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The Nonexistence of Certain Moufang Polygons

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Let $\Gamma=(V,E)$ be an arbitrary undirected graph, finite or infinite, V the vertex set of Γ, E the edge set, and let G be an arbitrary subgroup of $\mathrm{aut}(\Gamma)$. For each $x\in V$ we denote by $\Gamma(x)$ the set of vertices adjacent to x, by $G(x)$ the stabilizer of x in G and, for each $i\in\mathbb{N}$, by $G_i(x)$ the subgroup $\{a\in G(x)|\ a\in G(y)$ for all $y\in V$ with $\partial(x,y)\leqq i\}$ where $\partial(x,y)$ denotes the distance between x and y. An s-path (for $s\in\mathbb{N}$) is an $(s+1)$-tuple $(x_0,\dots,x_s)$ of vertices such that $x_i\in\Gamma(x_{i-1})$ if $1\leqq i\leqq s$ and $x_i\neq x_{i-2}$ if $2\leqq i\leqq s$. Let

$$G(x_0,\dots,x_s)=G(x_0)\cap\dots\cap G(x_s)$$

and

$$G_i(x_0,\dots,x_s)=G_i(x_0)\cap\dots\cap G_i(x_s)$$

for each s-path $(x_0,\dots,x_s)$ and each $i\in\mathbb{N}$. If H is a group acting on a set X, a an element of H, we denote by H^X the permutation group induced by H on X and by a^X the permutation of X induced by a.

We prove the following theorem:

Theorem 1. *Let $n\in\mathbb{N}$, $n\geqq 2$. Let $\Gamma=(V,E)$ be an undirected connected graph with $|\Gamma(x)|\geqq 3$ for every $x\in V$ and let G be a subgroup of* $\mathrm{aut}(\Gamma)$ *such that for each n-path $(x_0,\dots,x_n)$*

(i) *$G_1(x_1,\dots,x_{n-1})$ acts transitively on $\Gamma(x_n)-\{x_{n-1}\}$ and*

(ii) *$G_1(x_0,x_1)\cap G(x_0,\dots,x_n)=1$.*

Then $n=2$, 3, 4, 6 or 8. If G^V is transitive, then $n\neq 8$.

Suppose that $\Gamma=(V,E)$ is a thick generalized n-gon, in other words, a bipartite graph of girth $2n$ and diameter n with $|\Gamma(x)|\geqq 3$ for every $x\in V$, and that $G=\mathrm{aut}(\Gamma)$. It is easily verified that $G_1(x_0,x_1)\cap G(x_0,\dots,x_n)=1$ holds for every n-path $(x_0,\dots,x_n)$ (Theorem 2 below, a special case of [5, (4.1.1)]). By definition, condition (i) of Theorem 1 holds if and only if Γ is Moufang. Thus Theorem 1 implies Théorème 1 of [6]. Our proof of Theorem 1, however, is much shorter and simpler than the proof of Théorème 1 begun in [6]. For related results, see [3] and [7].

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Moufang n-gons actually exist for $n=2$, 3, 4, 6 and 8 (see, for instance, [4]). The covering construction described in [1, Chapter 19], when applied to these generalized n-gons, yields examples of graphs fulfilling the hypotheses of Theorem 1 which are not generalized n-gons. There are also examples which are not bipartite, for instance, a vertex-primitive trivalent graph fulfilling the hypotheses of Theorem 1 with $n=3$ and $G\cong PSL(2,p)$ where p is an arbitrary prime $\equiv \pm 1$ (mod 16) (see [8]), a vertex-primitive trivalent graph with $n=4$ and $G\cong \mathrm{aut}(SL(3,3))$ (see [8]) and a vertex-primitive 5-valent graph with $n=3$ and $G\cong J_3$ (see [2]).

