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Is Gauss Quadrature Optimal for Analytic Functions?

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Summary. We consider the problem of optimal quadratures for integrands $f\colon [-1,1]\to\mathbb{R}$ which have an analytic extension $\bar{f}$ to an open disk D_r of radius r about the origin such that $|\bar{f}|\leqq 1$ on $\bar{D}_r$. If $r=1$, we show that the penalty for sampling the integrand at zeros of the Legendre polynomial of degree n rather than at optimal points, tends to infinity with n. In particular there is an "infinite" penalty for using Gauss quadrature. On the other hand, if $r>1$, Gauss quadrature is almost optimal. These results hold for both the worst-case and asymptotic settings.

Subject Classifications: AMS(MOS): 65D30; CR: G1.4.

1. Introduction

This paper deals with approximations to $\int_{-1}^{1} f(x)\,dx$ by algorithms whose sole knowledge of f consists of samples at points from the interval $[-1,1]$. We assume that integrands belong to the class $F(D_r)$ of functions $f\colon [-1,1]\to\mathbb{R}$ having an analytic extension to $D_r=\{z\in\mathbb{C}\colon |z|<r\}$ whose modulus is bounded by unity on $\bar{D}_r$.

One of the best-known methods of approximating such integrals is Gauss quadrature. This algorithm is derived by requiring the approximation to be exact for polynomials of as high a degree as possible. There are many papers dealing with the error analysis of Gauss quadratures, see for instance [1, 6, 7, 8, 9, 12]. In particular, sharp error estimates are known for analytic functions on the ellipse with foci ± 1 and the sum of semi-axes q, where $q>1$. The behavior of Gauss quadrature for the class $F(D_r)$ with $r>1$ follows easily. We know no previous results for the class $F(D_1)$.

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The goal of this paper is not the study of Gauss quadrature per se. We are actually interested in the intrinsic error of Gauss information, i.e., the minimal error among all algorithms which evaluate the integrand at Gauss nodes. (There is no a priori reason to believe that Gauss quadrature uses Gauss information optimally, i.e., that the error of Gauss quadrature equals the intrinsic error of Gauss information.) Our aim is to compare the intrinsic error of Gauss information using n nodes to the n-th minimal error, i.e., the minimal error among *all* algorithms which evaluate the integrand at n points. When the former is worse than the latter, this tells us that not only is Gauss quadrature bad, but it is bad precisely because *any* algorithm using Gauss information is bad.

A number of papers [3, 10, 11] show that for large values of r Gauss quadrature is almost optimal. Note that as r increases the class $F(D_r)$ looks more like a class of polynomials, while $F(D_\infty)$ consists of constants. In this paper we address the question:

Is Gauss quadrature close to optimal for *all* r?

We pursue our results in both a worst-case and an asymptotic setting [14, 15]. When $r=1$, there is no "breathing room" between the interval of integration and the region of analyticity of integrands. We first consider the case $r=1$ in the worst-case setting. Due to [4, 5], the n-th minimal worst-case error is roughly $\exp(-c\sqrt{n})$ for some $c>0$. On the other hand, we show that in the worst-case setting the error of *any* algorithm using n Gauss nodes is at least about n^{-2}.

We next consider the case $r=1$ in the asymptotic setting. We apply general results of [15] to show that the n-th minimal asymptotic error is roughly $\exp(-c\sqrt{n})$ for some $c>0$. We prove that the minimal (asymptotic) error of any algorithm using n Gauss nodes is roughly n^{-2}.

Hence in both settings there is an unbounded penalty for using Gauss nodes when $r=1$. We stress that this is a bad property of Gauss nodes rather than Gauss quadrature. That is, this holds for *any* algorithm using Gauss nodes.

We finally discuss the case $r>1$ which allows some "breathing room" between the interval of integration and the region of analyticity of integrands. We show that in both the worst-case and asymptotic settings the n-th minimal error is roughly $\exp(-cn)$ for some $c>0$ and that Gauss quadrature is almost optimal.

