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Removing Handles from Non-Singular Algebraic Hypersurfaces in CP_{n+1}

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If X is a non-singular algebraic hypersurface in CP_{n+1} of odd complex dimension n, then by general results of surgery theory it is possible to decompose X as a differentiable connected sum, $X = N \# a(S^n \times S^n)$, where the middle Betti number of N is 0 or 2. Here aM denotes the connected sum of a copies of M. Exactly how many handles can be removed depends on the degree of X.

Theorem 1. *Let X be a non-singular hypersurface in CP_{n+1}, n odd, defined by a polynomial of degree d.*

(i) *If $d \not\equiv \pm 3\,(8)$, then $X = M \# a(S^n \times S^n)$ where $H^n(M) = 0$.*

(ii) *If $d \equiv \pm 3\,(8)$, then $X = N \# a(S^n \times S^n)$ where $H^n(N) = \mathbb{Z} \oplus \mathbb{Z}$ and the Kervaire invariant of N is 1.*

Note that d even is included in case (i). For $n = 1, 3$, or 7 the first type of decomposition is always possible. In case (ii) there is a further decomposition in the topological category $N = M_1 \# M_0$ where $H_n(M_1) = 0$ and M_0 is $(n-1)$-connected.

But M_0 has no smooth structure (as a closed manifold) at least for $n \neq 2^k - 1$, see [1, p. 157]. The separating sphere is non-zero in bP_{2n} and is topologically embedded in codimension one in the smooth algebraic variety X.

The manifold M with $H^n(M) = 0$ has the homology module of CP_n. There is a map $f\colon M \to CP_n$ classifying the generator of $H^2(M)$ which induces an isomorphism on rational homology, on π_1, and on $\pi_*(\ )\otimes Q$. Thus M has the rational homotopy type of projective space.

When d is odd this result depends on the computation of the Kervaire invariant.

Theorem 2. *The Kervaire invariant of a hypersurface X of degree d is*

0 if $d \equiv \pm 1\,(8)$

1 if $d \equiv \pm 3\,(8)$.

Theorem 3. *If the degree of X is even, there is an embedded sphere, $S^n \subset X$, which is homologically trivial but has non-trivial normal bundle* $(n \neq 1, 3, 7)$.

Most cases of these results have been obtained independently by Shigeyuki Morita [4] by studying the affine variety which is the complement in X of a

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hyperplane section. The Kervaire invariant was first computed by Dennis Sullivan and the proof given here follows his suggestion to use branched coverings. William Browder has made the computation from the functional Steenrod square viewpoint. The proofs here are offered for their simplicity. I thank Ravindra Kulkarni, Dennis Sullivan, and Emery Thomas for helpful conversations.

§ 1. Removing Handles

The wedge $S^n \vee S^n$ sits in $S^n \times S^n$ so that both spheres have trivial normal bundle and so that the complement of a neighborhood of $S^n \vee S^n$ is a $2n$-ball. Conversely if $S^n \vee S^n \subset M$ with trivial normal bundles, then $M = S^n \times S^n \# M_1$. Thus to remove handles from M we embed spheres with trivial normal bundle to give geometrically a symplectic basis for (a summand of) $H_n(M)$.

(1.1) For a hypersurface $X \subset CP_{n+1}$, $H_n(X)$ is spherical. Let $L = p^{-1}(X)$ where $p\colon S^{2n+3} \to CP_{n+1}$ is the Hopf fibration. L is $(n-1)$-connected [3, p. 45] so we have

$$\begin{array}{ccc} \pi_n L & \xrightarrow{\cong} & H_n L \\ {\scriptstyle\cong}\downarrow & & \downarrow \\ \pi_n X & \longrightarrow & H_n X \end{array}$$

where the right hand map is surjective by the Gysin sequence, since n is odd.

Since X is simply connected, Haefliger's extension [2] of the Whitney embedding theorem shows that any map $S^n \to X$ is homotopic to an embedding and further by the Whitney process we find (cf. [7]):

(1.2) There are embeddings of $S^n \vee S^n \to X$ giving a symplectic basis for $H_n(X)$.

(1.3) The normal bundle of any embedded $S^n \subset X$ is stably trivial. We have $\nu(S^n \subset X) \oplus \nu(X \subset CP_{n+1}) | S^n = \nu(S^n \subset CP_{n+1})$. But a 2-plane bundle over S^n is trivial and, since S^n is null-homotopic in CP_{n+1}, Haefliger's result [2] shows its normal bundle is trivial.

