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A Cancellation Theorem for Projective Modules in the Metastable Range

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If A is an affine ring of dimension d and P, Q are finitely generated projective A-modules which are stably isomorphic, i.e., $P \oplus A^s \approx Q \oplus A^s$ for some $s < \infty$, then $P \approx Q$ provided $\operatorname{rk} P \geqq d+1$. This follows from the cancellation theorem of Bass [1, Ch. IV, Cor. 3.5]. The object of this paper is to show that the bound $d+1$ can be reduced under certain additional hypotheses. Throughout the paper (except for § 5) all rings will be commutative with unit. The notation $A = B[x_1, \ldots, x_e]$ will always be understood to mean that A is a polynomial ring over B in the indeterminates $x_1, \ldots, x_e$. The only exception is that the notation $A[a^{-1}]$ will, as usual, denote the localization of A with respect to the powers of a. The term "variety" will always mean a geometrically integral scheme of finite type. Here is the main result.

Theorem 1. *Let $V = \operatorname{Spec} A$ be a smooth affine variety of dimension d over an infinite field k. Suppose that $A = B[x_1, \ldots, x_e]$. Let P and Q be finitely generated projective A-modules which are stably isomorphic. Then $P \approx Q$ provided* $\operatorname{rk} P \geqq d + 1 - \frac{e}{2}$.

As an immediate consequence of this we get

Corollary 2. *Let A, B, d, e be as in Theorem 1. Let P be a finitely generated projective A-module with* $\operatorname{rk} P \geqq d + 1 - \frac{e}{2}$. *Then $P \approx A \otimes_B P_0$ where P_0 is a finitely generated projective B-module.*

To see this we observe that $K_0(B) \xrightarrow{\approx} K_0(A)$ since A, and hence B, is regular [4, Cor. 17.5.2] [1, Ch. XII, Th. 2.2] [1, Ch. XII, Th. 3.1]. Therefore, if $P_0 = B \otimes_A P$ then P and $A \otimes_B P_0$ are stably isomorphic and Theorem 1 applies.

In particular, if $B = k$, we obtain the following result which was announced by Suslin [14].

Corollary 3 (Suslin). *If k is an infinite field, all projective $k[x_1, \ldots, x_n]$-modules of* rank $\geqq \frac{n}{2} + 1$ *are free.*

I have not seen any details of Suslin's proof. The argument used here is based on one due to Roitman [10] who proved Corollary 3 for rank $\geqq n$. The results obtained here are obtained by replacing Roitman's algebraic general position argument by a Bertini type theorem. This theorem also gives an immediate proof of the theorem of Kleinman used in [9].

Recently, Vaserštein und Suslin have announced further results on Serre's problem obtained by making use of symplectic methods [14]. This suggested applying the methods of the present paper to obtain a cancellation theorem for symplectic modules similar to Theorem 1. As a consequence we get the following: If k is an infinite field with char $k \neq 2$, then all projective $k[x_1, \ldots, x_n]$-modules are free for $n \leqq 4$. This is a special case of results of Vaserštein and Suslin. I do not know how to extend the methods used here to cover all cases considered in [14]. In particular I do not know how to extend the methods of this paper to finite fields.

Profound thanks are due to H. Bass and M.P. Murthy for their contributions to this paper. The first stages of this work were done in collaboration with Murthy, following a talk by Bass on Roitman's work. In particular, two key ideas of this paper, the use of Bertini's theorem and the idea of using general linear combinations of the coordinates as coefficients, are both entirely due to Murthy. My original intention was to include him as joint author of this paper and it was only after considerable protest on his part that I was dissuaded from doing so. The final form of the paper is largely due to Bass' suggestions. After I showed him the proof of Theorem 1.3 in the free case and the deduction of Suslin's theorem, he remarked that the same argument gives Theorem 1 in the stably free case and that one would need something like Theorem 1.3 to prove Theorem 1 in general. He also pointed out a variant of Roitman's argument which led to Theorem 3.2. Thanks are also due to Bass, Murthy, and Serre for information concerning the results of Vaserštein and Suslin and further thanks are due to Bass for showing me a detailed exposition of Vaserštein's work.

1. The Bertini Theorems

Although Bertini's theorem is one of the best known results in algebraic geometry, I was unable to find a statement of it in exactly the form needed here. Therefore I will begin by giving the required form of this theorem. The version dealing with sections by linear subspaces will suffice. Let $\mathscr{P}$ be a property of linear subspaces of the n-dimensional affine space $\mathbf{A}^n$ over a field k. Let $C \subset k^N$, $N = (n+1)(n-r)$, be the set of all (λ_{ij}, μ_i), $i = 1, \ldots, n-r$, $j = 1, \ldots, n$ such that the equations $\sum \lambda_{ij} x_j + \mu_i = 0$ define an r-dimensional subspace of $\mathbf{A}^n$ with the property $\mathscr{P}$. As usual we say that a general r-dimensional linear subspace has the property $\mathscr{P}$ if C contains a non-empty Zariski open set of k^N.

Theorem 1.1 (Bertini). *Let V be a locally closed subvariety of $\mathbf{A}^n =$ Spec $k[x_1, \ldots, x_n]$ of dimension d. Then for a general linear subspace L of $\mathbf{A}^n$ of dimension r,*

(1) *$L \cap V$ is geometrically reduced of dimension $d - r$ everywhere or is empty.*

(2) *$L \cap V$ is a variety if* $\dim L \cap V \neq 0$.

(3) *$L \cap V$ is smooth if V is.*

If $\dim L \cap V = 0$, (1) means that $L \cap V = \operatorname{Spec}(K_1 \times \cdots \times K_r)$ where the K_i are finite separable extensions of k. The notation $L \cap V$ means $L \times_n V$. In other words, if $V = \operatorname{Spec} A$, then $L \cap V = \operatorname{Spec} A/(\sum \lambda_{ij} x_j + \mu_i)$.

Proof. Let s_{ij}, t_i be indeterminates and let

$$X = \operatorname{Spec} A[s_{ij}, t_i]/(\sum s_{ij} x_j + t_i)$$

and $S = \operatorname{Spec} k[s_{ij}, t_u]$. If s is a closed point of S given by $s_{ij} \mapsto \lambda_{ij} \in k$, $t_i \mapsto \mu_i \in k$, then the fiber of $X \to S$ at s is obviously $L \cap V$. By [4, 9.2.6, 9.9.5], the set C' of points s of S where the fibre X_s satisfies any of the conditions of Theorem 1.1, is constructible. It follows that the set $C \subset k^N$ of all (λ_{ij}, μ_i) such that $L \cap V$ satisfies the required conditions is also constructible because the set of closed points of S is k^N with the Zariski topology. Using this observation, we can first reduce to the case where L is a hyperplane. Let H be given by $\sum \lambda_{1j} x_j + \mu_1 = 0$ and L' by $\sum \lambda_{ij} x_j + \mu_i = 0$ for $i \geqq 2$. By induction we can assume that $L' \cap V$ satisfies the required conditions for general $\lambda_{ij}, \mu_i, i \geqq 2$. For any fixed value of these λ, μ, we will then know that $L \cap V = H \cap L' \cap V$ satisfies the conditions for general λ_{1j}, μ_1. Now apply the following elementary lemma.

Lemma 1.2. *Let $C \subset k^{p+q}$ be any set. Let $f\colon C \to k^q$ be the projection on the last q factors. Let D be the set of $x \in k^q$ such that $f^{-1}(x)$ is dense in k^p. If D is dense in k^q, then C is dense in k^{p+q}. Therefore if C is constructible, it will contain a dense open set of k^{p+q}.*

Proof. Let $\bar{C}$ be the closure of C. If $x \in D$ then $\overline{f^{-1}(x)} = k^p \times x$ but $\overline{f^{-1}(x)} \subset \bar{C}$. Therefore $k^p \times D \subset \bar{C}$. But $k^p \times D$ is dense in k^{p+q}. This is obvious if k is finite. If k is infinite, we must show that if $f \in k[x_1, \ldots, x_p, y_1, \ldots, y_q]$ is zero on $k^p \times D$ then $f = 0$. Write $f = \sum g_i(y) m_i$ where the m_i are monomials in the x_i. If $b \in D$, then $f(a, b) = 0$ for all $a \in k^p$. Therefore all $g_i(b) = 0$. Since D is dense in k^q, all $g_i(y) = 0$. The last statement is clear if k is finite. If k is infinite write $C = \bigcup E_i$ where E_i is locally closed. Then $\bar{C} = k^{p+q} = \bigcup \bar{E}_i$. Since k is infinite, k^{p+q} is irreducible (since $fg = 0$ on k^{p+q} implies that $fg = 0$ as a polynomial) so some $\bar{E}_i = k^{p+q}$ and E_i is the required open set.

We can now assume that L is a hyperplane. To prove Theorem 1.1, it will suffice to prove that the set C' above contains a dense open set U of S since then $U \cap k^N \subset C$ will be a dense open set of k^N. Since C' is constructible, it will suffice to show that the generic point of S lies in C'. This reduces Theorem 1.1 to the case of a *generic* hyperplane section. If V is closed, this case is treated in detail in [6, Ch. VIII, § 6]. If V is only locally closed, apply the results of [6, Ch. VIII, §6] to its closure $\bar{V}$. This is still a variety and $L\cap V=L\cap \bar{V}-(\bar{V}-V)$ is an open set of $L\cap \bar{V}$ so (1) and (2) for V obviously follow from (1) and (2) of $\bar{V}$. For (3) note that $\bar{V}$ need not be smooth but [6, Ch. VIII, § 6, Prop. 13] hows that L meets the nonsingular set of $\bar{V}$ in a smooth subscheme so (3) follows.