We begin the proof of Theorem 1. Suppose that Γ and G fulfill the hypotheses (but not that G^V is transitive) and that $n\geqq 3$. Let $x\in V$ be arbitrary. Choose two $(n-1)$-paths $(x_0,\ldots,x_{n-1})$ and $(y_0,\ldots,y_{n-1})$ with $x_0=y_0=x$ and $x_1\neq y_1$. By condition (i), $\langle G_1(x_1,\ldots,x_{n-1}), G_1(y_1,\ldots,y_{n-1})\rangle$ acts transitively (in fact 2-transitively) on $\Gamma(x)$ since $|\Gamma(x)|\geqq 3$. Thus $G(x)^{\Gamma(x)}$ is transitive for every $x\in V$. For each $\{x,y\}\in E$, $\langle G(x), G(y)\rangle$ thus acts transitively on E since Γ is connected. Hence G acts transitively on E.

Lemma 1. *Let $(x_0,\ldots,x_{n-1})$ and $(y_0,\ldots,y_{n-1})$ be two $(n-1)$-paths with $x_0=y_0$ and $x_1=y_1$. Then $G_1(x_1,\ldots,x_{n-1})$ and $G_1(y_1,\ldots,y_{n-1})$ induce the same permutation group on $\Gamma(x_0)$.*

Proof. By condition (i) there exists an element $a\in G_1(x_0)$ mapping $(x_0,\ldots,x_{n-1})$ to $(y_0,\ldots,y_{n-1})$. □

For each 1-path (x,y) we denote by $H(x,y)$ the permutation group $G_1(x_1,\ldots,x_{n-1})^{\Gamma(x)}$ where $(x_1,\ldots,x_{n-1})$ is any $(n-2)$-path with $x_1=y$ and $x_2\neq x$. Lemma 1 implies that $H(x,y)$ is well defined.

Lemma 2. *For each edge $\{x,y\}$ and each $w\in\Gamma(x)-\{y\}$, $G_1(x,y)^{\Gamma(w)}=H(w,x)$.*

Proof. $G_1(x,y)^{\Gamma(w)}\geqq H(w,x)$ by definition. Let $(x_0,\ldots,x_{n+1})$ be an arbitrary $(n+1)$-path with $x_0=w$, $x_1=x$ and $x_2=y$. Let $a\in G_1(x,y)$ be arbitrary. By condition (i) there exists an element $b\in G_1(w,x,y)$ such that $ab^{-1}\in G(x_0,\ldots,x_n)$ and then an element $c\in G_1(x_1,\ldots,x_{n-1})$ such that $ab^{-1}c^{-1}\in G(x_0,\ldots,x_{n+1})$. Thus $ab^{-1}c^{-1}\in G_1(x,y)\cap G(x_0,\ldots,x_{n+1})$ so that $ab^{-1}c^{-1}=1$ by condition (ii). Hence $a^{\Gamma(w)}=c^{\Gamma(w)}$. But $c^{\Gamma(w)}\in H(w,x)$ by definition. □

Let $(x_0,\ldots,x_t)$ be an arbitrary t-path, $t\geqq 1$. We define $U(x_0,\ldots,x_t)$ to be $\{a\in G_1(x_1,\ldots,x_{t-1})|a^{\Gamma(x_0)}\in H(x_0,x_1)$ and $a^{\Gamma(x_t)}\in H(x_t,x_{t-1})\}$ if $t\geqq 2$ and $\{a\in G(x_0,x_1)|a^{\Gamma(x_0)}\in H(x_0,x_1)$ and $a^{\Gamma(x_1)}\in H(x_1,x_0)\}$ if $t=1$. Lemma 2 implies that $U(x_0,\ldots,x_t)=G_1(x_1,\ldots,x_{t-1})$ if $t\geqq 3$.

