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The Structure of Crossed Product C^*-Algebras: A Proof of the Generalized Effros-Hahn Conjecture*

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Abstract. If G is a second countable locally compact group acting continuously on a separable C^*-algebra A, then every primitive ideal of the crossed product $C^*(G, A)$ is contained in an induced primitive ideal, and if G is amenable, equality holds. Thus if G is amenable and acts freely on Prim(A), the "generalized Effros-Hahn conjecture" holds: there is a canonical bijection between primitive ideals of $C^*(G, A)$ and G-quasi-orbits in Prim(A). Applications to the "Mackey machine" for a non-regularly embedded normal subgroup of a locally compact group are discussed. The proof of the theorem is based on a "local cross-section" result together with Mackey's original methods.

One of the oldest and most natural problems in group representation theory is to determine information about irreducible representations of a group G from knowledge of the irreducible representations of some normal subgroup N and of the action of G on N and its dual. In the context of unitary representations of locally compact groups, this problem was attacked, and to a large extent solved, in an important paper of Mackey [18]. However, a reasonable answer to the problem seemed to require that N be "regularly embedded" – otherwise difficult ergodic-theoretic difficulties arise and one is unlikely to be able to classify all the irreducible representations of G up to unitary equivalence (see, e.g., [22] for extensive discussion of these issues). Work of many investigators (beginning with Guichardet [14]) suggested, however, that even when N is not regularly embedded, it should nevertheless be possible to classify the irreducible representations of G up to *weak* equivalence. This, in turn, suggested that representations of group extensions should be viewed in terms of the structure of "crossed product" and "twisted crossed product" C^*-algebras.

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Given a C^*-algebra A and a locally compact group G acting continuously by *-automorphisms on A, one can construct in a natural way an associated "crossed-product" C^*-algebra $C^*(G, A)$ (for various versions of the construction, see [11, 27, 8, 6, 26, 13]), representations of which correspond to covariant pairs of representations of G and of A. For instance, when A is the group C^*-algebra $C^*(H)$ of a locally compact group H on which G acts by automorphisms, $C^*(G, A)$ is just the group C^*-algebra of the semidirect product group $G \times_s H$. In the simplest case, when A is the abelian C^*-algebra of continuous functions vanishing at infinity on some homogeneous space $X = G/H$ of G, the Mackey imprimitivity theorem [16, 18] asserts that every representation of $C^*(G, A)$ is "induced" from a representation of H. More generally, when A is type I and G acts "smoothly" on the dual $\hat{A}$ of A (i.e., A is "regularly embedded" – there are several ways [17, part 4; 18; 11; 23] of phrasing this precisely), one still knows [26] that every irreducible representation of $C^*(G, A)$ is induced from an irreducible representation of $C^*(H, A)$, where H is the stabilizer in G of some point in $\hat{A}$. In particular, *every primitive ideal of* $C^*(G, A)$ *is the kernel of an induced representation* (namely, one induced from a representation lying over a single primitive ideal of A).

In their memoir [8], E.G. Effros and F. Hahn suggested that this last statement may be true for completely arbitrary "non-smooth" group actions as well, at least if G is amenable and A is abelian. Thus one would have a substitute for the imprimitivity theorem (on the C^*-algebra level) that would apply even in the puzzling case of "non-transitive quasi-orbits". The purpose of this paper is to prove their conjecture, at least in the separable case, without placing any restrictions on G, A, or the nature of the action of G. We should mention that considerable progress had already been made in this direction over the years (see [14, 27, 8, 12, 24, 25]). Sauvageot [25] has proven our theorem in full for the case of G discrete, along with the "easy direction" of the theorem (the fact that every primitive ideal of $C^*(G, A)$ contains an induced primitive ideal) for arbitrary amenable G. However, virtually all of the existing literature on the "hard direction" of the theorem (the reverse containment) assumes either that G is close to being discrete, or else that G is close to being abelian and one has extra information about A. Our interest had therefore been focused on methods that would work when G is, say, a connected non-compact Lie group. A key motivating example here was that of an irrational flow on a torus, which occupies a central role in the representation theory of non-type I Lie groups (see Sect. 4.4 at the end of this paper). The methods of [12, § 6] and the extensive work of Pukanszky [20, 21] (see also [13]) had shown that the Effros-Hahn conjecture was true for this transformation group, but the proofs depended on special circumstances that do not hold in general.

Our method of proof is based on a "local cross-section theorem", perhaps of independent interest in the study of transformation groups, along with a local version of Mackey's original proof [15, 16, 18] of the imprimitivity theorem. The cross-section theorem was suggested by recent work in the theory of measure groupoids [9] and also by the observation that local cross-sections exist for the irrational flow on the torus, and perhaps were the ingredient needed for the general case.

One word about terminology: all topological transformation groups are assumed jointly continuous. If (G, X) is a topological transformation group, a G-

quasi-orbit in X is an equivalence class for the relation $\sim$ on X defined by $x \sim y \Leftrightarrow Gx$ and Gy have the same closure in X. (The reader is cautioned that this definition, which is the one given in [8], is *not* the same as that used in [18], [22] or [1], where a quasi-orbit is an equivalence class of ergodic measures.) For $x \in X$, G_x denotes the stabilizer of x in G. The term *representation* always means strongly continuous unitary representation when applied to groups and continuous *-representation on a Hilbert space when applied to C^*-algebras, and all Hilbert spaces used are separable.

The main results of this paper are Theorem 3.1 and its Corollaries 3.2 and 3.3. Section 1 contains our local cross-section theorem (Theorem 1.4) as well as various preliminaries on cross-sections for homogeneous spaces (1.5) and the "regularized" topology on the primitive ideal space of a C^*-algebra (1.6). Section 2 is largely technical and contains a measure-theoretic version of Theorem 1.4 that is needed for the proof of the main theorem. The reader may prefer to skip this section on first reading and proceed directly to Sect. 3. Section 4.1 contains an explicit counterexample (which also appears, but only implicitly, in [27]) showing that amenability of G is necessary in 3.2 and 3.3. Our theorem has as an immediate consequence a version of the generalized Effros-Hahn conjecture for the "twisted covariance algebras" of [13] – we are grateful to Philip Green for pointing this out to us. As a consequence, one has a very general version of the "Mackey machine" [18, 23] for group extensions in which the normal subgroup need not be type I or regularly embedded. This is discussed in Sect. 4.2 and 4.3. Finally, Sect. 4.4 illustrates the discussion of 4.3 in the case of connected Lie groups, for which Pukanszky [20, 21] has obtained much more complete information (see also [13]).

§1. A "Local Cross-Section" Theorem for Polish Transformation Groups

We begin with a lemma which, although easy to prove, does not seem to have been used before in the context suggested below.

Lemma 1.1. *Let (G, X) be a topological transformation group with G locally compact and second countable and with X Polish, and let Q be a G_δ G-invariant subset of X. Equip the space Σ of all closed subgroups of G with the Fell topology* (*see* [1, *Ch. II*, §2]), *and let $S: Q \to \Sigma$ be the map assigning to each point y of Q its stabilizer G_y. Then there exists a point of continuity for S in Q.*

Proof. Since Q is a G_δ subset of X, it itself is Polish [2, Ch. IX, §6.1] and we may as well assume $Q = X$. By [1, Ch. II, Prop. 2.1 and 2.3], there exist countably many lower semi-continuous functions f_i on X, with values in [0, 1], such that a point of X is a point of continuity for S if and only if it is a point of continuity for each f_i. Every Polish space is Baire, so by [2, Ch. IX, §2.7 and §5, exercise 22a], the complement of the set of continuity points for each f_i in X is meagre, hence so is the complement of the set of joint continuity points for all the f_i. This complement cannot be all of the Baire space X, and thus S has a point of continuity.