If Q is an A-module and $x\in Q$, the ideal $\mathfrak{o}_Q(x)$ of A consists of all $h(x)$ for $h\in \operatorname{Hom}_A(Q,A)$ [1, Ch. IV, § 1]. Thus x is unimodular if and only if $\mathfrak{o}_Q(x)=A$.

Theorem 1.3. *Let $V=\operatorname{Spec} A$ be a smooth affine variety over an infinite field k. Let Q be a finitely generated projective A-module of* rank r. *Let $(q,a)\in Q\oplus A$ be a unimodular element. Then there is a $y\in Q$ such that $I=\mathfrak{o}_Q(q+ay)$ has the following properties.*

(1) *The subscheme $U=\operatorname{Spec} A/I$ of V is smooth over k and* $\dim U=\dim V-r$ *unless $U=\emptyset$.*

(2) *If* $\dim U\neq 0$ *then U is a variety.*

Note that Q has constant rank since V is connected [13, Th. 7.8]. If V is not connected, we can, of course, apply Theorem 1.3 to each component of V.

I will actually prove a stronger form of this.

Theorem 1.4. *In the situation of Theorem* 1.3, *there is a finite set $S\subset Q$, depending only on Q and A, such that if $T=\{t_1,\ldots,t_m\}$ is any finite subset of Q containing S, then for general λ_j in k, the element $y=\lambda_1 t_1+\cdots+\lambda_m t_m$ has the properties required in Theorem* 1.3.

As usual, "for general λ_j in k" means "for all $(\lambda_1,\ldots,\lambda_m)$ in a non-empty Zariski open set U of k^m". Of course, U will depend on q, a, and T.

We begin by exhibiting the set S. Let $x_1,\ldots,x_n\in A$ be a finite set of elements such that the map $\xi: V\to \mathbb{A}^n$ maps V isomorphically onto a locally closed subvariety of $\mathbb{A}^n$. We can find a finite covering of V by special affine open sets, i.e. $V=\bigcup V_i$ with $V_i=\operatorname{Spec} A[a_i^{-1}]$ and $\sum Aa_i=A$, such that each $Q_i=A[a_i^{-1}]\otimes_A Q$ is free. Since V is compact, it is enough to do this locally. If $\mathfrak{p}$ is a prime ideal of A, $Q_{\mathfrak{p}}$ is free so we can find a free A-module F with $f: F\to Q$ inducing $F_{\mathfrak{p}}\approx Q_{\mathfrak{p}}$. Since $\ker f$, $\operatorname{ck} f$ are finitely generated they are annihilated by some $a\notin \mathfrak{p}$. Therefore $A[a^{-1}]\otimes_A Q\approx A[a^{-1}]\otimes_A F$ is free. Let $e_{i1},\ldots,e_{ir}\in Q$ map into a base for Q_i. We choose S to consist of the elements e_{ij} and all $x_k e_{ij}$, $1\leqq k\leqq n$.

With this choice of S, we can immediately reduce to the case where Q is free. If Q is free on $e_1,\ldots,e_r$ we can choose S to consist of the e_i and the $x_k e_i$. Now, the image of T in Q_i will contain such a set. Note that the x_i still give a locally closed embedding of V_i. This is the reason why we did not insist on a closed embedding. If Theorem 1.4, with our choice of S, holds when Q is free, the subscheme U_i of V_i defined by $\mathfrak{o}_{Q_i}(q+ay)=A[a_i^{-1}]\,\mathfrak{o}_Q(q+ay)$ will satisfy the conditions of Theorem 1.3 for general

λ_j. Since these conditions are local and there are only a finite number of V_i, it is clear that Theorem 1.4 will hold for Q.

We must now prove Theorem 1.4 for the case where Q is free with base $e_1, \ldots, e_r$ and S is the set of all e_i and $x_k e_i$. Let $T = \{t_1, \ldots, t_m\}$. Let $q = q_1 e_1 + \cdots + q_r e_r$ and $t_i = \sum t_{ij} e_j$. Then $y = \sum_{i,j} \lambda_i t_{ij} e_j$ and $q + ay = \sum (q_j + a\lambda_i t_{ij}) e_j$ so $I = \mathfrak{o}_Q(q + ay)$ is the ideal generated by the $q_j + a \sum_i \lambda_i t_{ij}$. Our first aim is to eliminate the elements in $T - S$.

Let u_i be indeterminates, $X = \operatorname{Spec} A[u_i]/(q_j + a \sum u_i t_{ij})$, and $S = \operatorname{Spec} k[u_i]$. If $\lambda = (\lambda_i)$ is a closed point of S, the fiber X_λ is Spec A/I. As in the proof of Theorem 1.1 we see that the set C of $\lambda \in k^m$ such that $y = \sum \lambda_i t_i$ satisfies the conclusion of Theorem 1.3, is constructible. Therefore, by Lemma 1.2, it will suffice to show that if $\lambda_{p+1}, \ldots, \lambda_m$ are given for some fixed p then y satisfies the conclusion of Theorem 1.3 for general $\lambda_1, \ldots, \lambda_p$. We choose p and the numbering of the t_i so that $S = \{t_1, \ldots, t_p\}$ and $T - S = \{t_{p+1}, \ldots, t_m\}$.

Let $y' = \lambda_1 t_1 + \cdots + \lambda_p t_p$ and $y'' = \lambda_{p+1} t_{p+1} + \cdots + \lambda_m t_m$. Then $q + ay = (q + ay'') + ay'$. Now $(q + ay'', a)$ is unimodular so it follows that $q + ay'' + ay'$ has the required properties for general $\lambda_1, \ldots, \lambda_p$.

Now suppose $T = S$ where Q is free on $e_1, \ldots, e_r$ and S is the set of e_j e_j and $x_i e_j$. Let $q = q_1 e_1 + \cdots + q_r e_r$. Let $\varphi\colon V \to \mathbb{P}^{r+n}$ have homogeneous coordinates $(q_1, \ldots, q_r, a, x_1 a, \ldots, x_n a)$. This is well defined since $(q_1, \ldots, q_r, a)$ is unimodular so some entry is non-zero at each point of V. Let W be the closed subset of V defined by $a = 0$. Then φ maps $V - W$ isomorphically onto a locally closed subset of $\mathbb{P}^{n+r}$. In fact, on $V - W$, φ is given by $(q_1 a^{-1}, \ldots, q_r a^{-1}, 1, x_1, \ldots, x_n)$ so $\varphi(V - W)$ lies in an affine subspace of $\mathbb{P}^{n+r}$ and embeds $V - W$ as the graph of the map $(q_1 a^{-1}, \ldots, q_r a^{-1})$: $V - W \to \mathbb{A}^r$. Now $\varphi(W) \subset \{(a_1, \ldots, a_r, 0, \ldots, 0) = \mathbb{P}^{r-1} \subset \mathbb{P}^N$. Let L be a general linear subspace of $\mathbb{P}^N$ of dimension n. Then $\mathbb{P}^{r-1} \cap L = \emptyset$. By Bertini's Theorem 1.1, $L \cap \varphi(V - W)$ will satisfy the conditions imposed on U in Theorem 1.3 and hence so will $U' = \varphi^{-1}(\varphi(V) \cap L)$. Let the homogeneous coordinates in $\mathbb{P}^{n+r}$ be $(u_1, \ldots, u_r, z_0, \ldots, z_n)$ and suppose L is given by equations $\sum \mu_{ij} u_j + \sum v_{ik} z_k = 0$, $i = 1, \ldots, r$, or, in matrix form $\mu u + v z = 0$. For general μ, v we have $\det \mu \neq 0$ so L is defined by $u + \mu^{-1} v z = 0$. Let $\lambda = \mu^{-1} v$. Then U' is clearly defined by the equations $q_i + \lambda_{i0} a + \sum_1^n \lambda_{ij} a x_j = 0$. But these are just the coordinates of $q + ay$ where $y = \sum \lambda_i t_i$ with $S = \{t_1, \ldots, t_m\}$ and the λ's are suitably reindexed so $U' = U$ for this choice of λ. Let $\theta\colon \mathrm{GL}_r(k) \times k^{r(n+1)} \cong \mathrm{GL}_r(k) \times k^{r(n+1)}$ by $\theta(\mu, v) = (\mu, \mu^{-1} v)$. We know that U' satiesfies the conclusion of Theorem 1.3 if (μ, v) lies in some non-empty open set O of $\mathrm{GL}_r(k) \times k^{r(n+1)}$. Therefore the set of (μ, λ) such that U has the required properties contains an open set $\theta(O)$. If we fix a μ such that $(\mu \times k^{r(n+1)}) \cap$

$\theta(O)=\emptyset$, the λ with $(\mu, \lambda)\in\theta(O)$ form a non-empty open set with the required property.

By using a stepwise procedure as in Roitman's paper, one can prove a bit more. A strictly lower triangular matrix will mean one with 1's on the diagonal and 0's above the diagonal.

