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On the Characters of the Discrete Series

The Hermitian Symmetric Case

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§ 1. Introduction

An irreducible, unitary representation of a locally compact, unimodular group is said to be *square-integrable* if it can be realized on an invariant subspace for the left regular representation. The *discrete series* is the set of equivalence classes of such representations. In Harish-Chandra's work on the Plancherel formula for semisimple Lie groups, the discrete series plays the central role. Thus, among the representations of a semisimple Lie group, the discrete series representations are of particular interest.

According to Harish-Chandra's criterion [13], a connected semisimple Lie group G has a non-empty discrete series exactly when it contains a compact Cartan subgroup. If G does contain a compact Cartan subgroup H, Harish-Chandra parameterizes the discrete series by, roughly speaking, the dual group $\hat{H}$ of H, modulo the action of the normalizer of H in G. To be more precise, I denote the Lie algebras of G, H by $\mathfrak{g}$, $\mathfrak{h}$, and their complexifications by $\mathfrak{g}^{\mathbb{C}}$, $\mathfrak{h}^{\mathbb{C}}$. Via exponentiation, $\hat{H}$ becomes isomorphic to a lattice $\Lambda \subset i\mathfrak{h}^*$ ($\mathfrak{h}^*$ = dual space of $\mathfrak{h}$); the lattice Λ contains the root system Φ of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$. For simplicity, assume that G has a complexification $G^{\mathbb{C}}$, which is simply connected. Then, according to Harish-Chandra's fundamental results on the discrete series [13], for every nonsingular [1] $\lambda \in \Lambda$, there exists a unique tempered [2] invariant eigendistribution Θ_λ, such that

$$(1.1) \qquad \Theta_\lambda|_H = (-1)^q \frac{\sum_{w \in W} \varepsilon(w)\, e^{w\lambda}}{\prod_{\alpha \in \Phi, (\alpha, \lambda) > 0} (e^{\alpha/2} - e^{-\alpha/2})};$$

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[1] i.e. $(\lambda, \alpha) \neq 0$ for all $\alpha \in \Phi$.

[2] A distribution is called tempered if it extends continuously to the space of rapidly decreasing functions [13].

here W denotes the Weyl group of H in G, q is one half the dimension of the symmetric space corresponding to G, and $\varepsilon(w)$ = sign of w. Every such Θ_λ is the character of a discrete series representation, and conversely.

Harish-Chandra's character formula (1.1) completely determines the discrete series characters on the set of elliptic elements. However, it does not give a direct, global description. For a number of reasons, it would be desirable to know the characters also outside the elliptic set. Harish-Chandra's construction of the Θ_λ is analytic, intricate, and difficult. On the other hand, his proof of the uniqueness suggests that one ought to be able to construct the Θ_λ with essentially combinatorial arguments.

In this paper, I shall look at the following four problems:

a) finding an essentially combinatorial construction of the discrete series characters;

b) to describe the Θ_λ globally, in as explicit a manner as possible;

c) to prove Blattner's conjecture about the decomposition of a discrete series representation under the action of a maximal compact subgroup; and

d) obtaining concrete realizations of all discrete series representations (not just "most" of them, as in [31, 32, 35]).

At present, I can attack these problems only if the symmetric space of G carries a Hermitian symmetric structure, and if, in addition, G has a faithful finite dimensional representation[3]. However, it seems likely that at least a) and b) can be treated by similar methods, even in the absence of a Hermitian symmetric structure.

The basic tool is a certain relationship between the various discrete series characters which belong to the same infinitesimal character. Let Φ^c and Φ^n denote the sets of, respectively, compact and noncompact roots in Φ. I now fix a system of positive roots Ψ in Φ, and I look at the invariant eigendistributions Θ_λ parameterized by the set

(1.2) $$\{\lambda\in\Lambda \mid (\lambda,\alpha)>0 \text{ for all } \alpha\in\Psi\}.$$

It is known in principle that Θ_λ, with λ restricted to lie in the set (1.2), is given by a formula which depends on the parameter λ in a coherent manner. The formula makes sense whether or not λ lies in the set (1.2). Hence, by letting λ wander over the larger set

(1.3) $$\{\lambda\in\Lambda \mid (\lambda,\alpha)>0 \text{ for all } \alpha\in\Psi\cap\Phi^c\},$$

one obtains a family of invariant eigendistributions $\Theta(\Psi, \lambda)$, which depends on the choice of the system of positive roots Ψ, and which is parameterized by the set (1.3). By construction,

(1.4) $$\Theta(\Psi,\lambda)=\Theta_\lambda, \quad \text{provided} \quad (\lambda,\alpha)>0 \quad \text{for all } \alpha\in\Psi.$$

Moreover, for any $w\in W$ = Weyl group of H in G,

(1.5) $$\Theta(w\Psi, w\lambda)=\Theta(\Psi,\lambda).$$

[3] I have made this latter assumption in order to avoid some unpleasant, but presumably minor, technical complications.

Suppose that $\beta\in\Phi^n$ is a simple root, relative to the system of positive roots Ψ, and let s_β denote the reflection about the root β. Then, as will be shown in this paper,

$$\Theta(\Psi,\lambda)+\Theta(s_\beta\Psi,\lambda)=\Theta. \tag{1.6}$$

Here Θ denotes an invariant eigendistribution, which is induced from a maximal cuspidal parabolic subgroup; the inducing character, restricted to a Levi component M of the parabolic subgroup, belongs to the class of invariant eigendistributions $\Theta(\dots,\dots)$ of M. This is the relationship between the various discrete series characters, which was alluded to before.

The identity (1.6) can be used as a vehicle for certain inductive arguments. If one wants to prove any given statement about the invariant eigendistributions $\Theta(\Psi,\lambda)$ – and hence, about the discrete series characters – it suffices to verify the statement for only one choice of a system of positive roots Ψ, provided one can show that the statement is compatible with the relationship (1.6). Thus, in order to get such inductive arguments going, one must understand the invariant eigendistributions $\Theta(\Psi,\lambda)$ corresponding to at least one system of positive roots Ψ. If the symmetric space of G admits a Hermitian symmetric structure, for some choices of Ψ, the $\Theta(\Psi,\lambda)$ are fairly simple from many points of view, as will be explained next.

Let K be the maximal compact subgroup of G which contains H. I shall assume that G/K is Hermitian symmetric. In a natural fashion [9], the invariant complex structures on G/K are in one-to-one correspondence with those systems of positive roots Ψ, which have the property

$$\alpha_1,\alpha_2\in\Psi\cap\Phi^n\Rightarrow\alpha_1+\alpha_2\notin\Phi. \tag{1.7}$$

Fix such a Ψ, and equip G/K with the corresponding invariant complex structure. In [9], Harish-Chandra has constructed a class of representations T_λ, on spaces of holomorphic sections of homogeneous vector bundles on G/K; the parameter λ runs over the set (1.3). The family of these representations is known, somewhat informally, as the "holomorphic discrete series". In general, the representations T_λ are neither unitary nor irreducible. However, if λ lies in the set (1.2), T_λ turns out to be a discrete series representation, and its character is Θ_λ [9]. The characters of the representations T_λ have been globally computed by Martens [28] and Hecht [16]. In particular, their computations provide explicit and global formulas for the invariant eigendistributions $\Theta(\Psi,\lambda)$ corresponding to any system of positive roots Ψ which satisfies (1.7). Beginning with such a system of positive roots, and using the identity (1.6), one can then study the $\Theta(\Psi,\lambda)$ by inductive arguments.

I continue to assume that G/K is Hermitian symmetric. For reference purposes, I fix a system of positive roots Ψ_0, subject to the condition (1.7). Under the action of the Weyl group W of H in G, every system of positive roots is conjugate to one which satisfies

$$\Psi\cap\Phi^c=\Psi_0\cap\Phi^c. \tag{1.8}$$

In view of (1.5), if one wants to understand the $\Theta(\Psi,\lambda)$, it suffices to consider systems of positive roots Ψ as in (1.8). For any such Ψ, there exists a chain of positive root systems

$$\Psi_0,\Psi_1,\dots,\Psi_m=\Psi, \tag{1.9}$$

such that each Ψ_j is obtained from the preceding Ψ_{j-1} by reflection about a non-compact, simple root. According to the relationship (1.6), there exists a definite induced invariant eigendistribution Θ_j, with

$$\Theta_j = \Theta(\Psi_j, \lambda) + \Theta(\Psi_{j-1}, \lambda). \tag{1.10}$$

Summing over j, with alternating signs, one finds

$$\Theta(\Psi, \lambda) = (-1)^m\, \Theta(\Psi_0, \lambda) + \textstyle\sum_{j=1}^m (-1)^{m-j}\, \Theta_j. \tag{1.11}$$

The character formula of Martens and Hecht completely determines $\Theta(\Psi_0, \lambda)$. Each Θ_j is an induced invariant eigendistribution, and the inducing character in turn belongs to the class of invariant eigendistributions $\Theta(\ldots, \ldots)$, for a lower dimensional subgroup of G. Thus, at least in principle, the formula (1.11) allows one to compute $\Theta(\Psi, \lambda)$; the computation proceeds by induction on the dimension of G.

According to the description which was given above, the invariant eigendistributions $\Theta(\Psi, \lambda)$ were obtained in terms of Harish-Chandra's discrete series characters. However, it is possible to construct them directly, by induction on the dimension of G. As follows from the explicit description of the Θ_j, coupled with the inductive hypothesis, the formula (1.11) describes an invariant eigendistribution. Without using Harish-Chandra's results on the discrete series characters, I shall show that the definition (1.11) does not depend on the choice of the chain (1.9), and that $\Theta(\Psi, \lambda)$ is tempered whenever λ lies in the set (1.2). In particular, these arguments prove the existence of the discrete series characters, provided, of course, G/K carries a Hermitian symmetric structure. Except for the computation of the characters of the "holomorphic discrete series", which uses some function theory, the arguments are essentially combinatorial.

As follows from the formula (1.11), the invariant eigendistributions $\Theta(\Psi, \lambda)$ can be built up from the characters of the "holomorphic discrete series": every $\Theta(\Psi, \lambda)$ is a linear combination of eigendistributions, which are induced up from cuspidal parabolic subgroups[4], such that the inducing characters, on the Levi components of the parabolic subgroups, are characters of "holomorphic discrete series representations". In [37], such a formula is worked out explicitly for the group $Sp(n, \mathbb{R})$ (Theorem 1 of [37]); it has a rather straightforward appearance. Analogous formulas exist for any simple matrix group which operates on a Hermitian symmetric space. However, as will be explained below, it is only necessary to deal with the case of $Sp(n, \mathbb{R})$.

The characters of the "holomorphic discrete series" are known completely, and the process of inducing an invariant eigendistribution from a parabolic subgroup can be carried out explicitly. The type of formula which was just mentioned therefore leads to a concrete and a global description of the discrete series characters (cf. Theorem 2 of [37]). Unfortunately, this description, which amounts to a semi-explicit formula, is highly complicated; for example, it is impossible to tell from the formula that the discrete series characters are tempered. As is argued in [37] by means of concrete examples, the complicated appearance of

[4] For the purpose of this statement, G itself should be viewed as a parabolic subgroup.

the formula merely reflects the relatively complicated nature of the discrete series characters. For a general group G, it will be very difficult to express the discrete series characters by a completely explicit global formula in closed form – if it can be done at all.

On the other hand, the methods of this paper give a reasonably direct algorithm for computing the discrete series characters, provided, of course, G has a Hermitian symmetric quotient G/K. Let B be an arbitrary Cartan subgroup of G, and B^j a particular connected component of B. To B^j one can attach a certain semisimple subgroup G' of G, which contains both a split and a compact Cartan subgroup, and which has the following property: the discrete series characters of G, restricted to B^j, can be expressed in a simple manner in terms of discrete series characters of G', restricted to the identity component of a split Cartan subgroup[5]. When G/K carries a Hermitian symmetric structure, G' turns out to be, up to covering, a product of copies of $Sp(k, \mathbb{R})$, for various integers k. Moreover, if G is simple, but not locally isomorphic to $Sp(n, \mathbb{R})$, G' is either $Sp(2, \mathbb{R})$ or a product of copies of $SL(2, \mathbb{R})$, again up to covering. For $Sp(2, \mathbb{R})$ and $SL(2, \mathbb{R})$, the discrete series characters can be computed easily enough. Hence, if G is simple, with Hermitian symmetric quotient G/K, but not locally isomorphic to $Sp(n, \mathbb{R})$, a computation of the discrete series characters of G presents no major problems. By the same reasoning, for $G = Sp(n, \mathbb{R})$, it really suffices to compute the discrete series characters on the identity component of a split Cartan subgroup. For any given n, and for any particular discrete series character, this can be done by means of the semi-explicit formula which was mentioned above.

Unless λ lies in the set (1.2), the invariant eigendistribution $\Theta(\Psi, \lambda)$ need not be the character of an irreducible representation. However, using the identity (1.11), one can show that $\Theta(\Psi, \lambda)$ is always the character of a virtual representation, i.e. of a formal, finite, integral linear combination of irreducible G-modules. Thus, in the obvious manner, for each irreducible K-module, one can define the multiplicity of the given K-module in the virtual representation corresponding to $\Theta(\Psi, \lambda)$. Blattner's conjecture predicts the K-decompositions of the discrete series representations. Formally, at least, the conjecture makes sense for all of the $\Theta(\Psi, \lambda)$, not just for the discrete series characters. In order to prove Blattner's conjecture, it therefore suffices to verify this extended version of the conjecture for a single choice of positive root system Ψ, and to show that it is consistent with the identity (1.6). As will be demonstrated in this paper, the conjecture is indeed consistent with (1.6), whether or not G/K carries a Hermitian symmetric structure. If G/K is Hermitian symmetric, and if Ψ satisfies the condition (1.7), every $\Theta(\Psi, \lambda)$ arises as the character of a representation of the "holomorphic discrete series". The K-multiplicities of these representations were computed by Harish-Chandra [9]. Hence, for every connected, semisimple matrix group G, with Hermitian symmetric quotient G/K, Blattner's conjecture becomes a consequence of the relationship (1.6).

In analogy to the Borel-Weil-Bott theorem about compact Lie groups, Langlands [27] had conjectured that the discrete series representations of a

[5] This feature is an important ingredient of the construction of the invariant eigendistributions $\Theta(\Psi, \lambda)$. As was pointed out to me by Zuckerman, it can also be deduced directly from Harish-Chandra's construction of the discrete series characters.

connected, semisimple Lie group G can be realized on L^2-cohomology groups of holomorphic line bundles over the quotient of G by a compact Cartan subgroup H. Narasimhan-Okamoto [31] obtained a similar statement involving vector bundles over G/K, provided G/K carries a Hermitian symmetric structure, but they could prove it only for "most" discrete series representations. The original conjecture, again only for "most" discrete series representations, was proven in [35]. Other realizations of discrete series representations appear in [21, 32, 36]. In all cases, the arguments depend on an alternating sum formula and a vanishing theorem. The proofs of the vanishing theorems are based on what amounts to curvature computations. Such curvature computations cannot be pushed far enough to give sharp vanishing theorems: they work equally well on the quotient of G by a uniform discrete subgroup; in this setting, the sharp versions of the vanishing theorems are demonstrably false.

Among the various concrete realizations of discrete series representations, perhaps the most attractive is Parthasarathy's construction, in terms of the Dirac operator [32]. As will be shown below, the sharp vanishing theorem for the Dirac operator can be derived from Blattner's conjecture. In particular, if G/K admits a Hermitian symmetric structure, all discrete series representations of G can be realized on suitable L^2 kernels of the Dirac operator. With slight modifications, the arguments which deduce the sharp vanishing theorem for the Dirac operator from Blattner's conjecture also work in the settings of [21, 31, 35]. I do not know whether Blattner's conjecture implies the complete Langlands conjecture [27].

In order to put the results of this paper into perspective, I shall briefly discuss the previous status of problems a)–d), which were raised above. Harish-Chandra's proof of the uniqueness of the Θ_λ amounts to an algorithm, which makes it possible to compute the discrete series characters globally, at least in principle. Whenever this algorithm can be carried out concretely, it provides a direct, combinatorial constructions of the Θ_λ, which is then quite independent of Harish-Chandra's existence proof. The algorithm proceeds by induction over the set of conjugacy classes of connected components of Cartan subgroups, equipped with a certain partial ordering. In general, this ordered set has a rather complicated structure. Even for low dimensional groups, it becomes quite difficult to work out the algorithm explicitly. As far as I know, only the groups of real rank one [14], the indefinite unitary groups [19], and $Sp(2, \mathbb{R})$ [20] have been treated by this method. It has been suggested that one might be able to guess a general, global formula for the discrete series characters, once enough special cases are known. If one had a conjectured explicit formula, it might not be too hard to verify it, by checking it against Harish-Chandra's algorithm. However, as was already mentioned, a completely explicit, global formula cannot be written down so easily.

Instead of looking at each Θ_λ individually, one may ask whether the sum of the Θ_λ, extended over all λ in an orbit of the Weyl-group of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$, has a simple global formula. From many points of view, such a formula would accomplish as much as a solution of problem b). According to a suggestion of Zuckerman, for every matrix group G, the sum of the Θ_λ should be expressible in terms of characters induced from finite dimensional characters of parabolic subgroups. Harish-Chandra has proven that this can be done, at least in principle. In the case of groups of real rank one and for the indefinite unitary groups, Zuckerman has carried out the computations explicitly [41] (also cf. [29]); he is now investigating the various series of classical groups. In spirit, his approach is similar to the approach taken in this paper.