Hence optimality of Gauss quadrature for analytic functions requires a strong assumption on analyticity of integrands f, i.e., $f\in F(D_r)$ with $r>1$. The integration problem for the class $F(D_r)$ with $r>1$ is essentially easier than the corresponding problem for the class $F(D_1)$.

2. How bad is Gauss Quadrature when $r=1$?

2.1. Worst-Case Setting

We want to approximate $\int_{-1}^{1} f(x)\,dx$ for $f\in F=F(D_1)$ using the following *information* about f

$$N_n(f)=[f(x_1),f(x_2),\ldots,f(x_n)]$$

where $x_i \in [-1,1]$. By an *algorithm* φ we mean amy mapping such that φ: $N_n(F) \to \mathbb{R}$. If φ is *linear*, i.e., $\varphi(N_n(f)) = \sum_{k=1}^{n} a_k f(x_k)$ for some a_k, we will refer to φ as a *quadrature rule* (or, more briefly, a *quadrature*).

In the *worst-case setting* we measure the *error* $e(N_n, \varphi)$ of an algorithm φ using N_n by

$$e(N_n, \varphi) := \sup \left\{ \left| \int_{-1}^{1} f(x)\,dx - \varphi(N_n(f)) \right| : f \in F \right\}.$$

It is well-known, see [2] and also [14, Thm. 3.1, p. 54], that

$$e(N_n) := \inf_{\varphi} e(N_n, \varphi) = \sup \left\{ \int_{-1}^{1} f(x)\,dx : f \in F, N_n(f) = 0 \right\} \tag{1}$$

and the infimum in (1) is attained for a linear algorithm (quadrature). Bojanov [4, 5] proved that

$$e(n) := \inf_{x_1, x_2, \ldots, x_n} e(N_n) \geqq \exp(-5\pi\sqrt{n/2}) \tag{2}$$

and found information N_n^* and a quadrature Q_n^*,

$$\begin{aligned} N_n^*(f) &= [f(x_1^*), f(x_2^*), \ldots, f(x_n^*)], \\ Q_n^*(N_n^*(f)) &= \sum_{k=1}^{n} a_k^* f(x_k^*), \end{aligned} \tag{3}$$

such that

$$e(N_n^*, Q_n^*) = \exp(-\pi\sqrt{n/2}). \tag{4}$$

Due to [2], see also [14, Thm. 7.1, p. 48], the estimate (2) is valid even if the points x_i are chosen *adaptively*, i.e., x_i is allowed to depend on $f(x_1)$, $f(x_2)$, ..., $f(x_{i-1})$. Hence N_n^* and Q_n^* are almost optimal, i.e., the choice of x_k^* and a_k^* nearly minimizes the error of any algorithm using n samples.

We compare the almost optimal information N_n^* with *Gauss* information N_n^G,

$$N_n^G(f) = [f(\zeta_1), f(\zeta_2), \ldots, f(\zeta_n)], \tag{5}$$

where ζ_k is the k-th zero of the Legendre polynomial P_n. We now prove that the choice of nodes ζ_k is very poor.

Theorem 1.

$$e(N_n^G) = \Theta(n^{-2}). \quad \square$$

Here we use the Θ-notation, which may be thought of as a "two-sided" O-notation. That is, $f = \Theta(g)$ iff $f = O(g)$ and $g = O(f)$.

Proof. Let $h(z) := \prod_{i=1}^{n} \frac{z - \zeta_i}{1 - z\zeta_i} \left(= \frac{P_n(z)}{z^n P_n(1/z)} \right)$. Note that $h \in F$, $N_n^G(h) = 0$ and $|h(z)| = 1$ for $|z| = 1$. We first prove that

$$I_n := \int_{-1}^{1} |h(x)|^2\,dx \leqq e(N_n^G) \leqq \int_{-1}^{1} |h(x)|\,dx =: J_n. \tag{6}$$

