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Meromorphic Mappings onto Compact Complex Spaces of General Type

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To the memory of Yotaro Tsukamoto

1. Introduction

The main theorem of this paper is the following:

Theorem 1. *Let X be a Moišezon space and Y a compact complex space of general type. Then the set of surjective meromorphic mappings of X onto Y is finite.*[1, 2]

We recall that a compact complex space X is called a *Moišezon space* if the transcendence degree of its field of meromorphic functions is equal to the dimension of X. A compact complex manifold Y of dimension n is said to be of *general type* if

$$\sup_{m\to+\infty} \lim \frac{1}{m^n} \dim \Gamma(K_Y^m) > 0,$$

where K_Y is the canonical line bundle of Y and $\Gamma(K_Y^m)$ is the space of holomorphic sections of the line bundle K_Y^m. A compact complex space Y is said to be of general type if any (and hence every) non-singular model of Y is of general type.

If $\dim Y=1$, Theorem 1 reduces to the classical theorem of de Franchis [3; p. 139]. Thus our result settles one of the conjectures of Lang [9]. In the theorem of de Franchis, Y is a curve of genus at least 2. The condition on Y in higher dimension suggested by Lang is that Y be hyperbolic. We assume instead that Y is of general type. It is very likely that a compact hyperbolic space is necessarily of general type.

The proof, easily reduced to the case where both X and Y are non-singular, projective-algebraic, is by a combination of arguments in [2, 3, 6, 12].

Theorem 1 implies that the group of bimeromorphic automorphisms of a compact complex space of general type is finite (see [2, 5] for a direct transcendental proof and [10] for a direct algebraic proof). It is, of course, desirable to find an algebraic proof of Theorem 1.

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1 Throughout the paper, X and Y denote reduced irreducible complex spaces.
2 It has been pointed out by Professor Y. Namikawa that Theorem 1 holds for any compact complex space X which is not necessarily a Moišezon space. See the argument given at the end of the paper.

Improving on the result in [6], we show the following

Theorem 2. *Let X be a complex space and A a complex subspace of X. Let Y be an n-dimensional compact complex space of general type. Then every meromorphic map $f: X-A \to Y$ of maximal rank n extends to a meromorphic map $f: X \to Y$.*

From Theorems 1 and 2, we obtain

Corollary. *Let X and Y be as in Theorem 1. Let A be a complex subspace of X. Then the set of meromorphic maps $f: X-A \to Y$ of maximal rank n (where $n = \dim Y$) is finite.*

We say that $f: X-A \to Y$ is of rank n if its differential is of rank n at some regular point of $X-A$ where f is holomorphic. Theorem 1 is proved in §§ 2~6. The proof of Theorem 2 is indicated in § 7.

2. Reduction to the Case where X and Y are Non-Singular, Projective Algebraic

Every Moišezon space X is bimeromorphic to a projective algebraic variety (see Theorem 1 of [11]). By Hironaka's theorem on resolutions of singularities, X is bimeromorphic to a non-singular projective algebraic variety.

Let Y be a compact complex manifold of general type. Let $\Gamma(K_Y^m)^*$ be the dual space of $\Gamma(K_Y^m)$ and $P(\Gamma(K_Y^m)^*)$ the projective space of lines in $\Gamma(K_Y^m)^*$. The natural map $f_m: Y \to P(\Gamma(K_Y^m)^*)$ is, in general, meromorphic and its image is a variety of dimension $n (= \dim Y)$ for some $m > 0$, (see [1] for details). By pulling back meromorphic functions of $f_m(Y)$, we obtain n algebraically independent meromorphic functions on Y. We have shown that if Y is of general type, it is a Moišezon space. Hence, Y is bimeromorphic to a non-singular projective algebraic variety.

3. Schwarz Lemma

In this section we same that X is a compact complex manifold of dimension n and Y is an n-dimensional projective algebraic manifold of general type. As before, K_Y denotes the canonical line bundle of Y.

Then we have a positive integer m, a subspace W of $\Gamma(K_Y^m)$ and a line bundle L over Y with a holomorphic section $a \in \Gamma(L)$ such that

(1) $L^{-1} K_Y^m$ is very ample,

(2) $s \in \Gamma(L^{-1} K_Y^m) \to a s \in \Gamma(K_Y^m)$ defines an isomorphism of $\Gamma(L^{-1} K_Y^m)$ onto W.