Let π_λ be a particular discrete series representation, with character Θ_λ. According to the version of Langlands' conjecture proven in [35], if the parameter λ is sufficiently nonsingular, i.e. if

$$(1.12) \qquad |(\lambda, \alpha)| \geqq c \quad \text{whenever} \quad \alpha \in \Phi,$$

for some suitably chosen constant c, π_λ can be realized on an L^2 cohomology group of a homogeneous, holomorphic line bundle $\mathscr{L}_\lambda$ over G/H. After adjusting the constant c in (1.12), one can show that the natural mapping from the L^2 cohomology group in question to the corresponding sheaf cohomology group is injective. It is also possible to analyze the sheaf cohomology group with the methods of complex analysis: under the action of K, the sheaf cohomology group breaks up in the manner predicted by Blattner's conjecture for π_λ. Since the representation space of π_λ injects into this sheaf cohomology group, the K-multiplicities of π_λ must be bounded by the predicted multiplicities. By these arguments one half of Blattner's conjecture was proven in [34, 36], but of course only for discrete series representations with sufficiently nonsingular parameter λ. Hotta and Parthasarathy have succeeded in simplifying the arguments of [34, 36] and sharpening the hypotheses slightly, by working with the Dirac operator on G/K, rather than with cohomology [22]. Except for the results of [22, 36], Blattner's conjecture was known only in the cases of $SL(2, \mathbb{R})$ [2], the de Sitter group [5], and the "holomorphic discrete series" [9], where it comes up as a by-product of explicit constructions of discrete series representations.

The difficulties of realizing all discrete series representations of a given group were already described. Of the various vanishing theorems based on curvature computations, Parthasarathy's [32] is the sharpest, and perhaps even the best possible. The fact that Blattner's conjecture, if proven, allows one to realize all discrete series representations has been known to a number of people for some time. It has not appeared in writing, presumably because Blattner's conjecture was not available.

To conclude the introduction, I shall give a quick guide to the organization of this paper. Besides establishing notation, section two deals with preliminaries about Cartan subgroups and Weyl groups. Most of the material is familiar to experts, but there seems to be no convenient reference for it. Section three recalls the results of Martens and Hecht on the characters of the "holomorphic discrete series". The existence and the main properties of the invariant eigendistributions $\Theta(\Psi, \lambda)$, for groups G with Hermitian symmetric quotient G/K, are stated in § 4. Sections five and six contain the proofs of these results. In § 7, which begins with a general discussion of Blattner's conjecture, I prove the conjecture, provided G/K carries a Hermitian symmetric structure. The precise vanishing theorem for the Dirac operator is deduced from Blattner's conjecture in § 8; I have also used this opportunity to show how Parthasarathy's construction [32] can be simplified by using some of the methods of [35, 36]. In the final section, I speculate about the possibilities of extending the arguments of this paper to the case of an arbitrary semisimple matrix group.

§ 2. Cartan Subalgebras and Subgroups

It is the purpose of this section to dispose of various preliminaries about semisimple Lie groups, and to establish notation. I intend to use much of the material also in a future paper. For this reason, some of the statements are presented in greater generality than is needed here. The major point will be a reformulation of Kostant's and Sugiura's classification of conjugacy classes of Cartan subalgebras, some results on the Weyl group of a general Cartan subgroup, an investigation of the connected components of a Cartan subgroup, and finally some simple observations about certain reductive subgroups. Many of the arguments below are straightforward, or are known to experts. I have therefore tried to keep the proofs succinct; sometimes only a sketch of a proof will be given.

Although semisimple Lie groups will really be the objects of interest, the usual reasons make it advantageous to consider also reductive groups. Throughout, G will stand for a connected, reductive Lie group, which contains a maximal compact subgroup K of the same rank as G. In order to avoid some unpleasant technicalities, I shall also assume that G is a matrix group. Because of the hypotheses, one can choose a compact Cartan subgroup H of G, with $H \subset K$. It is known that

(2.1) H is connected, and K contains the normalizer of H.

The Lie algebras of the groups G, K, H will be denoted by the corresponding small German letters $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{h}$, and their complexifications by $\mathfrak{g}^{\mathbb{C}}$, $\mathfrak{k}^{\mathbb{C}}$, $\mathfrak{h}^{\mathbb{C}}$; as a general notational convention, the superscript $\mathbb{C}$ shall always mean "complexification".

The differentials of the characters of H form a lattice $\Lambda \subset i\mathfrak{h}^*$. Each $\lambda \in \Lambda$ belongs to the character e^{λ}, which is given by

(2.2) $$e^{\lambda}(\exp X) = e^{\langle \lambda, X \rangle}, \qquad \text{for } X \in \mathfrak{h}.$$

The set of nonzero roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ lies in Λ. Depending on the particular context, it will be referred to as $\Phi(G, H)$, $\Phi(H)$, $\Phi(\mathfrak{h})$, or simply as Φ. A finite covering of G need not be a matrix group again; the following simple lemma provides a criterion for deciding whether certain coverings are linear. In order to state it, I let $(\, , \,)$ be the positive definite inner product on $i\mathfrak{h}^*$ induced by the trace form of a faithful finite dimensional representation of G.

(2.3) **Lemma.** *If an integral multiple of $\lambda \in i\mathfrak{h}^*$ belongs to Λ, and if $2(\lambda, \alpha)(\alpha, \alpha)^{-1} \in \mathbb{Z}$ for all $\alpha \in \Phi$, then there exists a finite covering of G by a matrix group $\tilde{G}$, such that λ lifts to a character e^{λ} on the inverse image $\tilde{H}$ of H.*

For semisimple Lie groups, this statement is well-known. The general case can be reduced to the semisimple case; details are left to the reader.

Since K contains the center of G, G/K is a noncompact symmetric space. Hence $\mathfrak{g}$ has a Cartan decomposition

(2.4) $$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

and a Cartan involution

(2.5) $$\theta: \mathfrak{g} \to \mathfrak{g}, \quad \text{with} \quad \theta|_{\mathfrak{k}} = 1, \quad \theta|_{\mathfrak{p}} = -1.$$

Because $\mathfrak{h}$ is θ-stable, θ preserves the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$,

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus (\textstyle\bigoplus_{\alpha\in\Phi} \mathfrak{g}^{\alpha}). \tag{2.6}$$

A root $\alpha\in\Phi$ is called compact or noncompact, depending on whether $\mathfrak{g}^{\alpha}\subset\mathfrak{k}^{\mathbb{C}}$ or $\mathfrak{g}^{\alpha}\in\mathfrak{p}^{\mathbb{C}}$. I let Φ^c (or $\Phi^c(\mathfrak{h})$, $\Phi^c(H)$, etc.) denote the set of compact roots, and Φ^n the set of noncompact roots. Equivalently, Φ^c can be described as the root system of $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$.

For each $\alpha\in\Phi$, I once and for all choose nonzero elements $Y_\alpha\in\mathfrak{g}^{\alpha}$, $Z_\alpha\in i\mathfrak{h}$, subject to the following conditions:

$$\begin{gathered} [Z_\alpha, Y_\alpha] = 2Y_\alpha, \quad [Z_\alpha, Y_{-\alpha}] = -2Y_{-\alpha}, \\ [Y_\alpha, Y_{-\alpha}] = Z_\alpha, \quad \bar{Z} = -Z_\alpha = Z_{-\alpha}, \\ \bar{Y}_\alpha = -Y_{-\alpha} \quad \text{if } \alpha\in\Phi^c, \quad \bar{Y}_\alpha = Y_{-\alpha} \quad \text{if } \alpha\in\Phi^n \end{gathered} \tag{2.7}$$

(barring denotes complex conjugation, relative to the real form $\mathfrak{g}\subset\mathfrak{g}^{\mathbb{C}}$). Every triple Y_α, $Y_{-\alpha}$, Z_α spans a copy of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ in $\mathfrak{g}^{\mathbb{C}}$, which is invariant under complex conjugation, and which can therefore be regarded as the complexification $\mathfrak{s}_\alpha^{\mathbb{C}}$ of a subalgebra $\mathfrak{s}_\alpha\subset\mathfrak{g}$. Then $\mathfrak{s}_\alpha\simeq\mathfrak{su}(2)$ if $\alpha\in\Phi^c$, and $\mathfrak{s}_\alpha\simeq\mathfrak{sl}(2,\mathbb{R})$ if $\alpha\in\Phi^n$.

Following Koranyi-Wolf [23], I introduce the Cayley transform corresponding to a noncompact root α: it is the automorphism $c_\alpha\colon\mathfrak{g}^{\mathbb{C}}\to\mathfrak{g}^{\mathbb{C}}$ given by

$$c_\alpha = \operatorname{Ad}\exp\frac{\pi}{4}(Y_{-\alpha} - Y_\alpha). \tag{2.8}$$

As direct consequences of the definition,

$$\begin{aligned} &\bar{c}_\alpha = c_\alpha^{-1}, \\ &c_\alpha = 1 \quad \text{on } \{X\in\mathfrak{h}^{\mathbb{C}} \mid \langle\alpha, X\rangle = 0\}. \end{aligned} \tag{2.9}$$

By an explicit computation in $SL(2,\mathbb{C})$, one can check that

$$\begin{aligned} &c_\alpha Z_\alpha = Y_\alpha + Y_{-\alpha}, \\ &c_\alpha^2 Z_\alpha = -Z_\alpha. \end{aligned} \tag{2.10}$$

Two roots in an abstract root system are said to be strongly orthogonal if they are not proportional, and if neither their sum nor their difference is a root. For α, $\beta\in\Phi(\mathfrak{h})$, this amounts to the relation $[\mathfrak{s}_\alpha, \mathfrak{s}_\beta] = 0$. Hence

$$c_\alpha c_\beta = c_\beta c_\alpha, \quad \text{provided } \alpha, \beta\in\Phi^n \text{ are strongly orthogonal}. \tag{2.11}$$

Now let $S\subset\Phi^n(\mathfrak{h})$ be a strongly orthogonal subset (i.e. a subset consisting of pairwise strongly orthogonal roots). In view of (2.11),

$$c_S = \textstyle\prod_{\alpha\in S} c_\alpha \tag{2.12}$$

is well defined. According to (2.9) and (2.10), the Cartan subalgebra $c_S\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$ is preserved by complex conjugation. Hence $c_S\mathfrak{h}^{\mathbb{C}} = \mathfrak{b}_S^{\mathbb{C}}$, for some Cartan subalgebra $\mathfrak{b}_S$ of the real Lie algebra $\mathfrak{g}$.

The elements $Y_\alpha \in \mathfrak{g}^\alpha$ are not completely determined by the requirements (2.7), so that the construction of the Cartan subalgebra $\mathfrak{b}_S \subset \mathfrak{g}$ depends not only on the set S, but also on the choice of the Y_α. However:

(2.13) *Remark.* Let c_S and $\tilde{c}_S$ be the Cayley transforms corresponding to the same strongly orthogonal set of noncompact roots S, but to different choices of the Y_α. Then $c_S \circ \operatorname{Ad} h = \operatorname{Ad} h \circ \tilde{c}_S$, for a suitably chosen $h \in H$. In particular, the conjugacy class of $\mathfrak{b}_S$ depends only on S.

The verification of this remark, which can be reduced to the case of $G = SL(2, \mathbb{R})$, is left to the reader.

Now let $\mathfrak{b} \subset \mathfrak{g}$ be an arbitrary Cartan subgroup. Then $\mathfrak{b}$ has a splitting

(2.14) $$\mathfrak{b} = \mathfrak{b}_+ \oplus \mathfrak{b}_-,$$

such that, under every finite dimensional representation of G, the elements of $\mathfrak{b}_+$ act with purely imaginary eigenvalues, and the elements of $\mathfrak{b}_-$ with real eigenvalues. If $\mathfrak{b}$ happens to be θ-invariant, one can also describe $\mathfrak{b}_+$ as the $(+1)$-eigenspace of θ and $\mathfrak{b}_-$ as the (-1)-eigenspace.

(2.15) **Lemma.** *Every Cartan subalgebra of the form* $\mathfrak{b}_S$, *with* S *a strongly orthogonal subset of* $\Phi^n(\mathfrak{h})$, *remains invariant under the Cartan involution* θ. *Moreover,*

$$\mathfrak{b}_{S,+} = \mathfrak{b}_S \cap \mathfrak{k} = \{X \in \mathfrak{h} \mid \langle \alpha, X \rangle = 0 \text{ for all } \alpha \in S\},$$

$$\mathfrak{b}_{S,-} = \textstyle\bigoplus_{\alpha \in S} \mathbb{R}(Y_\alpha + Y_{-\alpha}).$$

Essentially, the following observations constitute a proof: the common kernel of the element of S, viewed as linear functionals of $\mathfrak{h}$, is left pointwise fixed by c_S; the span of the iZ_α, $\alpha \in S$, is a complement of this kernel in $\mathfrak{h}$; and finally $c_S Z_\alpha = Y_\alpha + Y_{-\alpha}$ lies in $\mathfrak{p}$ whenever $\alpha \in S$ (cf. (2.10)).

According to (2.1), the Weyl group of H in G (i.e. the normalizer of H modulo H) coincides with the Weyl group of H in K. It will be referred to as $W(G, H)$, or simply as $W(H)$. I can now restate Kostant's [24] and Sugiura's [38] classification of conjugacy classes of Cartan subalgebras, in terms of data attached to the compact Cartan subgroup H. The first part of the proposition is due Martens [28].

(2.16) **Proposition.** *Every Cartan subalgebra of* $\mathfrak{g}$ *is conjugate to one of the form* $\mathfrak{b}_S$, *for some strongly orthogonal subset* S *of* $\Phi^n(\mathfrak{h})$. *Two Cartan subalgebras* $\mathfrak{b}_{S_1}$, $\mathfrak{b}_{S_2}$, *corresponding to two such sets* S_1, S_2, *are conjugate if and only if some element of* $W(H)$ *maps* $S_1 \cup (-S_1)$ *onto* $S_2 \cup (-S_2)$.

For the proof, I shall need some lemmas about abstract root systems. It will be convenient to have the following definitions: a subset S of an abstract (reduced) root system Φ is a strongly orthogonal spanning set if it consists of pairwise strongly orthogonal roots, and if every $\alpha \in \Phi$ can be represented as a $\mathbb{Q}$-linear combination of the elements of S. An abstract root system Φ has the property SO if it contains a strongly orthogonal spanning set. Clearly Φ has this property precisely when every irreducible component of Φ does.

(2.17) **Lemma.** *Let* Φ *be an irreducible root system with the property* SO, *and* $S \subset \Phi$ *a strongly orthogonal spanning set. If roots of two different lengths occur in* Φ, *then not all roots in* S *can be short.*

Proof by contradiction. Let $S=\{\gamma_1,\dots,\gamma_s\}$, with all γ_i being short roots. I renormalize the inner product on the underlying vector space, so that $(\gamma_i,\gamma_i)=1$. I now pick a long root $\alpha\in\Phi$. Since the case of the root system G_2 can be eliminated by a simple check, one must have $(\alpha,\alpha)=2$; also, $(\alpha,\gamma_i)=\pm 1$ or 0, for all i. Hence, if α is expressed as $\alpha=\sum c_i\gamma_i$, each c_i equals 0 or ± 1. On the other hand, $\sum c_i^2=(\alpha,\alpha)=2$, which immediately leads to a contradiction.

(2.18) **Lemma** (cf. Proposition II.3 of [28]). *Let Φ be an abstract root system, Φ' a sub-root system, and $\Phi''=\{\alpha\in\Phi|\alpha\perp\Phi'\}$. If Φ and Φ' both have the property SO, then so does Φ''. Moreover, the sum of the ranks of Φ' and Φ'' is then equal to the rank of Φ.*

Proof. A simple inductive argument reduces the problem to the case when $\Phi'=\{\pm\alpha\}$, for some $\alpha\in\Phi$. Also, I may assume that Φ is irreducible. By assumption, Φ contains a strongly orthogonal spanning set S. If a conjugate of α under the action of the Weyl group lies in S, there is nothing more to be done. Otherwise, since the Weyl group of an irreducible root system operates transitively on the set of roots of a given length, α must be short and every $\gamma\in S$ must be long; at this point, (2.17) has to be invoked, of course. I enumerate S as $\{\gamma_1,\dots,\gamma_s\}$, and I renormalize the inner product on the underlying vector space, so that $(\gamma_i,\gamma_i)=1$. By simple inspection, the root system G_2 can be excluded. Hence, and because α is short, $(\alpha,\alpha)=\frac{1}{2}$. If one expresses α as $\sum c_i\gamma_i$, one now finds that $\sum c_i^2=\frac{1}{2}$, and $c_i=\pm\frac{1}{2}$ or 0. Thus, after renumbering the γ_i and replacing some by their negatives, if necessary, $\alpha=\frac{1}{2}(\gamma_1-\gamma_2)$. But then $\beta=\frac{1}{2}(\gamma_1+\gamma_2)$ is also a root. For $i\neq 1, 2$, $\beta\pm\gamma_i$ is longer than γ_i and cannot be a root. Consequently, Φ'' contains $\{\beta,\gamma_3,\dots,\gamma_s\}$ as a strongly orthogonal spanning set.