Corollary 1.2. *Let (G, X), Σ and S be as in Lemma* 1.1, *and let Q be a quasi-orbit of G in X. Then there exists a point of continuity in Q for the map $S: Q \to \Sigma$.*

Proof. By the proof of [8, Lemma 1.1], Q is a G_δ subset of X.

The statement of the following lemma is more general than needed for the proof of the cross-section theorem, but will be used in the proof of the main theorem of this paper.

Lemma 1.3. *Let (G, X) be a Hausdorff topological transformation group with G locally compact, and let $S\colon X \to \Sigma$ be the map assigning to $x \in X$ its stabilizer G_x. Let C be a compact subset of G, M a symmetric neighborhood of the identity in G, and Y any subset of X. Then given $y \in Y$ and any neighborhood V of y in Y, there exists a neighborhood $W \subseteq V$ of y such that if $v \in V$, $s, t \in C$ and $s\,v$, $t\,v \in W$, then $s\,t^{-1} \in MG_y$. In addition, if y is a continuity point for the restricted map $S|_Y\colon Y \to \Sigma$, then there exists a neighborhood U of y in Y such that for all $u \in U$, $C \cap G_y \subseteq MG_u$.*

Proof. If the first statement were false, we could choose a neighborhood base $W_n \subseteq V$ of y, and points $v_n \in V$, s_n, $t_n \in C$ such that $s_n v_n$, $t_n v_n \in W_n$, but $s_n t_n^{-1} \notin MG_y$. By compactness of C, we may pass to a subsequence and assume $s_n \to s$, $t_n \to t \in C$. Then $v_n = s_n^{-1}(s_n v_n) = t_n^{-1}(t_n v_n)$ converges to $s^{-1} y$ and $t^{-1} y$, so $s t^{-1} \in G_y$. As M is a neighborhood of the identity, $s_n t_n^{-1} \in M s t^{-1}$ eventually, and we have a contradiction. If the second statement were false, we could choose a neighborhood base W_n of y in Y and points $w_n \in W_n$, $c_n \in C \cap G_y$, with $c_n \notin MG_{w_n}$. By compactness of C again, we may pass to a subsequence and assume $c_n \to c \in C \cap G_y$. Let O be a symmetric open neighborhood of the identity in G, with $O^2 \subseteq M$. Then $Oc \cap G_y \neq \emptyset$, and by the continuity of $S|_Y$ at y, $Oc \cap G_{w_n} \neq \emptyset$, $\forall n \geqq n_0$. Thus there are $o_n \in 0$, $t_n \in G_{w_n}$ with $o_n c = t_n$, $\forall n \geqq n_0$. Thus $c_n = c_n c^{-1} c = (c_n c^{-1})(o_n^{-1} t_n) \in O^2 G_{w_n}$ eventually, which is a contradiction.

Theorem 1.4. *Let (G, X) be a topological transformation group with G locally compact and second countable and with X Polish. Let S be as in Lemma 1.3 and let $x \in X$ be a point of continuity of S. Let M be a compact symmetric neighborhood of the identity in G and let V be a neighborhood of x in X. Then there is a Borel set $T \subseteq V$ which is a "local transversal" for the action of M at x, in the sense that $M \cdot T$ is a neighborhood of x in which every element can be written in the form $m \cdot t$, with $m \in M$ and $t \in T$; furthermore, the t is unique and the Borel structure of T is identical to the quotient Borel structure induced by the natural map of $M \cdot T$ onto T.*

Proof. Let P be a compact symmetric neighborhood of the identity with $P^2 \subseteq M$, and let $N = M^6$. By the first part of Lemma 1.3, applied to the compact set N and the symmetric neighborhood P of the identity, there exists a neighborhood $W_1 \subseteq V$ of x which in particular satisfies the property that if $w \in W_1$, $n \in N$, $n\,w \in W_1$, then $n \in PG_x$. By the second part of Lemma 1.3, there exists a neighborhood W_2 of x such that for all $y \in W_2$, $PN \cap G_x \subseteq PG_y$.

Let W be an open neighborhood of x with $W \subseteq W_1 \cap W_2$. We claim that on W, the relation $\sim$, where $y \sim z$ if and only if $y = n\,z$ for some $n \in N$, is an equivalence relation. It is clearly reflexive and symmetric. If $y, z, w \in W$ with $y \sim z$ and $z \sim w$, then $y = n\,z$ for some $n \in N$. Thus $n = p_1 g_1$ with $p_1 \in P$, $g_1 \in G_x$. $g_1 = p_1^{-1} n \in G_x \cap PN \subseteq PG_z$, so $g_1 = p_2 g_2$, $p_2 \in P$, $g_2 \in G_z$. Thus $y = n\,z = p_1 p_2 g_2 \cdot z = p_1 p_2 \cdot z$. So $y \in P^2 z$, and similarly $z \in P^2 w$; hence $y \in P^4 w \subseteq N\,w$. As N is compact, the equivalence class $N\,w \cap W$ of a point $w \in W$ is closed in W, while the saturation $NO \cap W$ of an open set

$O \subseteq W$ is open, since W was chosen open in X and O is thus open in X. Lemma 2 of [4] thus applies to give a Borel set $T \subseteq W$ which intersects each equivalence class in one and only one point. The argument above shows $W \subseteq M \cdot T$, so that $M \cdot T$ is a neighborhood of x, while if $m_1 t_1 = m_2 t_2$ for $m_i \in M$, $t_i \in T$, then $t_1 = (m_1^{-1} m_2) t_2 \sim t_2$, hence $t_1 = t_2$.

Finally, we must show that a subset $B \subseteq T$ is Borel if and only if MB is Borel in MT. But T is Borel in X, hence is a standard Borel space, and if B is Borel in T, it follows that the analytic space MT is a disjoint union of the two analytic Borel spaces MB and $M(B^c \cap T)$, and thus that MB is Borel in MT [1, Ch. I, Prop. 2.4]. For the converse, note that the map $M \times T \to MT$ sending $(m, t) \mapsto mt$ is continuous in the relative topologies of the spaces involved, thus Borel. The inverse image of the Borel set $MB \subseteq MT$ is the Borel set $M \times B \subseteq M \times T$, and thus B is Borel in T.

Remark. The above argument required merely that N be chosen equal to M^2. The choice $N = M^6$ implies the stronger result that if $m_1 m_2 m_3 t = m_4 m_5 m_6 s$, with $m_i \in M$ and $s, t \in T$, then $s = t$. This will be used in Proposition 2.2 and Lemma 3.8.