For the proof of this assertion, see [7, 6]. Let $s_0, s_1, \ldots, s_N$ be a basis for $\Gamma(L^{-1} K_Y^m)$ and $t_0, t_1, \ldots, t_N$ the corresponding basis for W. Thus, $t_i = a s_i$ for $i = 0, 1, \ldots, N$. The common zeroes of $t_0, t_1, \ldots, t_N$ is equal to the zeroes of the section a, which may or may not be empty.

We define a volume element ω_Y on Y by

$$\omega_Y := \left(\sum |t_j \wedge \bar{t}_j|\right)^{1/m}.$$

This formal expression stands for the following $2n$-form. In terms of a local coordinate system $y^1, \ldots, y^n$ of Y, write

$$t_j = h_j(dy^1 \wedge \cdots \wedge dy^n)^m,$$

where h_j is a locally defined holomorphic function. Then

$$\omega_Y = (\sum |h_j|^2)^{1/m} (\sqrt{-1})^n dy^1 \wedge d\bar{y}^1 \wedge \cdots \wedge dy^n \wedge d\bar{y}^n$$

is a globally defined $2n$-form on Y vanishing at the zeroes of the section $a \in \Gamma(L)$ and positive elsewhere.

Considering L locally as product $U \times \mathbf{C}$ where U is a small open subset of Y, we can represent the section a by a holomorphic function a_U on U. Then the function

$$\frac{H}{|a_U|^2}, \quad \text{where } H := \sum |h_j|^2,$$

is everywhere positive since the zeroes of H cancel out with the zeroes of $|a_U|^2$. Since $\partial\bar{\partial}(\log H) = \partial\bar{\partial}(\log H/|a_U|^2)$, the Ricci tensor

$$R_{\alpha\bar{\beta}} = -\frac{1}{m}\partial^2 \log H/\partial y^\alpha \partial \bar{y}^\beta$$

is well defined everywhere on Y (even at the zeroes of H) and is negative definite. (The fact that the Ricci tensor is negative definite follows from the condition that $L^{-1} K_Y^m$ is very ample. In fact, from the definition of $R_{\alpha\bar{\beta}}$ above one sees immediately that $-R_{\alpha\bar{\beta}}$ is the metric tensor induced from the Fubini-Study metric of $P_N(\mathbf{C})$ by the imbedding $Y \to P_N(\mathbf{C})$ defined by $(s_0, s_1, \ldots, s_N)$.) We may summarize this by saying that the Ricci form associated to the volume form ω_Y is negative-definite everywhere.

Lemma 1. *Let ω be the Poincaré-Bergman volume element of the unit polydisk Δ^n of dimension n. Then there exists a positive constant c such that*

$$f^*\omega_Y \leqq c \cdot \omega \quad \textit{for every meromorphic map } f\colon \Delta^n \to Y.$$

Proof. This is an equi-dimensional generalization of the Schwarz-Pick-Ahlfors lemma. Following Yau [12], we observe that $f^* t_j$, which is holomorphic outside the singularity set (i.e., the set of points of indeterminacy) $A \subset \Delta^n$, extends to a holomorphic "form" on Δ^n by the theorem of Hartogs since codim $A \geqq 2$. It follows that $f^*\omega_Y$ extends also to a smooth $2n$-form on Δ^n. The usual proof of the equi-dimensional Schwarz lemma for holomorphic mappings can be applied to the function $u = f^*\omega_Y/\omega$ on Δ^n. See [3, 4] for details.

Lemma 2. *There exists an everywhere positive volume form ω_X on X such that*

$$f^*\omega_Y \leqq \omega_X \quad \textit{for every meromorphic map } f\colon X \to Y.$$

Proof. We cover X by a (locally) finite open cover $\{D_i\}$ such that each D_i is biholomorphic to the polydisk Δ^n. Let ω_i be the volume form on D_i corresponding to the volume form on Δ^n. Let $\{\rho_i\}$ be a partition of 1 subordinate to $\{D_i\}$. We set

$$\omega_X := \sum \rho_i \omega_i.$$

Then there is a positive constant c such that $f^* \omega_Y \leqq c \cdot \omega_X$ for every meromorphic map $f: X \to Y$. By normalizing ω_X, we may assume that $c=1$.