This normal bundle is determined by an element of $\ker\{\pi_n BSO_n \to \pi_n BSO\}$. If $n = 1$, 3, or 7 the kernel is 0, the normal bundles of the embeddings in (1.2) are all trivial, and the decomposition (i) results. For other odd n the kernel is $\mathbb{Z}/2$. In case d is even Theorem 3 provides a sphere which we may take disjoint from the basis (1.2) and connect along a thin tube to a sphere in the basis. Under this process the invariant in $\mathbb{Z}/2$ of the normal bundle adds. Thus we obtain a symplectic basis with trivial normal bundles and hence the decomposition (i).

(1.4) When d is odd the normal bundle of an embedded $S^n \subset X$ depends only on the homology class S^n represents.

Given two homotopic embeddings of S^n in X, there is an S^2 embedded in the complement of the image of the homotopy and generating $\pi_2 X$; in fact, S^2 is the 2-skeleton of X. The Chern class $c_1(X) = (n+2-d)$ generator, and since $n-d$ is even, $w_2 X = 0$ so $\nu(S^2 \subset X)$ is trivial. Doing surgery on this S^2 we obtain a 2-connected manifold Y containing the same homotopic embeddings of S^n. But Haefliger's theorem [2] applies to Y to show the embeddings are smoothly isotopic. Hence the normal bundle depends only on the homotopy class.

The Gysin sequence used in (1.1) shows that $\ker\{\pi_n X \to H_n X\} = \mathbb{Z}/d$. Since d is odd, while the normal bundle is determined by a $\mathbb{Z}/2$ invariant, a null-homologous S^n must also have trivial normal bundle. Finally, homologous embeddings of S^n may be made disjoint by the Whitney process without changing their normal bundles and (1.4) follows.

As a result we can define the Kervaire invariant of an odd degree hypersurface X geometrically as the Arf invariant of the quadratic function $\psi\colon H_n(X;\mathbb{Z}/2) \to \mathbb{Z}/2$ defined by $\psi(x)=1$ if and only if x is represented by a sphere embedded with non-trivial normal bundle. The argument in [7, pp. 167–168] shows that ψ is associated to the cup product pairing:

$$\psi(x+y) = \psi(x) + \psi(y) + (x \cup y)[X].$$

The remaining case, d odd, of Theorem 1 follows from the computation of this invariant (Theorem 2) and the general result that there is a geometric symplectic basis if and only if the Arf invariant of ψ is 0.

§ 2. The Kervaire Invariant

Any nonsingular algebraic hypersurface X is diffeomorphic to the Fermat hypersurface

$$X(d) = \{[z] \in CP_{n+1} : z_0^d + \cdots + z_{n+1}^d = 0\}.$$

An action of the group $\mathbb{Z}/d$ on $X(d)$ is generated by the map $[z] \to [\omega z_0, z_1, \ldots, z_{n+1}]$ where $\omega = e^{2\pi i/d}$. The fixed point set is the Fermat hypersurface in the hyperplane $CP_n = \{[z] \in CP_{n+1} : z_0 = 0\}$. The projection of $X(d)$ to this hyperplane defined by $\pi([z]) = [z_1, \ldots, z_{n+1}]$ identifies CP_n as $X(d)/\mathbb{Z}/d$.

Thus up to diffeomorphism we have a d-fold, cyclic branched cover $\pi\colon X \to CP_n$ with branch set a hypersurface K of degree d in CP_n.

Further the quadratic form ψ is invariant under the covering group, that is $\psi(x) = \psi(t_* x)$ for $t \in \mathbb{Z}/d$, since the diffeomorphism t preserves the normal bundle of an embedded S^n representing x.

Consider the general situation of a branched, d-fold cyclic cover $\pi\colon X \to Z$ between $2n$-manifolds branched over a codimension 2 submanifold K. Assume d and n are both odd. Let $q\colon H^n(X;\mathbb{Z}/2) \to \mathbb{Z}/2$ be a quadratic form associated to the cup product pairing. Since π^* maps $H^*(Z;\mathbb{Z}/2)$ injectively, q restricts to such a form on Z. We denote the Arf invariant by $a(X)$ and $a(Z)$ in the two cases.