Theorem 1.5. *Let $V=\operatorname{Spec} A$ be a smooth affine variety over an infinite field k. Let $f=(f_1, \ldots, f_{r+1})$ be a unimodular row over A. Then there is a strictly lower triangular matrix T over A such that $fT=(g_1, \ldots, g_{r+1})$ has the following properties.*

(1) *For each i, the subscheme U_i of V defined by the ideal $Ag_1+\cdots+Ag_i$ is smooth over k and* $\dim U_i=\dim V-i$ *unless $U_i=\emptyset$.*

(2) *If* $\dim U_i\neq 0$, *U_i is a variety.*

Proof. Let $x_1, \ldots, x_n$ be as above and let $\varphi: V\to\mathbb{P}^N$ have homogeneous coordinates $f_1, \ldots, f_{r+1}$ and all $x_i f_j$ with $2\leq j\leq r+1$. Let $W\subset V$ be defined by $f_2=\cdots=f_{r+1}=0$. Again φ maps $V-W$ isomorphically onto a locally closed subset of $\mathbb{P}^N$ since we can recover the x_i by taking ratios of coordinates of $\varphi(x)$. Also $\varphi(W)$ is the point $(1, 0, \ldots, 0)$. Let H be a general hyperplane of $\mathbb{P}^N$. Then $\varphi(W)\cap H=\emptyset$ and by Bertini's theorem $\varphi(V-W)\cap H$ has the properties required of U_1. If H is defined by $h=\sum\lambda_i f_i+\sum\lambda_{ij}x_i f_j=0$ we can set $g_1=\lambda_1^{-1}h$ since $\lambda_1\neq 0$ in general. Now repeat the same argument on U_1 and the unimodular row $f_2|U_1, \ldots, r_{r+1}|U_1$ to get g_2 (still defined on V). Continuing in this way we get $g_1, \ldots, g_r$ and set $g_{r+1}=f_{r+1}$.

Remark. Using Lemma 1.2 it is easy to see that we can take T to be a strictly lower triangular matrix whose subdiagonal entries are general linear combinations of $1, x_1, \ldots, x_n$. In fact, let T have this form. Let X_1 be the linear space of coefficients of $1, x, \ldots, x_n$ occurring in the first column of T, X_2 that for the second column, etc. Let $C_i\subset X_1\times\cdots\times X_i$ be the set of coefficients such that for the corresponding T, $U_1, \ldots, U_i$ have the required properties. As before, C_i is constructible. The proof of Theorem 1.5 shows that C_1 contains a non-empty open set O_1. Suppose $O_i\subset C_i$ is a non-empty open set. Apply Lemma 1.2 to $C_{i+1}\cap O_i\times X_{i+1}$ in $O_i\times X_{i+1}$. The proof of Theorem 1.5 shows that C_{i+1} contains a non-empty open set.

I will conclude this section by showing that Theorem 1.3 can be used to replace the theorem of Kleiman used in [9].

Corollary 1.6. *Let $V=\operatorname{Spec} A$ be a connected smooth affine scheme of dimension d over an infinite field k. Let P be a finitely generated projective A-module of* rank *r. Then there is a map $f: P\to A$ such that if $I=f(P)$ then $U=\operatorname{Spec} A/I$ is a smooth subscheme of V of dimension $d-r$ unless $U=\emptyset$. Furthermore, we can assume that U is geometrically integral if V is and* $\dim U>0$.

This is exactly the consequence of Kleiman's theorem used in [9]. To prove it, apply Theorem 1.3 to $(0,1)\in Q\oplus A$ where $Q=P^*=\mathrm{Hom}_A(P,A)$. We get $y\in Q$ such that $o_Q(y)=I$ has the required properties. Now $y\in Q=P^*$ so $y\colon P\to A$ and the definition of $o_Q(y)$ shows that $I=y(P)$ since $P\approx Q^*$.

Remark. We can get more information using Theorem 1.5 if we know that $P\oplus A\approx A^{r+1}$ which is the case in the application in [9]. In this case we have $0\to P\to A^{r+1}\xrightarrow{f}A\to 0$ where $f(x)=\sum f_i x_i$ and $(f_1,\dots,f_{r+1})$ is unimodular. By Theorem 1.5 we can make a change of base in A^{r+1} getting $0\to P\to A^{r+1}\xrightarrow{g}A\to 0$ where $g=(g_1,\dots,g_{r+1})$ satisfies the conclusions of Theorem 1.5. Let $p\colon P\to A$ be projection on the last factor of A^{r+1}. Since $P=\{x\in A^{r+1}|\sum g_i x_i=0\}$ we see that $\mathrm{im}\,p=\{x_{r+1}\in A|g_{r+1}x_{r+1}\in I\}$ where $I=Ag_1+\dots+Ag_r$. Clearly $I\subset\mathrm{im}\,p$. Conversely, let $x\in\mathrm{im}\,p$. Since $g_1,\dots,g_{r+1}$ is unimodular, $I+Ag_{r+1}=A$. Write $1=i+ag_{r+1}$ with $i\in I$. Then $x=ix+ag_{r+1}x\in I$ so $\mathrm{im}\,p=I$ and we recover the conclusion of Corollary 1.6. The advantage of this approach is that we also get quite a bit of information about the kernel of p. It is clear that $\ker p=\left\{x\in A^r\middle|\sum_1^r g_i x_i=0\right\}$. Now, if $r<d$, the irreducibility of the U_i shows that $(g_1,\dots,g_r)$ is an A-sequence and hence its associated Koszul complex E is exact, i.e.

$$0\to E_r\to E_{r-1}\to\cdots\to E_2\to E_1\to E_0\to A/I\to 0 \tag{1}$$

is exact. Now $E_0=A$, $E_1=A^r$ and $E_1\to E_0$ is given by $(x_1,\dots,x_n)\mapsto\sum_1^r g_i x_i$. Therefore $\ker p\approx\ker[E_1\to E_0]$ and we have an exact sequence

$$0\to E_r\to E_{r-1}\to\cdots\to E_2\to P\xrightarrow{p}A\to A/I\to 0. \tag{2}$$

In particular, if $r=2$ then $\ker p\approx E_r\approx A$. It is well known that if I is generated by an A-sequence then $\mathrm{Ext}^i(A/I,A)=0$ for $i\neq r$ and

$$\mathrm{Ext}^r(A/I,A)\approx A/I$$

generated by the Yoneda class of (1). Since (2) is also a projective resolution of A/I, its Yoneda class also generates $\mathrm{Ext}^r(A/I,A)$. We deduce easily that the class of the extension $0\to\ker p\to P\to I\to 0$ generates $\mathrm{Ext}^1(I,\ker p)$ since it corresponds to the class of (2) under the isomorphism $\mathrm{Ext}^1(I,\ker p)\approx\mathrm{Ext}^r(A/I,A)\approx A/I$ derived from (2). However, I do not know of any applications beyond the one in [9].

2. The General Projection Theorem

The following theorem is very well known but I was unable to find any reference for it in the form needed here. Murthy pointed out the paper of

Lluis [15] which treats the projective case. Lluis' argument can easily be modified to cover the case needed here. I will give a direct argument here. It is based on the same ideas as [15].

Theorem 2.1. *Let V be a smooth closed subscheme of $\mathbb{A}^N$ over an infinite field k with* $\dim V = d$. *Let $p: \mathbb{A}^N \to \mathbb{A}^r$ be a general projection with $r \geq 2d+1$. Then p maps V isomorphically onto a closed subscheme of $\mathbb{A}^r$.*

Proof. Let Δ be the diagonal of $V \times V$ and let $\varphi: V \times V - \Delta \to \mathbb{A}^N - \{0\}$ by $\varphi(x, y) = x - y$. Let $\pi: \mathbb{A}^N - \{0\} \to \mathbb{P}^{N-1}$ be the canonical map and let X be the closure of $\pi\varphi(V \times V - \Delta)$. Then $\dim X \leq 2d$. Let $L^{N-r-1} \subset \mathbb{P}^{N-1}$ be the image of $\ker p$ in $\mathbb{P}^{N-1}$. For general p, $L^{N-r-1} \cap X = \emptyset$ since $N - r - 1 + 2d < N - 1$. This means that if $x, y \in V$, $x \neq y$ then $p(x) \neq p(y)$ since $x - y \notin L$. Now let $T(V)$ be the tangent bundle of V. If V is defined by the equations $f_j = 0$, then $T(V) \subset \mathbb{A}^N \times \mathbb{A}^N$ is the set of (x, y) satisfying $f_j(x) = 0$ and $\sum_i \frac{\partial f_j}{\partial x_i} y_i = 0$. Since V is smooth, the fiber $T_x(V)$ of pr_1: $T(V) \to V$ has dimension d. Define $\psi: T(V) - V \times 0 \to \mathbb{A}^N - \{0\}$ by $\psi(x, y) = y$. Let Y be the closure of $\pi\psi(T(V) - V \times 0)$ in $\mathbb{P}^{N-1}$. Since $\dim T(V) = 2d$, the above argument shows that $L^{N-r-1} \cap Y = \emptyset$ for general p and so $p(y) \neq 0$ for $(x, y) \in T(V)$. In other words, the map $T_x(V) \to T_{p(x)}(\mathbb{A}^r)$ is injective.

To see that $p(V)$ is closed in general, we use the following well known result.