(2.19) **Lemma.** *In an abstract root system with the property SO, any two strongly orthogonal spanning sets are conjugate under the action of the Weyl group.*

Proof by induction of the rank. I may assume that the root system Φ in question is irreducible. Let S_1, S_2 be two strongly orthogonal spanning sets. Because of (2.17), and because the Weyl group acts transitively on the long roots, S_1 and some conjugate of S_2 have a root α in common. But this reduces the problem to the orthogonal complement of α in Φ, whose rank is lower by one.

I return to the setting of Proposition (2.16). The next few lemmas, besides entering the proof of (2.16), will be needed later.

(2.20) **Lemma.** *Let S_1, S_2 be two strongly orthogonal subsets of $\Phi^n(\mathfrak{h})$, such that some element w of $W(H)$ maps $S_1\cup(-S_1)$ onto $S_2\cup(-S_2)$. Then, for a suitably chosen representative $k\in K$ of w, $\mathrm{Ad}\,k\circ c_{S_1}=c_{S_2}\circ\mathrm{Ad}\,k$.*

Proof. The problem can be subdivided into two special cases, namely $wS_1=S_2$ on the one hand, and $w=1$, $S_1\cup(-S_1)=S_2\cup(-S_2)$ on the other. To deal with the first situation, I begin with an arbitrary representative $k\in K$ of w. As follows from the definition of the Cayley transform, $\mathrm{Ad}\,k\circ c_{S_1}\circ\mathrm{Ad}\,k^{-1}$ is the Cayley transform corresponding to S_2, but a possibly different choice of the Y_α. Thus one can quote (2.13): if k is modified by some $h\in H$, one obtains $\mathrm{Ad}\,k\circ c_{S_1}\circ\mathrm{Ad}\,k^{-1}=c_{S_2}$. In the second special case, when $w=1$, there exists an $h\in H$, such that $\mathrm{Ad}\,h(Y_\alpha)=Y_\alpha$ whenever $\pm\alpha\in S_1\cap S_2$, and $\mathrm{Ad}\,h(Y)=-Y_\alpha$ whenever $\pm\alpha\in S_1\cap(-S_2)$. But then $k=h$ has the desired property.

I now consider an arbitrary Cartan subalgebra $\mathfrak{b}\subset\mathfrak{g}$. The root system of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$ will be denoted by $\Phi(\mathfrak{b})$. In terms of the decomposition $\mathfrak{b}=\mathfrak{b}_+\oplus\mathfrak{b}_-$ of (2.14), I define the two sub-root systems

$$\Phi(\mathfrak{b})_+=\{\alpha\in\Phi(\mathfrak{b})\,|\,\langle\alpha,\mathfrak{b}_-\rangle=0\},\qquad \Phi(\mathfrak{b})_-=\{\alpha\in\Phi(\mathfrak{b})\,|\,\langle\alpha,\mathfrak{b}_+\rangle=0\}. \tag{2.21}$$

Equivalently, $\Phi(\mathfrak{b})_+$ consists of all roots which assume purely imaginary values on $\mathfrak{b}$, and $\Phi(\mathfrak{b})_-$ of all roots with real values on $\mathfrak{b}$. At this point, it should be remembered that G has a compact Cartan subgroup.

(2.22) **Lemma.** *The root system $\Phi(\mathfrak{b})_-$ has the property SO. Its elements span the dual space of $\mathfrak{b}_-$.*

Proof. Let $\mathfrak{a}\subset\mathfrak{g}$ be a maximally split Cartan subalgebra, with $\mathfrak{b}_-\subset\mathfrak{a}$. If the splitting $\mathfrak{a}=\mathfrak{a}_+\oplus\mathfrak{a}_-$ is defined in analogy to (2.14), $\mathfrak{b}_-$ must lie in $\mathfrak{a}_-$. As is asserted by [38], $\Phi(\mathfrak{a})$ contains a strongly orthogonal subset $\{\gamma_1,\dots,\gamma_s\}$, such that

$$\mathfrak{a}_+\oplus\mathfrak{b}_-=\{X\in\mathfrak{a}\,|\,\langle\gamma_i,X\rangle=0,\ 1\leqq i\leqq s\}. \tag{2.23}$$

Corresponding to each γ_i, since γ_i assumes real values on $\mathfrak{a}$, there exists a copy of $\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{g}$, which is spanned by real generators of the $\pm\gamma_i$-root spaces and by their commutator. Because the γ_i are strongly orthogonal, any two of these copies of $\mathfrak{sl}(2,\mathbb{R})$ commute. Their product is a subalgebra $\mathfrak{s}\subset\mathfrak{g}$, of rank s, which centralizes $\mathfrak{a}_+\oplus\mathfrak{b}_-$, and which contains $\mathfrak{s}\cap\mathfrak{a}_-$ as a split Cartan subalgebra. Evidently $\mathfrak{s}$ has a toroidal Cartan subalgebra $\mathfrak{t}$, whose direct sum with $\mathfrak{a}_+\oplus\mathfrak{b}_-$ becomes a Cartan subalgebra of $\mathfrak{g}$. The conjugacy class of a Cartan subalgebra depends only on its split part [38]. Hence one can replace $\mathfrak{b}$ by one of its conjugates, without altering $\mathfrak{b}_-$, so as to arrange $\mathfrak{b}_+=\mathfrak{a}_+\oplus\mathfrak{t}$. In $\mathfrak{s}^{\mathbb{C}}$, $\mathfrak{t}^{\mathbb{C}}$ and $\mathfrak{s}^{\mathbb{C}}\cap\mathfrak{a}_-^{\mathbb{C}}$ are conjugate. A suitable inner automorphism of $\mathfrak{g}^{\mathbb{C}}$ therefore provides an isomorphism $\mathfrak{a}^{\mathbb{C}}\simeq\mathfrak{b}^{\mathbb{C}}$, which acts as the identity on $\mathfrak{b}_-^{\mathbb{C}}$, and which maps $\mathfrak{b}_+^{\mathbb{C}}$ into $\mathfrak{a}_+^{\mathbb{C}}\oplus\mathfrak{s}^{\mathbb{C}}$. Under this isomorphism, $\Phi(\mathfrak{b})_-$ corresponds to a sub-root system of $\Phi(\mathfrak{a})_-$, namely to

$$\{\alpha\in\Phi(\mathfrak{a})_-\,|\,\alpha\perp\gamma_i,\ 1\leqq i\leqq s\} \tag{2.24}$$

(cf. (2.23)). Since $\mathfrak{g}$ contains a toroidal Cartan subalgebra, $\Phi(\mathfrak{a})_-$ has the property *SO* [38]. According to (2.18), the root system (2.24), and thus $\Phi(\mathfrak{b})_-$, must also have this property; moreover,

$$rk\,\Phi(\mathfrak{b})_-=rk\,\Phi(\mathfrak{a})_- - s=\dim\mathfrak{a}_- - s=\dim\mathfrak{b}_-,$$

as was to be shown.

(2.25) **Lemma.** *Let $W(\mathfrak{b})_-$ be the subgroup of the Weyl group of $(\mathfrak{g}^{\mathbb{C}},\mathfrak{b}^{\mathbb{C}})$ generated by the reflections about the root hyperplanes corresponding to the roots in $\Phi(\mathfrak{b})_-$. Then every $w\in W(\mathfrak{b})_-$ can be realized by an element of the normalizer of $\mathfrak{b}$ in G.*

Proof. Every $\alpha\in\Phi(\mathfrak{b})_-$ is real on $\mathfrak{b}$. The α-root space is therefore spanned by some $E_\alpha\in\mathfrak{g}$. The E_α's, together with $\mathfrak{b}_-$, generate a semisimple (because of (2.22)) subalgebra $\mathfrak{g}_-\subset\mathfrak{g}$; by construction, $\mathfrak{g}_-$ centralizes $\mathfrak{b}_+$ and contains $\mathfrak{b}_-$ as a split Cartan subalgebra. To $\mathfrak{g}_-$, there corresponds a connected subgroup G_- of G.

According to standard results, the normalizer of $\mathfrak{b}_-$ in G_- acts on $\mathfrak{b}_-$ as the full Weyl group of $(\mathfrak{g}^{\mathbb{C}}_-, \mathfrak{b}^{\mathbb{C}}_-)$. Hence the lemma.

Since the roots in $\Phi(\mathfrak{b})_-$ assume real values on $\mathfrak{b}$, for each $\alpha \in \Phi(\mathfrak{b})_-$ one can choose a real generator $E_\alpha \in \mathfrak{g}$ of the α-root space of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$, and an element $F_\alpha \in \mathfrak{b}_-$, subject to the following conditions:

$$(2.26) \qquad \begin{aligned} &[F_\alpha, E_\alpha] = 2E_\alpha, \qquad [F_\alpha, E_{-\alpha}] = -2E_{-\alpha}, \\ &[E_\alpha, E_{-\alpha}] = F_\alpha = -F_{-\alpha}. \end{aligned}$$

According to (2.22), $\Phi(\mathfrak{b})_-$ contains a strongly orthogonal spanning set $\{\alpha_1, \ldots, \alpha_s\}$. For $i \neq j$, $[E_{\pm\alpha_i}, E_{\pm\alpha_j}] = 0$. Hence the product

$$\prod_{j=1}^s \exp \frac{i\pi}{4} (E_{\alpha_j} + E_{-\alpha_j})$$

is well-defined. In order to state the next lemma, I assume that $\mathfrak{b} = \mathfrak{b}_S$, for some strongly orthogonal subset $S \subset \Phi^n(\mathfrak{h})$. As usual, c_S denotes the Cayley transform (2.12).

(2.27) **Lemma.** *Under the hypotheses just mentioned, there exists an element b of the normalizer of* $\mathfrak{b}$ *in G, which operates on $\mathfrak{b}^{\mathbb{C}}$ as an element of $W(\mathfrak{b})_-$, such that*

$$c_S \circ \mathrm{Ad}\, b = \mathrm{Ad}\, b \circ \prod_{j=1}^s \mathrm{Ad} \exp \frac{i\pi}{4} (E_{\alpha_j} + E_{-\alpha_j}).$$

It should be emphasized that the statement holds regardless of the particular choices which were made in defining the various ingredients of the lemma.

Proof. The isomorphism $c_S: \mathfrak{h}^{\mathbb{C}} \simeq \mathfrak{b}^{\mathbb{C}}$ induces an isomorphism of the dual spaces

$$c_S^*: \mathfrak{b}^{\mathbb{C}*} \xrightarrow{\sim} \mathfrak{h}^{\mathbb{C}*},$$

whose inverse maps S onto a strongly orthogonal spanning set of $\Phi(\mathfrak{b})_-$. Taking into account (2.19), one can enumerate the elements of S as $\gamma_1, \ldots, \gamma_s$, and one can find some $w \in W(\mathfrak{b})_-$, so that

$$(2.28) \qquad c_S^* w \alpha_j = \gamma_j, \qquad \text{for } 1 \leqq j \leqq s.$$

Because of (2.25), w can be realized by an element b_1 of the normalizer of $\mathfrak{b}$ in G. For each j, $\tilde{F}_{\alpha_j} = \mathrm{Ad}\, b_1 (c_S Z_{\gamma_j})$ lies in the Lie product of the root spaces of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$ corresponding to α_j and $-\alpha_j$, and $\langle \alpha_j, \tilde{F}_{\alpha_j} \rangle = 2$, as follows from (2.27), (2.28), and the fact that c_S is an automorphism of $\mathfrak{g}^{\mathbb{C}}$. Hence, in the notation of (2.26), $\tilde{F}_{\alpha_j} = F_{\alpha_j}$; i.e.

$$(2.29) \qquad \mathrm{Ad}\, b_1 (c_S Z_{\gamma_j}) = F_{\alpha_j}, \qquad 1 \leqq j \leqq s.$$

Next, I introduce

$$(2.30) \qquad \tilde{E}_{\alpha_j} = i\, \mathrm{Ad}\, b_1 (c_S Y_{\gamma_j}), \qquad \tilde{E}_{-\alpha_j} = -i\, \mathrm{Ad}\, b_1 (c_S Y_{-\gamma_j}).$$

By an explicit computation in $SL(2, \mathbb{C})$, one checks that $c_S^{-1} Y_{-\gamma_j} = -c_S Y_{\gamma_j}$. Hence the complex conjugate of $\tilde{E}_{\alpha_j}$ is

$$-i\, \mathrm{Ad}\, b_1 (\bar{c}_S Y_{-\gamma_j}) = -i\, \mathrm{Ad}\, b_1 (c_S^{-1} Y_{-\gamma_j}) = \tilde{E}_{\alpha_j},$$

so that $\tilde{E}_{\alpha_j} \in \mathfrak{g}$; similarly, $\tilde{E}_{-\alpha_j} \in \mathfrak{g}$. In view of their definition and of (2.29), the $\tilde{E}_{\pm\alpha_j}$ are therefore real generators of the $\pm\alpha_j$-root spaces of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$, and in place of $E_{\pm\alpha_j}$ they satisfy the identity (2.26), for $\alpha = \alpha_j$. But this is possible only if $\tilde{E}_{\alpha_j} = a_j E_{\alpha_j}$, $\tilde{E}_{-\alpha_j} = a_j^{-1} E_{-\alpha}$, with a suitable nonzero constant $a_j \in \mathbb{R}$. Modifying b_1 by an element of $\exp \mathfrak{b}_-$, one can arrange $a_j = \pm 1$. Thus

$$(2.31) \qquad i \operatorname{Ad} b_1 (c_S Y_{\gamma_j}) = \varepsilon_j E_{\alpha_j}, \qquad i \operatorname{Ad} b_1 (c_S Y_{-\gamma_j}) = -\varepsilon_j E_{-\alpha_j},$$

with $\varepsilon_j = \pm 1$. For each j, $\operatorname{Ad} \exp \frac{\pi}{2} (E_{\alpha_j} - E_{-\alpha_j})$ maps F_{α_j} to its negative, E_{α_j} to $-E_{-\alpha_j}$, $E_{-\alpha_j}$ to $-E_{\alpha_j}$, and it acts as the identity on the α_j-hyperplane in $\mathfrak{b}$; this can again be checked by a computation in $SL(2, \mathbb{R})$. If one defines b_2 as the product of $\operatorname{Ad} \exp \frac{\pi}{2} (E_{\alpha_j} - E_{-\alpha_j})$, extended over those j for which $\varepsilon_j = -1$, then

$$\operatorname{Ad} b_2 (E_{\alpha_j} + E_{-\alpha_j}) = \varepsilon_j (E_{\alpha_j} + E_{-\alpha_j}), \qquad 1 \leqq j \leqq s.$$

Also, b_2 normalizes $\mathfrak{b}$, and $\operatorname{Ad} b_2$ operates on $\mathfrak{b}$ as an element of $W(\mathfrak{b})_-$. I now set $b = b_1^{-1} b_2^{-1}$. As a consequence of (2.31) and the definition of c_S, for $1 \leqq j \leqq s$,

$$i \operatorname{Ad} b (E_{\alpha_j} + E_{-\alpha_j}) = c_S (Y_{-\gamma_j} - Y_{\gamma_j}) = (Y_{-\gamma_j} - Y_{\gamma_j}).$$

This finally implies the identity concerning c_S. The assertion about b has already been verified individually for both b_1 and b_2.

Proof of (2.16). Let $\mathfrak{b} \subset \mathfrak{g}$ be an arbitrary Cartan subalgebra. According to (2.22), $\Phi(\mathfrak{b})_-$ has a strongly orthogonal spanning set $\{\alpha_1, \ldots, \alpha_s\}$, with $s = \dim \mathfrak{b}_-$. Using these roots, one can construct a subalgebra $\mathfrak{s} \subset \mathfrak{g}$, which centralizes $\mathfrak{b}_+$, and which contains $\mathfrak{b}_-$ as a split Cartan subalgebra. The direct sum of $\mathfrak{b}_+$ with a toroidal Cartan subalgebra of $\mathfrak{s}$ is a toroidal Cartan subalgebra of $\mathfrak{g}$, and hence conjugate to $\mathfrak{h}$. Replacing $\mathfrak{b}$ by one of its conjugates, one can therefore arrange that $\mathfrak{h} \subset \mathfrak{b}_+ \oplus \mathfrak{s}$, $\mathfrak{b} \subset \mathfrak{b}_+ \oplus \mathfrak{s}$. This reduces the first statement of the proposition to the special case $\mathfrak{g} \simeq \mathfrak{sl}(2, \mathbb{R})$, for which it is easy to check.