4. Compactness of the Set of Surjective Meromorphic Mappings

Throughout this section we assume again that X is a compact complex manifold of dimension n and Y is an n-dimensional projective algebraic manifold of general type. We set

$M(X, Y) :=$ the set of surjective meromorphic mappings of X onto Y.

Let W be the subspace of $\Gamma(K_Y^m)$ defined in § 3. As pointed out in the proof of Lemma, the theorem of Hartogs implies that every meromorphic mapping $f: X \to Y$ induces a linear map $f^*: W \to \Gamma(K_X^m)$.

In both W and $\Gamma(K_X^m)$ we define a norm $\| \ \|$, (see [2]).

$$\|s\|^2 := \int_X |s \wedge \bar{s}|^{1/m} \quad \text{for } s \in \Gamma(K_X^m),$$

$$\|t\|^2 := \int_Y |t \wedge \bar{t}|^{1/m} \quad \text{for } t \in W.$$

The symbolic expressions $|s \wedge \bar{s}|^{1/m}$ and $|t \wedge \bar{t}|^{1/m}$ should be interpreted in the same way as $(\sum |t_j \wedge \bar{t}_j|)^{1/m}$ in the definition of ω_Y; they are $2n$-forms on X and Y, respectively. Although these norms do not satisfy the convexity condition $\|s+s'\| \leqq \|s\| + \|s'\|$, we shall call them "norms" by abuse of language.

Lemma 3. *There is a positive constant c such that $\|f^* t\| \leqq c\|t\|$ for every $t \in W$ and every meromorphic mapping $f: X \to Y$.*

Proof. Since $\|\lambda t\| = |\lambda| \cdot \|t\|$ for every complex number λ, it suffices to show that there is a constant c such that $\|f^* t\| \leqq c$ whenever $\|t\| \leqq 1$. Let $t_0, t_1, \ldots, t_N$ be the basis for W chosen in § 3. Every t in W can be written uniquely as

$$t = \sum \lambda_j(t) \cdot t_j,$$

where $\lambda_0(t), \ldots, \lambda_N(t)$ are complex numbers which depend linearly on t. Let c' be the maximum of $\sum |\lambda_j(t)|^2$ as t runs through the compact subset $\|t\| \leqq 1$ of W. By Schwarz's inequality, we have

$$|t \wedge \bar{t}| \leqq c' \cdot \sum |t_j \wedge \bar{t}_j| \quad \text{on } Y \text{ if } \|t\| \leqq 1.$$

Hence,

$$|f^* t \wedge f^* \bar{t}| \leqq c' \cdot f^* \left(\sum |t_j \wedge \bar{t}_j|\right) \quad \text{on } X \text{ if } \|t\| \leqq 1.$$

Integrating the m-th roots of both sides, we obtain

$$\|f^* t\|^2 \leqq c'^{1/m} \int_X f^* \omega_Y \quad \text{if } \|t\| \leqq 1.$$

Because of Lemma 2, it suffices to set

$$c = c'^{1/m} \int_X \omega_X.$$

Lemma 4. *$\|f^* t\| \geqq \|t\|$ for every surjective $f \in M(X, Y)$ and $t \in W$.*

Proof. We have

$$\|f^* t\|^2 = \int_X f^* |t \wedge \bar{t}|^{1/m} = (\deg f) \int_Y |t \wedge \bar{t}|^{1/m} = (\deg f) \|t\|^2 \geqq \|t\|^2.$$

Two remarks are in order. Although the $2n$-form $|t \wedge \bar{t}|^{1/m}$ vanishes on a subvariety of Y, it defines a nonzero element of $H^{2n}(Y; \mathbf{R})$. The formula

$$\int_X f^* |t \wedge \bar{t}|^{1/m} = (\deg f) \int_Y |t \wedge \bar{t}|^{1/m},$$

where $\deg f$ denotes the mapping degree of f, is well known when f is smooth. But $\deg f$ can be defined even when f is meromorphic and the formula above is still valid. Let $\hat{X}$ be (a desingularization of) the graph of f and let $\hat{f}: \hat{X} \to Y$ be the holomorphic lift of f. Then we define $\deg f = \deg \hat{f}$. The meromorphic mapping f is surjective if and only if $\deg f \geqq 1$.