Lemma 2.2. *Let V be a closed subscheme of $\mathbb{A}^N$ over an infinite field k with* $\dim V = d$. *Let $p: \mathbb{A}^N \to \mathbb{A}^r$ be a general projection with $r > d$. Then $W = p(V)$ is a closed subscheme of $\mathbb{A}^r$ and V is finite over W.*

This follows immediately from the proof of the normalization lemma in [11, Ch. I, § 3, n° 4]. Since the proof is so short I will repeat it here. Let $p(x) = y$ where $y_i = \sum a_{ij} x_j$. Extend p to a projection p' of $\mathbb{P}^N$ on $\mathbb{P}^r$ by sending $(x_0, x_1, \ldots, x_N)$ to $(x_0, y_1, \ldots, y_r)$. The center of this projection will be $L^{N-r-1} \subset \mathbb{P}^{N-1} = \{x_0 = 0\}$ where L is as above. Let $\bar{V}$ be the closure of V in $\mathbb{P}^N$. Then $\dim \bar{V} \cap \mathbb{P}^{N-1} < d$ so $L \cap \bar{V} = \emptyset$ for general p. Therefore p' is defined everywhere on $\bar{V}$ so $p'(\bar{V})$ is complete and hence closed in $\mathbb{P}^r$. It follows that $W = p(V) = p'(\bar{V}) \cap \mathbb{A}^r$ is closed in $\mathbb{A}^r$. To see that $V \to W$ is finite, observe as in [11] that if I is the ideal defining $\bar{V}$, then $(I, x_0, \sum a_{ij} x_j)$ defines $\emptyset$ and hence contains all monomials of degree $\geq q$ say. Now set $x_0 = 1$. If M is the set of all monomials of degree $< q$ in $k[x_1, \ldots, x_n]$, it follows that $k[x_1, \ldots, x_n]/I \cap k[x_1, \ldots, x_n]$ is generated as a $k[y_1, \ldots, y_r]$-module by the image of M. Therefore V is finite over W.

To complete the proof of Theorem 2.1, it will be convenient to have k algebraically closed. Suppose that the theorem is true in this case. Let $V = \operatorname{Spec} A$. The theorem says that for general c_{ij}, the map θ:

$k[y_1, \ldots, y_r] \to A$ sending y_i to $\sum c_{ij} x_j$ is onto. Let k' be the algebraic closure of k. By our assumption we can find a polynomial $G(c_{ij})$ over k' so that $k' \otimes \theta$ is onto for $G(c_{ij}) \neq 0$. By taking the product of the conjugates of G raised to a suitable power we can assume that G has coefficients in k. But if $k' \otimes \theta$ is onto so is θ. Thus the theorem for k follows from the theorem for k'.

We can now assume that k is algebraically closed. Since $p|V$ is injective and p maps each component of V to a closed set (for general p), it will suffice to consider the case where V is irreducible. Since k is algebraically closed, we can identify $T_x(V)$ with the Zariski tangent space $\mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$ where $\mathfrak{m}_x$ is the maximal ideal of $O_{V,x}$. In fact, suppose $x=0$ and let V be defined by $f_j=0$. Let $f_j = l_j + h_j$ where l_j consists of the terms of degree 1 in f_j. Then $\sum (\partial f_j/\partial x_i)_{x=0}\, y_i = l_j(y)$ so $T_0(V)$ is defined by $l_j(y)=0$. But $\mathfrak{m}_x/\mathfrak{m}_x^2 = (k x_1 + \cdots + k x_n)/\sum k l_j(x)$. Since $p\colon T_x(V) \to T_{p(x)}(\mathbb{A}^r)$ is injective, this implies that $\mathfrak{m}_{\mathbb{A}^r, p(x)}/\mathfrak{m}^2_{\mathbb{A}^r, p(x)} \to \mathfrak{m}_{V,x}/\mathfrak{m}^2_{V,x}$ is onto and hence so is $\mathfrak{m}_{W,p(x)}/\mathfrak{m}^2_{W,p(x)} \to \mathfrak{m}_{V,x}/\mathfrak{m}^2_{V,x}$. Now let $V = \mathrm{Spec}\, A$ and $W = \mathrm{Spec}\, B$ where $B \subset A$ is the image of $k[y_1, \ldots, y_r] \to A$. Let $\mathfrak{m}$ be a maximal ideal of B. Since $p\colon V \to W$ is $1-1$ and onto, there is a unique maximal ideal $\mathfrak{n}$ of B with $\mathfrak{n} \supset \mathfrak{m}$. If $\mathfrak{n}$ corresponds to $x \in V$ then $O_{V,x} = A_{\mathfrak{n}}$ and $\mathfrak{m}/\mathfrak{m}_x^2 \approx \mathfrak{n}/\mathfrak{n}^2$ and similarly for B. Therefore $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2$ is onto. This shows that $\mathfrak{n} = A\mathfrak{m} + \mathfrak{n}^2$ and it follows that $\mathfrak{n} = A\mathfrak{m}$ since it is sufficient to check this locally. In $A_{\mathfrak{n}}$, it follows from Nakayama's lemma, while at other maximal ideals of A it is trivial because such ideals do not contain $\mathfrak{m}$. Now let $M = A/B$ as a B-module and tensor $B \to A \to M \to 0$ with $B/\mathfrak{m}$ getting $B/\mathfrak{m}B \to A/\mathfrak{m}A \to M/\mathfrak{m}M \to 0$. We have just seen that $\mathfrak{m}A = \mathfrak{n}$. Since k is algebraically closed, $B/\mathfrak{m} = k = A/\mathfrak{n}$ and it follows that $M/\mathfrak{m}M = 0$. Since M is a finitely generated B-module, Nakayama's lemma shows that $M_{\mathfrak{m}} = 0$. Since this holds for all $\mathfrak{m}$, $M = 0$ and $A = B$.

Remark. If V is not smooth, we can use the same argument to obtain the affine analogue of Lluis' theorem [15]. The map $\psi\colon T(V) - V \times 0 \to \mathbb{P}^{N-1}$ above clearly factors through the associated projective bundle $\mathbb{P}T(V)$. Therefore $\dim Y \leqq \dim \mathbb{P}T(V) = \dim T(V) - 1$ and the argument works provided $r \geqq \max(2d+1, \dim T(V))$. As in [15] observethat the part of $T(V)$ lying over the non-singular part of V has dimension $2d$ while the rest has dimension $\leqq z + \dim V_s \leqq z + d - 1$ where V_s is the singular set of V and z is the maximum dimension of the Zariski tangent spaces of V. This gives the condition $r \geqq \max(2d+1, z+d-1)$ as in [15]. For the case $d=1$ this is the best possible result. For example, let $V = \mathrm{Spec}\, A$ where $A = k[x^n, x^{n+1}, \ldots] \subset k[x]$. If $\mathfrak{m}$ is the maximal ideal at $x=0$, then $\dim \mathfrak{m}/\mathfrak{m}^2 = 2n$. If V can be embedded in $\mathbb{A}^r$, then $A = k[x_1, \ldots, x_r]/I$. Since every maximal ideal of $k[x_1, \ldots, x_r]$ has r generators the same will be true for A and so $r \geqq 2n = z + d - 1$.

We now apply Theorem 2.1 to obtain the following technical result which will be needed in § 3.

Lemma 2.3. *Let* $V = \operatorname{Spec} A$ *be an affine scheme of finite type over an infinite field* k *such that* $V = V_0 \times \mathbb{A}^e$ *where* $\mathbb{A}^e$ *is an affine e-space, i.e.,* $A = B[x_1, \ldots, x_e]$ *where* $V_0 = \operatorname{Spec} B$. *Let* $W = \operatorname{Spec} A/I$ *be a closed subscheme of* V *which is smooth over* k. *Then*

(1) *If* $\dim W \leqq \frac{e}{2} - 1$, *there is a subring* C *of* A *with* $B \subset C$, $A = C[X]$ *such that* $A = C + I$.

(2) *If* $\dim W \leqq \frac{e}{2}$, *then* $A = k[x_1, \ldots, x_e] + I$.

Proof. Suppose $V_0 \subset \mathbb{A}^n$ so $V = V_0 \times \mathbb{A}^e \subset \mathbb{A}^{n+e}$. Since W is smooth and $2 \dim W + 1 \leqq e - 1$ (in case (1)), a general projection of $\mathbb{A}^{n+e}$ on $\mathbb{A}^{e-1}$ will map W isomorphically onto a closed subscheme W' of $\mathbb{A}^e$ by Theorem 2.1. Now any sufficiently general projection $\psi: \mathbb{A}^n \times \mathbb{A}^e \to \mathbb{A}^{e-1}$ will map $\mathbb{A}^e$ onto $\mathbb{A}^{e-1}$ and therefore will factor as $\mathbb{A}^n \times \mathbb{A}^e \xrightarrow{\lambda, \mu} \mathbb{A}^e \xrightarrow{p} \mathbb{A}^{e-1}$ where p is the projection on the last $e-1$ factors and μ: $\mathbb{A}^e \approx \mathbb{A}^e$. Let θ: $\mathbb{A}^n \times \mathbb{A}^e \approx \mathbb{A}^n \times \mathbb{A}^e$ by $\theta(y, x) = (y, \lambda(y) + \mu(x))$. Then $\psi = p\theta$ where p is again the projection on the last $e-1$ factors. Clearly $V = V_0 \times \mathbb{A}^e$ is stable under θ. Replacing V by its image under θ (which amounts to changing the choice of isomorphism $V \approx V_0 \times \mathbb{A}^e$) we can assume that p itself maps W isomorphically onto W'. Now

$$p: \; V = \operatorname{Spec} B[x_1, \ldots, x_e] \to \mathbb{A}^e = \operatorname{Spec} k[x_2, \ldots, x_e]$$

corresponds to the inclusion

$$k[x_2, \ldots, x_e] \to B[x_1, \ldots, x_e].$$

Under this map,

$$W = \operatorname{Spec} B[x_1, \ldots, x_e]/I \xrightarrow{\cong} W' = \operatorname{Spec} k[x_2, \ldots, x_e]/J$$

say, so

$$k[x_2, \ldots, x_e]/J \xrightarrow{\approx} B[x_1, \ldots, x_e]/I.$$

If we let $C = B[x_2, \ldots, x_e]$ it follows immediately that $A = C + I$. Clearly $B \subset C$ and $A = C[x_1]$.