Lemma 3. *Let $(x_0,\ldots,x_{n+1})$ be an arbitrary $(n+1)$-path. For every t with $2\leqq t\leqq n$, $U(x_1,\ldots,x_t)=\langle U(x_0,\ldots,x_t), U(x_1,\ldots,x_{n+1})\rangle$.*

Proof. $\langle U(x_0,\ldots,x_t), U(x_1,\ldots,x_{n+1})\rangle\leqq U(x_1,\ldots,x_t)$ by definition. Let $a\in U(x_1,\ldots,x_t)$ be arbitrary. By definition there exists an element $b\in U(x_1,\ldots,x_{n+1})$ such that $ab^{-1}\in G_1(x_1)$. We have $ab^{-1}\in G_1(x_1,\ldots,x_{t-1})$. If $t\geqq 3$, $G_1(x_1,\ldots,x_{t-1})=U(x_0,\ldots,x_t)$ as observed above. Suppose that $t=2$. Since $a^{\Gamma(x_2)}\in H(x_2,x_1)$ and $b\in G_1(x_2)$, there exists an element $c\in G_1(x_0,x_1)$ such that $ab^{-1}c^{-1}\in G_1(x_1,x_2)$. By Lemma 2, $(ab^{-1}c^{-1})^{\Gamma(x_0)}\in H(x_0,x_1)$. But $(ab^{-1}c^{-1})^{\Gamma(x_0)}=(ab^{-1})^{\Gamma(x_0)}$. Thus $ab^{-1}\in U(x_0,x_1,x_2)$. □

The main idea behind the proof of the next lemma is borrowed from the proofs of Lemmas 6 and 7 of [6].

Lemma 4. *For each $\{x, y\} \in E$, the center $ZU(x, y)$ of $U(x, y)$ is nontrivial.*

Proof. For each i with $1 \leqq i \leqq n-1$ and each n-path $(x_0, \ldots, x_n)$ we set $U_i(x_0, \ldots, x_n) = \{a \in U(x_0, \ldots, x_n) | a \in U(x'_0, \ldots, x'_n)$ for each n-path $(x'_0, \ldots, x'_n)$ such that $x'_j = x_j$ whenever $i \leqq j \leqq n-1\}$. If $(x_0, \ldots, x_n)$ and $(x'_0, \ldots, x'_n)$ are n-paths with $x_j = x'_j$ for $i \leqq j \leqq n-1$ but $x_{i-1} \neq x'_{i-1}$, then $(x_0, \ldots, x_i, x'_{i-1}, \ldots, x'_0)$ is a $2i$-path and $U_i(x_0, \ldots, x_n) \leqq U(x_0, \ldots, x_i, x'_{i-1}, \ldots, x'_0)$; by condition (ii), $U_i(x_0, \ldots, x_n) = 1$ if $2i \geqq n+1$. On the other hand, $U_1(x_0, \ldots, x_n) = U(x_0, \ldots, x_n)$ for each n-path $(x_0, \ldots, x_n)$. Thus we may choose $m \leqq n-1$ minimal such that there exists an n-path $(y_0, \ldots, y_n)$ with $U(y_0, \ldots, y_n) \neq U_m(y_0, \ldots, y_n)$. Extend $(y_0, \ldots, y_n)$ to a $(n+m)$-path $(y_0, \ldots, y_n, y_{n+1}, \ldots, y_{n+m})$ and let

$$\begin{aligned} A &= [U(y_0, \ldots, y_n), U(y_m, \ldots, y_{n+m})] \\ &= \langle a b a^{-1} b^{-1} | a \in U(y_0, \ldots, y_n), b \in U(y_m, \ldots, y_{n+m}) \rangle. \end{aligned}$$