The "if" part of the second statement is a direct consequence of (2.20). In order to prove the "only if" part, I consider two strongly orthogonal subsets S_1, S_2 of $\Phi^n(\mathfrak{h})$, such that the Cartan subalgebras $\mathfrak{b}_{S_1}$ and $\mathfrak{b}_{S_2}$ are conjugate, say

$$(2.32) \qquad \mathfrak{b}_{S_2} = \operatorname{Ad} g \, \mathfrak{b}_{S_1},$$

with $g \in G$. An application of (2.27) to $\mathfrak{b} = \mathfrak{b}_{S_1}$ gives the identity

$$c_{S_1} \circ \operatorname{Ad} b_1 = \operatorname{Ad} b_1 \circ \textstyle\prod_{j=1}^{s} \operatorname{Ad} \exp \frac{i\pi}{4} (E_{\alpha_j} + E_{-\alpha_j}),$$

for a suitable b_1 in the normalizer of $\mathfrak{b}_{S_1}$, which operates on $\mathfrak{b}_{S_1}$ as an element of $W(\mathfrak{b}_{S_1})_-$. On the other hand, since

$$\begin{aligned} \mathfrak{b}_{S_1}^{\mathbb{C}} &= \operatorname{Ad} g^{-1} \mathfrak{b}_{S_2}^{\mathbb{C}} = \operatorname{Ad} g^{-1} \circ c_{S_2}(\mathfrak{h}^{\mathbb{C}}) \\ &= \operatorname{Ad} g^{-1} \circ c_{S_2} \circ \operatorname{Ad} g (\operatorname{Ad} g^{-1} \mathfrak{h}^{\mathbb{C}}), \end{aligned}$$

one may view $\operatorname{Ad} g^{-1} \circ c_{S_2} \circ \operatorname{Ad} g$ as a Cayley transform, where now $\operatorname{Ad} g^{-1} H$ plays the role of H. Thus another application of (2.27) leads to the identity

$$\operatorname{Ad} g^{-1} \circ c_{S_2} \circ \operatorname{Ad} g \circ \operatorname{Ad} b_2 = \operatorname{Ad} b_2 \circ \textstyle\prod_{j=1}^{s} \operatorname{Ad} \exp \frac{i\pi}{4} (E_{\alpha_j} + E_{-\alpha_j}),$$

with the usual conditions on b_2. The two identities about c_{S_1} and c_{S_2} together imply

$$c_{S_2} \circ \operatorname{Ad} g \circ \operatorname{Ad} b = \operatorname{Ad} g \circ \operatorname{Ad} b \circ c_{S_1};$$

here $b = b_2 b_1^{-1}$ normalizes $\mathfrak{b}_{S_1}$ and operates on $\mathfrak{b}_{S_1}$ as an element of $W(\mathfrak{b}_{S_1})_-$. I set $k = gb$, so that the preceeding identity becomes $c_{S_2} \circ \operatorname{Ad} k = \operatorname{Ad} k \circ c_{S_1}$. Because of (2.32),

$$\begin{aligned} \operatorname{Ad} k\, \mathfrak{h}^{\mathbb{C}} &= \operatorname{Ad} k \circ c_{S_1}^{-1}(\mathfrak{b}_{S_1}^{\mathbb{C}}) = c_{S_2}^{-1} \circ \operatorname{Ad} k(\mathfrak{b}_{S_1}^{\mathbb{C}}) \\ &= c_{S_2}^{-1} \circ \operatorname{Ad} g \circ \operatorname{Ad} b(\mathfrak{b}_{S_1}^{\mathbb{C}}) = c_{S_2}^{-1}\, \mathfrak{b}_{S_2}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}}. \end{aligned}$$

Thus k normalizes H (hence $k \in K$, cf. (2.1)). As can be read off from (2.15), because $\operatorname{Ad} k\, b_{S_1} = \mathfrak{b}_{S_2}$, $\operatorname{Ad} k$ must map $\bigoplus_{\pm\alpha \in S_1} \mathfrak{g}^{\alpha}$ onto $\bigoplus_{\pm\alpha \in S_2} \mathfrak{g}^{\alpha}$. This is possible only if the element $w \in W(H)$ which $\operatorname{Ad} k$ represents maps $S_1 \cup (-S_1)$ onto $S_2 \cup (-S_2)$. The proof of (2.16) is now complete.

To each Cartan subalgebra $\mathfrak{b} \subset \mathfrak{g}$, there corresponds a Cartan subgroup, namely the centralizer of $\mathfrak{b}$ in G. Since G was assumed to be a matrix group, every Cartan subgroup B is Abelian. Clearly two Cartan subgroups are conjugate precisely when the associated Cartan subalgebras are; hence (2.16) also provides a classification of conjugacy classes of Cartan subgroups.

It will be necessary to have certain information about the Weyl group of a general Cartan subgroup B. This Weyl group, i.e. the normalizer of B in G modulo B, will be denoted by $W(B)$ or $W(G, B)$. In a natural manner, $W(B)$ can be identified with a subgroup of the Weyl group of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$, for which I shall use the symbol $W(B)_{\mathbb{C}}$:

(2.33) $$W(B) \subset W(B)_{\mathbb{C}}.$$

I recall the definition of $W(\mathfrak{b})_-$ in (2.25). To have consistent notation, this group will now be referred to as $W(B)_-$. As follows from its definition,

(2.34) $W(B)_-$ is a normal subgroup of $W(B)$.

In order to understand $W(B)$ better, it is helpful to relate it to the Weyl group $W(H)$ of a compact Cartan subgroup H. For this purpose, without loss of generality, I assume that $\mathfrak{b} = \mathfrak{b}_S$, for some strongly orthogonal subset S of $\Phi^n(\mathfrak{h})$; I now write B_S instead of B. Via the Cayley transform c_S: $\mathfrak{h}^{\mathbb{C}} \simeq \mathfrak{h}_S^{\mathbb{C}}$, one can transfer $W(H)$ and its subgroups to subgroups of $W(B_S)_{\mathbb{C}}$. In particular,

(2.35) $$U(B_S) = \{c_S \circ w \circ c_S^{-1} \mid w \in W(H), wS \subset S \cup (-S)\}$$

is a subgroup of $W(B_S)_{\mathbb{C}}$ (here $W(B_S)_{\mathbb{C}}$ is viewed as a transformation group on $\mathfrak{b}_S^{\mathbb{C}}$). The notation $U(B_S)$ is not entirely satisfactory, because the group (2.35) depends not only on B_S, but also on the choices of H and S; however, bringing out this dependence in the notation would lead to complications.

(2.36) **Proposition.** *The Weyl group $W(B_S)$ contains $U(B_S)$. Every $w \in W(B_S)$ can be expressed as a product $w_1 w_2$, with $w_1 \in U(B_S)$, $w_2 \in W(B_S)_-$.*

In order to have a more complete description of $W(B_S)$, one must determine $W(B_S)_- \cap U(B_S)$; this will be done in (2.43) below.

Proof. Suppose $w \in W(H)$ maps S into $S \cup (-S)$, and let $k \in K$ be an element of K which represents w. I now apply (2.20), with $S_1 = S$, $S_2 = \operatorname{Ad} k^{-1}(S)$, $c_{S_2} = \operatorname{Ad} k^{-1} \circ c_S \circ \operatorname{Ad} k$, and for the trivial element of $W(H)$. Conclusion: if k is properly chosen, one can arrange that $c_S \circ \operatorname{Ad} k = \operatorname{Ad} k \circ c_S$. Hence $c_S \circ w \circ c_S^{-1}$ can be realised as $\operatorname{Ad} k$, which implies $c_S \circ w \circ c_S^{-1} \in W(B_S)$. To prove the second statement, I let $w \in W(B_S)$ be given, and I represent it as $\operatorname{Ad} g$, $g \in G$. In particular, $\mathfrak{b}_S = \operatorname{Ad} g\, \mathfrak{b}_S$. One can now proceed just as in the proof of (2.16), with $S_1 = S_2 = S$. It is argued there that $g = k b^{-1}$, for some k in the normalizer of H and some b in the normalizer of B; moreover,

a) $\operatorname{Ad} b$ operates on $\mathfrak{b}_S$ as an element of $W(B_s)_-$;

b) $\operatorname{Ad} k$, viewed as element of $W(H)$, maps S into $S \cup (-S)$;

c) $c_S \circ \operatorname{Ad} k \circ c_S^{-1} = \operatorname{Ad} k$.

Hence, if one sets w_1 and w_2 equal to the elements of $W(B_S)$ determined by, respectively, $\operatorname{Ad} k$ and $\operatorname{Ad} b^{-1}$, one obtains the desired decomposition of w.

As a matrix group, G has a complexification $G^{\mathbb{C}}$; $G^{\mathbb{C}}$ is a connected, reductive, complex Lie group, which contains G as a real form. When the Lie algebra of $G^{\mathbb{C}}$ is identified with $\mathfrak{g}^{\mathbb{C}}$, the subgroup $G \subset G^{\mathbb{C}}$ corresponds to the subalgebra $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$. Any given Cartan subgroup B of G can be complexified to a Cartan subgroup $B^{\mathbb{C}}$ of $G^{\mathbb{C}}$. The exponential map of $G^{\mathbb{C}}$ maps the Lie algebra $\mathfrak{b}^{\mathbb{C}}$ of $B^{\mathbb{C}}$ onto $B^{\mathbb{C}}$. In particular, every $b \in B$ can be written as $b = \exp X$, for some $X \in \mathfrak{b}^{\mathbb{C}}$; of course, X need not lie in $\mathfrak{b}$. This fact will be used in the statement of (2.38) below.

I continue to consider an arbitrary Cartan subgroup $B \subset G$, with Lie algebra $\mathfrak{b}$. The definition of the splitting $\mathfrak{b} = \mathfrak{b}_+ \oplus \mathfrak{b}_-$ in (2.14) should be recalled. For the purpose of stating the next proposition, it is necessary to introduce the weight lattice $\Lambda(\Phi(B))$ of the root system $\Phi(B)$; one must be slightly careful here, because G need not be semisimple:

$\lambda \in i\mathfrak{b}_+^* \oplus \mathfrak{b}_-^*$ belongs to $\Lambda(\Phi(B))$ if and only if

(2.37) a) the linear functional λ vanishes on the center of $\mathfrak{g}$, and
b) $2(\lambda, \alpha)(\alpha, \alpha)^{-1} \in \mathbb{Z}$, for all $\alpha \in \Phi(B)$.

(In b), $(\ ,\ )$ denotes an inner product on $i\mathfrak{b}_+^* \oplus \mathfrak{b}_-^*$ coming from the trace form of faithful finite dimensional representation of G).

Although the following statement is certainly not unknown, I could nor find a convenient reference for it (however, cf. 1.4.1.3 of [39]).

(2.38) **Proposition.** *There exists a unique direct product decomposition $B = B_+ \cdot B_-$, such that*

a) *B_+ is compact, and its identity component B_+^0 is a torus, with Lie algebra $\mathfrak{b}_+$;*

b) *B_- is connected and is isomorphic, via the exponential map, to its Lie algebra $\mathfrak{b}_-$.*

Moreover, B_+/B_+^0 can be generated by

$$\{\exp(iX) \mid X \in \mathfrak{b}_-, \langle \lambda, X \rangle \in \pi \mathbb{Z}, \textit{for all } \lambda \in \Lambda(\Phi(B))\}.$$

The latter is a finite subgroup of B_+, in which every element has order two.

Proof. With respect to a given faithful finite dimensional representation of G, every $b \in \exp \mathfrak{b}_-$ has positive eigenvalues, whereas the elements of a compact subgroup have eigenvalues of modulus one. Also, B can be diagonalized over $\mathbb{C}$. Hence the decomposition, of it is known to exist at all, is necessarily direct and uniquely determined. In order to verify the existence of the decomposition, it should be observed that G is the product of a connected semisimple subgroup and a central torus; this central torus is contained in $\exp \mathfrak{b}_+$. Hence, without loss of generality, I may assume G to be semisimple. If the decomposition of B, including the statement about B_+/B_+^0, has been established in some finite covering of G, it then also follows for G. Thus I can further reduce the problem, by requiring that $G^{\mathbb{C}}$ be simply connected. Now let $b \in B$ be given. According to the remarks above the proposition, $b = \exp(X + iY)$, with $X, Y \in \mathfrak{b}$. Clearly $\exp \mathfrak{b}$ lies in B, so that

$$B = \exp \mathfrak{b}_- \cdot \exp \mathfrak{b}_+ \cdot (G \cap \exp(i\mathfrak{b}));$$

here the first factor is isomorphic to $\mathfrak{b}_-$ via exp, and the second is a torus. It remains to identify the last factor.

Complex conjugation in $\mathfrak{g}^{\mathbb{C}}$ lifts to an automorphism of $G^{\mathbb{C}}$, which will also be denoted by $g \mapsto \bar{g}$. Since $G^{\mathbb{C}}$ is, by assumption, semisimple and simply connected, the group of real points in $G^{\mathbb{C}}$ is known to be connected (e.g. (4.7) of [4]); it therefore coincides with G. Thus, if $X \in \mathfrak{b}$,

$$\exp(iX) \in G \Leftrightarrow \exp(iX) = \exp(\overline{iX})$$
$$\Leftrightarrow \exp(iX) = \exp(-iX) \Leftrightarrow \exp(2iX) = 1.$$

In the simply connected group $G^{\mathbb{C}}$, any $Z \in \mathfrak{b}^{\mathbb{C}}$ exponentiates to the identity precisely when

(2.39) $$\langle \lambda, Z \rangle \in 2\pi i \mathbb{Z}, \quad \text{for all } \lambda \in \Lambda(\Phi(B)).$$

On $\mathfrak{b}_+$, every weight λ assumes purely imaginary values, on $\mathfrak{b}_-$ real values. For $Z = 2iX$, with $X \in \mathfrak{b}$, (2.39) now implies $X \in \mathfrak{b}_-$. Hence

$$G \cap \exp(i\mathfrak{b}) = \{\exp(iX) \mid X \in \mathfrak{b}_-, \langle \lambda, X \rangle \in \pi \mathbb{Z} \text{ for all } \lambda \in \Lambda(\Phi(\mathrm{B}))\}.$$

As was already shown, every element of this group has order two. Since it lies in the torus $\exp(i\mathfrak{b}_- \oplus \mathfrak{b}_+)$, it must also be finite. The proposition follows.

More delicate information about the connected components of a Cartan subalgebra B can be obtained by considering the centralizer of B_+^0 in G.

(2.40) **Lemma.** *The centralizer of B_+^0 has a factorization (not necessarily direct) as $B_+^0 \cdot G_B$, such that G_B is a connected, semisimple subgroup of G. In G_B, $B \cap G_B$ lies as a split Cartan subgroup. The rank of a maximal compact subgroup K_B of G_B agrees with the rank of G_B. If G/K is Hermitian symmetric, then so is G_B/K_B. In this situation, every simple factor of G_B is isomorphic, up to covering, to $Sp(n, \mathbb{R})$, for some n.*

Proof. Together with $\mathfrak{b}_-$, the E_α corresponding to $\alpha \in \Phi(\mathfrak{b})_-$ (notation of (2.26)) span a subalgebra $\mathfrak{g}_B \in \mathfrak{g}$. Because of (2.22), $\mathfrak{g}_B$ is semisimple; it contains $\mathfrak{b}_-$ as a split Cartan subalgebra. Let G_B be the connected subgroup with Lie algebra $\mathfrak{g}_B$. Virtually by construction, $B_+^0 \cdot G_B$ is at least the identity component of the centralizer of B_+^0. To prove the connectedness of the centralizer, I look at a typical element g. Since $\operatorname{Ad} g(\mathfrak{g}_B) = \mathfrak{g}_B$, and since any two split Cartan subalgebras of $\mathfrak{g}_B$ are conjugate, modifying g by a suitable element of G_B, one can arrange that $\operatorname{Ad} g(\mathfrak{b}_-) = \mathfrak{b}_-$, i.e. $g \in B$. The connectedness of the centralizer thus comes down to the containment $B \subset B_+^0\, G_B$. In analogy to $\Lambda(\Phi(B))$, I set

$$\Lambda(\Phi(\mathfrak{b})_-) = \{\lambda \in \mathfrak{b}_-^* \mid 2(\lambda, \alpha)(\alpha, \alpha)^{-1} \in \mathbb{Z} \text{ for all } \alpha \in \Phi(\mathfrak{b})_-\}.$$

Applying (2.38) to both B in G and to $B \cap G_B$ in G_B, one would be able to deduce $B \subset B_+^0\, G_B$ from the following statement:

(2.41)
$$\text{for every } \mu \in \Lambda(\Phi(\mathfrak{b})_-), \text{ there exists some } \lambda \in \Lambda(\Phi(B)), \quad \text{such that } \mu = \lambda|_{\mathfrak{b}_-}.$$

To verify (2.41), I choose a system of simple roots $\{\alpha_1, \dots, \alpha_r\}$ in $\Phi(B)$, such that $\{\alpha_1, \dots, \alpha_s\}$, $s \leqq r$, forms a system of simple roots for the sub-root system $\Phi(\mathfrak{b})_-$; this can be done because $\mathfrak{g}_B^{\mathbb{C}}$ is the derived algebra of the centralizer in $\mathfrak{g}^{\mathbb{C}}$ of a subalgebra of the Cartan algebra $\mathfrak{b}^{\mathbb{C}}$. If $\lambda_1, \dots, \lambda_r \in \Lambda(\Phi(B))$ are the fundamental weights dual to $\alpha_1, \dots, \alpha_r$, then $\lambda_1, \dots, \lambda_s$ restrict to generators of $\Lambda(\Phi(\mathfrak{b})_-)$. This implies (2.41), and hence also the connectedness of the centralizer of B_+^0. As for the remaining assertions, since $\mathfrak{b}_-$ is a split Cartan subalgebra of $\mathfrak{g}_B$, B must intersect G_B in a split Cartan subgroup. Conjugating B if necessary, one may assume $B_+^0 \subset H$. But then H is a Cartan subgroup of the centralizer of B_+^0, and hence $H \cap G_B$ turns out to be a compact Cartan subgroup of G_B. Again if $B_+^0 \subset H$, G_B remains stable under the Cartan involution θ. Thus $K \cap G_B$ becomes a maximal compact subgroup of G_B. The quotient $G_B/K \cap G_B$ can be naturally identified with the $B_+^0 \cdot G_B$-orbit of the identity coset in G/K. This orbit inherits an invariant complex structure from that of G/K, provided G/K is Hermitian symmetric, because the group $B_+^0 \cdot G_B$ contains the center of K: at the tangent space of the identity coset in G/K, the complex structure tensor is given by an element of the center of K (see Chapter VIII of [17]). Finally, every simple factor of G_B contains both a split and a compact Cartan subgroup. If it is also known to operate on a Hermitian symmetric space, this narrows down the possibilities to $Sp(n, \mathbb{R})$ or its adjoint group, as can be deduced from Moore's [30] description of the root system of a Hermitian symmetric space, or in a number of other ways.