Lemma 5. *If $f \in M(X, Y)$, then the induced linear map $f^*: W \to \Gamma(K_X^m)$ is injective.*

Proof. Let t be a nonzero element of W and U an open subset of X in which f is regular and non-degenerate. Being a holomorphic section of K_Y^m, t cannot vanish identically on the open subset $f(U)$ of Y. Hence, $f^* t$ cannot vanish identically on U.

To state the next lemma, let

$S := $ the dual space of $\Gamma(K_X^m)$,

$T := $ the dual space of $W (\subset \Gamma(K_Y^m))$,

$i := $ the natural meromorphic mapping $X \to P(S)$,

$j := $ the natural imbedding $Y \to P(T)$,

where $P(S)$ and $P(T)$ denote the projective spaces consisting of complex lines through the origin in S and T respectively.

Let $f \in M(X, Y)$. Since $f^*: W \to \Gamma(K_X^m)$ is injective by Lemma 5, its dual map $f_*: S \to T$ is surjective. Let $\tilde{f}: P(S) \to P(T)$ be the meromorphic map induced by f_*. The proof of the following lemma is straightforward.

Lemma 6. *Let $f \in M(X, Y)$. Then the diagram*

$$\begin{array}{ccc} P(S) & \xrightarrow{\tilde{f}} & P(T) \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle j} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

Lemma 7. *Two distinct elements $f, g \in M(X, Y)$ give rise to distinct meromorphic mappings $\tilde{f}$, $\tilde{g}$ of $P(S)$ into $P(T)$.*

Proof. This follows from Lemma 6 and the fact that $j: Y \to P(T)$ is an imbedding.

Lemma 8. *$\{f^*; f \in M(X, Y)\}$ is a compact subset of* $\mathrm{Hom}(W, \Gamma(K_X^m))$.

Proof. Let $M^* = \{f^*; f \in \mathrm{M}(\mathrm{X}, \mathrm{Y})\}$. Let c be a fixed positive constant. Then the set

$$\Phi := \{\varphi \in \mathrm{Hom}(W, \Gamma(K_X^m)); \|t\| \leqq \|\varphi(t)\| \leqq c\|t\| \text{ for all } t \in W\}$$

is a compact subset of $\mathrm{Hom}(W, \Gamma(K_X^m))$. Let c be the constant given in Lemma 3. Then $M^* \subset \Phi$ by Lemmas 3 and 4. Let $\{f_k\}$ be an infinite sequence of elements in $M(X, Y)$. By taking a subsequence if necessary, we may assume that $\{f_k^*\}$ converges to an element φ in Φ since Φ is compact. Since φ satisfies $\|\varphi(t)\| \geqq \|t\|$, φ is an injective homomorphism $W \to \Gamma(K_X^m)$. The dual map of φ is a surjective homomorphism $S \to T$ and induces a meromorphic map $\tilde{\varphi}: P(S) \to P(T)$. Since $\varphi = \lim f_k^*$ and $j(Y)$ is closed in $P(T)$, it follows that $\tilde{\varphi} \circ i(x) \in j(Y)$ if x is a point in X (not belonging to the singularity set of $\tilde{\varphi} \circ i$). This shows that $\tilde{\varphi} \circ i: X \to P(T)$ is a meromorphic map of X into $j(Y)$. If we set $f = j^{-1} \circ \tilde{\varphi} \circ i$, then $\varphi = f^*$. Since f^* is in Φ, it satisfies $\|f^*(t)\| \geqq \|t\|$ and hence f is of maximal rank. This shows that f is in $M(X, Y)$.

Remark. The proof of Lemma 5 shows that if $f \in M(X, Y)$, then $f^*: \Gamma(K_Y^m) \to \Gamma(K_X^m)$ is injective. Hence, $\dim \Gamma(K_X^m) \geqq \dim \Gamma(K_Y^m)$ if $M(X, Y)$ is nonempty. This shows that X must be of general type if there is a surjective meromorphic mapping $f: X \to Y$.