Indeed, let $N(f):=[f(\zeta_1),f'(\zeta_1),\ldots,f(\zeta_n),f'(\zeta_n)]$. Obviously $e(N_n^G)\geqq e(N)$. Due to [4] we have $e(N)=\int_{-1}^{1}|h(x)|^2\,dx$ which proves the left inequality of (6). Choose now an arbitrary $g\in F$ such that $N_n^G(g)=0$. Since the function $g(z)/h(z)$ is analytic in D_1, by the maximum modulus principle we get

$$\sup_{z\in D_1}|g(z)/h(z)|=\sup_{|z|=1}|g(z)/h(z)|=\sup_{|z|=1}|g(z)|\leqq 1.$$

Hence $g(x)\leqq|g(x)|\leqq|h(x)|$ for every $x\in[-1,1]$. Since g is arbitrary, this and (1) yield $e(N_n^G)\leqq J_n$. The proof of (6) is completed.

By

$$P_n(\xi)=\pi^{-1}\int_0^{\pi}(\xi+\sqrt{\xi^2-1}\cos\theta)^n\,d\theta$$

(see [13, p. 87]), we get

$$\begin{aligned} x^n P_n(1/x) &= \pi^{-1}\int_0^{\pi/2}[(1+t\cos\theta)^n+(1-t\cos\theta)^n]\,d\theta \\ &= \pi^{-1}\sum_{k=0}^{n}\binom{n}{k}[t^k+(-t)^k]\int_0^{\pi/2}\cos^k\theta\,d\theta \end{aligned}$$

where $t=\sqrt{1-x^2}$ and $x\in[0,1]$. For $n\geqq 4$ we have

$$\begin{aligned} \left(1+\frac{(n-2)(n-3)}{8}t^2\right)^2 &\leqq 1+\frac{n(n-1)}{4}t^2+\frac{n(n-1)(n-2)(n-3)}{64}t^4 \\ &\leqq \pi^{-1}\sum_{k=0}^{n}\binom{n}{k}[t^k+(-t)^k]\int_0^{\pi/2}\cos^k\theta\,d\theta\leqq\sum_{k=0}^{n}\binom{n}{k}t^k \\ &= (1+t)^n\leqq\exp(nt). \end{aligned}$$

We thus proved

$$\left[1+\frac{(n-2)(n-3)}{8}(1-x^2)\right]^2\leqq x^n P_n(1/x)\leqq\exp(n\sqrt{1-x^2}). \tag{7}$$

For $|x|\leqq 1$ we have $P_n(x)=1+P_n'(\xi(x))(x-1)$ where $\xi(x)\in[-1,1]$. Due to Markov's inequality we have for $|\zeta|\leqq 1$, $|P_n'(\zeta)|\leqq n^2\max_{|x|\leqq 1}|P_n(x)|=n^2$. Therefore for $|x|\leqq 1$,

$$|P_n(x)|\geqq 1-n^2(1-x). \tag{8}$$

We are now ready to estimate I_n and J_n. We first show that $I_n\geqq c_1 n^{-2}$ for a positive c_1 which does not depend on n. By the right inequality of (7) we get

$$\begin{aligned} I_n=2\int_0^1\left[\frac{P_n(x)}{x^n P_n(1/x)}\right]^2 dx &\geqq 2\int_0^1|P_n(x)|^2\exp(-2n\sqrt{1-x^2})\,dx \\ &\geqq 2\int_a^1|P_n(x)|^2\exp(-2n\sqrt{1-x^2})\,dx=:\tilde{I}_n, \end{aligned}$$

where $a=1-1/(2n^2)$.

Since $\exp(-2n\sqrt{1-x^2})$ monotonically increases, due to (8) we have

$$\tilde{I}_n \geqq 2\exp(-2n\sqrt{1-a^2})\int_a^1 (1-n^2(1-x))^2\,dx$$

$$\geqq 2\exp(-2)\frac{1}{3n^2}[1-(1-n^2(1-a))^3] = \frac{7\exp(-2)}{12n^2}.$$

Thus

$$I_n \geqq \tfrac{7}{12}\exp(-2)n^{-2}.$$

To complete the proof it is enough to show that $J_n \leqq c_2 n^{-2}$ where c_2 does not depend on n. By the left inequality of (7), for $w=(n-2)(n-3)/8$ we get