(2.42) *Remark.* If $B = B_S$, for some strongly orthogonal subset $S \subset \Phi^n(\mathfrak{h})$, I shall write G_S instead of G_{B_S}. In this case, $H \cap G_S$ lies in G_S as a compact Cartan subgroup, whose Lie algebra is spanned by $\{iZ_\alpha \mid \alpha \in S\}$ (cf. (2.7)). Moreover, $H = B_{S,+}^0 \cdot (G_S \cap H)$.

Indeed, H contains $B_{S,+}^0$, and this implies that $H \cap G_S$ is a Cartan subgroup of G_S. For every $\alpha \in S$, iZ_α belongs to the derived algebra of the centralizer of $\mathfrak{b}_{S,+}$ in $\mathfrak{g}^{\mathbb{C}}$. On the other hand, $iZ_\alpha \in \mathfrak{h}$. Hence the Lie algebra of $G_S \cap H$ at least contains the span of $\{iZ_\alpha\, \alpha \in S\}$. The dimension of the latter, namely the cardinality of S, agrees with the rank of G_S, so that the containment must be an equality.

Finally, as a count of dimensions shows, $B^0_{S,+} \cdot (G_S \cap H)$ — which is obviously a subtorus of H — cannot be smaller than H.

At this point, it is easy to complete the description of the Weyl group which was given in (2.36). Since G_S centralizes $B^0_{S,+}$, every g in the normalizer of $G_S \cap H$ in G_S commutes with $B^0_{S,+}$; in particular, g normalizes H. Hence the Weyl group $W(G_S, G_S \cap H)$ can be embedded, in a natural manner, into $W(G, H)$. In analogy to the group $U(B_S)$, I define

$$U(G_S, G_S \cap B_S) = \{c_s \circ w \circ c_s^{-1} \mid w \in W(G_S, G_S \cap H), wS \subset S \cup (-S)\}.$$

(2.43) **Corollary.** *In the notation of* (2.36),

$$U(B_S) \cap W(B_S)_- = U(G_S, G_S \cap B_S).$$

Proof. According to the proof of (2.36), if $w \in W(G, H)$ maps S into $S \cup (-S)$, then w has a representation as $\operatorname{Ad} k$, such that $c_S \circ \operatorname{Ad} k \circ c_S^{-1} = \operatorname{Ad} k$. Hence $c_S \circ w \circ c_S^{-1}$, viewed as an element of $W(B_S)$, is also represented by $\operatorname{Ad} k$. If $c_S \circ w \circ c_S^{-1} \in W(B_S)_-$, $\operatorname{Ad} k$ must operate trivially on $B^0_{S,+}$, so that $k \in B^0_{S,+} \cdot G_S$. But then $w \in W(G_S, G_S \cap H)$, which proves the containment $U(B_S) \cap W(B_S)_- \subset U(G_S, G_S \cap B_S)$. The reverse containment follows from the argument which shows that $U(B_S) \subset W(B_S)$, applied to G_S and $G_S \cap B_S$ instead of G and B_S.

I return to the setting of (2.40), and I let B^j be a connected component of the Cartan subgroup B. For every $\alpha \in \Phi(\mathfrak{b})_-$, e^α assumes nonzero, real values on B, and hence either only positive, or only negative values on B^j. Those $\alpha \in \Phi(\mathfrak{b})_-$, for which $e^\alpha > 0$ on B^j, form a sub-root system Φ^j of $\Phi(\mathfrak{b})_-$. Since $\Phi(\mathfrak{b})_-$ can be naturally identified with $\Phi(G_B, B \cap G_B)$, one may think of Φ^j also as a sub-root system of the latter. The E_α corresponding to $\alpha \in \Phi^j$ (notation of (2.26)) and $\mathfrak{b}_-$ together span a reductive subalgebra of $\mathfrak{g}_B$, which contains $\mathfrak{b}_-$ as a split Cartan subalgebra. The connected subgroup which this Lie algebra determines shall be denoted by $G(B^j)$.

(2.44) **Proposition.** *The group $G(B^j)$ is semisimple; it contains both a split and a compact Cartan subgroup. The identity component B^0 of B lies in $B^0_+ \cdot G(B^j)$, and $B^j \subset B^0 \cdot Z(G(B^j))$ ($Z(\ldots)$ = center of ...). If G/K happens to be Hermitian symmetric, every simple factor of $G(B^j)$ is isomorphic, up to covering, to $Sp(n, \mathbb{R})$, for some n.*

With the help of this proposition, it is possible to make an important reduction in the problem of explicitly computing the characters of the discrete series (cf. (4.22) below). For most of this paper, G/K will be assumed to be Hermitian symmetric. In that case, the proof of the proposition becomes even simpler. I have included the general case here, because it will be needed in § 5, where some of the arguments are given in greater generality than necessary for the purposes of this paper.

Proof. Since G_B meets every component of B, the statement does not really concern G, but only G_B. In other words, I may assume that B is a split Cartan subgroup of a connected, semisimple Lie group G, which is also known to contain a compact Cartan subgroup. I set $\Phi = \Phi(\mathfrak{b})$; then Φ is an abstract, reduced root system with the property SO (cf. (2.22)). Now let $\Lambda = \Lambda(\Phi)$ be the weight lattice of the root system Φ, i.e.

$$\Lambda = \{\lambda \in \mathfrak{b}^* \mid 2(\lambda, \alpha)(\alpha, \alpha)^{-1} \in \mathbb{Z} \text{ for all } \alpha \in \Phi\},$$

and $\Lambda' \subset \Lambda$ the sublattice generated by 2Λ, Φ^j, and all sums $\alpha_1 + \alpha_2$, with $\alpha_i \in \Phi$, $\alpha_i \notin \Phi^j$, $i=1, 2$. Because of (2.38), the sub-root system Φ^j satisfies

(2.45) $$\Lambda' \cap \Phi \subset \Phi^j.$$

In particular, this implies

(2.46) $$\alpha_1, \alpha_2 \in \Phi, \quad \alpha_1, \alpha_2 \notin \Phi^j, \quad \alpha_1 + \alpha_2 \in \Phi \Rightarrow \alpha_1 + \alpha_2 \in \Phi^j.$$

I now claim the following: if Φ is a reduced, abstract root system with the property SO, and if Φ^j is a sub-root system of Φ satisfying (2.45), then

(2.47) $$\Phi^j \text{ has the property } SO, \text{ and } rk\, \Phi^j = rk\, \Phi.$$

Assuming (2.47), which will be verified presently, it is a simple matter to prove the proposition. Because of the assertion about the rank of Φ^j, $G(B^j)$ must be semisimple; $G(B^j)$ contains a compact Cartan subgroup since Φ^j has the property SO [38]. With the simplifying assumptions which were made, the Lie algebra of $G(B^j)$ contains $\mathfrak{b}$, so that $B^0 \subset G(B^j)$. For any $b \in B^j$, there exists some $b_0 \in B^0$, such that $e^\alpha(b) = e^\alpha(b_0)$ whenever $\alpha \in \Phi^j$. But this means $b b_0^{-1} \in Z(G(B^j))$, and hence $B^j \subset B^0 \cdot Z(G(B^j))$. If G/K has a Hermitian symmetric structure, each irreducible component of the root system Φ must be of type C_n (cf. (2.40)). As can then be deduced directly from (2.45) and (2.46), each irreducible component of Φ^j must also be of type C_k, for some k. Hence each simple factor of $G(B^j)$ is isomorphic, up to covering, to $Sp(k, \mathbb{R})$.

It remains to prove (2.47); the argument will proceed by induction on the rank. If Φ has rank one, 2Λ contains Φ, so that $\Phi = \Phi^j$. Now let Φ have rank greater than one. If Φ fails to be irreducible, each irreducible component of Φ can be assumed to satisfy (2.47), and hence Φ itself also satisfies (2.47). Otherwise, for an irreducible root system of rank greater than one, (2.46) guarantees the existence of at least one root $\beta \in \Phi^j$. In terms of β, I define

$$\tilde{\Phi} = \{\alpha \in \Phi \mid \alpha \perp \beta\}, \quad \tilde{\Phi}^j = \tilde{\Phi} \cap \Phi^j, \quad \tilde{\Lambda} = \text{weight lattice of}$$
$$\text{the root system } \tilde{\Phi} = \{\mu \in \mathfrak{b}^* \mid \mu \perp \beta,\ 2(\mu, \alpha)(\alpha, \alpha)^{-1} \in \mathbb{Z} \text{ if } \alpha \in \tilde{\Phi}\}.$$

According to (2.18), $\tilde{\Phi}$ has the property SO, and the rank of $\tilde{\Phi}$ is exactly one less than the rank of Φ. For any $\mu \in \tilde{\Lambda}$, there exists some $\lambda \in \Lambda$, such that λ differs from μ by a multiple of β; this statement is analogous to, and can be verified in the same manner as, (2.41). Since $\lambda \in \Lambda$, and since $\mu \perp \beta$, $2\lambda - 2\mu$ must be an integral multiple of β. Thus

(2.48) $$2\tilde{\Lambda} \subset \Lambda'.$$

Now let $\tilde{\Lambda}'$ be the sublattice of $\tilde{\Lambda}$ generated by $2\tilde{\Lambda}$, $\tilde{\Phi}^j$, and all sums $\alpha_1 + \alpha_2$, with $\alpha_i \in \tilde{\Phi}$, $\alpha_i \notin \tilde{\Phi}^j$, $\alpha_1 + \alpha_2 \in \tilde{\Phi}$. Because of (2.48), Λ' contains $\tilde{\Lambda}$, which implies

(2.49) $$\tilde{\Lambda}' \cap \tilde{\Phi} \subset \tilde{\Phi}^j.$$

As was already mentioned, $\tilde{\Phi}$ has the property SO. In view of (2.49), and by induction, one can now infer the statement (2.47) for $\tilde{\Phi}^j$.

It must still be shown that Φ^j has the property SO. If β is strongly orthogonal to every root in $\tilde{\Phi}^j$, there is no problem. Otherwise, I select $\gamma\in\tilde{\Phi}^j$, such that $\beta\pm\gamma$ are roots. Then both $\beta+\gamma$ and $\beta-\gamma$ belong to Φ^j, and they must both be strongly orthogonal to any root which is orthogonal to them. Because of (2.18), the orthogonal complement of γ in $\tilde{\Phi}^j$ contains a strongly orthogonal spanning set. Combined with $\beta\pm\gamma$, it becomes a strongly orthogonal spanning set of Φ^j. This demonstrates (2.47) and completes the proof of the proposition.

The final remarks of this section will be devoted to the centralizer of the split part B_- of a Cartan subgroup B. Without loss of generality, I assume $B=B_S$, for some strongly orthogonal subset $S\subset\Phi^n(\mathfrak{h})$. On the Lie algebra level, the centralizer in $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{b}_{S,-}$ can be described as $\mathfrak{m}_S^{\mathbb{C}}\oplus\mathfrak{b}_{S,-}^{\mathbb{C}}$, where

(2.50) $\mathfrak{m}_S^{\mathbb{C}}=$ span of $\mathfrak{b}_{S,+}^{\mathbb{C}}$ and all root spaces of $(\mathfrak{g}^{\mathbb{C}},\mathfrak{b}_S^{\mathbb{C}})$ corresponding to roots in $\Phi(\mathfrak{b}_S)_+$

(cf. (2.21)). Any root in $\Phi(\mathfrak{b}_S)_+$ assumes purely imaginary values on $\mathfrak{b}_S$. Hence the Lie algebra $\mathfrak{m}_S^{\mathbb{C}}$ defined by (2.50) arises as the complexification of a real subalgebra $\mathfrak{m}_S\subset\mathfrak{g}$. Clearly,

(2.51) $$\mathfrak{m}_S\oplus\mathfrak{b}_{S,-} \text{ is the centralizer of } \mathfrak{b}_{S,-} \text{ in } \mathfrak{g}.$$

The Cartan involution $\theta\colon\mathfrak{g}\to\mathfrak{g}$ lifts to an involutive automorphism of G, which shall also be referred to as θ. Then K is the group of fixed points of θ. According to (2.15), θ leaves $\mathfrak{b}_S$ invariant, hence also B_S, $B_{S,+}$, etc. The following statements are fairly standard; see §4 of [40], for example.

(2.52) **Lemma.** *The centralizer of $B_{S,-}$ can be uniquely factored into the direct product $M_S\cdot B_{S,-}$, with $\theta M_S=M_S$. The identity component M_S^0 of M_S is reductive, it contains $M_S^0\cap K$ as a maximal compact subgroup and $B_{S,+}^0$ as a compact Cartan subgroup, and it corresponds to the Lie algebra $\mathfrak{m}_S$. For every $m\in M_S$, $\operatorname{Ad}m:\mathfrak{m}_S^{\mathbb{C}}\to\mathfrak{m}_S^{\mathbb{C}}$ belongs to the adjoint group of the complexified Lie algebra $\mathfrak{m}_S^{\mathbb{C}}$.*

In particular, M_S^0 again satisfies all hypotheses which were originally imposed on G. For the study of M_S/M_S^0, it is helpful to introduce the intermediate normal subgroup

(2.53) $$M_S^{\dagger}=\{m\in M_S\,|\,\operatorname{Ad}m\colon M_S^0\to M_S^0 \text{ is inner}\}$$

(cf. [40], §3). I shall also consider the finite group

(2.54) $$F_S=\{\exp(iX)\,|\,X\in\mathfrak{b}_{S,-},\ \langle\lambda,X\rangle\in\pi\mathbb{Z}, \text{ for all } \lambda\in\Lambda(\Phi(B_S))\},$$

which first came up in (2.38).

(2.55) **Lemma.** *The finite group F_S is central in $M_S^{\dagger}$, and $M_S^{\dagger}=M_S^0\cdot F_S$.*

Proof. As follows from its definition, F_S is Abelian and commutes with M_S^0; hence it is central in $M_S^{\dagger}$. Every coset in $M_S^{\dagger}/M_S^0$ has a representative g which centralizes M_S^0. Since g also commutes with $B_{S,-}$, it must lie in B_S. According to (2.38), for some $f\in F_S$, gf^{-1} lies in B_S^0. But then $gf^{-1}\in M_S\cap B_S=B_{S,+}^0$, so that f and g determine the same coset in $M_S^{\dagger}/M_S^0$.

There is one simple observation in this context which should be recorded for future use. If the set S consists of a single noncompact root β, the group G_β of

(2.42) is isomorphic to $SL(2,\mathbb{R})$ or its adjoint group. In either case, F_β is precisely the center of G_β. Since H intersects G_β in a compact Cartan subgroup, it must contain F_β. Thus:

(2.56) *Remark.* If $S=\{\beta\}$, for some $\beta\in\Phi^n(\mathfrak{h})$, F_β has order two or is trivial, and $F_\beta\subset H$.

Every element of the normalizer of $B^0_{S,+}$ in M^0_S centralizes $\mathfrak{b}_{S,-}$. Hence the Weyl group $W(M^0_S, B^0_{S,+})$ may be viewed as a subgroup of

$$(2.57)\qquad \{w\in W(G,B_S)\,|\,w|_{\mathfrak{b}_{S,-}}=1\};$$

a normal subgroup, in fact, as is not hard to check.

(2.58) **Lemma.** *$M_S/M^\dagger_S$ is isomorphic to the quotient of the group* (2.57) *modulo* $W(M^0_S, B^0_{S,+})$.

Proof. If an element w of the group (2.57) is represented as $\mathrm{Ad}\,g$, with $g\in G$, then g must centralize $B_{S,-}$, so that $g\in M_S\cdot B_{S,-}$. In particular, the representative g can be chosen to lie in M_S. Subject to the condition $g\in M_S$, g becomes determined up to a factor in $M_S\cap B=B_{S,+}\subset M^\dagger_S$. If $\mathrm{Ad}\,g$ represents an element of $W(M^0_S, B^0_{S,+})$, one could have picked g from M^0_S, which lies in $M^\dagger_S$. These remarks describe a well-defined homomorphism of the quotient of the two Weyl groups into $M_S/M^\dagger_S$. To show surjectivity, I pick $g\in M_S$. Up to conjugacy, M^0_S contains only a single compact Cartan subgroup. Hence, if g is modified by a suitable factor in M^0_S, g can be made to normalize $\mathfrak{b}_{S,+}$. Since g also centralizes $\mathfrak{b}_{S,-}$, $\mathrm{Ad}\,g$ represents an element of the group (2.57). The injectivity of the mapping, finally, follows directly from the definition of $M^\dagger_S$.