For use in § 3, we prove the following:

Proposition 1.5. *Let G, X and Σ be as in Lemma 1.1. For each $H \in \Sigma$, let c_H be a cross-section $G/H \to G$, and let d_H: $G \to G$ be given by $d_H(g) = c_H(gH)$. Then the c_H can be chosen so that the function d: $G \times \Sigma \to G$ given by $d(g, H) = d_H(g)$ is Borel. Also, for each $x \in X$ let c_x be a cross-section $G/G_x \to G$, let d_x: $G \to G$ be given by $d_x(g) = c_x(g G_x)$, and let d: $G \times X \to G$ be given by $d(g, x) = d_x(g)$. Then the c_x can be chosen so that d is Borel.*

Proof. The second part of the proposition follows from the first and the fact that S: $X \to \Sigma$, $S(x) = G_x$, is Borel. For the first part, define the following equivalence relation on $G \times \Sigma$:

$$(g, H) \sim (l, K) \Leftrightarrow H = K \quad \text{and} \quad gH = lH.$$

It is easy to see that the saturation of each compact subset (and in particular of each point) of $G \times \Sigma$ is closed. Since each closed subset of $G \times \Sigma$ is a K_σ, it follows that the saturation of every closed subset of $G \times \Sigma$ is Borel, and thus by [2, Ch. IX, §6.8] there exists a Borel transversal T for the equivalence relation. The equivalence relation is a closed subset of $(G \times \Sigma) \times (G \times \Sigma)$, so as in the proof of 1.4, the saturation of any Borel set $B \subseteq T$ is both analytic and coanalytic, thus Borel. Hence the natural map $G \times \Sigma \to T$ is Borel, and we may take d to be this map followed by projection onto G.

Now let G be a second countable locally compact group acting continuously as a group of $*$-automorphisms of a separable C^*-algebra A. For the purpose of studying the primitive ideal structure of the crossed product C^*-algebra $C^*(G, A)$, the natural object of investigation is the topological transformation group $(G, \mathrm{Prim}(A))$, where $\mathrm{Prim}(A)$ denotes the primitive ideal space of A with its usual hull-kernel topology. However, this topology need not be Hausdorff, hence the previous results do not immediately apply. It is known, however, that one can introduce a new topology on $\mathrm{Prim}(A)$, with respect to which it becomes a Polish space. For reference, we summarize what we need in the following proposition.

Proposition 1.6. Prim(A) *can be endowed with a new topology with respect to which it becomes a Polish space. This new topology, called the regularized topology, is stronger than, but generates the same Borel structure as, the hull-kernel topology. Furthermore, in the regularized topology,* $(G, \mathrm{Prim}(A))$ *is a topological transformation group.*

Proof. Prim(A) is in one-to-one correspondence with a subset X of the compact Hausdorff space $\mathcal{N}(A)$ of C^*-pseudo-norms on A [10, §2]. X is Polish [5, Théorème 7] and the natural map $\varphi: X \to \mathrm{Prim}(A)$ is a Borel isomorphism [7, Lemma 2.3]. If $N_i \to N$ in X, then $N_i(a) = \|a + \varphi(N_i)\|_{A/\varphi(N_i)} \to N(a) = \|a + \varphi(N)\|_{A/\varphi(N)}$ for each $a \in A$. If I is a closed two-sided ideal in A with $I \nsubseteq \varphi(N)$, then $N(x) \neq 0$ for some $x \in I$, hence $N_i(x) \neq 0 \; \forall i \geqq i_0$, and $I \nsubseteq \varphi(N_i)$, $i \geqq i_0$, so $\varphi(N_i) \to \varphi(N)$. Finally, if $g_i \to g$ in G and $N_i \to N$ in X, then for $a \in A$,

$$\begin{aligned} |g_i \cdot N_i(a) - g \cdot N(a)| &= |N_i(g_i^{-1} a) - N(g^{-1} a)| \\ &\leqq |N_i(g_i^{-1} \cdot a) - N_i(g^{-1} \cdot a)| + |N_i(g^{-1} \cdot a) - N(g^{-1} \cdot a)| \\ &\leqq \|g_i^{-1} \cdot a - g^{-1} \cdot a\| + |N_i(g^{-1} \cdot a) - N(g^{-1} \cdot a)| \to 0, \end{aligned}$$

so $g_i \cdot N_i \to g \cdot N$ in X.

§2. A Measurable "Local Cross-Section" Theorem

Throughout this section, let (G, X) be a topological transformation group with G locally compact and second countable and with X Polish, and let μ be a finite quasi-invariant measure on X with support all of X. For the purposes of the main theorem, we need a measurable version of Theorem 1.4, in which the transversal lies in a prescribed μ-conull subset of X. We carry through the details in this section and end with a description of the fiber measures obtained from the integral decomposition of μ over the transversal.

Lemma 2.1. *With G, X and μ as above, let M be a compact symmetric neighborhood of the identity in G, let $x_0 \in X$ (chosen, say, as in* 1.1), *and let T be a Borel subset of X which is a "local transversal" for the neighborhood $M \cdot T$ of x_0, as described in Theorem* 1.4. *Let μ_1 denote the restriction of μ to $M \cdot T$, and ν the image of μ_1 on T under the natural map $\theta: M \cdot T \to T$, so that for Borel $B \subseteq T$, $\nu(B) = \mu(M \cdot B)$. If X_0 is any μ-conull Borel subset of X, one can find a ν-conull analytic set $S \subseteq T$ and a μ_1-conull analytic set $D \subseteq X_0 \cap MT$ such that θ maps D onto S, and for which there exists a Borel cross-section $\sigma: S \to D$.*

Proof. As MT is analytic, so is $X_0 \cap MT$, and one can find a standard Borel subset $F \subseteq X_0 \cap MT$ on which μ_1 is concentrated. $\theta(F)$ is an analytic subset of T on which ν is concentrated, and one can apply [1, Ch. I, Proposition 2.15] to $\theta: F \to (\theta(F), \nu)$.

As σ is one-to-one, S is Borel isomorphic with its image $Y \subseteq X_0 \cap MT$, and Y is an analytic transversal for a μ_1-conull subset $D \subseteq X_0 \cap MT$. Thus, for each $d \in D$, $\theta(d) = \theta(y)$ for one and only one $y \in Y$. The measurable analog of Theorem 1.4 is the following:

Proposition 2.2. *MY is an analytic subset of X of positive μ-measure, the map ψ of MY onto Y given by $\psi(my) = y$ is well-defined and Borel, and the Borel structure of Y is* *identical to the quotient Borel structure induced by ψ. Furthermore, if $m_1 m_2 y_1 = m_3 m_4 y_2$, for $m_i \in M$ and $y_j \in Y$, then $y_1 = y_2$.*

Proof. MY is clearly analytic. To see that it is of positive measure, observe first that for $d = mt \in D$, with $m \in M, t \in T$, there exists $y \in Y$ of the form $y = m' t$, with $m' \in M$. Thus $D \subseteq M^2 Y$ and $\mu(M^2 Y) > 0$. But M^2 can be covered by a finite number of translates of M, and the result follows by quasi-invariance of μ. Now pick $y_1 = m_5 t_1$ and $y_2 = m_6 t_2$, with $y_i \in Y$, $m_j \in M$, and $t_k \in T$. If $m_1 m_2 y_1 = m_3 m_4 y_2$ with $m_j \in M$, then $m_1 m_2 m_5 t_1 = m_3 m_4 m_6 t_2$ and $t_1 = t_2$ by the remark following Theorem 1.4. Thus $\theta(y_1) = \theta(y_2)$, so $y_1 = y_2$ by the construction of Y. Thus ψ is well-defined, and the last statement is verified. The remaining statements follow exactly as in the proof of the corresponding part of Theorem 1.4.