Since $U(y_0, \ldots, y_n) \neq U_m(y_0, \ldots, y_n)$, there exists an n-path $(y'_0, \ldots, y'_n)$ with $y'_j = y_j$ whenever $m \leqq j \leqq n-1$ such that $U(y_0, \ldots, y_n) \nleqq U(y'_0, \ldots, y'_n)$. By condition (i) there exists an element $b \in U(y_m, \ldots, y_{m+n})$ such that $b(y'_{m-1}) = y_{m-1}$. Hence $U_{m-1}(y_0, \ldots, y_n) \leqq U(b(y'_0), \ldots, b(y'_n))$, i.e., $b^{-1} U_{m-1}(y_0, \ldots, y_n) b \leqq U(y'_0, \ldots, y'_n)$. By the choice of m, $U_{m-1}(y_0, \ldots, y_n) = U(y_0, \ldots, y_n)$. It follows that

$$b^{-1} U(y_0, \ldots, y_n) b \neq U(y_0, \ldots, y_n);$$

in particular, $A \neq 1$. Since $U(y_0, \ldots, y_n) \leqq G_1(y_m)$ and $U(y_m, \ldots, y_{n+m}) \leqq G(y_m)$, $A \leqq G_1(y_m)$. Analogously, $A \leqq G_1(y_n)$ and thus $A \leqq G_1(y_m, \ldots, y_n) = U(y_{m-1}, \ldots, y_{n+1})$. Since $U(y_0, \ldots, y_n) = U_{m-1}(y_0, \ldots, y_n)$, we have $U(y_0, \ldots, y_n) \leqq U(c(y_0), \ldots, c(y_n))$ for each element $c \in U(y_{m-1}, \ldots, y_{n+1})$ and so $[U(y_0, \ldots, y_n), U(y_{m-1}, \ldots, y_{n+1})] \leqq U(y_0, \ldots, y_{n+1}) = 1$. Since $U(y_{n+m}, \ldots, y_m) = U_{m-1}(y_{n+m}, \ldots, y_m)$ (by the choice of m), $[U(y_{m-1}, \ldots, y_{n+1}), U(y_m, \ldots, y_{n+m})] \leqq U(y_{m-1}, \ldots, y_{n+m}) = 1$. Thus $1 \neq A \leqq ZU(y_{m-1}, \ldots, y_{n+1})$.

We may thus choose $t \geqq 1$ minimal such that there exists a t-path $(x_0, \ldots, x_t)$ with $ZU(x_0, \ldots, x_t) \neq 1$. If $t=1$ then $ZU(x, y) \neq 1$ for every edge $\{x, y\}$ since G^E is transitive. Thus we may suppose that $t \geqq 2$. Extend $(x_0, \ldots, x_t)$ to an $(n+1)$-path $(x_0, \ldots, x_t, x_{t+1}, \ldots, x_{n+1})$ and choose $s \geqq t$ maximal such that there exists a nontrivial element, say a, in $ZU(x_0, \ldots, x_t) \cap U(x_0, \ldots, x_s)$. (By condition (ii), $s \leqq n$.) Since $U(x_1, \ldots, x_{n+1}) \leqq G(x_1, \ldots, x_{t-1})$, $U(x_1, \ldots, x_{n+1})$ normalizes $U(x_0, \ldots, x_t)$. Hence $U(x_1, \ldots, x_{n+1})$ normalizes $ZU(x_0, \ldots, x_t)$ and so $[U(x_1, \ldots, x_{n+1}), a] \leqq ZU(x_0, \ldots, x_t)$. But $[U(x_1, \ldots, x_{n+1}), a] \leqq U(x_t, \ldots, x_{s+1})$ since $U(x_1, \ldots, x_{n+1}) \leqq G_1(x_s)$ and $a \in G(x_s)$. By the choice of s, $[U(x_1, \ldots, x_{n+1}), a] = 1$. By Lemma 3, $a \in ZU(x_1, \ldots, x_t)$. This contradicts the choice of t. □

Lemma 5. *Let $k = (n-2)/2$ if n is even and $k = (n-1)/2$ if n is odd. Then $G_k(x, y) \neq 1$ for every edge $\{x, y\}$.*