I identify the root system $\Phi(\mathfrak{b}_S)_+$ in (2.21) with the root system of $(\mathfrak{m}^{\mathbb{C}}_S, \mathfrak{b}^{\mathbb{C}}_{S,+})$, i.e.

$$(2.59)\qquad \Phi(\mathfrak{b}_S)_+\simeq\Phi(M^0_S, B^0_{S,+}).$$

On the other hand, since the Cayley transform $c_S\colon \mathfrak{h}^{\mathbb{C}}_S\simeq\mathfrak{b}^{\mathbb{C}}_S$ acts as the identity on $\mathfrak{b}^{\mathbb{C}}_{S,+}$, $\Phi(\mathfrak{b}_S)_+$ can also be naturally identified with a sub-root system of $\Phi(\mathfrak{h})$:

$$(2.60)\qquad \Phi(\mathfrak{b}_S)_+\simeq\{\alpha\in\Phi(\mathfrak{h})\,|\,\alpha\perp S\}.$$

The Cartan involution preserves $\mathfrak{m}^{\mathbb{C}}_S$ and acts as the identity on $\mathfrak{b}^{\mathbb{C}}_{S,+}$. Consequently, the root spaces of $(\mathfrak{m}^{\mathbb{C}}_S, \mathfrak{b}^{\mathbb{C}}_{S,+})$ must be θ-stable. Just as in the case of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$, a root $\alpha\in\Phi(M^0_S, B^0_{S,+})$ is called compact or noncompact, depending on whether the corresponding root space lies in $\mathfrak{k}^{\mathbb{C}}$ or $\mathfrak{p}^{\mathbb{C}}$. Via the identification (2.59), for each $\alpha\in\Phi(\mathfrak{b}_S)_+$, one has the notion of compactness or noncompactness of α in terms of the pair $(\mathfrak{m}^{\mathbb{C}}_S, \mathfrak{b}^{\mathbb{C}}_{S,+})$, and via the identification (2.60), in terms of the pair $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$.

(2.61) **Lemma.** *Under the identification* (2.60), *a root $\alpha\in\Phi(\mathfrak{b}_S)_+$ is either strongly orthogonal to all roots in S, or to all roots in S with exactly one exception. In the first case, the two notions of compactness or noncompactness in terms of* $(\mathfrak{m}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}}_{S,+})$ *and in terms of* $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ *agree, and in the second case, they are opposed.*

Proof. If $\alpha\in\Phi(\mathfrak{b}_S)_+$ fails to be strongly orthogonal to two roots $\gamma_1, \gamma_2\in S$, then $\alpha\pm\gamma_1$, $\alpha\pm\gamma_2$ must all be roots, because $\alpha\perp\gamma_i$. Since $(\alpha+\gamma_1, \alpha+\gamma_2)=(\alpha,\alpha)>0$, $\gamma_1-\gamma_2$ is also a root; contradiction! I now let $B(\,,)$ denote the trace form of a faithful finite dimensional representation of G; it is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. Hence, if X is a generator of the α-rootspace of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$, the compactness or noncompactness of α relative to $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ can be read off from the sign of $B(X, \bar{X})$. Similarly, the sign of

$$B(c_S X, \overline{c_S X}) = B(c_S X, c_S^{-1} \bar{X}) = B(c_S^2 X, \bar{X})$$

determines the compactness or noncompactness of α relative to $(\mathfrak{m}_S^{\mathbb{C}}, \mathfrak{b}_{S,+}^{\mathbb{C}})$. It therefore suffices to show that $c_S^2 X = X$ if α is strongly orthogonal to all $\gamma \in S$, and that otherwise $c_S^2 X = -X$. In the first situation, the identity $c_S^2 X = X$ is essentially clear. In the second situation, one may as well assume that S consists of a single root γ, which is orthogonal to α, but not strongly orthogonal. In the notation of (2.7), Z_γ, Y_γ, $Y_{-\gamma}$ span a copy of $\mathfrak{sl}(2, \mathbb{C})$ in $\mathfrak{g}^{\mathbb{C}}$. Because of the hypotheses, X lies in an irreducible, three-dimensional module of this copy of $\mathfrak{sl}(2, \mathbb{C})$, in the zero-eigenspace of Z_γ. The identity $c_\gamma^2 X = -X$ now follows from a computation with $SL(2, \mathbb{C})$.

Just as in the case of G/K, the quotient $M_S^0/M_S^0 \cap K$ may be regarded as a symmetric space, of the noncompact type.

(2.62) **Lemma.** *If G/K is Hermitian symmetric, and if no simple factor of G is isomorphic, up to covering, to $SO(2, 2n+1)$, for some $n \geq 2$, then $M_S^0/M_S^0 \cap K$ is again Hermitian symmetric. If G is isomorphic to $SO(2, 2n+1)$ or its twofold linear covering, then $M_S^0/M_S^0 \cap K$ is also Hermitian symmetric, unless S consists of a single root, which is short. In this exceptional situation, $M_S^0 \simeq SO(1, 2n)$, at least up to covering.*

Proof. Let G/K be Hermitian symmetric. One can then introduce an ordering on $\Phi(\mathfrak{h})$, such that the sum of two positive noncompact roots is never a root. I shall apply the converse of this statement, which is also true, to $M_S^0/M_S^0 \cap K$. Specifically, I shall verify this property for the restricted positive root system in $\Phi(\mathfrak{b}_S)_+$, which the notion of compactness and noncompactness taken relative to $(\mathfrak{m}_S^{\mathbb{C}}, \mathfrak{b}_{S,+}^{\mathbb{C}})$; the exception mentioned in the lemma is excluded, of course. Without loss of generality, I assume G to be simple. If all roots in $\Phi(\mathfrak{b}_S)_+$, identified with roots in $\Phi(\mathfrak{h})$ via (2.60), are strongly orthogonal to S, there is nothing to be done, thanks to (2.61). Otherwise, there exist $\alpha \in (\mathfrak{b}_S)_+$ and $\gamma \in S$, which are not strongly orthogonal. Since $\alpha \perp \gamma$, $\alpha \pm \gamma$ must be roots, and α and γ must be short. Thus one is reduced to looking at a simple G, with G/K Hermitian symmetric, and such that there exist roots of two different lengths. As can be deduced either from Moore's results [30] or from the classification, this implies $G \simeq SO(2, 2n+1)$ or $G \simeq Sp(n, \mathbb{R})$, up to covering. The completion of the proof is now left to the reader.

§ 3. The "Holomorphic Discrete Series"

Throughout this section, and for most of the remainder of this paper, I shall consider a connected, reductive Lie group G, which admits a faithful finite dimensional representation, which has a compact center, and whose quotient G/K by a maximal compact subgroup K carries a Hermitian symmetric structure. The discussion in § 2 applies to such a group; I shall freely use the notation established there.

For groups of this type [6], Harish-Chandra [9] has constructed a certain class of representations in a very explicit manner. Some of the representations are unitary and belong to the discrete series, whereas others cannot be unitarized and are reducible. The class of square-integrable representations among these is often referred to, somewhat informally, as the holomorphic discrete series. As will be shown in the next few sections, the characters of Harish-Chandra's representations, including the non-unitary ones, can be used as the basic building blocks for the characters of the discrete series. Below, I shall briefly recall Harish-Chandra's construction, in the manner which is most convenient in the context of this paper.

Since G/K carries a Hermitian symmetric structure, there exists an ordering of the root system $\Phi(\mathfrak{h})$, such that the sum of two noncompact positive roots is never a root. For the time being, I fix such an ordering, and I denote the resulting set of positive roots by Ψ. For emphasis,

$$\alpha, \beta \in \Psi \cap \Phi^n(\mathfrak{h}) \Rightarrow \alpha + \beta \notin \Phi(\mathfrak{h}). \tag{3.1}$$

The choice of Ψ determines a splitting

$$\begin{aligned} &\mathfrak{p}^{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-, \quad \text{with} \\ &\mathfrak{p}_+ = \textstyle\bigoplus \mathfrak{g}^{\alpha}, \qquad \alpha \in \Psi \cap \Phi^n(\mathfrak{h}), \\ &\mathfrak{p}_- = \textstyle\bigoplus \mathfrak{g}^{-\alpha}, \qquad \alpha \in \Psi \cap \Phi^n(\mathfrak{h}) \end{aligned} \tag{3.2}$$

(cf. (2.6)). Both $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are $\operatorname{Ad} K$-invariant, Abelian subalgebras of $\mathfrak{g}^{\mathbb{C}}$, which are complex conjugate to each other. In the complexification $G^{\mathbb{C}}$ of G, they exponentiate to unipotent, Abelian subgroups P_+, P_-. The semidirect product $K^{\mathbb{C}} \cdot P_+$ turns out to be a parabolic subgroup of $G^{\mathbb{C}}$, and

$$G \cap (K^{\mathbb{C}} \cdot P_+) = K. \tag{3.3}$$

Hence G/K can be identified with the G-orbit of the identity coset in $G^{\mathbb{C}}/K^{\mathbb{C}} \cdot P_+$. For dimension reasons, the orbit is open, and so the embedding

$$G/K \subset G^{\mathbb{C}}/K^{\mathbb{C}} \cdot P_+ \tag{3.4}$$

induces an invariant complex structure on G/K; (3.4) is the usual embedding of the Hermitian symmetric space G/K in its compact dual.

I recall the definition of $\Lambda \subset i\mathfrak{h}^*$ as the lattice of differentials of characters of H (cf. (2.2)). With the usual notation, I set

$$\rho = \tfrac{1}{2} \textstyle\sum_{\alpha \in \Psi} \alpha. \tag{3.5}$$

Now let $\lambda \in i\mathfrak{h}^*$ be such that

$$\lambda + \rho \in \Lambda, \quad \text{and} \quad (\lambda, \alpha) > 0 \text{ for all } \alpha \in \Psi \cap \Phi^c(\mathfrak{h}) \tag{3.6}$$

($\Phi^c(\mathfrak{h})$ = set of nonzero, compact roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$). Since Λ contains the roots,

$$\mu = \lambda - \rho_c + \rho_n \tag{3.7}$$

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[6] Harish-Chandra does not assume that G is linear.

also lies in Λ; here ρ_c and ρ_n denote one half of the sum of the positive compact, resp. noncompact, roots. Because of the first condition in (3.6), $2(\lambda,\alpha)(\alpha,\alpha)^{-1}\in\mathbb{Z}$ for all roots α. In conjuction with the second condition, this implies $(\lambda-\rho_c,\alpha)\geqq 0$, whenever $\alpha\in\Psi\cap\Phi^c(\mathfrak{h})$. On the other hand, because $2\rho_n$ is the highest weight of a one dimensional K-module, namely of the highest exterior power of $\mathfrak{p}_+$, ρ_n is perpendicular to $\Phi^c(\mathfrak{h})$. Thus

(3.8) $$(\mu,\alpha)\geqq 0, \quad \text{for all } \alpha\in\Psi\cap\Phi^c(\mathfrak{h}),$$

and consequently there exists an irreducible, holomorphic $K^{\mathbb{C}}$-module V_μ, of highest weight μ. It extends uniquely to $K^{\mathbb{C}}\cdot P_+$, so that one may regard V_μ as a $K^{\mathbb{C}}\cdot P_+$-module.

Like any finite dimensional, holomorphic representation of $K^{\mathbb{C}}\cdot P_+$, V_μ associates a holomorphic vector bundle to the principle bundle

$$K^{\mathbb{C}}\cdot P_+\to G^{\mathbb{C}}\to G^{\mathbb{C}}/K^{\mathbb{C}}\cdot P_+,$$

which will be denoted by $\mathscr{V}_\mu\to G^{\mathbb{C}}/K^{\mathbb{C}}\cdot P_+$. The action of $G^{\mathbb{C}}$ on $G^{\mathbb{C}}/K^{\mathbb{C}}\cdot P_+$ lifts to this bundle. Hence the restriction $\mathscr{V}_\mu\to G/K$ is a G-homogeneous, holomorphic vector bundle. I set $\mathbf{F}_\lambda$ = space of holomorphic sections of $\mathscr{V}_\mu\to G/K$ (μ and λ are related by (3.7)). In a natural fashion, $\mathbf{F}_\lambda$ carries the structure of a Frechét space, on which G acts by translation. The G-action is continuous, viewed as a map from $G\times\mathbf{F}_\lambda$ to $\mathbf{F}_\lambda$.

As is not hard to show (cf. § 7), the space of K-finite vectors in the G-module $\mathbf{F}_\lambda$ is isomorphic, as K-module, to V_μ tensored with the symmetric algebra of $\mathfrak{p}_+$. In particular, the irreducible K-module V_μ occurs exactly once in $\mathbf{F}_\lambda$. Moreover, this K-submodule lies in every closed, G-invariant subspace of $\mathbf{F}_\lambda$. The center of $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ (= universal enveloping algebra of $\mathfrak{g}^{\mathbb{C}}$) therefore operates on $\mathbf{F}_\lambda$ according to a one-dimensional representation. By trivializing the vector bundle $\mathscr{V}_\mu$ over a neighborhood of the closure of G/K, the representation of G on the Frechét space $\mathbf{F}_\lambda$ can be made infinitesimally equivalent to a representation on a Hilbert space, namely on V_μ tensored with the space of L^2 boundary values of holomorphic functions on the Shilov boundary of G/K; details are given in [16]. Since the multiplicities of the various irreducible K-modules in $\mathbf{F}_\lambda$ satisfy the bound occurring in Harish-Chandra's definition of quasi-simplicity [10], the representation of G on $\mathbf{F}_\lambda$ is infinitesimally equivalent to a quasi-simple representation on a Hilbert space. Hence the usual proofs of the existence of a character apply to the G-module $\mathbf{F}_\lambda$: the character of $\mathbf{F}_\lambda$ is an invariant eigendistribution.

The construction of the G-module $\mathbf{F}_\lambda$ above depends not only on the choice of the parameter $\lambda\in i\mathfrak{h}^*$, but also on the choice of the system of the positive roots $\Psi\subset\Phi(\mathfrak{h})$; Ψ must satisfy (3.1), and Ψ and λ must be related by (3.7). In order to emphasize the dependence on both Ψ and λ, I define

(3.9) $$\Theta(\Psi,\lambda) = \text{character of the } G\text{-module } \mathbf{F}_\lambda.$$

In her thesis [28], Martens has investigated those of the G-modules $\mathbf{F}_\lambda$ which belong – up to infinitesimal equivalence – to the discrete series. In particular, she has explicitly and globally computed their characters. For the purposes of this paper, one needs the character formula for all of the $\mathbf{F}_\lambda$. The general case has been treated by Hecht [16], with quite different methods. I shall proceed to quote his result, after disposing of some preliminaries.

According to a deep theorem[7] of Harish-Chandra [11], each invariant eigendistribution on a reductive Lie group is a locally L^1 function, which is real analytic on the regular set – in more precise language, each invariant eigendistribution can be represented as integration against such a function. Henceforth, I shall slur over the distinction between an invariant eigendistribution and the function that represents it. Every regular element of the group G lies in a Cartan subgroup. Hence, in order to describe an invariant eigendistribution, it suffices to give a formula for it on one Cartan subgroup from each conjugacy class. I now consider a Cartan subgroup $B_S \subset G$, corresponding to a strongly orthogonal subset $S \subset \Phi^n(\mathfrak{b}_S)$. The Cayley transform c_S: $\mathfrak{h}^{\mathbb{C}} \simeq \mathfrak{b}_S^{\mathbb{C}}$ maps S onto a strongly orthogonal spanning set of $\Phi(\mathfrak{b}_S)_-$ (cf. (2.21)), say $\{\alpha_1, \ldots, \alpha_s\}$. Every e^{α_i} assumes real values on B_S. The reflection about α_i lies in the Weyl group of B_S; lifted to an automorphism of B_S, it maps the character e^{α_i}: $B_S \to \mathbb{R}^*$ to its inverse, whereas it preserves e^{α_j}, for $j \neq i$. Hence the conjugates of

$$\{b \in B_S \,|\, |e^{\alpha_i}(b)| < 1 \text{ for } 1 \leqq i \leqq s\} \tag{3.10}$$

cover a dense open subset of B_S. In passing, it should be remarked that e^α assumes only positive real values on B_S, provided that α is a member of a strongly orthogonal spanning set[8] of $\Phi(\mathfrak{b}_S)_-$, as follows from the arguments in the proof of (2.44). The absolute value signs in (3.10) are therefore irrelevant; some conjugate of every regular $b \in B_S$ lies in the set (3.10).

The Cartan subgroup B_S is wholly contained in its complexification $B_S^{\mathbb{C}}$. Since the Cayley transform establishes an isomorphism between $H^{\mathbb{C}}$ and $B_S^{\mathbb{C}}$, every function on B_S, or on a subset of B_S, can be pulled back via c_S to the appropriate subset of $H^{\mathbb{C}}$. The torus H contains the center of $G^{\mathbb{C}}$, so that every $\lambda \in \Lambda$ lifts to a character e^λ on $H^{\mathbb{C}}$. When this character is restricted to H, the notation agrees with that of (2.2). Using these conventions, the character formula of Martens and Hecht can be stated as follows:

(3.11) **Theorem.** (Martens [28], Hecht [16]). *Let S be a strongly orthogonal subset of $\Phi^n(\mathfrak{h})$, such that $S \subset \Psi$. Then $\Theta(\Psi, \lambda)$, restricted to the subset* (3.10) *of B_S, and pulled back to the corresponding subset of $H^{\mathbb{C}}$ via c_S, is given by*

$$(-1)^q \frac{\sum_{w \in W(H)} \varepsilon(w)\, e^{w\lambda}}{\prod_{\alpha \in \Psi}(e^{\alpha/2} - e^{-\alpha/2})};$$

here $\varepsilon(w)$, for $w \in W(H)$, denotes the sign of w, and $q = \frac{1}{2} \dim_{\mathbb{R}} G/K$.