In case 2, we apply the same argument to a general projection $\mathbb{A}^{n+e} \to \mathbb{A}^e$ getting $k[x_1, \ldots, x_e]/J \xrightarrow{\approx} A/I$.

3. The Metastable Range

If P is a finitely generated projective A-module, we will write "$\operatorname{rk} P \geqq s$" to mean $\operatorname{rk} P_{\mathfrak{p}} \geqq s$ for all prime ideals $\mathfrak{p}$ of A. The *projective stable range* of A is defined as follows.

Definition 3.1. We say $\operatorname{psr} A \leqq s$ if for every finitely generated projective A-module Q with $\operatorname{rk} Q \geqq s$ and every unimodular element $(q, a) \in Q \oplus A$, there is a $y \in Q$ such that $q + ay \in Q$ is unimodular.

The usual stable range $sr(A)$ is defined in the same way but with Q assumed to be free [1, Ch. V, Def. 3.1]. Clearly $sr(A) \leqq psr(A)$. I do not know any example where the strict inequality holds although there is no obvious reason why these two ranges should be equal.

If A is noetherian and $\dim m-\operatorname{Spec} A = d$, a special case of Bass' stable range theorem [1, Ch. IV, Th. 3.1] asserts that $psr(A) \leqq d+1$. Applying the definition of psr to $(a, q) = (1, 0)$ gives Serre's theorem: If $\operatorname{rk} Q \geqq psr A$ then $Q = A \oplus Q'$. An argument using transvections [1, Ch. IV, Th. 3.4] gives Bass's cancellation theorem: If $\operatorname{rk} Q \geqq psr A$ and $A \oplus Q \approx A \oplus P$ then $Q \approx P$.

In order to extend the cancellation theorem to lower ranks in certain cases, we will define a *projective metastable range*. This definition was suggested by the work of Roitman [10]. For technical reasons, it is necessary to consider pairs of rings $B \subset A$.

Definition 3.2. If $B \subset A$, we say $pmsr(A, B) \leqq s$ if for every finitely generated projective A-module Q with $\operatorname{rk} Q \geqq s$ and every unimodular element $(q, a) \in Q \oplus A$ we can find $y \in Q$ and a subring C of A such that $B \subset C \subset A$, $A = C[x]$, and $C + \mathfrak{o}_Q(q + a y) = A$.

We will write $pmsr(A)$ in case $B = \operatorname{im}[\mathbb{Z} \to A]$. Note that $pmsr(A) \leqq pmsr(A, B)$.

This definition is of course vacuous unless A does have the form $A = C[x]$ with $B \subset C$. If this is so, we clearly have $pmsr(A, B) \leqq psr(A)$. The following theorem shows that this can be considerably improved in certain cases.

Theorem 3.3. *Let* $\operatorname{Spec} A$ *be a smooth affine scheme of dimension* d *over an infinite field* k *such that* $A = B[x_1, \ldots, x_e]$ *with* $e > 0$. *Then* $pmsr(A, B) \leqq d + 1 - \frac{e}{2}$.

Proof. We can assume $\operatorname{Spec} A$ is connected. Let Q be a finitely generated projective A-module with $\operatorname{rk} Q = r$ and let $(a, q) \in A \oplus Q$ be unimodular. Choose $y \in Q$ so that $I = \mathfrak{o}_Q(q + a y)$ has the properties specified in Theorem 1.3. Then $\dim \operatorname{Spec} A/I = d - r$. If $d - r \leqq \frac{e}{2} - 1$, Lemma 2.3 gives us the required C.

Before stating the main cancellation theorem, I will give a special case which will show clearly the connection with the methods of Roitman [10] and with Theorem 5 of Vaserštein and Suslin [14]. The argument is taken directly from [10].

Theorem 3.4. *If* $pmsr(A) \leqq r$ *then* $E_{r+1}(A)$ *is transitive on the unimodular rows* $(a_1, \ldots, a_{r+1})$ *over* A.

Proof. Let $q = (a_1, \ldots, a_r) \in Q = A^r$ and $a = a_{r+1}$. By Definition 3.2 we can find $y = (y_1, \ldots, y_r) \in Q$ and $C \subset A$ so that $A = C[x]$ and $A =$

$C+\mathfrak{o}_Q(q+ay)$. By elementary transformations, change $(a_1, \ldots, a_{r+1})$ to $(b_1, \ldots, b_{r+1})=(a_1+a_{r+1}y_1, \ldots, a_r+a_{r+1}y_r, a_{r+1})$. Then $\mathfrak{o}_Q(q+ay)=\sum_1^r Ab_i$. Since $A=C+\sum_1^r Ab_i$ we can write $x-b_{r+1}=c+\sum_1^r d_i b_i$. By elementary transformations we can replace b_{r+1} by $b_{r+1}+\sum_1^r d_i b_i=x-c=z$ say. Now $A=C[x]=C[z]$. Let $b_i=c_i+ze_i$ for $i\leqq r$. By elementary transformations change $(b_1, \ldots, b_r, z)$ to $(c_1, \ldots, c_r, z)$. Taking this modulo z we see that $(c_1, \ldots, c_r)$ is unimodular. Therefore, by more elementary transformations we can take $(c_1, \ldots, c_r, z)$ to $(c_1, \ldots, c_r, 1)$ and then to $(0, \ldots, 0, 1)$.

We now come to the cancellation theorem.

Theorem 3.5. *Let B be a subring of A. Let P and Q be finitely generated projective A-modules such that Q has the form $A\otimes_B Q_0$ where Q_0 is a finitely generated projective B-module. If P and Q are stably isomorphic and* $\operatorname{rk} P \geqq pmsr(A, B)$, *then $P\approx Q$.*

Proof. We have $P\oplus A^s\approx Q\oplus A^s$ for some s. By induction on s, it will suffice to treat the case $s=1$. Let $\varphi\colon P\oplus A\approx Q\oplus A$ and let $\varphi(0,1)=(q,a)$. Apply Definition 3.2 to (q,a) getting $y\in Q$, $C\subset A$ with $A=C[x]$, $B\subset C$, $A=C+\mathfrak{o}_Q(q+ay)$. Let $\psi\colon Q\oplus A\approx Q\oplus A$ by $\psi(\xi,\eta)=(\xi+\eta y,\eta)$. By replacing φ by $\psi\varphi$, we can assume $A=C+\mathfrak{o}_Q(q)$. Write $a-x=c+i$, with $c\in C$, $i\in\mathfrak{o}_Q(q)$. By the definition of $\mathfrak{o}_Q$, there is some $h\colon Q\to A$ with $h(q)=i$. Let $\rho\colon Q\oplus A\approx Q\oplus A$ by $\rho(\xi,\eta)=(\xi,\eta-h(\xi))$. Then $\rho\varphi(0,1)=(q,a-i)=(q,z)$ where $z=a-i=x+c$ so that $A=C[z]$. Let $Q_1=C\otimes_B Q_0$. We use here the assumption that $B\subset C$. Then $Q=A\otimes_C Q_1=Q_1[z]=Q_1\oplus zQ$. Let $q=q_1+zq'$ with $q_1\in Q_1$, $q'\in Q$. By composing $\rho\varphi$ with the transvection $(\xi,\eta)\mapsto(\xi-\eta q',\eta)$ we can assume that $q=q_1$. Reducing modulo z shows that q_1 is unimodular in Q_1 and hence in Q. Therefore, by two more transvections we can change (q,z) to $(q,1)$ and then to $(0,1)$. This gives an isomorphism $P\oplus A\approx Q\oplus A$ which is the identity on A. Factoring out A gives $P\approx Q$.

As usual we say $B\subset A$ is a retract of A if there is a ring homomorphism $\rho\colon A\to B$ which is the identity on B. This is certainly true if A is a polynomial ring over B.

Corollary 3.6. *Let B be a retract of A. Let P and Q be finitely generated projective A-modules with* $\operatorname{rk} P\geqq pmsr(A,B)$. *If*

$$[P]=[Q]\in \operatorname{im}[K_0(B)\to K_0(A)],$$

then $P\approx Q$.

Proof. Let $Q_0=B\otimes_A Q$ using the retraction $\rho\colon A\to B$. Then $[Q_0]\in K_0(B)$ is the image of $[Q]\in K_0(A)$ under ρ. Since $K_0(B)\to K_0(A)$ is a monomorphism split by ρ we see that $[Q]=[A\otimes_B Q_0]$. By Theorem 3.5 we have $P\approx A\otimes_B Q_0$ and $Q\approx A\otimes_B Q_0$.

Corollary 3.7. *Let B be regular and $A=B[x_1,\ldots,x_e]$. Let P and Q be finitely generated projective A-modules with* $\operatorname{rk} P\geqq pmsr(A,B)$. *If P and Q are stably isomorphic then $P\approx Q$.*

We need only apply Corollary 3.6 since $K_0 B\approx K_0 A$ by regularity [1, Ch. XII, Th. 3.1]. Theorem 1 follows immediately from Corollary 3.7 and Theorem 3.3.

