $x_\alpha=0$, $x_{\alpha+\beta}\neq 0$; $x_\alpha=x_{\alpha+\beta}=0$, $x_{2\alpha+\beta}\neq 0$ and to X_3 or X_6 in the other cases. Using 5.11 with $T=T_{s_\alpha}$ we find, as in 7.15, that $\tau(s_\alpha)=1$, $\tau(s_\beta)=-1$.

To determine the λ_i we again use 5.7. The computation is straightforward and will be omitted. The final result is as follows ($p\neq 2, 3$, $q>5$):

$$Q_w(X_0)=1,\ Q_w(X_1)=(\chi(w)+2\tau(w))\,q+1,\ Q_w(X_1')=(\chi(w)-\tau(w))\,q+1,$$
$$Q_w(X_1'')=\chi(w)\,q+1,\ Q_w(X_2)=\chi\tau(w)\,q^2+(\chi(w)+\tau(w))\,q+1,$$
$$Q_w(X_3)=\varepsilon\tau(w)\,q^3+\chi\tau(w)\,q^2+\chi(w)\,q+1,$$
$$Q_w(X_6)=\varepsilon(w)\,(q^2-1)\,(q^6-1)/(q^2-\chi(w)\,q+\varepsilon(w)),$$

the notations being as in the similar formulas for type B_2. These polynomials are the ones occuring in the character tables of Chang and Ree [8] (the polynomials in question are found in the top part of the tables on pp. 409 and 410).

Note added in proof. The Weyl group representations in $H^*(\mathscr{B}_A^G, E)$ of 4.4 can be described in another way, which also makes sense in characteristic 0. This makes it possible to extend 6.10 to characteristic 0 and to deal with the questions raised in the last paragraph of no. 4. Details will appear elsewhere.

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