The compactness of $M(X, Y)$ has been obtained by P. Kiernan independently.

5. Analytic and Algebraic Structures on $M(X, Y)$

Let S and T be the dual spaces of $\Gamma(K_X^m)$ and W, respectively, as in § 4. Let

$$H := \mathrm{Hom}(S, T).$$

Each element of the projective space $P(H)$ induces, in a natural way, a meromorphic map $P(S) \to P(T)$. If $f \in M(X, Y)$, then f_* is a nonzero element of H by Lemma 5. (In fact, f_* is a surjective map $S \to T$.) Let $\hat{f}$ be the element of $P(H)$ represented by $f_* \in H$. Let

$$\hat{M} := \{\hat{f}; f \in M(X, Y)\} \subset P(H).$$

By Lemma 7, $\hat{M}$ is in a natural one-to-one correspondence with $M(X, Y)$. By Lemma 8, $\hat{M}$ is a compact subset of $P(H)$.

Let Z be the set of elements of $P(H)$ such that the induced meromorphic mappings $P(S) \to P(T)$ send $i(X)$ into $j(Y)$. Then Z is an algebraic variety in $P(H)$. Let $\xi = (\dots, \xi_\alpha, \dots)$ and $\eta = (\dots, \eta_\lambda, \dots)$ be homogeneous coordinate systems for $P(S)$ and $P(T)$, respectively. Let $\zeta = (\dots, \zeta_\lambda^\alpha, \dots)$ be the naturally induced homogeneous coordinate system for $P(H)$. If J is the ideal of homogeneous polynomials $Q(\eta)$ defining the variety $j(Y) \subset P(T)$, then Z is defined by the set of homogeneous polynomials

$$\{Q_a(\zeta) := Q(\zeta \cdot \xi(a)); Q \in J \text{ and } a \in i(X)\},$$

where $\zeta \cdot \xi$ denotes the matrix multiplication of ζ and ξ.

Clearly, $\hat{M}$ is the subset of Z consisting of those elements which map $i(X)$ surjectively onto $j(Y)$. This shows that $\hat{M}$ is an open subset of Z. On the other hand, $\hat{M}$ is compact. Hence, $\hat{M}$ is an algebraic subvariety of $P(H)$.

Now, we are in a position to prove the main lemma.

Lemma 9. *Let X be an n-dimensional compact complex manifold and Y an n-dimensional projective algebraic manifold of general type. Then $M(X, Y)$ is finite.*

Proof. The C^*-bundle $H-\{0\} \to P(H)$, restricted to any algebraic subvariety of $P(H)$, is (even topologically) non-trivial unless the subvariety is 0-dimensional. Over the subvariety $\hat{M}$ this bundle has a section, namely $\hat{f} \in \hat{M} \to f_* \in H-\{0\}$. Hence, $\hat{M}$ is 0-dimensional, i.e., finite. Being in one-to-one correspondance with $\hat{M}$, $M(X, Y)$ is finite.

6. The Case $\dim X > \dim Y$

In this section we assume that X is a p-dimensional projective algebraic manifold and Y is an n-dimensional projective algebraic manifold of general type. We continue to denote the set of surjective meromorphic mappings $f\colon X \to Y$ by $M(X, Y)$.

If $p<n$, then $M(X, Y)$ is empty. The case $p=n$ was settled in the preceding section. We assume therefore that $p>n$. Assuming that $M(X, Y)$ is infinite, we choose a countable infinite subset $\{f_1, f_2, \ldots\}$ of $M(X, Y)$ and fix it once and for all.

For each point x of X, let $G_n(X)_x$ be the Grassmannian of n-planes in the tangent space $T_x(X)$. Then $G_n(X) := \bigcup_{x \in X} G_n(X)_x$ is a bundle over X whose standard fibre is the Grassmannian of n-planes in $\mathbf{C}^p$. We say that an n-plane $V \in G_n(X)_x$ is *transversal* (with respect to $\{f_1, f_2, \ldots\}$) if every f_j induces an isomorphism $V \to T_{f_j(x)}(Y)$.