In case $e=1$, the hypothesis of Theorem 1 reads $\operatorname{rk} P\geqq d+\frac{1}{2}$ so in fact $\operatorname{rk} P\geqq d+1$ and Theorem 1 follows from Bass' cancellation theorem in this case. Bass has pointed out that a variant of Roitman's argument will permit us to reduce this bound by 1 if k is algebraically closed.

Theorem 3.8. *Let* Spec A *be a smooth affine variety of dimension d over an algebraically closed field k such that $A=B[x]$. Let P and Q be finitely generated projective A-modules with* $\operatorname{rk} P\geqq d$. *If P and Q are stably isomorphic then $P\approx Q$.*

Proof. As in the arguments given above we can reduce to the case where $P=A\otimes_B P_0$ and $\varphi\colon P\oplus A\approx Q\oplus A$. Let $\varphi(0,1)=(q,a)$. By applying Theorem 1.3 and using a transvection we can assume that $I=\mathfrak{o}_Q(q)$ is such that Spec A/I is smooth of dimension $\leqq 0$. By Lemma 2.3(2) we can write $A=C[z]$ where $B\subset C$ and $k[z]+I=A$. Let $a=f+i$ with $f\in k[z]$ and $i\in I$. Let $h\colon Q\to A$ with $h(q)=i$ and define $\rho\colon Q\oplus A\approx Q\oplus A$ by $\rho(\xi,\eta)=(\xi,\eta-h(\xi))$. Then $\rho\varphi\colon P\oplus A\approx Q\oplus A$ sends $(0,1)$ to (q,f). Since k is algebraically closed, f splits into linear factors. In the case considered by Bass, P was free and old result of Buchsbaum implies that Q is free. In the present case, the required generalization of Buchsbaum's Lemma is given in Theorem 5.7 below.

4. The Symplectic Case

Since the results of [2] are set in a much more general context, I will begin with a brief account of the results needed here. We consider finitely generated projective A-modules with an alternating bilinear form $\langle\ ,\ \rangle$. If P and Q are two such modules write $P\perp Q$ for $P\oplus Q$ with the form $\langle(p,q),(p',q')\rangle=\langle p,p'\rangle+\langle q,q'\rangle$. I will use the symbol $\cong$ to denote an isomorphism preserving the bilinear form.

Lemma 4.1. *Let P be a finitely generated projective A-module with an alternating form $\langle\ ,\rangle$. Then there is a bilinear form b on P such that $\langle x,y\rangle=b(x,y)-b(y,x)$.*

Proof. Let $P\oplus Q=F$ be free and finitely generated. Give Q the trivial form $\langle x,y\rangle=0$ and consider F as $P\perp Q$. If we can find a bilinear form b on F with $\langle x,y\rangle=b(x,y)-b(y,x)$, then $b|P\times P$ will do. Choose a base for F and let $\langle x, y\rangle = \sum x_i a_{ij} y_j$. Then $a_{ij} = -a_{ji}$ and $a_{ii}=0$. Let $b_{ij}=a_{ij}$ for $i>j$ and $b_{ij}=0$ for $i\leqq j$. Then $b(x, y)=\sum x_i b_{ij} y_j$ has the required properties.

Definition 4.2. A symplectic A-module is a finitely generated projective A-module P with an alternating form $\langle\ ,\ \rangle$ which is non-degenerate i.e., $P \xrightarrow{\approx} P^*$ by $x \mapsto \langle x, \cdot\rangle$.

If P is symplectic and b is any bilinear form on P then $b(x,\cdot)\in P^*$ and so has the form $\langle vx, \cdot\rangle$ for some $vx \in P$. Therefore, $b(x, y) = \langle vx, y\rangle$ where $v\colon P \to P$ is linear.

Corollary 4.3. *If P is symplectic, there is an endomorphism $v\colon P\to P$ such that $\langle x, y\rangle = \langle vx, y\rangle - \langle vy, x\rangle$.*

If Q is a finitely generated projective A-module, the hyperbolic module $H(Q)$ is defined to be $Q^* \oplus Q$ with the form $\langle (f, x), (g, y)\rangle = f(y) - g(x)$. This is clearly symplectic. Obviously $H(Q\oplus Q') = H(Q)\perp H(Q')$. We will write $H = H(A)$ and $H^n = H(A^n) = H \perp \cdots \perp H$. Thus H is $A \oplus A$ with $\langle (a, b), (c, d)\rangle = ad - bc$.

If Q has a non-degenerate bilinear form $\langle\ ,\ \rangle$ we can use it to identify Q with Q^*. Therefore, in this case $H(Q)$ can be identified with $Q\oplus Q$ with the form $\langle (a, b), (c, d)\rangle = \langle a, d\rangle - \langle c, b\rangle$.

Lemma 4.4. *Let P be a symplectic A-module. Let $\tilde{P}$ be P with the form $\langle x, y\rangle_{\tilde{P}} = -\langle x, y\rangle$. Then $H(P) \cong P \perp \tilde{P}$.*

Proof. By the previous remark, identify $H(P)$ with $P\oplus P$. Let $i\colon P\to P\oplus P$ by $i(x) = (vx, x)$. Let $\mu = 1 - v$ and let $j\colon \tilde{P}\to P\oplus P$ by $j(x) = (-\mu x, x)$. An elementary calculation shows that $i\oplus j\colon P\oplus\tilde{P}\to H(P)$ gives the required isomorphism. Note that $\langle \mu x, y\rangle = \langle x, vy\rangle$. For more details see [2, Ch. I, Prop. 3.7].

Corollary 4.5. *If P is a symplectic A-module, there is a symplectic A-module Q such that $P\perp Q \cong H^n$.*

Proof. Let P' be a finitely generated A-module such that $P\oplus P' \approx A^n$. Then $H(P)\perp H(P') \cong H(A^n) = H^n$. But $P \perp \tilde{P} \cong H(P)$ by Lemma 4.4.

An alternative proof of this result may perhaps be of interest. By looking at $P\perp P\perp H(P')$ we can reduce to the case where P is free. As in the proof of Lemma 4.1 we can choose a base and represent $\langle\ ,\ \rangle$ by an n by n matrix $M = (a_{ij})$. Let $N = (b_{ij})$ as in the proof of Lemma 4.1 so that $M = N - N^t$. Define

$$X = \begin{pmatrix} I & I \\ N & N^t \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \text{and } T = \begin{pmatrix} M & 0 \\ 0 & -M \end{pmatrix}$$

where all entries are n by n matrices, I = identity. Then T represents $P\perp\tilde{P}$, J represents H^n and an easy calculation gives $X^t J X = T$. Since T is invertible so is X and we get $P\perp\tilde{P}\cong H^n$.

As usual, define $KSp_0(A)$ to be the abelian group with generators $[P]$ for all symplectic A-modules P, with relations $[P]=[P']+[P'']$ if $P \cong P' \perp P''$. If $A \to B$ is a ring homomorphism, define $KSp_0(A) \to KSp_0(B)$ by sending $[P]$ to $[B \otimes_A P]$ with the induced bilinear form. This makes KSp_0 a functor in A.

Corollary 4.6. *Every element of $KSp_0(A)$ has the form $[P]-n[H]$ for some $n \in \mathbb{Z}$. Also $[P]=[Q]$ in $KSp_0(A)$ if and only if $P \perp H^m \cong Q \perp H^m$ for some m.*

Proof. The usual arguments used for $K_0(A)$ [1, Ch. VII, Prop. 1.3] show that every element has the form $[P]-[S]$ and that $[P]=[Q]$ if and only if $P \perp S \cong Q \perp S$. By Corollary 4.5 we can write $S \perp T \cong H^n$ for some T and the result follows immediately.

As usual, we say that P and Q are stably isomorphic if $P \perp S \cong Q \perp S$ for some symplectic S. By the preceding argument this is equivalent to $P \perp H^n \cong Q \perp H^n$ for some n.

Here is the symplectic analogue of Theorem 3.5.

Theorem 4.7. *Let B be a subring of A. Let P and Q be symplectic A-modules such that $Q = A \otimes_B Q_0$ where Q_0 is a symplectic B-module. If P and Q are stably isomorphic and* $\operatorname{rk} P \geqq pmsr(A, B) - 1$ *then $P \cong Q$.*

Proof. As in the proof of Theorem 3.5 it will suffice to treat the case where $P \perp H \cong Q \perp H$. The only new feature here is that we must use only transvections preserving the symplectic structure. If M is any symplectic module, Bass [2, Ch. I, § 5.1] has defined symplectic transvections to be automorphisms of M of the form $\sigma_{u,v,\alpha}(x) = x + \langle u, x \rangle v + \langle v, x \rangle u + \alpha \langle u, x \rangle u$ where $u, v \in M$ with $\langle u, v \rangle = 0$ and $\alpha \in A$. We will apply such transvections to $Q \perp H = Q \oplus A \oplus A$.

The only ones we will need are those with $u=(0,1,0)$, $v=(y,0,0)$ and those with $u=(0,0,-1)$ and $v=(y,0,0)$. These have the form

$$(q, b, a) \to (q + a y, b + \langle y, q \rangle + \alpha a, a), \tag{1}$$

$$(q, b, a) \to (q + b y, b, a + \langle y, q \rangle + \alpha b) \tag{2}$$

where we have written $-\alpha$ for α in (2). For convenience, I will arrange the notation to agree with (1) or (2) in each case. Thus α, y, etc. will constantly change their meanings throughout the following argument.