$$J_n = 2\int_0^1 \left|\frac{P_n(x)}{x^n P_n(1/x)}\right| dx \leqq 2\int_0^1 \frac{dx}{[1+w(1-x^2)]^2}$$

$$= 2(1+w)^{-2}\int_0^1 \frac{dx}{\left[1-\frac{w}{1+w}x^2\right]^2} \leqq 2(1+w)^{-2}\int_0^1 \frac{dx}{\left[1-\frac{w}{1+w}x\right]^2}$$

$$= 2(1+w)^{-1} = \frac{16}{n^2-5n+14}.$$

This gives the desired inequality and completes the proof. □

To understand the bad properties of Gauss information, suppose one needs to find an ε-approximation, i.e., to compute $I=I(f)$ such that $\left|\int_0^1 f(x)\,dx - I(f)\right| \leqq \varepsilon$ for all f from F. To get $I(f)$ we use n samples of f. From (2) and (4) we conclude that the minimal number of samples n has to be about $\ln^2 1/\varepsilon$. Using the information N_n^* of (3) with $n=\lceil 0.08\pi^{-2}\ln^2 1/\varepsilon\rceil$, the quadrature $Q_n^*(N_n^*(f))$ of (3) yields an ε-approximation since $e(N_n^*, Q_n^*) \leqq \varepsilon$. The cost of Q_n^* is proportional to n. From this we conclude that the ε-complexity, i.e., the minimal cost of computing an ε-approximation is given by

$$\mathrm{comp}(\varepsilon) = \Theta(\ln^2 1/\varepsilon)$$

and Q_n^* is an almost optimal complexity algorithm.

Suppose now one wants to find an ε-approximation using n Gauss nodes. Then, due to Theorem 1, n has to be of order $\varepsilon^{-1/2}$ and the ε-complexity (minimal cost) of Gauss information is given by

$$\mathrm{comp}_G(\varepsilon) = \Theta(\varepsilon^{-1/2}).$$

Let $\mathrm{pen}(\varepsilon) := \mathrm{comp}_G(\varepsilon)/\mathrm{comp}(\varepsilon)$ be the penalty of using Gauss information instead of the optimal one. From this we get

Theorem 2.

$$\mathrm{pen}(\varepsilon) = \Theta(\varepsilon^{-1/2}\ln^{-2} 1/\varepsilon),$$

so that

$$\lim_{\varepsilon\to 0}\mathrm{pen}(\varepsilon) = +\infty. \quad \square$$

2.2. Asymptotic Setting

In the worst-case setting the error of an algorithm is defined for fixed information N_n and for the worst integrand f. In some situations we prefer to fix f and apply to it information N_n with n tending to infinity. This is called the *asymptotic setting*. In this setting, *information* is an infinite sequence

$$\bar{N}(f)=[f(x_1),f(x_2),\ldots,f(x_k),\ldots].$$

We stress that the point x_i can be chosen *adaptively*, i.e., x_i can be an arbitrary function of $f(x_1)$, $f(x_2)$, ..., $f(x_{i-1})$. By an *algorithm* $\bar{\varphi}$ using $\bar{N}$ we now mean a sequence $\bar{\varphi}=\{\varphi_n\}_{n=1}^{\infty}$ where φ_n uses $N_n(f)=[f(x_1),\ldots,f(x_n)]$, i.e., φ_n: $N_n(F)\to\mathbb{R}$. The n-th *error of* $\bar{\varphi}$ *at* f is defined as

$$e_n(\bar{\varphi},f):=\left|\int_{-1}^{1} f(x)\,dx-\varphi_n(N_n(f))\right|.$$

In the asymptotic setting we wish to choose an algorithm $\bar{\varphi}$ as well as the nodes x_k for which the sequence $e_n(\bar{\varphi},f)$ goes to zero as fast as possible for all f from F.