To state the result we need a notation from number theory. The Jacobi symbol

$$(2|d) = \begin{cases} 1 & \text{for } d \equiv \pm 1 \bmod 8. \\ -1 & \text{for } d \equiv \pm 3 \bmod 8. \end{cases}$$

It is multiplicative:

$$(2|d)(2|m) = (2|dm).$$

For an odd prime p, $(2|p)$ is called a Legendre symbol and satisfies

$$2^{\frac{p-1}{2}} \equiv (2|p) \bmod p.$$

Theorem 4. *Let $\pi\colon X\to Z$ be a d-fold cover branched over K as above* (d odd). *Let q be a $\mathbb{Z}/d$-invariant quadratic form on $H^n(X;\mathbb{Z}/2)$. Assume*

$$\pi^*\colon H^j(Z;\mathbb{Z}/2)\to H^j(X;\mathbb{Z}/2)$$

is an isomorphism for $j\neq n$. Then

$$(-1)^{a(X)}=(-1)^{a(Z)}(2\,|\,d)^{\varepsilon}$$

where $\varepsilon=\chi(Z)-\chi(K)\bmod 2$.

The hypothesis on π_* is satisfied in the case $\pi\colon X\to CP_n$ of Theorem 2 or more generally whenever π is ramified over a taut embedding $K\subset Z$, see [6]. Also $\varepsilon=\chi(CP_n)-\chi(K)=\frac{1}{d}\{1-(1-d)^{n+1}\}$ is odd when d is odd. The Arf invariant $a(CP_n)=0$ since $H^n(CP_n)=0$ (compare the definition of a used below) so Theorem 4 implies Theorem 2.

The proof of Theorem 4 is by induction on the number of prime factors of d. If $d=pl$, p an odd prime, set $Y=X/\mathbb{Z}/l$. Then $\pi_1\colon X\to Y$ and $\pi_2\colon Y\to Z$ are p-fold and l-fold covers respectively, and $\pi=\pi_2\circ\pi_1$.

The quadratic form induced on $H^n(Y;\mathbb{Z}/2)$ is $\mathbb{Z}/l$-invariant since π_1^* maps $H^n(Y;\mathbb{Z}/2)$ injectively onto the subgroup of $H^n(X;\mathbb{Z}/2)$ fixed by $\mathbb{Z}/p\subset\mathbb{Z}/d$. The hypothesis on π^* for $j\neq n$ is satisfied since π_1^* and π_2^* are injective. Finally the Euler characteristic formula for a branched cover

$$\chi(Y)-\chi(K)=l(\chi(Z)-\chi(K))$$

shows $\varepsilon=\chi(Z)-\chi(K)=\chi(Y)-\chi(K)\bmod 2$. The inductive step is given by the formula

$$(-1)^{a(X)}=(-1)^{a(Y)}(2\,|\,p)^{\varepsilon}=(-1)^{a(Z)}(2\,|\,p)^{\varepsilon}(2\,|\,l)^{\varepsilon}=(-1)^{a(Z)}(2\,|\,d)^{\varepsilon}.$$

Now consider the case of a prime-fold cover $X\to Y$. Let $b(X)=\dim H^n(X;\mathbb{Z}/2)$ and let $\mathrm{Pos}(X)=\mathrm{Card}\,\{v\in H^n(X;\mathbb{Z}/2)\colon q(v)=1\}$. The Arf invariant satisfies [7, page 172].

$$2\,\mathrm{Pos}(X)=2^{b(X)}-(-1)^{a(X)}\,2^{b(X)/2}. \tag{2.1}$$

Since p is prime, $\mathbb{Z}/p$ acts freely on $H^n(X;\mathbb{Z}/2)-\pi_1^*H^n(Y;\mathbb{Z}/2)$, and this implies

$$\mathrm{Pos}(X)\equiv\mathrm{Pos}(Y)\bmod p. \tag{2.2}$$

The assumption on π^* implies

$$b(X)-b(Y)=\chi(X)-\chi(Y)$$

and, applying the Euler characteristic formula,

$$b(X)-b(Y)=(p-1)(\chi(Y)-\chi(K)).$$

Hence $2^{b(X)}\equiv 2^{b(Y)}\bmod p$.