Some remarks are in order. As is usual also with Weyl's character formula, both numerator and denominator should be multiplied through by e^ρ, or $e^{-\rho}$. As the formula stands, the exponential terms need not be well-defined on $H^{\mathbb{C}}$, since λ may not lie in Λ. If λ satisfies

$$(\lambda, \alpha) > 0 \quad \text{for all } \alpha \in \Psi, \tag{3.12}$$

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[7] A careful analysis of the arguments in [16] shows directly that the character of $\mathbf{F}_\lambda$ is a locally L^1 function, so that one does not really have to appeal to the theorem at this point.

[8] This is not a general fact; G/K must be Hermitian symmetric.

in addition to (3.7), the G-module $\mathbf{F}_\lambda$ belongs to the discrete series [13], up to infinitesimal equivalence, of course. In this case, the character $\Theta(\Psi, \lambda)$ is tempered [13, 39], as it must be. To deduce the temperedness from (3.11), one merely has to show that $(w\lambda, \gamma) \geqq 0$ whenever $w \in W(H)$, $\gamma \in S$. Since $W(H)$ is the Weyl group of H in K, and since $\operatorname{Ad} K$ normalizes $\mathfrak{p}_+$, $W(H)$ preserves $\Psi \cap \Phi^n(\mathfrak{h})$. Thus $w^{-1} S \subset \Psi$, for every $w \in W(H)$, and the inequality $(w\lambda, \gamma) \geqq 0$ follows from (3.12). Observe that the preceding argument proves the temperedness of $\Theta(\Psi, \lambda)$ as soon as $(\lambda, \alpha) \geqq 0$ for all $\alpha \in \Psi \cap \Phi^n$.

The global formula for $\Theta(\Psi, \lambda)$ has a very simple form. One might hope that this is typical for the characters of the discrete series, but unfortunately it is not. To begin with, in general the discrete series characters can be described by consistent formula only on a Weyl chamber of the root system $\Phi(\mathfrak{b}_S)_-$, rather than on a whole quadrant of the type (3.10). More seriously, the disconnectedness of a Cartan subgroup enters the general formula in an essential manner, as happens already for $Sp(2, \mathbb{R})$. Finally, the coefficients of the various exponential terms that could conceivably enter the formula, which are ± 1 or 0 in (3.11), may be arbitrarily large integers in general.

§ 4. The Characters of the Discrete Series

The main statements of this section involve the process of inducing an invariant eigendistribution from a cuspidal parabolic subgroup. It will therefore be necessary to introduce certain conventions about this process. I continue with the assumptions and with the notation of § 2; in particular, the group G is to have a faithful finite dimensional representation. At a later point, G/K will also be required to be Hermitian symmetric.

To begin with, I consider the Cartan subgroup B_S corresponding to a strongly orthogonal subset $S \subset \Phi^n(\mathfrak{h})$. As was described in § 2, the centralizer of $B_{S,-}$ can be factored as $M_S \cdot B_{S,-}$. It is possible to choose some $X \in \mathfrak{b}_{S,-}$, such that

$$\alpha \in \Phi(\mathfrak{b}_S),\ \langle \alpha, X \rangle = 0 \quad \Rightarrow \quad \alpha|_{\mathfrak{b}_{S,-}} = 0. \tag{4.1}$$

Those root spaces of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$ which belong to roots α with $\langle \alpha, X \rangle > 0$ span a nilpotent subalgebra $\mathfrak{n}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$; $\mathfrak{n}^{\mathbb{C}}$ is invariant under complex conjugation, and it is therefore the complexification of a subalgebra $\mathfrak{n} \subset \mathfrak{g}$. Evidently $M_S \cdot B_{S,-}$ normalizes the connected subgroup $N = \exp \mathfrak{n}$ of G. The semidirect product

$$M_S \cdot B_{S,-} \cdot N \tag{4.2}$$

turns out to be a cuspidal parabolic subgroup. This is just the standard construction which attaches to each Cartan subgroup a class of cuspidal parabolic subgroups.

In order to have a definite correspondance between invariant eigendistributions and the locally L^1 functions which represent them, I choose and keep fixed a particular normalization of the Haar measure dg of G. The Haar measure dk on K, dm on M_S, db on $B_{S,-}$, and dn on N can be chosen subject to the normalization

$$\int_G f(g)\, dg = \int_K \int_{M_S} \int_{B_{S,-}} \int_N f(kmbn)\, e^{2\rho_N}(b)\, dn\, db\, dm\, dk, \tag{4.3}$$

for every compactly supported, continuous function f. Here $\rho_N \in \mathfrak{b}^*_{S,-}$ is defined by

(4.4) $$\rho_N = \tfrac{1}{2} \sum \{\alpha \in \Phi(\mathfrak{b}_S) \mid \langle \alpha, X \rangle > 0\}\,|_{\mathfrak{b}_{S,-}};$$

equivalently, $e^{2\rho_N}$ can be described as the character by which $B_{S,-}$ operates on the top exterior power of $\mathfrak{n}^{\mathbb{C}}$.

I now consider an invariant eigendistribution ϕ on M_S (which will be tacitly identified with the locally L^1 function that represents it), and a particular $\nu \in \mathfrak{b}^*_{S,-}$. The formula

(4.5) $$\Theta(f) = \int_K \int_{M_S} \int_{B_{S,-}} \int_N f(kmbnk^{-1})\, \phi(m)\, e^{\nu-\rho_N}(b)\, dn\, db\, dm\, dk$$

defines a distribution Θ on G. If ϕ happens to be given as the character of a (quasi-simple) representation of M_S, Θ is the character of an induced representation [8, 18, 40]. In the context of unitary representations, ν is generally required to take imaginary, rather than real, values on $\mathfrak{b}_{S,-}$. Nevertheless, the usual arguments about the inducing of characters apply to this situation as well, and they show that

(4.6) Θ is an invariant eigendistribution, which is independent of the particular choice of N

(i.e. independent of the particular X in (4.1)).

When induced invariant eigendistributions come up below, ϕ will be given in terms of an invariant eigendistribution on the identity component M_S^0 of M_S, in a manner which is again suggested by the procedure of inducing representations. At this point, one should recall the definition (2.53) of the normal subgroup $M_S^\dagger \subset M_S$, the definition (2.54) of the finite group $F_S \subset M_S^\dagger$, and the statement of Lemma (2.55). I assume that the following data are given: an invariant eigendistribution ϕ_0 on M_S^0, and a character $\zeta: F_S \to \mathbb{C}^*$, which satisfy

(4.7) $$\phi_0(fm) = \zeta(f)\, \phi_0(m), \quad \text{whenever } f \in F_S \cap M_S^0,\ m \in M_S^0;$$

here ϕ_0 is viewed as a function. Because of (4.7), one can define a locally L^1 function ϕ_1 on $M_S^\dagger$, by setting

(4.8) $$\phi_1(fm) = \zeta(f)\, \phi_0(m), \quad \text{if } f \in F_S,\ m \in M_S^0.$$

One checks easily that ϕ_1 is again an invariant eigendistribution. For each $m \in M_S$, $\phi_1 \circ \mathrm{Ad}\, m$ depends only on the class of m in $M_S/M_S^\dagger$. Since this quotient is finite,

(4.9) $$\phi = \begin{cases} \sum_{m \in M_S^\dagger / M_S} \phi_1 \circ \mathrm{Ad}\, m & \text{on } M_S^\dagger \\ 0 & \text{on the complement of } M_S^\dagger \text{ in } M_S \end{cases}$$

describes a distribution ϕ on M_S.

(4.10) *Remark.* ϕ is an invariant eigendistribution.

Proof. By definition, ϕ remains invariant under inner automorphisms. For $m \in M_S$, $\mathrm{Ad}\, m$ operates on $\mathfrak{m}_S^{\mathbb{C}}$ as an element of the complex adjoint group; cf. (2.52). Hence $\mathrm{Ad}\, m$ acts trivially on the center of the universal enveloping algebra of $\mathfrak{m}_S^{\mathbb{C}}$. This makes ϕ an invariant eigendistribution.

To recapitulate, the invariant eigendistribution Θ of (4.5) can be put together from the following ingredients: a strongly orthogonal subset $S \subset \Phi^n(\mathfrak{h})$, a linear functional $\nu \in \mathfrak{b}_{S,-}^*$, an invariant eigendistribution ϕ_0 on M_S^0, and a character ζ of F_S, subject to the condition (4.7).

As in § 2, $\Lambda \subset i\mathfrak{h}^*$ will stand for the lattice of differentials of characters of H. I define

(4.11) $\lambda \in i\mathfrak{h}^*$ is *admissible* if $\lambda - \rho \in \Lambda$;

here ρ denotes one half of the sum of the positive roots, relative to some ordering of $\Phi(\mathfrak{h})$. The particular ordering does not matter: any two choices of ρ differ by a sum of roots. According to (2.15) and (2.52), H contains the compact Cartan subgroup $B_{S,+}^0$ of M_S^0. By restriction, each $\lambda \in i\mathfrak{h}^*$ thus determines an element of $i\mathfrak{b}_{S,+}^*$.

(4.12) *Remark.* If $\lambda \in i\mathfrak{h}^*$ is admissible, then the restriction of λ to $\mathfrak{b}_{S,+}$ is admissible, relative to M_S^0 and $B_{S,+}^0$.

Proof. I let X be an element of $\mathfrak{b}_{S,-}$ with the property (4.1), and I choose an ordering of $\Phi(\mathfrak{b}_S)$, such that $\langle \alpha, X \rangle > 0$ implies the positivity of α. Via the Cayley transform c_S, this ordering can be carried over to $\Phi(\mathfrak{h})$. The ordering also induces one on the root system of $(\mathfrak{m}_S^{\mathbb{C}}, \mathfrak{b}_{S,+}^{\mathbb{C}})$, compatibly with the identifications (2.59) and (2.60). Because of the choice of the ordering, and because of the manner in which the Cartan involution operates on $\Phi(\mathfrak{b}_S)$, if $\alpha \in \Phi(\mathfrak{h})$ is positive and does not vanish identically on $\mathfrak{b}_{S,+}$, then some other positive root has the same restriction to $\mathfrak{b}_{S,+}$ as $-\alpha$. Now let ρ be one half of the sum of the positive roots in $\Phi(\mathfrak{h})$, and $\rho_0 \in i\mathfrak{b}_{S,+}^*$ one half of the sum of the positive roots of $(\mathfrak{m}_S^{\mathbb{C}}, \mathfrak{b}_{S,+}^{\mathbb{C}})$. According to what was just said, the restriction of ρ to $\mathfrak{b}_{S,+}$ coincides with ρ_0; this gives the remark.

It is necessary to examine in some more detail the case when S consists of a single noncompact root, say β. One can then pick a system of positive roots in $\Phi(\mathfrak{h})$, such that $\alpha \in \Phi(\mathfrak{h})$ is positive whenever $(\alpha, \beta) > 0$. Now let $\lambda \in i\mathfrak{h}^*$ be admissible. With ρ set equal to one half of the sum of the roots in the positive root system chosen above, $\lambda - \rho$ lies in Λ, and hence lifts to a character $e^{\lambda-\rho}$ of H. According to (2.56), H contains F_β. Hence $e^{\lambda-\rho}$ restricts a character $\zeta_\lambda : F_\beta \to \mathbb{C}^*$.

(4.13) *Remark.* The character ζ_λ does not depend on the particular choice of the positive root system. Moreover, if β is replaced by $-\beta$, in which case $M_{-\beta} = M_\beta$ and $F_{-\beta} = F_\beta$, ζ_λ remains unchanged.

Proof. Any two legitimate choices of ρ differ by a sum of roots α, with $(\alpha, \beta) = 0$. Via (2.59) and (2.60), each such α may be viewed as a root of $(\mathfrak{m}_\beta^{\mathbb{C}}, \mathfrak{b}_{\beta,+}^{\mathbb{C}})$, so that $e^\alpha \equiv 1$ on the center of $M_\beta^\dagger$, which contains F_β. If $-\beta$ takes over the role of β, one can simply replace the positive root system used in the definition of ζ_λ by its negative. It therefore must be shown that $e^{2\rho} \equiv 1$ on F_β. By an argument similar to the one in the proof of (4.12), 2ρ must differ from a suitable multiple of β by a sum of roots of $(\mathfrak{m}_\beta^{\mathbb{C}}, \mathfrak{b}_{\beta,+}^{\mathbb{C}})$. On the other hand, $(2\rho, \beta)(\beta, \beta)^{-1}$ is an integer. This reduces the problem to showing that $e^\beta \equiv 1$ on F_β, which can be deduced from the definition of F_β.

In view of (2.59) and (2.60), the root system of $(\mathfrak{m}_\beta^{\mathbb{C}}, \mathfrak{b}_{\beta,+}^{\mathbb{C}})$ can be canonically identified with the set of roots in $\Phi(\mathfrak{h})$ which are perpendicular to β. As (2.61) demonstrates, such a root α may well be compact as a root of $(M_\beta^0, B_{\beta,+}^0)$, and noncompact as a root of (G, H).

(4.14) *Remark.* Let $\Psi \subset \Phi(\mathfrak{h})$ be a system of positive roots, for which β is a simple root. If $\lambda \in i\mathfrak{h}^*$ satisfies $(\lambda, \alpha) > 0$ for every $\alpha \in \Psi \cap \Phi^c(\mathfrak{h})$, then also $(\lambda, \alpha) > 0$, whenever $\alpha \in \Psi$ is perpendicular to β and compact, considered as a root of $(M_\beta^0, B_{\beta,+}^0)$.

Proof. When such a root α is compact also as a root of (G, H), there is nothing to be said. Otherwise, according to (2.61), $\alpha \pm \beta$ must be roots. As sums of two noncompact roots, both are compact. Since β is simple and α positive, $\alpha \pm \beta \in \Psi$. Thus $(\lambda, \alpha \pm \beta) > 0$, and hence $(\lambda, \alpha) > 0$.

The statement of Theorem (4.15) below is of an inductive nature. In order to get the induction going, one must agree to the following convention: if G is a connected, compact Lie group, i.e. if $G = K$, the one point space G/K is considered to be Hermitian symmetric. In this case, every root with respect to a Cartan subgroup $H \subset G$ is a compact root, and every positive root system has the property (3.1). The invariant eigendistributions $\Theta(\Psi, \lambda)$ of Theorem (3.11) are then precisely the characters of the finite dimensional, irreducible G-modules.

(4.15) **Theorem.** *Let G be a reductive Lie group, subject to the usual hypotheses of this paper, and such that G/K is Hermitian symmetric. For each positive root system $\Psi \subset \Phi(\mathfrak{h})$, and for each admissible $\lambda \in i\mathfrak{h}^*$* (cf. (4.11)), *which satisfies the condition*

$$(*) \qquad (\alpha, \lambda) > 0 \quad \textit{whenever} \quad \alpha \in \Psi \cap \Phi^c(\mathfrak{h}),$$

there exists an invariant eigendistribution $\Theta(\Psi, \lambda)$ on G, with the following properties:

a) *If the positive root system Ψ fulfills* (3.1), *$\Theta(\Psi, \lambda)$ agrees with the invariant eigendistribution described by* (3.11).

b) *Under translation by elements of the center $Z(G)$, $\Theta(\Psi, \lambda)$ transforms according to the rule*

$$\Theta(\Psi, \lambda)(zg) = e^{\lambda - \rho}(z)\, \Theta(\Psi, \lambda)(g),$$

for $z \in Z(G)$ and $g \in G$ (ρ = one half of the sum of the positive roots, relative to an arbitrary positive root system; $\Theta(\Psi, \lambda)$ is viewed as a function).

c) *Let β be a noncompact root, which is simple for Ψ. Removing β from Ψ and replacing it by $-\beta$, one obtains another positive root system, Ψ_1. The sum of $\Theta(\Psi, \lambda)$ and $\Theta(\Psi_1, \lambda)$ equals an induced invariant eigendistribution, which will now be described. The restriction of λ to $\mathfrak{b}_{\beta,+}$ determines a functional $\mu \in i\mathfrak{b}_{\beta,+}^*$, which is admissible for M_β^0, according to* (4.12). *In the root system of $(M_\beta^0, B_{\beta,+}^0)$, via the identifications* (2.59) *and* (2.60), *Ψ cuts out a system of positive roots Ψ_β. Because of* (4.14), *Ψ_β and μ satisfy $(*)$, relative to M_β^0 and $B_{\beta,+}^0$. By induction*[9], *Ψ_β and μ determine an invariant eigendistribution ϕ_0 on M_β^0. As follows from* b), *applied to M_β^0, ϕ_0 has the transformation property* (4.7), *with $\zeta = \zeta_\lambda$ as in* (4.13), *and with $S = \{\beta\}$. The inverse Cayley transform c_β^{-1} maps $\mathfrak{b}_{\beta,-}$ into $i\mathfrak{h}$, so that $\lambda \circ c_\beta^{-1}$ restricts to a functional $\nu \in \mathfrak{b}_{\beta,-}^*$. Let Θ be the induced invariant eigendistribution corresponding to the data $S = \{\beta\}$, ν, ϕ_0, and ζ_λ. Then*

$$\Theta(\Psi, \lambda) + \Theta(\Psi_1, \lambda) = \Theta.$$

The preceeding conditions a)–c) *determine the invariant eigendistributions $\Theta(\Psi, \lambda)$ completely. In addition, the distributions $\Theta(\Psi, \lambda)$ satisfy*

---

[9] At this point, remark (4.16) should be taken into account.

d) *For each* $w \in W(H)$, $\Theta(w\Psi, w\lambda) = \Theta(\Psi, \lambda)$.

e) *The restriction of* $\Theta(\Psi, \lambda)$ *to* H *is given by the formula*

$$(-1)^q \frac{\sum_{w \in W(H)} \varepsilon(w)\, e^{w\lambda}}{\prod_{\alpha \in \Psi} (e^{\alpha/2} - e^{-\alpha/2})}$$

($q = \frac{1}{2} \dim_{\mathbb{R}} G/K$; *to make the formula meaningful on* H, *both numerator and denominator may have to be multiplied through by* e^{ρ}).

f) *Under the stronger hypothesis*

$$(**) \qquad \begin{aligned} (\lambda, \alpha) &> 0 \quad \text{if } \alpha \in \Phi^c(\mathfrak{h}) \cap \Psi \\ (\lambda, \alpha) &\geqq 0 \quad \text{if } \alpha \in \Phi^n(\mathfrak{h}) \cap \Psi \end{aligned}$$

the distribution $\Theta(\Psi, \lambda)$ *is tempered* [13, 39].