Recently, Trojan [15] showed a surprising relation between the worst-case and asymptotic settings. For the integration problem his results can be summarized as follows. (The quantities $e(N_n)$ and $e(n)$ are defined as in Sect. 2.1.)

Given $f\in F$, let N_n^f denote the following *nonadaptive* information

$$N_n^f(g)=[g(\hat{x}_1),g(\hat{x}_2),\ldots,g(\hat{x}_n)],$$

where $\hat{x}_1=x_1$ and $\hat{x}_i=\hat{x}_i(f(\hat{x}_1),\ldots,f(\hat{x}_{i-1}))$ for $i=2,3,\ldots,n$.

(i) For any information $\bar{N}$, any algorithm $\bar{\varphi}$ using $\bar{N}$ and any nonnegative sequence $\{\delta_n\}_{n=1}^{\infty}$ converging to zero, the set F_0 of f for which

$$e_n(\bar{\varphi},f)=o(\delta_n e(N_n^f))$$

is boundary.

(ii) There exist information $\bar{N}^*$ and an algorithm $\bar{\varphi}^*$ using $\bar{N}^*$ such that

$$e_n(\bar{\varphi}^*,f)\leqq e(\lfloor n/4\rfloor).$$

Remark. The statement in (i) that F_0 is boundary means that $\overline{F-F_0}=F$. That is, for any nonnegative sequence $\{\delta_n\}_{n=1}^{\infty}$ converging to zero, the set of f for which

$$\limsup_{n\to\infty}\frac{e_n(\bar{\varphi},f)}{\delta_n e(N_n^f)}>0$$

is dense in F. □

From (i) with $\delta_n=\exp(-\sqrt{n})$ and from (2) it follows that for arbitrary information $\bar{N}$ and an arbitrary algorithm $\bar{\varphi}$ using $\bar{N}$ the set of f for which

$$e_n(\bar{\varphi},f)=o\left(\exp\left(-\left(1+\frac{5\pi}{\sqrt{2}}\right)\sqrt{n}\right)\right) \tag{9}$$

is boundary.

Let N_n^* and Q_n^* be given by (3). Define information

$$\bar{N}^*(f) := [N_1^*(f), N_2^*(f), N_4^*(f), \ldots, N_{2^k}^*(f), \ldots] \tag{10}$$

and the algorithm $\bar{\varphi}^* = \{\varphi_n^*\}$ using $\bar{N}^*$ as

$$\varphi_n^*(N_n(f)) := Q_{2^k-1}^*(N_{2^k-1}^*(f)), \qquad k = \lfloor \log_2(n+1) \rfloor, \tag{11}$$

where N_n consists of the first n samples of $\bar{N}^*$. From (4) we get

$$e_n(\bar{\varphi}^*, f) \leqq \exp(-\pi\sqrt{(n+1)/8}), \tag{12}$$

for all f from F. Due to (9), $\bar{N}^*$ and $\bar{\varphi}^*$ are almost optimal in the asymptotic setting.

We now compare $\bar{N}^*$ to corresponding Gauss information,

$$\bar{N}^G(f) := [N_1^G(f), N_2^G(f), N_4^G(f), \ldots, N_{2^k}^G(f), \ldots] \tag{13}$$

where $N_{2^k}^G$ is given by (5). We prove that Gauss information is also very poor in the asymptotic setting.

Theorem 3. *For any algorithm $\bar{\varphi}$ using $\bar{N}^G$ and any nonnegative sequence $\{\delta_n\}$ converging to zero the set of f for which $e_n(\bar{\varphi}, f) = o(n^{-2}\delta_n)$ is boundary.* □