For the rest of the paper, we will use only the "measurable" transversal Y, and will no longer mention T. With slight abuse of previous notation, let μ_1 be the restriction of μ to $M \cdot Y$ and let v be the measure $\psi_*(\mu_1)$ on Y. For $y \in Y$, $\psi^{-1}(\{y\}) = M \cdot y \subseteq G \cdot y$; $G \cdot y$ is Borel isomorphic to G/G_y and thus has a unique G-invariant measure class. Also, for each $y \in Y$ there exists a measure β_y on $M \cdot y$ such that for all Borel sets $S \subseteq M \cdot Y$,

$$\mu(S) = \int_Y \beta_y(S)\, dv(y).$$

The β_y are called fiber measures, and are unique up to v-null sets in Y ([17, §11] and [1, Ch. II, §2]). The following proposition is a "local version" of Lemma 11.5 of [17].

Proposition 2.3. *For v-almost all $y \in Y$, β_y is the restriction to $M \cdot y$ of a measure in the unique G-invariant measure class on $G \cdot y$.*

Proof. As in [17], let $t: G \times X \to G \times X$ be given by $t(g,x) = (g, gx)$, and let λ be any finite measure in the Haar measure class of G. Then $(t_*(\lambda \times \mu_1))|_{G \times MY} \ll \lambda \times \mu_1$, and again as in [17, §11], it follows that for v-almost all $y \in Y$,

$$S_y \equiv \{g \in G \mid (g_* \beta_y)|_{M \cdot y} \ll \beta_y\} \qquad \text{is } \lambda\text{-conull in } G.$$

We shall show that for any such y, β_y is equivalent to the restriction to My of a quasi-invariant measure on Gy. Define the measure $\lambda * \beta_y$ on $G \cdot y$ by $(\lambda * \beta_y)(A) = \int_G \beta_y(g^{-1} A)\, d\lambda(g)$ for all Borel sets $A \subseteq Gy$. Then $\lambda * \beta_y$ is quasi-invariant since λ is, while $(\lambda * \beta_y)|_{My} \ll \beta_y$ by our condition on y. As S_y^{-1} is λ-conull in G, it is dense, and for any symmetric neighborhood L of e in G, we have $M \subseteq \cup s_i L$ for some finite number of s_i in S_y^{-1}. Choose such an L with $L^2 \subseteq M$. If $A \subseteq Ly$ with $(\lambda * \beta_y)(A) = 0$, then $\beta_y(g^{-1} A) = 0$ a.e. $(d\lambda)$, so $\beta_y(l^{-1} A) = 0$ for some $l \in L \cap S_y^{-1}$. As $l^{-1} A \subseteq My$, we have $\beta_y(A) = (l_*^{-1} \beta_y)|_{My}(l^{-1} A) = 0$. Finally, for any $A \subseteq My$ with $(\lambda * \beta_y)(A) = 0$, $A = \cup (A \cap s_i Ly)$ with $(\lambda * \beta_y)(A \cap s_i Ly) = 0\, \forall i$. Thus $(\lambda * \beta_y)(s_i^{-1} A \cap Ly) = 0\, \forall i$, so $\beta_y(s_i^{-1} A \cap Ly) = 0\, \forall i$, by the previous comments. Similarly, $s_i^{-1} A \cap Ly \nsubseteq My$ and $s_i \in S_y^{-1}$, so $\beta_y(A \cap s_i Ly) = ((s_i^{-1})_* \beta_y)|_{My}(s_i^{-1} A \cap Ly) = 0\ \ \forall i$, and $\beta_y(A) = 0$. Thus $\beta_y \sim (\lambda * \beta_y)|_{My}$ and we are done.

§3. The Generalized Effros-Hahn Conjecture

Throughout this section, we assume G is a second countable locally compact group acting continuously by *-automorphisms on a separable C^*-algebra A. Before proving the main theorem, we first discuss the setting in which the results of §2 will be applied, and then describe the generalized Effros-Hahn conjecture, as formulated by Sauvageot [24, 25].

For measures on $\mathrm{Prim}(A)$, the notions of Borel measure, G-quasi-invariance and G-ergodicity depend only on the Borel structure of $\mathrm{Prim}(A)$, and are thus, by Proposition 1.6, independent of whether we are considering the hull-kernel or the regularized topology. The notion of G-quasi-orbit in $\mathrm{Prim}(A)$ is topological, of course, but clearly every quasi-orbit in the regularized (stronger) topology is contained in a quasi-orbit in the hull-kernel (weaker) topology. Each representation π of A determines (up to equivalence) a Borel measure μ on $\mathrm{Prim}(A)$, which may be assumed finite, and a μ-measurable field of representations $i \mapsto \pi_i$ based on $(\mathrm{Prim}(A), \mu)$ such that π_i is homogeneous with kernel i, and $\pi \cong \int_{\mathrm{Prim}(A)}^{\oplus} \pi_i \, d\mu(i)$ [7]. For a factor representation R of $C^*(G, A)$, let (V, π) be the associated covariant pair of representations of G and of A, respectively. The measure μ determined, as above, by π is then G-quasi-invariant and ergodic [25, §§3–4], and thus lives on a quasi-orbit X in $\mathrm{Prim}(A)$ with the *regularized* topology, precisely as in the proof of Lemma 1.1 of [8]. As a measure on $\mathrm{Prim}(A)$ with the regularized topology, μ has support $\bar{X}$, and thus as a measure on the regularized quasi-orbit X, μ has support all of X. It follows that all the results of §2 apply to the system (G, X, μ).

R also determines, of course, a quasi-orbit of $\mathrm{Prim}(A)$ equipped with the hull-kernel topology. Furthermore, this latter quasi-orbit (which contains X) depends only on the kernel of R, so there is a well-defined map

$$\mathrm{Prim}(C^*(G, A)) \to \{G\text{-quasi-orbits in } \mathrm{Prim}(A), \text{ with the hull-kernel topology}\}.$$

On the other hand, one has a natural construction of induced primitive ideals from hull-kernel G-quasi-orbits in $\mathrm{Prim}(A)$, arising from the notion of induced representations, due originally to Mackey [17] but generalized by numerous authors (e.g., [11, 8, 26]) over the last 25 years (for a very general treatment, far more powerful than anything we need here, see [13]). As defined by Sauvageot, an *induced primitive ideal* is the kernel of a representation of $C^*(G, A)$ induced from a homogeneous representation $L = \langle w, \tau \rangle$ of $C^*(H, A)$, where τ is a homogeneous representation of A with kernel J in $\mathrm{Prim}(A)$, and H is the stabilizer of J in G. The hull-kernel G-quasi-orbit determined by such an induced primitive ideal is exactly the hull-kernel G-quasi-orbit of J in $\mathrm{Prim}(A)$.

Our main theorem is

Theorem 3.1. *Every primitive ideal of $C^*(G, A)$ is contained in an induced primitive ideal.*

Corollary 3.2. *(the generalized Effros-Hahn conjecture). If G is amenable, every primitive ideal of $C^*(G, A)$ is an induced primitive ideal.*

Proof. Proposition 4.2 of [25] proves the reverse of the inclusion to be proven in Theorem 3.1.