Proof. We show first that $ZU(x, y) \leqq G_1(x, y)$. If this were not so, there would exist, say, a vertex $z \in \Gamma(y) - \{x\}$ and an element $a \in ZU(x, y)$ such that $a \notin G(z)$. By Lemma 2, $G_1(y, z) \leqq U(x, y)$. Hence $G_1(y, z) = {}^aG_1(y, z) = G_1(y, a(z))$ (where ${}^aG_1(y, z)$

denotes $a G_1(y, z) a^{-1}$), i.e. $G_1(z, y) \leqq G_1(a(z))$. This contradicts condition (i) since $n \geqq 3$. It follows that $ZU(x, y) \leqq G_1(x, y)$.

We conclude the proof of Lemma 5 by showing that in fact $ZU(x, y) \leqq G_k(x, y)$. Let $(x_0, \ldots, x_n)$ be an arbitrary n-path with $\{x_0, x_1\} = \{x, y\}$. It suffices to show that $ZU(x, y) \leqq G(x_0, \ldots, x_{k+1})$. Since $1 \neq ZU(x, y) \leqq G_1(x, y)$, by condition (ii) there exist indices s with $2 \leqq s \leqq n$ such that $ZU(x, y) \nleqq G(x_s)$. Choose s minimal and let a be an element of $ZU(x, y)$ not in $G(x_s)$. By the choice of s, $a \in G(x_{s-1})$. Since $U(x_0, \ldots, x_n) \leqq U(x, y)$, $U(x_0, \ldots, x_n) = {}^a U(x_0, \ldots, x_n) = U(a(x_0), \ldots, a(x_n))$. Thus $1 \neq U(x_0, \ldots, x_n) \leqq U(x_n, x_{n-1}, \ldots, x_{s-1}, a(x_s), a(x_{s+1}), \ldots, a(x_n))$. Since $a(x_s) \neq x_s$, $(x_n, x_{n-1}, \ldots, x_{s-1}, a(x_s), a(x_{s+1}), \ldots, a(x_n))$ is a $2(n-(s-1))$-path. By condition (ii), $2(n-(s-1)) \leqq n$, i.e. $s-1 \geqq k+1$. □

Now let $(x_0, \ldots, x_{3k+1})$ be an arbitrary $(3k+1)$-path and let $B = [ZU(x_{k-1}, x_k), ZU(x_{2k}, x_{2k+1})]$. By Lemma 5, $B \leqq [G_k(x_{k-1}, x_k), G_k(x_{2k}, x_{2k+1})]$. But clearly $[G_k(x_{k-1}, x_k), G_k(x_{2k}, x_{2k+1})] \leqq U(x_0, \ldots, x_{3k})$; for instance, if $2k \leqq i \leqq 3k-1$ and $x \in \Gamma(x_i)$ then $\partial(x_{2k}, a^{-1}(x)) = \partial(a^{-1}(x_{2k}), a^{-1}(x)) \leqq k$ for every $a \in G_k(x_{k-1}, x_k)$ and thus $ba^{-1}(x) = a^{-1}(x)$, i.e. $aba^{-1}b^{-1}(x) = aba^{-1}(x) = x$, for every $b \in G_k(x_{2k}, x_{2k+1})$. Suppose that $B \neq 1$. By condition (ii), $3k \leqq n$. If n is even, then $3(n-2)/2 \leqq n$, i.e. $n \leqq 6$. If n is odd, then $3(n-1)/2 \leqq n$, i.e. $n \leqq 3$. Thus we may suppose that $B = 1$.