Let $\theta: P \perp H = P \oplus A \oplus A \cong Q \perp H = Q \oplus A \oplus A$. As in the proof of Theorem 3.5 we will compose θ with symplectic transvections of $Q \perp H$ to make $\theta | H$ the identity. Let $\theta(0,0,1) = (q, b, a)$. By Definition 3.2 applied to $(Q \oplus A) \oplus A$ we can find $y \in Q$, $\alpha \in A$, and $B \subset C \subset A$ such that that $A = C[x]$ and

$$A = C + \mathfrak{o}_{Q \oplus A}(q + a y, b + \alpha a) = C + \mathfrak{o}_Q(q + a y) + A(b + \alpha a).$$

Now $\langle y, q\rangle = \langle y, q+ay\rangle \in \mathfrak{o}_Q(q+ay)$ so

$$\mathfrak{o}_Q(q+ay)+A(b+\langle y, q\rangle+\alpha a)=\mathfrak{o}_Q(q+ay)+A(b+\alpha a).$$

Therefore, if σ_1 is the transvection defined by (1) and we replace θ by $\sigma_1\theta$, we can assume that $A=C+\mathfrak{o}_Q(q)+Ab$. Now write $a-x=c+i+\alpha b$ where $c\in C$, $i\in\mathfrak{o}_Q(q)$, $\alpha\in A$ (not the same α as above). Since $\langle\ ,\ \rangle$ is non-degenerate, $\mathfrak{o}_Q(q)=\langle Q, q\rangle$ so we can find $y\in Q$ with $i=\langle y, q\rangle$. Let σ_2 be the transvection given by (2). Then $\sigma_2(q, b, a)=(q', b, x+c)$. By setting $z=x+c$ and replacing θ by $\sigma_2\theta$, we can assume that $\theta(0,0,1)=(q,b,z)$. Let $Q_1=C\otimes_B Q_0$ so $Q=A\otimes_C Q_1=Q_1\oplus zQ$. Let $q=q_1-zy$ with $q_1\in Q$, $y\in Q$. Let σ_3 be the transvection given by (1) with α to be determined. Then $\sigma_3(q,b,z)=(q_1, b'+\alpha z, z)$ where $b'=b-\langle y, q\rangle$. Write $b'=b_0+\beta z$ with $b_0\in C$, $\beta\in A$. Set $\alpha=-\beta$ so $b'+\alpha z=b_0$. By replacing θ by $\sigma_3\theta$ we can assume that $\theta(0,0,1)=(q,b,z)$ with $q\in Q_1$, $b\in C$. Since this is unimodular, reducing mod z shows that (q,b) is unimodular in $Q_1\oplus C$ and hence in $Q\oplus A$. Write $1-z=\beta+\alpha b$ where $\beta\in\mathfrak{o}_Q(q)$, and let $\beta=\langle y, q\rangle$ as above. Let σ_4 be given by (2). Then $\sigma_4(q,b,z)=(q',b',1)$. Replace θ by $\sigma_4\theta$ so that $\theta(0,0,1)=(q,b,1)$. Let σ_5 be given by (1) with $y=-q$, $\alpha=-b$ and replace θ by $\sigma_5\theta$. We now have $\theta(0,0,1)=(0,0,1)$. Let $\theta(0,1,0)=(q,b,a)$. Since $\langle(0,1,0),(0,0,1)\rangle=1$ we have $\langle(q,b,a),(0,0,1)\rangle=b=1$. Let σ_6 be given by (2) with $y=-q$, $\alpha=-a$ and replace θ by $\sigma_6\theta$. Then θ fixes $(0,0,1)$ and $(0,1,0)$ so $\theta(H)=H$. Now, in $P\perp H$ we have $P=H^\perp=\{x\in P\perp H \mid \langle x, b\rangle=0 \text{ for all } x\in H\}$. Similarly $Q=H^\perp$ in $Q\perp H$. Since $\theta(H)=H$ it follows that $\theta\colon P=H^\perp\cong H^\perp=Q$.

As in § 3, we can immediately deduce the following corollaries.

Corollary 4.8. *Let B be a retract of A. Let P and Q be symplectic A-modules with* $\operatorname{rk} P\geqq pmsr(A,B)-1$. *If* $[P]=[Q]\in\operatorname{im}[KSp_0(B)\to KSp_0(A)]$, *then* $P\approx Q$.

Corollary 4.9. *Let B be regular and $A=B[x_1,\ldots,x_e]$. Let P and Q be symplectic A-modules with* $\operatorname{rk} P\geqq pmsr(A,B)-1$. *If P and Q are stably isomorphic and if $\frac{1}{2}\in B$, then $P\approx Q$.*

If $\frac{1}{2}\in B$ and B is regular, a theorem of Karoubi [5, Ch. I, Th. 1.1] shows that $KSp_0(B)\approx KSp_0(A)$. I do not know if this is true when $\frac{1}{2}\notin B$.

We can use these results to improve the bound in Theorem 1 in certain cases.

Corollary 4.10. *Let B be a retract of A. Let P and Q be finitely generated projective A-modules with* $\operatorname{rk} P\geqq pmsr(A,B)-1$ *such that P and Q are stably isomorphic. If P and Q admit symplectic structures, $K_0(A)\approx K_0(B)$, and $KSp_0(A)\to K_0(A)$ is injective, then $P\approx Q$.*

If we assume that $KSp_0(B)\approx KSp_0(A)$, it will suffice to assume that $KSp_0(B)\to K_0(B)$ is injective. As Bass has remarked, this is so if all projective B-modules are free. In fact, all we really need is that every projective B-module of positive even rank has a summand isomorphic to A. In this case, if P is symplectic, we can find a unimodular $a \in P$. This means $\langle a, P\rangle = B$ so we can find $\langle a, b\rangle = 1$. Now $H = Ba + Bb$ is hyperbolic and $P = H \perp H^{\perp}$ so we can repeat the argument on $H^{\perp}$, finally getting $P \cong H^n$ (cf. [7, Ch. I, Cor. 3.5]).

As Bass has also remarked, any finitely generated projective module P of rank 2 will have a symplectic structure if $\operatorname{Pic} A = 0$ since then $\Lambda^2 P \approx A$. All these conditions are certainly satisfied if k is a field. We deduce that if k is an infinite field with $\operatorname{char} k \neq 2$, then all projective $k[x_1, \ldots, x_n]$-modules are free for $n \leqq 4$. This result was obtained by Vaseršteĭn and Suslin [14, Th. 1] without the hypothesis that k is finite. If k is infinite with $\operatorname{char} k = 2$ and $n \leqq 4$, the above arguments show that all projective $A = k[x_1, \ldots, x_n]$-modules will be free if and only if $KSp_0(A) \to K_0(A) = \mathbb{Z}$ is injective.

Remark. It is rather tempting to conjecture that the map $KSp_0(A) \to K_0(A)$ is always injective on the grounds that a stable invariant has no right to detect stably free modules which are not free. However, this not the case. It is easy to find counterexamples in topological K-theory. Since the group $Sp(n)$ is a maximal compact subgroup of $Sp_{2n}(C)$ the appropriate map to look at is $\widetilde{KSp}^0(X) \to \widetilde{KU}^0(X)$. For $X = S^5$ this is $\mathbb{Z}/2\mathbb{Z} \to 0$ so all complex vector bundles are stably free but $\widetilde{KSp}^0(S^5) \neq 0$. In the real case, the maximal compact subgroup of $Sp_{2n}(\mathbb{R})$ is $U(n)$ and the appropriate map is $\widetilde{KU}^0(X) \to \widetilde{KO}^0(X)$. For $X = S^2$, his is $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ while for $X = S^6$ it is $\mathbb{Z} \to 0$. The example over S^5 can easily be made algebraic. Consider S^{2n-1} as the unit sphere of $\mathbb{C}^n$ with its usual hermitian form. Define the "complex tangent bundle" of S^{2n-1} to be

$$\eta = \{(x, t) \in S^{2n-1} \times \mathbb{C}^n \mid t \perp \mathbb{C}x\}.$$

Clearly $\eta \oplus O_{\mathbb{C}} \approx O_{\mathbb{C}}$ where $O_{\mathbb{C}}$ is the trivial line bundle. Since $\mathscr{R}(x, y)$ is the usual real inner product we see also that $\eta \oplus O^1_{\mathbb{R}}$ is the usual real tangent bundle to S^{2n-1}. It is very easy to verify that the associated principal bundle of η is the canonical fibration $U(n-1) \to U(n) \to S^{2n-1}$ so by [3, Prop. 17.1] η is non-trivial for $n \geqq 3$. Now if $n = 3$, η is a rank 2 bundle over S^5 so $\Lambda^2 \eta$ is a complex line bundle. This is trivial since $\pi_4(U(1)) = 0$ so η has a symplectic structure over $\mathbb{C}$, unique up to a unit of $C_{\mathbb{C}}(S^5)$. Let $\beta \in \pi_4(Sp_2(\mathbb{C})) = \pi_4(Sp(1))$ be its canonical class. Clearly $\beta \notin 0$ otherwise η would be trivial as a symplectic bundle and hence as a complex bundle. But $\pi_4(Sp(1)) \xrightarrow{\approx} \pi_4(Sp)$ so η is stably non-trivial as a symplectic bundle. To get the required algebraic example, let $A = \mathbb{C}[x_1, \ldots, x_{2n}]/(x_1^2 + \cdots + x_{2n}^2 - 1)$ be the ring associated with S^{2n-1} [12]. Let $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4, \ldots, z_n = x_{2n-1} + ix_{2n}$. Let P be the projective A-module defined by the unimodular row $(z_1, \ldots, z_n)$. Then η is clearly the bundle associated with P. For $n=3$, $\operatorname{rk} P=2$ and $\Lambda^2 P \approx A$ because $K_0(A)=\mathbb{Z}$ [13, p. 123] and so $\operatorname{Pic} A=0$. Therefore P has a symplectic structure inducing the one considered for η. Since η is not stably trivial as a symplectic bundle, neither is P. Therefore $\widetilde{KSp}_0(A) \neq 0$ while $\tilde{K}_0(A)=0$ (as usual, the tilde indicates the kernel of the rank map).