Corollary 3.3. *If G is amenable and acts freely on $\mathrm{Prim}(A)$, then there is one and only one primitive ideal of $C^*(G,A)$ lying over each hull-kernel quasi-orbit in* $\mathrm{Prim}(A)$. *In particular, if every orbit is also hull-kernel dense, then $C^*(G,A)$ is simple.*

Proof. If G acts freely on $\mathrm{Prim}(A)$, all isotropy subgroups are trivial and there is only one induced primitive ideal over each quasi-orbit.

We shall prove Theorem 3.1 in a sequence of lemmas below, but first we must review Sauvageot's construction of an induced primitive ideal from a given irreducible representation of $C^*(G,A)$, and establish notation. Accordingly, let R be an irreducible representation of $C^*(G,A)$, let (V,π) be the corresponding covariant pair of representations of G and of A, respectively, and let $X \subseteq \mathrm{Prim}(A)$ be the regularized quasi-orbit on which the measure μ determined by the homogeneous decomposition of π lives. Henceforth we shall consider only the regularized topology on X.

By definition of the measure μ, R may be realized on the Hilbert space $L^2(X,\mu,\mathscr{H})$, with $\pi = \int_X^{\oplus} \pi_x \, d\mu(x)$, π_x being a homogeneous representation of A on $\mathscr{H}$ with kernel $x \in X (\subseteq \mathrm{Prim}(A))$. Let W be the "natural" representation of G on $L^2(X,\mu,\mathscr{H})$ by translations, so that

$$(W(s)f)(x) = (d\mu(s^{-1}x)/d\mu(x))^{\frac{1}{2}} f(s^{-1}x), \quad \text{for } s \in G, x \in X \quad \text{and} \quad f \in L^2(X,\mu,\mathscr{H}).$$

Then the operator $U(s) \equiv V(s)\, W(s^{-1})$, $s \in G$, is decomposable, and we may write

$$U(s) = \int_X U_0(s,x)\, d\mu(x),$$

where $U_0(s,x)$ is a unitary operator on $\mathscr{H}$, depending measurably on $x \in X$ for each fixed $s \in G$. For each fixed $s,t \in G$ we have the cocycle identity

$$U_0(s^{-1}t,x) = U_0(s^{-1},x)\, U_0(t,sx) \qquad \text{for } \mu\text{-almost all } x \in X,$$

while for fixed $s \in G$, $a \in A$, we have the intertwining relation

$$\pi_x(s\,a) = U_0(s,x)\, \pi_{s^{-1}x}(a)\, U_0(s,x)^{-1} \qquad \text{for } \mu\text{-almost all } x \in X.$$

Lemma 3.4. *We may choose a certain μ-conull Borel set $X_0 \subseteq X$ and replace U_0 by a "regularized version" U, satisfying the following properties:*

i) *for each fixed $s \in G$, $U(s,x) = U_0(s,x)$ for μ-almost all x.*

ii) *U is jointly Borel in G and X.*

iii) *$\forall\, s,t \in G$ and $x \in X_0 \cap s^{-1}X_0 \cap s^{-1}tX_0$, $U(s^{-1}t,x) = U(s^{-1},x)\, U(t,sx)$.*

iv) *$\forall\, x \in X_0$, $s \mapsto \sigma_x(s) \equiv U(s,x)$ is a unitary representation of G_x on $\mathscr{H}$, and (σ_x,π_x) is a covariant pair of representations of (G_x, A) on $\mathscr{H}$.*

v) *$\forall\, x \in X_0$, $U(s^{-1}t,x) = U(s^{-1},x)\, U(t,sx)$ for almost all $(s,t) \in G \times G$.*

Proof. (i)–(iv) are discussed in [24, §5] and [25, §3], but note that the X_0 in [24, 25] may have to be cut down further by a null set for (iv) to hold. (v) follows easily from the fact that U_0 can be chosen to satisfy (v) and that U can be chosen so that $U(s,x) = U_0(s,x)$ a.e. on $G\times X$. Again, X_0 may have to be cut down by another null set.

For $x\in X_0$, let r_x be the representation of $C^*(G_x, A)$ determined by the covariant pair (σ_x, π_x), and let $\tilde{r}_x = (\tilde{\sigma}_x \tilde{\pi}_x) = \mathrm{Ind}(\sigma_x, \pi_x)$ be the induced representation of $C^*(G,A)$. Sauvageot shows in [25, §§ 3–4] that $x\mapsto \tilde{r}_x$ is a μ-measurable field of representations of $C^*(G,A)$, and that for μ-almost all $x\in X_0$, the $\tilde{r}_x$ are homogeneous with the same kernel, and this kernel is an induced primitive ideal. Cutting down X_0 by a null set again, we may suppose all the $\tilde{r}_x$ are homogeneous with the same kernel. This kernel is the induced primitive ideal which we shall show contains kernel R. The key to our proof is that on a set $M\cdot Y$ (given by Proposition 2.2) which "looks like a rectangle", the cocycle U may be "untangled" (written locally as a coboundary) exactly as in Mackey's proof of the imprimitivity theorem.

Accordingly, let X_0 be a μ-conull Borel subset of X chosen as above, let M be a compact symmetric neighborhood of the identity in G, and let Y be an analytic subset of X_0 satisfying all the hypotheses of Proposition 2.2. Let μ_1 be the restriction of μ to MY and let $\nu = \psi_*(\mu_1)$, where $\psi: M\cdot Y\to Y$ is given by $\psi(my) = y$, exactly as in Proposition 2.2 and the subsequent discussion. Instead of working with Sauvageot's representation $\int_X^{\oplus} \tilde{r}_x\, d\mu(x)$, we shall work with the weakly equivalent homogeneous representation $\tilde{r} = (\tilde{\sigma}, \tilde{\pi}) = \int_Y^{\oplus} \tilde{r}_y\, d\nu(y)$, and show kernel $(\tilde{r}) \supseteq$ kernel R.

For $s\in G$ and $y\in Y$, let $\theta_y(s) \equiv U(s^{-1}, y)$, so that by property (v) of Lemma 3.4, we have, for $y\in Y$, $U(t, sy) = \theta_y(s)^{-1}\,\theta_y(t^{-1}s)$ for almost all $s,t\in G$. Note that for $s\in G_y$, $y\in Y$, $\theta_y(s) = U(s^{-1}, y) = \sigma_y(s^{-1})$. Recall that $\tilde{r}_y$ is usually realized on a space $\tilde{\mathscr{H}}_y$ of functions $\varphi_y: G\to\mathscr{H}$ which satisfy $\varphi_y(ts) = \theta_y(s)\,\varphi_y(t)$ for $s\in G_y$, $t\in G$, and which are square-integrable with respect to some quasi-invariant measure on G/G_y. For our purposes, it will be more convenient to realize $\tilde{r}_y$ on a space $\mathscr{H}_y$ of functions $\varphi_y: Gy\to\mathscr{H}$, and the ensuing discussion shows how to do this measurably in y.