Suppose that $ZU(x_{k-1}, x_k) \leqq G(x_{2k+1})$. Since $U(x_{k-1}, x_k)$ acts transitively on the set of all $(k+2)$-paths $(y_{k-1}, \ldots, y_{2k+1})$ with $y_{k-1} = x_{k-1}$ and $y_k = x_k$, we have $ZU(x_{k-1}, x_k) \leqq G_{k+1}(x_k)$. Suppose that $ZU(x_{k-1}, x_k) \nleqq G(x_{2k+1})$. Let a be an arbitrary element of $ZU(x_{k-1}, x_k)$ not in $G(x_{2k+1})$. Since $B = 1$, $ZU(x_{2k}, x_{2k+1}) = {}^a ZU(x_{2k}, x_{2k+1}) \leqq U(a(x_{2k}), \ldots, a(x_{2k+1}))$. Since $U(x_{2k}, x_{2k+1})$ acts transitively on the set of all $(k+2)$-paths $(y_{2k-1}, \ldots, y_{3k+1})$ such that $y_{2k-1} = x_{2k+1}$ and $y_{2k} = x_{2k}$, we have $ZU(x_{2k}, x_{2k+1}) \leqq G_{k+1}(x_{2k})$. Thus one way or the other we conclude that there exists a vertex x such that $G_{k+1}(x) \neq 1$. This implies that n is even since otherwise $2(k+1) = n+1$ and so $G_{k+1}(x) = 1$ for every vertex x by condition (ii).

If Γ is bipartite, its vertex set V is the union of two sets V_1 and V_2, the two equivalence classes of the equivalence relation $\{(x, y) \mid \partial(x, y) \text{ is even}\} \subseteq V \times V$. Since G^E is transitive, either G^V is transitive or Γ is bipartite and G acts transitively on both V_1 and V_2. Choose an arbitrary $(3k+4)$-path $(x_0, \ldots, x_{3k+4})$ such that $G_{k+1}(x_k) \neq 1$. Suppose first that G^V is transitive or that $n \equiv 2 \pmod 4$, i.e. that k is even. Since $(2k+2)-k$ is then also even, x_k and x_{2k+2} lie in the same G-orbit; thus $G_{k+1}(x_{2k+2}) \neq 1$. Let $C = [G_{k+1}(x_k), G_{k+1}(x_{2k+2})]$. Clearly $C \leqq U(x_1, \ldots, x_{3k+1})$. If $C \neq 1$ then $3k \leqq n$ by condition (ii) and hence $n \leqq 6$. Thus we may suppose that $C = 1$. Since $G_{k+1}(x_k) \leqq G_1(x_0, \ldots, x_{2k})$ and $G_1(x_0, x_1) \cap G(x_0, \ldots, x_{2k+2}) = 1$, we have $G_{k+1}(x_k) \nleqq G(x_{2k+2})$; let a be an element of $G_{k+1}(x_k)$ not in $G(x_{2k+2})$. But then, since $C = 1$,

$$G_{k+1}(x_{2k+2}) \leqq U(x_{3k+3}, \ldots, x_{2k+1}, a(x_{2k+2}), a(x_{2k+3}), \ldots, a(x_{3k+3})).$$

Since $(x_{3k+3}, \ldots, x_{2k+1}, a(x_{2k+2}), a(x_{2k+3}), \ldots, a(x_{3k+3}))$ is a path of length $2((3k+3)-(2k+1)) = n+1$, $G_{k+1}(x_{2k+2}) = 1$. This contradicts our earlier observation that $G_{k+1}(x_{2k+2}) \neq 1$.

Thus we may assume that $n \equiv 0 \pmod 4$, i.e. that k is odd (and that G^V is intransitive). This time we consider $D = [G_{k+1}(x_k), G_{k+1}(x_{2k+3})]$. Since $(2k+3)-k$ is even, x_k and x_{2k+3} lie in the same G-orbit; thus $G_{k+1}(x_{2k+3}) \neq 1$. Since $D \leqq U(x_3, \ldots, x_{3k})$, condition (ii) implies that $3k-3 \leqq n$, i.e. $n \leqq 12$, *if* $D \neq 1$. Suppose that $D=1$. Letting a again denote an element of $G_{k+1}(x_k)$ not in $G(x_{2k+2})$, we have $G_{k+1}(x_{2k+3}) \leqq U(x_{3k+4}, \ldots, x_{2k+1}, a(x_{2k+2}), \ldots, a(x_{3k+4}))$ where

$$(x_{3k+4}, \ldots, x_{2k+1}, a(x_{2k+2}), \ldots, a(x_{3k+4}))$$

is a path of length $2((3k+4)-(2k+1))=n+4$ and so $G_{k+1}(x_{2k+3})=1$. This contradicts our observation that $G_{k+1}(x_{2k+3}) \neq 1$.