Lemma 10. *There exists a transversal $V \in G_n(X)$ such that $Tf_1|V, Tf_2|V, \ldots$ are mutually distinct. (Here Tf_j denotes the differential of f_j and maps $T(X)$ into $T(Y)$.)*

Implicit in the statement is that $f_1, f_2, \ldots$ are all regular at x when $V \in G_n(X)_x$.

Proof. For each j, let S_j be the singularity set of the meromorphic mapping f_j. For each j, let N_j be the set of $V \in G_n(X)_x$ such that f_j is regular at x and $Tf_j\colon V \to T_{f_j(x)}(Y)$ is *not* an isomorphism. For each pair (i, j), let P_{ij} be the set of $V \in G_n(X)_x$ such that both f_i and f_j are regular at x and $Tf_i|V = Tf_j|V$. Clearly, $\pi^{-1}(S_j)$ is a subvariety of $G_n(X)$, where $\pi\colon G_n(X) \to X$ is the projection. The set N_j is a subvariety of $G_n(X) - \pi^{-1}(S_j)$. The set P_{ij} is a subvariety of $G_n(X) - \pi^{-1}(S_i \cup S_j)$. Hence

$$G := G_n(X) - ((\bigcup_j \pi^{-1}(S_j)) \cup (\bigcup_j N_j) \cup (\bigcup_{i,j} P_{ij}))$$

is dense in $G_n(X)$. (In fact, G is the intersection of countably many dense open subsets $G_n(X)-\pi^{-1}(S_j)$, $G_n(X)-(\pi^{-1}(S_j)\cup N_j)$ and $G_n(X)-\pi^{-1}(S_i\cup S_j)\cup P_{ij})$, $i,j=1,2,\ldots$.) Any element V in G satisfies the requirements of Lemma 10.

To complete the proof of Theorem 1, let $V\in G_n(X)_x$ be as in Lemma 10. Let X' be a subvariety of X passing through x such that $T_x(X')=V$. (This is where we use the assumption that X is projective algebraic.) By Lemma 10, $f_1|X', f_2|X', \ldots$ are mutually distinct elements of $M(X', Y')$. On the other hand, we know that $M(X', Y')$ is finite since $\dim X'=\dim Y'$. This contradiction arose from the assumption that $M(X, Y)$ is infinite.

7. Proof of Theorem 2

As in § 2, we may assume that Y is a non-singular projective algebraic manifold of general type and that X is non-singular. We may also assume that $f\colon X-A\to Y$ is holomorphic since the points of indeterminacy may be included in A.

We shall show that the proof can be reduced to the case where A is also non-singular. Let B be the singular locus of A so that $A-B$ is a non-singular subspace of $X-B$. Suppose that $f\colon X-A\to Y$ extends to a meromorphic map $f\colon X-B\to Y$. Fix an imbedding $Y\subset P_N(\mathbf{C})$ and let $w^0,\ldots,w^N$ be a homogeneous coordinate system for $P_N(\mathbf{C})$. The meromorphic functions $f^*(w^j/w^k)$ on $X-B$ extends to meromorphic functions on X since B has codimension at least 2 in X. Hence, f extends to a meromorphic map $f\colon X\to Y$.

Localizing f, we may further assume that X is a unit polydisk D^p in $\mathbf{C}^p$ and A is the polydisk $\{0\}\times D^{p-1}\subset D^p$. We denote the punctured disk $D-\{0\}$ by D^* so that $D^p-(\{0\}\times D^{p-1})=D^*\times D^{p-1}$. Since the second Cousin problem is solvable for the domain $D^*\times D^{p-1}$, we can lift the holomorphic map $f\colon D^*\times D^{p-1}\to Y\subset P_N(\mathbf{C})$ to a holomorphic map $\tilde{f}\colon D^*\times D^{p-1}\to\mathbf{C}^{N+1}$. Then $\tilde{f}$ is given by a system of $N+1$ functions $\varphi^0(z^1,\ldots,z^p),\ldots,\varphi^N(z^1,\ldots,z^p)$ holomorphic in $0<|z^1|<1$, $|z^2|<1,\ldots,|z^p|<1$.