Let $\mu_1(\cdot) = \int_Y \beta_y(\cdot)\, d\nu(y)$ be a decomposition of μ_1 into fiber measures over Y. By Proposition 2.3, we may assume that for all $y\in Y$, β_y is the restriction to $M\cdot y$ of a quasi-invariant measure γ_y on $G\cdot y$, and by the construction of γ_y in the proof of Proposition 2.3, it is clear that the γ_y can be measurably chosen, in the sense that for a bounded Borel function f on X, $y\mapsto \int_X f(x)\, d\gamma_y(x)$ is measurable and ν-integrable. Choose Borel cross-sections $c_y: G/G_y\to G$ such that $(y,s)\mapsto d_y(s) = c_y(sG_y)$ is Borel from $Y\times G$ to G (Proposition 1.5). Then if $\mathscr{H}_y = L^2(G\cdot y, \gamma_y, \mathscr{H})$, we have unitary operators $\tilde{\mathscr{H}}_y \to \mathscr{H}_y$ given by restriction of functions on G to the transversal $c_y(G/G_y)$ for G/G_y, followed by transportation of functions over to orbits in X. The realization of $\tilde{r}_y$ thereby obtained on $\mathscr{H}_y$ is given by the following formulas, where we use the same letters $\tilde{r}_y = (\tilde{\sigma}_y, \tilde{\pi}_y)$ for simplicity:

$$(\tilde{\pi}_y(a)\,\varphi)(ty) = \pi_y(d_y(t)^{-1}\cdot a)\,\varphi(ty)$$

and

$$(\tilde{\sigma}_y(s)\,\varphi)(ty) = \theta_y(d_y(s^{-1}t)^{-1}s^{-1}d_y(t))\,\varphi(s^{-1}ty)\,(d\gamma_y(s^{-1}ty)/d\gamma_y(ty))^{\frac{1}{2}},$$

where $a \in A$, $s, t \in G$, $y \in Y$ and $\varphi \in \mathscr{H}_y$. By the choice of γ_y and d_y, this realization of $\tilde{r}_y$ on $\mathscr{H}_y$ is clearly still ν-measurable in y.

Fix any non-zero vector

$$\varphi = \int_Y^{\oplus} \varphi_y \, d\nu(y)$$

in the Hilbert space of $\tilde{r}$, such that the associated vectors φ_y in $\mathscr{H}_y$ are supported in $M \cdot y$. To prove kernel $\tilde{r} \supseteq$ kernel R, it suffices to exhibit, for any compact set $K \subseteq G$, a sequence ξ_i of vectors in the Hilbert space of R, depending only on K and with $\Sigma \|\xi_i\|^2 = \|\varphi\|^2$, such that for any $g \in L^1(G)$ with support in K and for any $a \in A$, we have

$$(3.5) \quad \sum_i \langle R(g \otimes a^*) \xi_i, \xi_i \rangle = \langle \tilde{r}(g \otimes a^*) \varphi, \varphi \rangle.^1$$

Note that (3.5) is equivalent to

$$(3.6) \quad \sum_i \langle V(g) \xi_i, \pi(a) \xi_i \rangle = \langle \tilde{\sigma}(g) \varphi, \tilde{\pi}(a) \varphi \rangle = \int_Y \langle \tilde{\sigma}_y(g) \varphi_y, \tilde{\pi}_y(a) \varphi_y \rangle \, d\nu(y).$$

Fix K compact in G – we may assume without loss of generality that K contains the identity. To construct the ξ_i's, we need the following

Lemma 3.7. *Y contains a countable disjoint collection $\{Y_i\}$ of Borel subsets, whose union is ν-conull in Y, such that, for each i,*

$$s Y_i \cap Y_i \neq \emptyset \quad \text{for } s \in MKM \text{ implies } s \in M^2 G_y, \quad \text{for any } y \in Y_i.$$

Proof. We first show Y contains a countable collection of subsets T_i, whose union is ν-conull in Y, such that on each T_i, the map $S: T_i \to \Sigma$, sending a point in T_i to its stabilizer group, is continuous, and then apply Lemma 1.3 to each T_i. By [1, Ch. II, Prop. 2.1 and 2.3], there exist countably many real-valued Borel functions f_j on Y such that for any $T \subseteq Y$, $S|_T: T \to \Sigma$ is continuous if and only if $f_j|_T: T \to \Sigma$ is continuous for each j. An obvious iteration of Lemma 4.1, Ch. 2, of [19] (a variant of Lusin's theorem) allows us to find, for each $i \geqq 1$, a Borel subset T_i of Y such that $\nu(Y - T_i) \leqq 1/i$ and all the $f_j|_{T_i}$ are continuous (recall that Y and all its subsets are metric spaces since X is Polish).

We now apply Lemma 1.3 to each T_i. For each $x \in T_i$, there exists a neighborhood W of x in T_i such that for all $u \in W$, $M^2 KM \cap G_x \subseteq MG_u$, and such that if $sW \cap W \neq \emptyset$ for $s \in MKM$, then $s \in MG_x$. Each T_i can be covered by countably many such W's, say $T_i = \bigcup_j W_{ij}$.

For a fixed W_{ij} and $s \in MKM$, $s W_{ij} \cap W_{ij} \neq \emptyset$ implies $s \in M^2 G_w$ for any $w \in W_{ij}$, since $s = mg$ for some $m \in M$ and $g \in G_x$, with $x \in W_{ij}$, and thus $g = m^{-1} s \in G_x \cap M^2 KM \subseteq MG_w$ for any $w \in W_{ij}$, so that $s = mg \in M^2 G_w$. The above

[1] Linear combinations of elements of $L^1(G, A)$ of the form $g \otimes a$, $g \in L^1(G)$ and $a \in A$, are dense in $C^*(G, A)$ – see, for instance, the preliminary comments in § 1 of [13]

implication is clearly true for any subset of W_{ij}, so we can denote by $\{Y_i\}$ the (countable) collection of sets obtained by considering W_{11}, $W_{12}-W_{11}$, ..., $W_{ij}-\bigcup_{k<i}\bigcup_{l=1}^{\infty} W_{kl}-\bigcup_{l<j} W_{il}, \ldots$.

Lemma 3.8. *Let* $\mathcal{K}_y$ *denote the subspace of* $\mathcal{H}_y$ *consisting of functions with support* $\subseteq M\cdot y$, *and let* $\mathcal{K}_i=\int_{Y_i}^{\oplus}\mathcal{K}_y\,dv(y)$. *Let* $\mathcal{L}_i$ *denote the subspace of* $L^2(X,\mu,\mathcal{H})$ *consisting of functions with support* $\subseteq MY_i$. *Then* $\mathcal{K}_i$ *and* $\mathcal{L}_i$ *are invariant, respectively, under the representations* $\tilde{\pi}$ *and* π, *and if, for* $\varphi=\int_{Y_i}^{\oplus}\varphi_y\,dv(y)\in\mathcal{K}_i$, *we define* $(\psi_i\varphi)(m\,y)=(\theta_y(d_y(m)))^{-1}\varphi_y(m\,y)$, *then* ψ_i *is a unitary operator from* $\mathcal{K}_i$ *to* $\mathcal{L}_i$ *which intertwines* $\tilde{\pi}$ *and* π. *Furthermore, if* P_i *and* Q_i *denote, respectively, the projection from the Hilbert space of* $\tilde{r}$ *onto* $\mathcal{K}_i$ *and the projection from the Hilbert space of* R *onto* $\mathcal{L}_i$, *we have*

$$Q_i V(s)\psi_i\varphi=\psi_i P_i\tilde{\sigma}(s)\varphi, \quad \textit{for } \varphi\in\mathcal{K}_i \quad \textit{and} \quad s\in K.$$