To conclude the proof of Theorem 1, we need only eliminate the case $n=12$.

Lemma 6. *Suppose $n=12$ (so that $k=5$). Let $(x_0, \ldots, x_{12})$ be an arbitrary 12-path such that $G_6(x_2) \neq 1$. Let a be an arbitrary nontrivial element in $G_6(x_2)$ (so that $a \notin G(x_9)$). Then:*

(i) *For each $f \in U(x_8, \ldots, x_{13})$ mapping $(a(x_9), a(x_{10}))$ to (x_7, x_6) and for each $b \in G_6(x_{10})$, $[a,b] = {}^{fa}b \in G_6(x_6)$.*

(ii) *For each $c \in G_6(x_6)$ there exists an element $b \in G_6(x_{10})$ such that $[a,b]=c$.*

Proof. Let b and f be as in part (i). Clearly $[a,b] \in U(x_0, \ldots, x_{12})$. Since $fa(x_{10})=x_6$, ${}^{fa}b \in G_6(x_6)$. Since b and f lie in $G_1(x_{12})$, $[a,b]$ and ${}^{fa}b$ induce the same permutation on $\Gamma(x_{12})$. Thus ${}^{fa}b \cdot [a,b]^{-1} \in U(x_0, \ldots, x_{12}) \cap G_1(x_{12})=1$. To prove part (ii), simply set $b=(fa)^{-1} c (fa)$. □

Suppose $n=12$. Let $(x_0, \ldots, x_{20})$ be an arbitrary 20-path such that $G_6(x_2) \neq 1$. Let a be a nontrivial element in $G_6(x_2)$, b a nontrivial element of $G_6(x_{10})$ and $f \in U(x_8, \ldots, x_{17})$ an element mapping $(a(x_9), a(x_{10}))$ to (x_7, x_6) (which exists by condition (i)). Let $c=[b,a]$. By part (i) of Lemma 6, $c={}^{fa}(b^{-1})$. We have $[f,b] \in [U(x_8, \ldots, x_{17}), G_6(x_{10})] \leqq U(x_4, \ldots, x_{17})=1$ and so

$$[c,f]=[[b,a],f]=[b,a]\,f\,a\,b\,a^{-1}\,b^{-1}f^{-1}=[b,a]\cdot {}^{fa}b \cdot b^{-1}=b^{-1}.$$

By part (ii) of Lemma 6, we can choose an element $d \in G_6(x_{14})$ such that $[c,d]=b^{-1}$. Since $[f,b]=1$, $[c,df^{-1}]=[c,d]\,[c,f]^{-1}=(b^{-1})(b^{-1})^{-1}=1$. Thus $c \in G_6(x_6) \cap {}^{df^{-1}}G_6(x_6)$. But $G_6(x_6) \cap {}^{df^{-1}}G_6(x_6)=1$ unless $df^{-1} \in G(x_7, x_6)$. Since $c \neq 1$, we conclude that $(d^{-1}(x_7), d^{-1}(x_6))=(f^{-1}(x_7), f^{-1}(x_6))=(a(x_9), a(x_{10}))$.