Proof. For each positive integer n, let $k = \lfloor \log_2(n+1) \rfloor$ and

$$\tilde{N}_n^G(f) := [N_1^G(f), N_2^G(f), \ldots, N_{2^k-1}^G(f)].$$

We first estimate $e(\tilde{N}_n^G)$. The same arguments as in the proof of (6) with

$$h(z) = \prod_{j=0}^{k-1} \left[\frac{P_{2^j}(z)}{z^{2^j} P_{2^j}(1/z)}\right]^2$$

lead to the inequality

$$e(\tilde{N}_n^G) \geqq \int_{-1}^{1} h^2(x)\,dx =: I_k. \tag{14}$$

Let $b := \sqrt{1-2^{-2k+1}}$. Due to the right hand side of (7), we have for $x \in [b, 1]$

$$\left[\prod_{j=0}^{k-1} x^{2^j} P_{2^j}(1/x)\right]^2 \leqq \exp\left(2\sum_{j=0}^{k-1} 2^j \sqrt{1-b^2}\right) \leqq \exp(2\sqrt{2}).$$

From (8) for $x \in [b, 1]$ we get

$$\prod_{j=0}^{k-1} P_{2^j}^2(x) \geqq \prod_{j=0}^{k-1} [1 - 2^{2j}(1-\sqrt{1-2^{-2k+1}})]^2 \geqq \prod_{j=0}^{k-1} (1-2^{2j-2k+1})^2$$

$$= \prod_{j=1}^{k-1} (1-2^{-2j-1})^2 \geqq \prod_{j=0}^{\infty} (1-2^{-2j-1})^2 =: c.$$

Observe that c exists and is positive. Hence $h^2(x) \geqq c\exp(-2\sqrt{2})$ for $b \leqq x \leqq 1$. We now estimate I_k.

$$I_k = 2\int_0^1 h^2(x)\,dx \geqq 2\int_b^1 h^2(x)\,dx \geqq 2(1-b)c\exp(-2\sqrt{2})$$
$$= 2(1-\sqrt{1-2^{-2k+1}})c\exp(-2\sqrt{2}) \geqq 2^{-2k+1}c\exp(-2\sqrt{2}).$$

Since $2^{-2k} \geqq n^{-2}/4$, (14) implies

$$e(\tilde{N}_n^G) \geqq c_1 n^{-2}$$

for a positive c_1 which does not depend on n. Note that this estimate is sharp. Indeed, Theorem 1 yields $e(N_{2^k-1}^G) = \Theta(n^{-2})$. Obviously $e(N_{2^k-1}^G) \geqq e(\tilde{N}_n^G)$. Therefore

$$e(\tilde{N}_n^G) = \Theta(n^{-2}).$$

This and (i) complete the proof. □

Theorem 3 states that the speed of convergence of algorithms using Gauss information is at most n^{-2} whereas (9) and (12) state that the optimal speed is roughly $\exp(-c\sqrt{n})$ where $c>0$.

We now show the superiority of the optimal algorithm. Let $s_1(n) = n^{-2}$ and $s_2(n) = \exp(-c\sqrt{n})$. Assume that one wants to choose the minimal n such that $s_i(n) \leqq \varepsilon$. Then for the function s_1 we have $n = n_1(\varepsilon) = \Theta(\varepsilon^{-1/2})$ whereas for the function s_2 we have $n = n_2(\varepsilon) = \Theta(ln^2\, 1/\varepsilon)$. The penalty function

$$\operatorname{pen}(\varepsilon) := \frac{n_1(\varepsilon)}{n_2(\varepsilon)} = \Theta(\varepsilon^{-1/2} \ln^{-2} 1/\varepsilon)$$

goes to $+\infty$ as ε goes to zero.

3. Gauss Quadrature is Almost Optimal when $r>1$

In this section we show that Gauss quadrature is almost optimal for the class $F(D_r)$ with $r>1$ in both the worst-case and asymptotic settings.

We begin with the worst-case setting. The quantities $e(N_n, \varphi)$, $e(N_n)$, and $e(n)$ are defined as in Sect. 2.1, except that now $F = F(D_r)$. To show that Gauss quadrature is almost optimal, we need some auxiliary results for the integration problem for a different class of integrands. Let E_q be an ellipse whose foci are ± 1 and sum of semi-axes is $q>1$. By $F(E_q)$ we mean the set of functions f: $[-1,1] \rightarrow \mathbb{R}$ having an analytic extension $\bar{f}$ to E_q such that $|\bar{f}| \leqq 1$ on $\bar{E}_q$. For the class $F(E_q)$ Bakhvalov [1] proved (see also [6, 7, 8, 12]) that the minimal worst-case error of algorithms using n samples of f is $\Theta(q^{-2n})$. Furthermore he showed that the worst-case error of Gauss quadrature is of order q^{-2n}. Thus Gauss quadrature is almost optimal in the worst-case setting for the class $F(E_q)$.