Proof. ψ_i is clearly an isometry by virtue of the relationship between the measures involved. The invariance of $\mathcal{K}_i$ and $\mathcal{L}_i$ under $\tilde{\pi}$ and π, respectively, is clear. That ψ_i intertwines $\tilde{\pi}$ and π follows from the relationship $\pi_x(s\,a)=U_0(s,x)\,\pi_{s^{-1}x}(a)\,U_0(s,x)^{-1}$. All the details thus far are just as in the proof of the imprimitivity theorem – the cochain of which U is (locally) a coboundary provides us with the intertwining isometry ψ_i. The only novelty is in the last statement, which we proceed to check. For simplicity we drop the Radon-Nikodym derivatives. For $s\in K$ and $\varphi\in\mathcal{K}_i$,

$$\begin{aligned} Q_i V(s)\psi_i\varphi|_{my} &= V(s)\psi_i\varphi|_{my}=V(s)\,W(s^{-1})\,W(s)\psi_i\varphi|_{my} \\ &= U(s,m\,y)\,(\psi_i\varphi)(s^{-1}m\,y)=U(s,m\,y)\,\chi_{MY_i}(s^{-1}m\,y)\,(\psi_i\varphi)(s^{-1}m\,y), \end{aligned}$$

for $m\,y\in MY_i$, while

$$\begin{aligned} \psi_i P_i\tilde{\sigma}(s)\varphi|_{my} &= (\theta_y(d_y(m)))^{-1}\,(P_i\tilde{\sigma}(s)\varphi)_y(m\,y) \\ &= (\theta_y(d_y(m)))^{-1}\,(\tilde{\sigma}(s)\varphi)_y\,(m\,y) \\ &= (\theta_y(d_y(m)))^{-1}\cdot\theta_y(d_y(s^{-1}m)^{-1}s^{-1}d_y(m))\cdot\varphi_y(s^{-1}m\,y). \end{aligned}$$

If, for $s^{-1}my\in MY_i$, $(\psi_i\varphi)(s^{-1}my)=(\theta_y(d_y(s^{-1}m)))^{-1}\varphi_y(s^{-1}my)$, we have equality exactly as in the imprimitivity theorem. The last thing to check, then, is that if $m\,y\in MY_i$, $s\in K$ and $s^{-1}m\,y\in MY_i$, then in fact $s^{-1}m\,y\in My$. But if $s^{-1}m\,y=m'\,y'$ for some $m'\in M$, $y'\in Y_i$, then $m^{-1}s\,m'\in MKM$ and sends $y'\in Y_i$ to $y\in Y_i$. By Lemma 3.7, $m^{-1}s\,m'\in M^2G_{y'}$, so $y=m^{-1}s\,m'\,y'\in M^2y'$, and $y=y'$ by the last statement of Proposition 2.2.

We now finish the proof of the main theorem. Pick $0\neq\varphi\in\int_Y\varphi_y\,dv(y)$ and write $\varphi=\sum\oplus\varphi_i$ with $\varphi_i\in\mathcal{K}_i$. Lemma 3.8 implies that if $\xi_i=\psi_i\varphi_i$, then $\langle V(s)\xi_i, \pi(a)\xi_i\rangle=\langle\tilde{\sigma}(s)\varphi_i\ \tilde{\pi}(a)\varphi_i\rangle$ for all i and all $s\in K$, $a\in A$. Clearly, then, $\|\varphi\|^2=\sum\|\xi_i\|^2$. Also, $\mathcal{K}_i\subseteq\int_{Y_i}^{\oplus}\mathcal{H}_y\,dv(y)$, and $\int_{Y_i}^{\oplus}\mathcal{H}_y\,dv(y)$ is $\tilde{r}$-invariant, so for $i\neq j$,

$\tilde{r}(C^*(G,A))\varphi_i \perp \tilde{r}(C^*(G,A))\varphi_j$. It follows that for any $a \in A$ and $g \in L^1(G)$ with $\operatorname{supp} g \subseteq K$,

$$\begin{aligned}\langle \tilde{\sigma}(g)\varphi, \tilde{\pi}(a)\varphi\rangle &= \sum_i \langle \tilde{\sigma}(g)\varphi_i, \tilde{\pi}(a)\varphi_i\rangle \\ &= \sum_i \int_K \langle g(s)\tilde{\sigma}(s)\varphi_i, \tilde{\pi}(a)\varphi_i\rangle\, ds \\ &= \sum_i \int_K \langle g(s)V(s)\xi_i, \pi(a)\xi_i\rangle\, ds \\ &= \sum_i \langle V(g)\xi_i, \pi(a)\xi_i\rangle,\end{aligned}$$

and we are done.

§4. Concluding Remarks

4.1. The amenability of G is necessary in Corollaries 3.2 and 3.3. Indeed, suppose G is a non-amenable group which acts freely on a compact Hausdorff space X, preserving a finite measure. (For an example, let G be the free group on two generators embedded as a non-closed discrete subgroup of a non-commutative compact Lie group X. Then G acts freely by left translation and leaves Haar measure invariant.) By the Krein-Milman theorem, there exists a G-invariant ergodic measure μ on X. Form the representation π of $C^*(G,X)=C^*(G,C(X))$ on $L^2(X,\mu)$, in which G acts by translations and $C(X)$ acts by multiplication operators. Since μ is ergodic, this is a factor representation, whose kernel will be a primitive ideal of $C^*(G,X)$ (provided that G and X are second countable). This primitive ideal does not contain an induced primitive ideal, since if it did, the restriction V of π to G would be weakly contained in the left regular representation λ of G (which is the restriction to G of any irreducible induced representation from an isotropy subgroup, since G acts freely). However, V contains the trivial one-dimensional representation of G on constant functions, which is not weakly contained in λ since G was assumed non-amenable.[2]

4.2. To see how our results apply to the "Mackey machine" for group extensions, it is convenient to generalize them first to the "twisted covariance algebras" of [13]. (These are essentially equivalent to the "verallgemeinerte L^1-Algebren" of H. Leptin, the "twisted group algebras" of R.C. Busby and H.A. Smith, and the "produits croisés restreints" of N. Dang Ngoc. We have adopted P. Green's notation here since it agrees most closely with our own.) Such an algebra $C^*(G,A,\mathcal{T})$ is a quotient of the ordinary crossed product $C^*(G,A)$, so we easily obtain from 3.1 and 3.2 the following theorem.

Theorem 4.2. *Every primitive ideal of a twisted covariance algebra $C^*(G,A,\mathcal{T})$ (with G and A separable) is contained in an induced primitive ideal. If $G/N_{\mathcal{T}}$ is amenable, every primitive ideal of $C^*(G,A,\mathcal{T})$ is an induced primitive ideal.*

Proof. The first statement follows from 3.1, since one sees from Corollary 5 of [13] that if R is an irreducible representation of $C^*(G,A)$ that "preserves $\mathcal{T}$,"

[2] We note that this example is implicit in [27], théorème 4.20 and Proposition 5.2, but we thought it best to make it more explicit.

then the $\tilde{r}$ constructed from R in Sect. 3 also preserves $\mathcal{T}$. (We thank Phil Green for pointing this out to us.)