Now choose an arbitrary element $e \in U(x_1, \ldots, x_{13})$. We have $[d,e] \in [G_6(x_{14}), U(x_1, \ldots, x_{13})] \leqq U(x_7, \ldots, x_{18})$. Since $(da(x_9), da(x_{10}))=(x_7, x_6)$ and $e \in G_1(x_9, x_{10})$, ${}^{da}e \in G_1(x_6, x_7)$. Since $[a,e] \in [G_6(x_2), U(x_1, \ldots, x_{13})] \leqq G_6(x_2) \cap G_1(x_8)=1$, ${}^{da}e = {}^{d}e$. Since $e \in G_1(x_6, x_7)$, $[d,e]={}^{d}e \cdot e^{-1} \in G_1(x_6, x_7)$. Thus $[d,e] \in U(x_5, \ldots, x_{18})=1$. Hence $d \in G_6(x_{14}) \cap {}^{e}G_6(x_{14})$. But $G_6(x_{14}) \cap {}^{e}G_6(x_{14})=1$ unless $e \in G(x_{14})$. It follows that $e \in U(x_1, \ldots, x_{13}) \cap G(x_{14})=1$ for all $e \in U(x_1, \ldots, x_{13})$. This contradicts condition (i) since $|\Gamma(x_{13})-\{x_{12}\}|>1$.

The proof of Theorem 1 is now complete. □

As mentioned in the introduction, we need the following result in order to conclude that Theorem 1 implies Théorème 1 of [6]. The result is just a special case of [5, (4.1.1)], but we include a short proof (also due to J. Tits) for the convenience of the reader.

Theorem 2. *Let Γ be a thick generalized n-gon and $G=\mathrm{aut}(\Gamma)$. Then $G_1(x_0, x_1) \cap G(x_0, \ldots, x_n)=1$ for each n-path $(x_0, \ldots, x_n)$.*

Proof. Let $(u_0, \ldots, u_n)$ be an arbitrary n-path. For every neighbor v of u_n there is an $(n-2)$-path $(v_0, \ldots, v_{n-2})$ with $v_0=v$ and $v_{n-2} \in \Gamma(u_0)$. Since the girth of Γ is $2n$, it follows that $G_1(u_0) \cap G(u_0, \ldots, u_n) \leqq G_1(u_n)$.

Now let $(w, x_0, \ldots, x_n)$ be an arbitrary $(n+1)$-path. To prove Theorem 2, it suffices, since Γ is connected, to show that $G_1(x_0, x_1) \cap G(x_0, \ldots, x_n) \leqq G_1(w, x_0) \cap G(w, x_0, \ldots, x_{n-1})$. There exists a $2n$-path $(x_0, \ldots, x_n, x_{n+1}, \ldots, x_{2n})$ extending $(x_0, \ldots, x_n)$ with $x_{2n-1} = w$ and $x_{2n} = x_0$; let $H = G(x_0, \ldots, x_{2n})$. We have $G_1(x_0, x_1) \cap G(x_0, \ldots, x_n) \leqq G(x_{2n-1}, x_0, \ldots, x_n) \leqq H$. As observed in the previous paragraph, $G_1(u_0) \cap G(u_0, \ldots, u_n) \leqq G_1(u_n)$ for every n-path $(u_0, \ldots, u_n)$. It follows that $G_1(x_0) \cap G(x_0, \ldots, x_n) \leqq G_1(x_n)$ and thus $G_1(x_0, x_1) \cap G(x_0, \ldots, x_n) \leqq G_1(x_1) \cap G(x_1, \ldots, x_n, y) \leqq G_1(y)$ for every $y \in \Gamma(x_n) - \{x_{n-1}\}$. Choose such a vertex $y \neq x_{n+1}$ (using the hypothesis that Γ is thick). Then $G_1(y) \cap G(y, x_n, x_{n+1}, \ldots, x_{2n-1}) \leqq G_1(x_{2n-1})$. Thus $G_1(x_0, x_1) \cap G(x_0, \ldots, x_n) \leqq G_1(y) \cap H \leqq G_1(x_{2n-1})$. □

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