We shall use Bakhvalov's results to show that Gauss quadrature is also almost optimal for $F(D_r)$ with $r>1$. Let $q_1 = r + \sqrt{r^2+1}$ and $q_2 = r + \sqrt{r^2-1}$.

Since $E_{q_2} \subset D_r \subset E_{q_1}$, we have $F(E_{q_1}) \subset F(D_r) \subset F(E_{q_2})$. Therefore there exist positive constants c_1 and c_2, independent of n, such that

$$c_1 q_1^{-2n} \leqq e(n) \leqq e(N_n^G, G_n) \leqq c_2 q_2^{-2n}. \tag{15}$$

Here N_n^G, G_n are Gauss information and Gauss quadrature respectively. Observe that $q_2/q_1 < 1$ and for large r the ratio q_2/q_1 is close to one. Since $r > 1$, $(q_2/q_1)^{-2n}$ goes to zero as n tends to infinity. Thus there is a large gap in the bounds of the estimate (15). In contrast to this, the ε-complexity is known to within a constant. (The ε-complexity is defined as at the end of Sect. 2.1, except that now $F = F(D_r)$.) Since $\text{comp}(\varepsilon) = \Theta(\inf\{n : e(n) \leqq \varepsilon\})$, (15) yields

Theorem 4. *For the class $F(D_r)$ with $r > 1$ the ε-complexity of the integration problem in the worst-case setting is*

$$\text{comp}(\varepsilon) = \Theta(\ln 1/\varepsilon).$$

Furthermore, Gauss quadrature G_n with $n = \lceil \ln(c_2/\varepsilon)/(2 \ln q_2) \rceil$ yields an ε-approximation with almost minimal complexity. □

We now turn to the asymptotic setting. From Trojan's result (i) with $F = F(D_r)$ and from (15) we get

Theorem 5. *For arbitrary information $\bar{N}$, any algorithm $\bar{\varphi}$ using $\bar{N}$ and any nonnegative sequence $\{\delta_n\}$ converging to zero, the set of f for which*

$$e_n(\bar{\varphi}, f) = o(\delta_n q_1^{-2n})$$

is boundary.

Furthermore, for the algorithm $\bar{\varphi}^G = \{\varphi_n^G\}$ defined as the following sequence of Gauss quadratures

$$\varphi_n^G(N_n(f)) := G_{2^k-1}(N_{2^k-1}(f)), \qquad k = \lfloor \log_2(n+1) \rfloor,$$

where N_n consists of n samples of $\bar{N}^G$ from (12), *we have*

$$e_n(\bar{\varphi}^G, f) = O(q_2^{-2n}), \qquad \forall f \in F(D_r). \quad \square$$

Theorem 5 states that the speed of convergence of Gauss quadratures is at least q_2^{-2n} whereas the speed of convergence of *any* algorithm is at most q_1^{-2n}. We stress that there is no practical difference between the functions q_1^{-2n} and q_2^{-2n}. Indeed, suppose we choose the minimal n such that $q_i^{-2n} \leqq \varepsilon$. Then

$$n = n_i(\varepsilon) = \left\lceil \frac{\ln 1/\varepsilon}{2 \ln 1/q_i} \right\rceil = \Theta(\ln 1/\varepsilon).$$

Since $n_1(\varepsilon)$ is of the same order as $n_2(\varepsilon)$, we have to perform roughly the same number of function evaluation to make the error smaller than ε, whether the speed of convergence is q_1^{-2n} or q_2^{-2n}. This establishes that Gauss quadrature is almost optimal in the asymptotic setting for the class $F(D_r)$ with $r > 1$.

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