From (2.1) and (2.2) we have

$$(-1)^{a(X)}2^{b(X)/2}\equiv(-1)^{a(Y)}2^{b(Y)/2}\bmod p$$

and hence

$$(-1)^{a(X)} \equiv (-1)^{a(Y)} \left(2^{\frac{p-1}{2}}\right)^{\varepsilon} \bmod p.$$

Theorem 4 follows replacing $2^{\frac{p-1}{2}}$ by the Legendre symbol.

Remark. In the case of an odd degree hypersurface X, $\pi: X \to CP_n$ is a $\mathbb{Z}/2$-degree 1 normal map and Theorem 4 shows that its surgery obstruction is $(2|d)$. Note however that the proof requires the quadratic form ψ to be defined on all of $H^n(X; \mathbb{Z}/2)$.

§ 3. Proof of Theorem 3

As in § 2 it suffices to consider the Fermat hypersurface

$$X = \{[z]: z_0^d + \cdots + z_{n+1}^d = 0\}$$

and to find in it, when d is even, an embedded, homologically trivial S^n with non-trivial normal bundle. The hyperplane $CP_n = \{[z]: z_{n+1} = 0\}$ meets X transversely in X_0. Let $n = 2q+1$ and set

$$CP_q = \{[z]: z_{n+1} = 0, z_0 + \omega z_1 = 0, z_2 + \omega z_3 = 0, \ldots, z_{n-1} + \omega z_n = 0\}$$

where $(-\omega)^d = -1$. It is easy to check that $CP_q \subset X_0$. The normal bundle of CP_n in CP_{n+1} restricts to be the normal bundle of X_0 in X and the restriction to CP_q is the Hopf bundle. The associated S^1-bundle is the desired S^n:

$$\begin{array}{ccccc}
S^n & \hookrightarrow & X & \hookrightarrow & CP_{n+1} \\
{\scriptstyle p}\downarrow & & \cup\!\uparrow & & \cup\!\uparrow \\
CP_q & \hookrightarrow & X_0 & \hookrightarrow & CP_n.
\end{array}$$

S^n is homologically trivial since it bounds the disk bundle.

There are two bundle equations:

$$\tau S^n = p^* \tau CP_q \oplus \varepsilon$$

$$\nu(S^n \subset X) = p^* \nu(CP_q \subset X_0) \oplus \varepsilon$$

where ε is the trivial real line bundle. The first uses the fact that τS^1 is trivial and the second that $\nu(S^1 \subset D^2)$ is trivial. Let $\tau = \tau CP_q \oplus \varepsilon$ and $\nu = \nu(CP_q \subset X_0) \oplus \varepsilon$. The cofibration $S^n \xrightarrow{p} CP_q \xrightarrow{j} CP_{q+1}$ gives the diagram

$$\begin{array}{ccccc}
[S^n, BSO_n] & \xleftarrow{p^*} & [CP_q, BSO_n] & \xleftarrow{j^*} & [CP_{q+1}, BSO_n] \\
\downarrow & & \downarrow{\scriptstyle =} & & \downarrow \\
[S^n, BSO] & \longleftarrow & [CP_q, BSO] & \xleftarrow{j^*} & [CP_{q+1}, BSO].
\end{array}$$

We must show $p^*\tau = p^*\nu \in [S^n, BSO_n]$. Both bundles are stably trivial, hence lie in $\mathbb{Z}/2$, so it is enough to show $p^*\nu$ is non-trivial for $n \neq 1, 3, 7$.

In general $p^*\omega = 0$ for $\omega \in [CP_q, BSO_n]$ iff there is an $\eta \in [CP_{q+1}, BSO_n]$ with $j^*\eta = \omega$, hence iff there is an $\tilde{\eta} \in [CP_{q+1}, BSO]$ with $j^*\tilde{\eta} = \omega$ and Stiefel-Whitney class $w_{2q+2}\tilde{\eta} = 0$.

Let γ be the Hopf bundle over CP_q. The unitary bundles τCP_q and $\nu(CP_q \subset X_0) \oplus \gamma^d$ have the same Chern classes and hence by results of Peterson [5] are stably equivalent. Therefore as stable real bundles $\tau = \nu + \gamma^d$.

Hence if $j^*\tilde{\eta} = \nu$, then $j^*(\eta + \gamma^d) = \tau$. But $w_{2q+2}(\eta) = w_{2q+2}(\eta + \gamma^d)$ for d even, and this is non-zero if τ is non-trivial. Therefore ν is also non-trivial. This completes the proof of Theorem 3.

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