The second statement follows similarly from 3.2, except for the difficulty that G may not be amenable even if $G/N_{\mathcal{T}}$ is, hence Proposition 4.2 of [25] need not apply directly. However, one can modify Sauvageot's proof by applying Lemma 6.2 of [24] with $G/N_{\mathcal{T}}$ in place of G (note that $N_{\mathcal{T}}$ operates trivially on Prim(A), so that the action of $G/N_{\mathcal{T}}$ on Prim(A) is well-defined) and by using amenability of $G/N_{\mathcal{T}}$.

Now we immediately get our version of the "Mackey machine".

Theorem 4.3. *Let G be a second countable locally compact group and let N be a closed normal subgroup of G. Then every irreducible unitary representation of G weakly contains a representation induced from a homogeneous representation σ of the stabilizer G_J of some primitive ideal J of $C^*(N)$, such that the restriction of σ to N is homogeneous with kernel J. If G/N is amenable, every irreducible representation of G is weakly equivalent to an induced representation of this form.*

Proof. By the Corollary to Proposition 1 of [13], we may view $C^*(G)$ as $C^*(G, C^*(N), \mathcal{T})$ for some twisting map $\mathcal{T}$ with $N_{\mathcal{T}} = N$.

Note that when N is type I, G_J may be identified with G_ρ where ρ is the (unique up to unitary equivalence) irreducible representation of N with kernel J. In this case, ρ extends to a projective representation of G_ρ as in the Mackey theory [18, 23] and when G/N is amenable, Theorem 4.3 says that every irreducible representation of G is weakly equivalent to a representation obtained by Mackey's procedure (using projective representations of the "little groups" G_ρ/N). When G/N is not amenable, however, 4.1 indicates that there may be "bad" primitive ideals of $C^*(G)$ not obtainable by inducing from stability groups. (The example may be modified so as to occur in a purely group theoretic context. For instance, let G be the semidirect product of $H = SL(2, \mathbb{Z})$ acting on $N = \mathbb{Z}^2$ in the usual way. Then H acts freely on a dense quasi-orbit in $\hat{N} = \mathbb{T}^2$ and preserves Haar measure.)

4.4. Theorem 4.3 has many potential applications to the unitary representation theory of groups with non-regularly embedded normal subgroups, for which the ordinary "Mackey machine" fails. Many interesting examples are provided by connected Lie groups, especially the solvable ones. Although there is nothing new to be accomplished for these, since the primitive ideal spaces of such groups have already been determined by Pukanszky [20, 21] along with considerable extra information, our theorem provides an alternative proof for some of Pukanszky's results. If G is a connected, simply connected Lie group and N is its commutator subgroup, then N is locally algebraic and type I by [21, Lemma 1.1.1]. (When G is solvable, N is nilpotent and the Kirillov orbit method provides a very explicit picture of $\hat{N}$. For more general G, our understanding of $\hat{N}$ is somewhat less precise.) By 4.3, every irreducible representation of G is therefore weakly equivalent to a representation induced from a homogeneous representation σ of some stability group G_ρ, where $\rho \in \hat{N}$ and $\sigma|_N$ is a multiple of ρ. (In particular, this proves [21, Lemma 1.2.5] – that every irreducible representation of G is weakly equivalent to one living over a single orbit in $\hat{N}$.) Since

$G_\rho \supseteq N$, G_ρ is normal and all points in the same G-orbit have the same stabilizer. In fact, in this case G_ρ is the same for all ρ in any G-quasi-orbit Q in N [21, Lemmas 1.1.4 and 1.1.5]. Therefore we may apply 4.3 again with G_ρ in place of N and conclude that the set of primitive ideals of $C^*(G)$ lying over Q is parametrized by the set of G-quasi-orbits in

$$\{J \in \operatorname{Prim} C^*(G_\rho) \colon J \text{ lies over } Q\}.$$

This result should be compared with [21, Proposition 1], which is slightly different since Pukanszky works with a group K that is in general smaller than G_ρ. (His method has the advantage that the Mackey obstruction to extending ρ to K vanishes, so that the relevant part of $C^*(K)$ is type I and one may use $\hat{K}$ in place of Prim $C^*(K)$. That one may use K in place of G_ρ in the analysis of the part of Prim $C^*(G)$ lying over Q is a consequence of the very special nature of the projective dual of the abelian "little group" G_ρ/N.)

In some special cases, direct application of 3.2 may also be of help in computing the primitive ideal space of a connected group G. If G and N are as above, we may take some sequence of subgroups $N = N_0 \subseteq N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n = G$ (the inverse image of a flag in G/N) in which N_{i+1} is the semi-direct product of N_i by an action of $\mathbb{R}$, so that $C^*(N_{i+1}) \cong C^*(\mathbb{R}, C^*(N_i))$. By 3.2, we conclude that every primitive ideal of $C^*(N_{i+1})$ is induced. (The problem, of course, is that this statement is vacuous for ideals lying over one-point orbits in $\operatorname{Prim}(N_i)$.) For instance, let G be the "Dixmier group" [3] with Lie algebra spanned by basis elements $e_1, \ldots, e_7$ with non-trivial brackets

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = e_5, \quad [e_1, e_5] = -e_4, \quad [e_2, e_6] = e_7, \quad [e_2, e_7] = -e_6,$$

and let N_1 denote the connected normal subgroup with Lie algebra spanned by $e_2, \ldots, e_7$. Then $N_1 \cong \mathbb{R}^3 \times H$, where H is the universal covering group of the motion group of the plane. Points in $\hat{N}_1 (\cong \operatorname{Prim}(N_1))$ in general position have quasi-orbits that look topologically like 2-tori on which $\exp(\mathbb{R}e_1)$ acts by irrational flows. Since such flows are free, one knows by 3.2 and 3.3 that there is a unique primitive ideal of $C^*(G) \cong C^*(\exp(\mathbb{R}e_1), C^*(N_1))$ lying over each such quasi-orbit. Similar phenomena occur for many other non-type I solvable Lie groups, or for type I solvable groups with non-regularly embedded connected normal subgroups.

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Note Added in Proof

Finally, we mention one more application of Theorem 4.3 – namely, one gets a quick proof of the theorem of Roger Howe (in "The Fourier transform for nilpotent locally compact groups, I", Pacific J. Math. **73**, 307–327 (1977), Proposition 5, p. 321) that if G is a second countable, locally compact nilpotent group, then every irreducible unitary representation of G is weakly equivalent to a monomial representation. Indeed, let $\pi \in \hat{G}$. One may as well assume π is faithful on the center Z of G. Let A be a maximal abelian subgroup of $Z^{(2)}(G)$. One quickly checks that if Q is the quasi-orbit in $\hat{A}$ determined by π, then every point in Q has the same stabilizer H in G, where H is the centralizer of A in G. Thus π is weakly equivalent to a representation induced from H (Theorem 4.3). But H has shorter nilpotent length than G, since $Z^{(2)}(G) \cap H = A$ (recall A is maximal abelian in $Z^{(2)}(G)) \subseteq Z(H)$. So induction on the length of the central series finishes the argument. (Note: the above is just Howe's proof restated, not a new argument. But the proof of the general Effros-Hahn conjecture makes it much clearer where Howe's facts about weak containment are coming from.)