There is also an analogue of η in the quaternionic case. Take $S^{4n-1} \subset \mathbb{H}^n$ with the inner product $(a, b)=\sum a_i \bar{b}_i$ and let

$$\eta=\{(x, t) \in S^{4n-1} \times \mathbb{H}^n \mid (t, \mathbb{H} x)=0\}.$$

The associated principal bundle is $Sp(n-1) \rightarrow Sp(n) \rightarrow S^{4n-1}$ so η is non-trivial for $n \geqq 2$. For $n=2$ it is clearly non-trivial as a complex bundle since $\pi_6(Sp(1)) \approx \pi_6(U(2))$. However η is stably trivial. In fact $\eta \oplus O_{\mathbb{H}}^1=O_{\mathbb{H}}^n$.

5. An Elementary Cancellation Theorem

In this Section I will prove the cancellation theorem used in proving Theorem 3.8. The results of this section are elementary and do not require commutativity or finite generation. We begin by recalling an argument used in connection with the fundamental theorem of K-theory.

Lemma 5.1. *Let R be a subring of A. Let P and Q be projective A-modules and let α: $Q \rightarrow P$, β: $P \rightarrow Q$ be monomorphisms. If A, $P/\alpha\beta P$, and $Q/\beta\alpha Q$ are projective over R, then so are $P/\alpha Q$ and $Q/\beta P$.*

Proof. Clearly P and Q are projective over R since A is and P and Q are direct summands of free A-modules. Since $0 \rightarrow Q \xrightarrow{\alpha} P \rightarrow P/\alpha Q \rightarrow 0$ we have $\operatorname{pd}_R P/\alpha Q \leqq 1$ (pd = projective dimension) and similarly $\operatorname{pd}_R Q/\beta P \leqq 1$. Now α: $Q \approx \alpha Q$ induces $Q/\beta P \approx \alpha Q/\alpha\beta P$. Since $0 \rightarrow \alpha Q/\alpha\beta P \rightarrow P/\alpha\beta P \rightarrow P/\alpha Q \rightarrow 0$ we have $\operatorname{pd}_R Q/\beta P=0$ and similarly $\operatorname{pd}_R P/\alpha Q=0$.

The following corollary applies in particular to $A=R[x]$.

Corollary 5.2. *Let R be a subring of A and let $a \in A$ be a central non-zero-divisor. Let P and Q be projective A-modules with $aP \subset Q \subset P$. If A and A/aA are projective over R then so is P/Q.*

This follows immediately from Lemma 5.1 with $\alpha(x)=x$, $\beta(x)=ax$. Note $P/\alpha\beta P=P/aP$ and $Q/\beta\alpha Q=Q/aQ$ are projective over A/aA and hence over R.

It is also worth noting that if P is finitely generated over A and A/aA is finitely generated over R, then P/Q is also finitely generated over R as a quotient of P/aP.

As an application we can give an elementary proof of a result proved in [8, Th. 1.3] using K-theory.

Theorem 5.3. *Let P and Q be finitely generated projective modules over $A=R[x]$. Suppose that $fP\subset Q\subset P$ where f is a monic polynomial with coefficients in the center of R. Then P and Q are stably isomorphic.*

Proof. By Corollary 5.2, $M=P/Q$ is finitely generated and projective over R. By [1, Ch. XII, § 1] we have the characterisitic sequence $0\to M[x]\to M[x]\to M\to 0$ where $M[x]=R[x]\otimes_R M$. Since $0\to Q\to P\to M\to 0$, Schanuel's Lemma [1, Ch. I, 6.3] shows that

$$P\oplus M[x]\approx Q\oplus M[x].$$

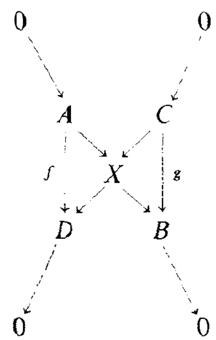
In case $f=x$ we can do much better. It would be interesting to know whether the hypothesis that F is extended can be omitted.

Theorem 5.4. *Let F and P be projective $R[x]$-modules such that F is extended, i.e., $F=R[x]\otimes_R F_0$. If $xF\subset P\subset F$, then $P\approx F$.*

Proof. As R-modules, $F=F_0\oplus xF$ and $P=P_0\oplus xF$ where $P_0=P\cap F_0$ (cf. [9, §1]). By Corollary 5.2, $F_0/P_0=F/P$ is projective over R so $F_0=P_0\oplus Q_0$ for some Q_0. Now $F=P_0[x]\oplus Q_0[x]$ and $P=P_0\oplus xP_0[x]\oplus xQ_0[x]=P_0[x]\oplus xQ_0[x]$. Thus $F\approx P$ since $Q_0[x]\approx xQ_0[x]$.

The following is a slight generalization of a well-known elementary lemma.

Lemma 5.5. (The X-Lemma.) *If*



commutes and the two sequences are exact, then $\ker f\approx\ker g$ *and* $\operatorname{ckr} f\approx\operatorname{ckr} g$.

Proof. The snake lemma on

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & X & \longrightarrow & B & \longrightarrow & 0\\
 & & \Big\downarrow f & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & D & \longrightarrow & D & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

gives $0\to\ker f\to C\xrightarrow[g]{} B\to\operatorname{ckr} f\to 0$.

We next give a generalization of an old result of Buchsbaum.

Lemma 5.6. *Let $\mathscr{P}$ be any class of A-modules. Let S be the set of central non-zero divisors of A with the following property: If* $0\to Q\to P\to A/sA\to 0$ *with* $s\in S$ *and* $P\in\mathscr{P}$, *then* $Q\approx P$. *Then S is closed under multiplication.*

Proof. If $s,t\in S$ then $0\to A/sA\xrightarrow{t}A/stA\to A/tA\to 0$. Let $\mathscr{M}$ be the class of A-modules such that if $0\to Q\to P\to M\to 0$ with $P\in\mathscr{P}$, $M\in\mathscr{M}$ then $P\approx Q$. All A/sA with $s\in S$ lie in $\mathscr{M}$ so it will suffice to show that $\mathscr{M}$ is closed under extensions. Let $0\to M'\to M\to M''\to 0$ with $M',M''\in\mathscr{M}$. Let $0\to Q\to P\xrightarrow{f}M\to 0$ with $P\in\mathscr{P}$. If $Q'=f^{-1}(M')$, then $0\to Q'\to P\to M''\to 0$ so $P\approx Q'$. Using this in $0\to Q\to Q'\to M'\to 0$ we get $0\to Q\to P\to M'\to 0$ so $Q\approx P$.

Buchsbaum considered the case where $\mathscr{P}$ is the class of finitely generated free modules and formulated his result in terms of unimodular rows. If s is a central non-zero divisor and $0\to Q\to A^n\xrightarrow{f}A/sA\to 0$ where f is represented by the matrix $(f_1,\dots,f_n)$ then $(f_1,\dots,f_n,s)$ is a unimodular row and

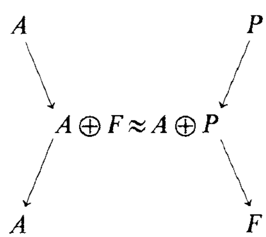
$$Q=\{(x_1,\dots,x_n)|\textstyle\sum x_if_i\equiv 0 \bmod s\}\approx\{(x_1,\dots,x_n,y)|\sum x_if_i+ys=0\}.$$

Note y is determined by $x_1,\dots,x_n$ since s is a non-zero-divisor. Therefore S is the set of central non-zero-divisors such that every unimodular row with some element in S defines a free module. The fact that this S is closed under multiplication was also discovered independently by Towber.

We can now prove the cancellation theorem.

Theorem 5.7. *Let F and P be projective* $A=R[x]$*-modules such that F is extended. Suppose* $A\oplus F\approx A\oplus P$ *and the composition* $A\to A\oplus F\approx A\oplus P\to A$ *is given by* $a\mapsto af$. *If* $f=u(x-a_1)\dots(x-a_n)$ *where u is a unit of A and the* a_i *are central elements of R, then* $P\approx F$.

Proof. By composing the given isomorphism with $A\oplus P\approx A\oplus P$ by $(a,p)\to(au^{-1},p)$, we can assume that $u=1$. By Lemma 5.5 on



we get $0\to P\to F\to A/fA\to 0$. By Lemma 5.6, it will suffice to do the case $f=x-a_i$. Let $z=x-a_i$. Then $A=R[z]$ and $zF\subset P\subset F$ so $P\approx F$ by Theorem 5.4.

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