Now we use the particular imbedding $Y\subset P_N(\mathbf{C})$ constructed in § 3. We recall that the imbedding was defined using a certain $(N+1)$-dimensional subspace W of $\Gamma(K_Y^m)$. It suffices to prove the following

Lemma 11. *Let*

$$\varphi^j(z^1,\ldots,z^p)=\sum_{h=-\infty}^{\infty}A_h^j(z^2,\ldots,z^p)(z^1)^h,\qquad j=0,1,\ldots,N,$$

be Laurent expansions with respect to the variable z^1 with holomorphic coefficients $A_h^j(z^2,\ldots,z^p)$. Then $A_h^j(z^2,\ldots,z^p)=0$ for $h\leqq -m$.

Proof. This lemma was proved when $p=n$ and the map $f\colon D^*\times D^{n-1}\to Y$ is of maximal rank in our earlier paper [6], where the particular construction of the imbedding $Y\subset P_N(\mathbf{C})$ was used strongly. We note that the integer m appearing in Lemma 11 is the exponent in K_Y^m.

Assume that $p>n$. Since $f\colon D^*\times D^{p-1}\to Y$ is of rank n, there exists an n-dimensional plane P in $\mathbf{C}^p$ (not necessarily through the origin) such that the restriction of f to the intersection $P\cap(D^*\times D^{p-1})$ is of rank n.

By moving P slightly if necessary, we may assume that P intersects the hyperplane $z^1=0$ transversally. By a linear change of the coordinate system in $\mathbf{C}^p$, we may further assume that P is defined by

$$z^{n+1}=a^{n+1}, \dots, z^p=a^p,$$

where $a^{n+1}, \dots, a^p$ are constants. We define

$$\alpha=(a^{n+1}, \dots, a^p),$$
$$f_\alpha(z^1, \dots, z^n)=f(z^1, \dots, z^n, a^{n+1}, \dots, a^p),$$
$$\varphi_\alpha^j(z^1, \dots, z^n)=\varphi^j(z^1, \dots, z^n, a^{n+1}, \dots, a^p).$$

Then $(\varphi_\alpha^0, \dots, \varphi_\alpha^N)$ gives the lift of f_α. The Laurent expansions of φ_α^j are given by

$$\varphi_\alpha^j(z^1, \dots, z^n)=\sum_{h=-\infty}^{\infty} A_h^j(z^2, \dots, z^n, a^{n+1}, \dots, a^p)(z^1)^h.$$

Since Lemma 11 holds for $p=n$ and hence for f_α, we obtain

$$A_h^j(z^2, \dots, z^n, a^{n+1}, \dots, a^p)=0 \quad \text{for } h \leqq -m.$$

Since f_α remains to be of rank n when $\alpha=(a^{n+1}, \dots, a^p)$ is moved slightly, we have

$$A_h^j(z^2, \dots, z^n, z^{n+1}, \dots, z^p)=0 \quad \text{for } h \leqq -m$$

for $(z^{n+1}, \dots, z^p)$ in a neighborhood of $(a^{n+1}, \dots, a^p)$ and hence for all $(z^{n+1}, \dots, z^n)$. This completes the proof of Lemma 11.

As stated in Footnote (2), we shall extend Theorem 1 to an arbitrary compact complex space X. Let $\mathfrak{M}(X)$ and $\mathfrak{M}(Y)$ be the fields of meromorphic functions on X and Y, respectively. Let X^* be a projective algebraic variety with $\mathfrak{M}(X)=\mathfrak{M}(X^*)$. Then

$$\begin{aligned}\mathrm{Mer}(X, Y) &\subset \{\varphi: \mathfrak{M}(Y) \to \mathfrak{M}(X); \text{ injective morphism}\}\\ &=\{\varphi: \mathfrak{M}(Y) \to \mathfrak{M}(X^*); \text{ injective morphism}\}\\ &=\mathrm{Mer}(X^*, Y).\end{aligned}$$

Since we have shown that $\mathrm{Mer}(X^*, Y)$ is finite, we may conclude that $\mathrm{Mer}(X, Y)$ is also finite.

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