In particular, the distributions $\Theta(\Psi, \lambda)$, *with* Ψ *and* λ *subject to the condition* $(\lambda, \alpha) > 0$ *for all* $\alpha \in \Psi$, *are precisely the characters of the discrete series* [13].

(4.16) *Remark.* In order to inductively apply the theorem itself in c), one must know that $M_\beta^0 / M_\beta^0 \cap K$ is again Hermitian symmetric. According to (2.62), this is almost always true; but it fails if β is a short, noncompact root in a simple factor of G, which is isomorphic, up to covering, to $SO(2, 2n+1)$, $n \geqq 2$. There are two possible ways of dealing with the exception. For $G = SO(2, 2n+1)$, any positive root system $\Psi \subset \Phi(\mathfrak{h})$ can be connected to one of the type (3.1) by a chain of positive root systems $\Psi = \Psi_0, \Psi_1, \ldots, \Psi_N$, such that each Ψ_i is obtained from Ψ_{i-1} by changing the sign of a long, noncompact, simple root. In c), one can therefore simply exclude the case of any noncompact, simple root β for which $M_\beta^0 / M_\beta^0 \cap K$ fails to be Hermitian symmetric. With this restriction in c), the existence and uniqueness statements of the theorem remain correct; I shall prove this version of the theorem. Alternatively, it is possible to widen the scope of the theorem slightly, so as to make the class of groups G which it treats closed under the passage from G to M_β^0: a group G belongs to the larger class precisely when all of its noncompact, simple factors are, up to covering, either copies of $SO(1, 2n)$, or automorphism groups of Hermitian symmetric spaces. For groups with factors isomorphic to $SO(1, 2n)$, the statement of condition a) must then be modified, of course (cf. § 9).

(4.17) *Remarks.* I) The condition c) is symmetric in Ψ and Ψ_1, as follows from (4.13) and the fact that μ can be changed into $-\mu$ by an element of $W(B_\beta)$, which operates trivially on $B_{\beta, +}$.

II) Let $G = T \cdot G_1 \cdot \cdots \cdot G_N$ be a factorization of G, such that the G_i are connected semisimple subgroups, intersecting each other only in the center of G, if at all, and T a central torus. Then Θ can be expressed as a product $\Theta_1 \cdot \cdots \cdot \Theta_N$, multiplied by the restriction of the character $e^{\lambda - \rho}$ to T; here Θ_i denotes the invariant eigendistribution on G_i, parameterized in the manner of the theorem by Ψ and λ, restricted to G_i. This is a consequence of the uniqueness statement in the theorem; each application of the transition process c) only involves one of the factors G_i.

III) Under a finite covering $\tilde{G} \to G$ by another linear group $\tilde{G}$, $\Theta(\Psi, \lambda)$ pulls back to the invariant eigendistribution on $\tilde{G}$ which corresponds to Ψ and λ. Again, one can infer this from the uniqueness statement; the condition c) is compatible with going to the covering group.

The proof of the theorem will be given in § 5 and § 6. It is elementary—if somewhat lengthy—, and almost entirely combinatorial, except for the use of (3.11), which has a function-theoretic proof [16]. The main purpose of the theorem is to serve as a vehicle for inductive arguments: in order to verify a property of the distributions $\Theta(\Psi, \lambda)$, it suffices to verify it whenever the positive root system Ψ satisfies (3.1)—this is usually a simple matter—, and then to show that the property is consistent with statement c) in (4.15). The proofs of the theorem itself, of the other theorems in this section, and of Blattner's conjecture are all based on such inductive arguments.

From Theorem (4.15), one can deduce a global formula for the distributions $\Theta(\Psi, \lambda)$, in terms of those described by (3.11) and the process of inducing invariant eigendistributions. Since the inducing process of inducing is computable [8, 18, 40], and since the distributions of (3.11) are completely understood, the formula can be used, at least in principle, to explicitly write down the character of any given discrete series representation on any given Cartan subgroup. Special cases of this formula will be discussed, with only a sketch of a proof, in [37]. For actual computations, Theorems (4.21) and (4.22) below are much more useful, however.

In order to state those two theorems, I must take care of some preliminaries. Let $G' \subset G$ be a connected, reductive subgroup, which contains the compact Cartan subgroup H of G. For reference purposes, I choose a positive root system $\Psi_0 \subset \Phi(G, H)$; it restricts to a positive root system in $\Phi(G', H)$. As usual, ρ and ρ' shall denote one half of the sum of the positive roots in, respectively, $\Phi(G, H)$ and $\Phi(G', H)$. According to (2.3), there exists a finite covering $\tilde{G}' \to G'$ of G' by a matrix group $\tilde{G}'$, such that both ρ and ρ' lift to characters on the inverse image $\tilde{H}$ of H in $\tilde{G}'$. In passing, one should observe that

(4.18) if $\lambda \in i\mathfrak{h}^*$ is admissible relative to G, then it is also admissible relative to $\tilde{G}'$

(cf. (4.11)).

(4.19) *Remark.* There exists a unique function $\Delta_{G,G'}$ on $\tilde{G}'$, which can be expressed as the difference of two characters of finite dimensional representations of $\tilde{G}'$, and such that the restrictions of $\Delta_{G,G'}$ to $\tilde{H}$ is

$$\prod (e^{\alpha/2} - e^{-\alpha/2}),$$

with α running over all roots in Ψ_0 outside of $\Phi(G', H)$.

Proof. Since e^{ρ} and $e^{\rho'}$ make sense on $\tilde{H}$, the product multiplies out as an integral linear combination of characters of $\tilde{H}$. The center of the complexification of $\tilde{G}'$ lies in $\tilde{H}$. It therefore suffices to show that the product remains invariant under the action of every w in the Weyl group of the root system $\Phi(G', H)$. For any such w, $\varepsilon(w)$ is plus or minus one, depending on whether w alters the sign of an even or of an odd number of positive roots in $\Phi(G', H)$. Of course, the statement also holds with $\Phi(G, H)$ in place of $\Phi(G', H)$. Hence w flips the sign of an even number of roots in $\Psi_0 - (\Psi_0 \cap \Phi(G', H))$, as was to be shown.

The definition of $\Delta_{G,G'}$ depends on the choice of the reference positive root system Ψ_0. For any positive root system $\Psi \subset \Phi(G, H)$, I set $\varepsilon_{G,G'}(\Psi) = \pm 1$, so that

$$\Delta_{G,G'}|_{\tilde{H}} = \varepsilon_{G,G'}(\Psi) \prod (e^{\alpha/2} - e^{-\alpha/2}), \tag{4.20}$$

with α now running over $\Psi - (\Psi \cap \Phi(G', H))$. To distinguish the invariant eigendistributions of Theorem (4.15) on G and $\tilde{G}'$, I shall denote them by $\Theta_G(\ldots, \ldots)$ and $\Theta_{\tilde{G}'}(\ldots, \ldots)$. Let B be a Cartan subgroup of G. Replacing B by one of its conjugates, one can arrange that $H \subset B_+^0 \cdot G_B$ (cf. (2.40)). In order to simplify the notation, I set $G' = B_+^0 \cdot G_B$. If Ψ and λ satisfy the conditions of (4.15), then λ is also admissible relative to $\tilde{G}'$, and hence, for any $w \in W(G, H)$,

$$\varepsilon_{G,G'}(w\Psi) \cdot \Theta_{\tilde{G}'}(w\Psi \cap \Phi(G', H), w\lambda)$$

is a well-defined invariant eigendistribution on $\tilde{G}'$. As can be checked, it depends only on the coset of w in $W(G', H) \setminus W(G, H)$.

(4.21) **Theorem.** *Under the hypothesis stated above,*

$$\Theta_G(\Psi, \lambda)|_{G'} = (-1)^{q-q'} (\Delta_{G,G'})^{-1} \cdot \sum_{w \in W(G', H) \setminus W(G, H)} \varepsilon_{G,G'}(w\Psi) \cdot \Theta_{\tilde{G}'}(w\Psi \cap \Phi(G', H), w\lambda),$$

in the sense that the right hand side can be pulled down from $\tilde{G}'$ to G', where it then agrees with the left hand side; $q = \frac{1}{2}\dim_{\mathbb{R}} G/K$, $q' = \frac{1}{2}\dim_{\mathbb{R}} G'/G' \cap K$.

When B is a split Cartan subgroup, the statement (4.21) becomes vacuous. At the other extreme, for $B = H$, it becomes equivalent to (4.15e). For a general B, however, since $B \subset B_+^0 \cdot G_B$, it reduces the problem of computing the distributions $\Theta(\Psi, \lambda)$ on B to the analogous problem with G_B taking the place of G, and a split Cartan subgroup of G_B taking the place of B. The significance of this reduction stems from the fact that each simple factor of G_B is isomorphic, up to covering, to $Sp(n, \mathbb{R})$, for some n; moreover, n is always less than or equal to 2, unless a simple factor of G is itself isomorphic to $Sp(n, \mathbb{R})$. The proof of (4.21) is a key ingredient of the proof of (4.15), and will be given in §§ 5, 6.

One more unpleasant matter remains to be dealt with: a general Cartan subgroup B may have several connected components; on two distinct components of B, the explicit formula for one of the distributions $\Theta(\Psi, \lambda)$ may look entirely different. The next theorem will dispose of this difficulty. I enumerate the connected components of B as $B^0, B^1, \ldots, B^N$ (B^0 = identity component). For any particular B^j, I consider the group $G(B^j)$ of Proposition (2.44). Possibly after replacing B by one of its conjugates, one has $H \subset B_+^0 \cdot G(B^j)$. In the discussion above (4.21), I now let $B_+^0 \cdot G(B^j)$ play the role of G'.

(4.22) **Theorem.** *Under the hypotheses stated above,*

$$\Theta_G(\Psi, \lambda)|_{B^j} = (-1)^{q-q'} (\Delta_{G,G'})^{-1} \cdot \sum_{w \in W(G', H) \setminus W(G, H)} \varepsilon_{G,G'}(w\Psi)\, \Theta_{\tilde{G}'}(w\Psi \cap \Phi(G', H), w\lambda);$$

the same explanations as in (4.21) *apply.*

Remark. For the characters of the discrete series, the conclusions of Theorems (4.21) and (4.22) can be deduced directly from Harish-Chandra's construction[10].

The statement of Theorem (4.22) looks formally identical to that of Theorem (4.21). There is an important difference, however: the formula in (4.22) may hold only on B^j; on some other connected component of B, it may be totally false. Just as (4.21), (4.22) is a crucial ingredient of the proof of Theorem (4.15). It will be proven in §§ 5, 6.

Thanks to Theorems (4.21) and (4.22), the problem of explicitly computing the invariant eigendistributions $\Theta(\Psi, \lambda)$ reduces to one rather special case; it suffices to know the $\Theta(\Psi, \lambda)$ on the identity component of a split Cartan subgroup, for all those simple groups which operate on a Hermitian symmetric space, and which contain a split Cartan subgroup. As was pointed out before, up to covering, only the real symplectic groups have these properties.

Actually, in many cases, one can compute the invariant eigendistributions $\Theta(\Psi, \lambda)$ in terms of the discrete series characters of $SL(2, \mathbb{R})$. To see this, I consider a simple matrix group G, with G/K Hermitian symmetric, and such that G is not locally isomorphic to $Sp(n, \mathbb{R})$ or $SO(2, 2n+1)$. All roots of G must then have the same length. For any subgroup G' of G which can come up in an application of Theorems (4.21) and (4.22), every simple factor has roots of only one length. On the other hand, each such simple factor of G' has a Hermitian symmetric quotient and contains a split Cartan subgroup. Conclusion: up to covering, G' is a product of copies of $SL(2, \mathbb{R})$ and a central torus. For the group $SL(2, \mathbb{R})$, the distributions $\Theta(\Psi, \lambda)$ are of course well known. Hence, for any given connected component B^j of a Cartan subgroup $B \subset G$, there is a simple, explicit formula for the restriction of $\Theta(\Psi, \lambda)$ to B^j. It will be left to the reader to write out the concrete formula.

One can deal with the group $SO(2, 2n+1)$ by very similar arguments. In this situation, if G' is a subgroup arising from an application of (4.21) or (4.22), the semisimple part of G' is locally isomorphic to $Sp(2, \mathbb{R})$ or a product of at most two copies of $SL(2, \mathbb{R})$. Since the discrete series characters of $Sp(2, \mathbb{R})$ have been computed [18, 37], one again obtains explicit global formulas for the invariant eigendistributions $\Theta(\Psi, \lambda)$. These remarks leave open only the case of $G = Sp(n, \mathbb{R})$, which is considerably more difficult, and which is considered in [37].

§ 5. Some Inductive Arguments

This section is devoted to some key arguments in the proof of Theorem (4.15). Roughly speaking, they will show that (4.21) and (4.22), as well as other statements, are compatible with (4.15c). These arguments do not depend on the Hermitian symmetric structure of G/K; I therefore shall not specifically assume that G/K is Hermitian symmetric.

To begin with, it is necessary to have an explicit formula for the process of inducing an invariant eigendistribution from a maximal cuspidal parabolic subgroup. Thus let β be a noncompact root of (G, H). I consider an invariant eigendistribution ϕ_0 on M_β^0, a character $\zeta\colon F_\beta \to \mathbb{C}^*$ which satisfies (4.7), and a

[10] This was pointed out to me by Zuckerman.

linear functional $\nu \in \mathfrak{b}^*_{\beta,-}$. These data give rise to an invariant eigendistribution Θ, as described in the beginning of § 4. I shall now use J.A. Wolf's computation of induced characters (Theorem 4.3.8 of [40]; as stated there, the theorem contains an error, which will be rectified below), to determine the restriction of Θ to the various Cartan subgroups of G.

Let then $B \subset G$ be a Cartan subgroup. I consider the set of $M_\beta \cdot B_{\beta,-}$-conjugacy classes of Cartan subgroups of $M_\beta \cdot B_{\beta,-}$, which are conjugate under G to B. From each such conjugacy class, I pick a representative, and I enumerate the representatives as $B_1, \ldots, B_n$. Of course it may happen that no conjugate of B lies in $M_\beta \cdot B_{\beta,-}$; in this case $n=0$. For each B_i, I choose a particular $g_i \in G$ so that $B = \operatorname{Ad} g_i(B_i)$. In terms of ϕ_0, ζ, and ν, one can define a function[11] ψ on $M_\beta \cdot B_{\beta,-}$ as follows:

$$
\begin{aligned}
&\psi(g)=0 && \text{if } g \notin M_\beta^\dagger \cdot B_{\beta,-}, \\
&\psi(mfb)=\phi_0(m)\,\zeta(f)\,e^\nu(b), && \text{if } m \in M_\beta^0,\ f \in F_\beta,\ b \in B_{\beta,-}
\end{aligned}
\tag{5.1}
$$

(cf. (2.54) and (2.55)). Now let X be a generator of $\mathfrak{b}_{\beta,-}$. Since each B_i contains $B_{\beta,-}$, X also lies in $\mathfrak{b}_i$. Consequently $\operatorname{Ad} g_i(X) \in \mathfrak{b}$, for $1 \leqq i \leqq n$.

(5.2) **Lemma.** *For* $b \in B$,

$$
\Theta(b) = a \cdot \sum_{i=1}^n \frac{1}{c_i} \sum_{w \in W(G,B)} \psi(\operatorname{Ad} g_i^{-1}(wb)) \cdot \Big|\prod_{\alpha \in \Phi(G,B),\, \langle \alpha, \operatorname{Ad} g_i(X)\rangle > 0} (e^{\alpha/2} - e^{-\alpha/2})(wb)\Big|^{-1},
$$

with $c_i = \# \{w \in W(G,B) \mid w(\operatorname{Ad} g_i(X)) = \operatorname{Ad} g_i(X)\}$,

$$
a = \frac{\# \{w \in W(G,B) \mid wX = X\}}{\# W(M_\beta^0, B_{\beta,+}^0)}
$$

Remark. The expression

$$
\prod_{\alpha \in \Phi(G,B),\, \langle \alpha, \operatorname{Ad} g_i(X)\rangle > 0} (e^{\alpha/2} - e^{-\alpha/2}) \tag{5.3}
$$

may make sense only on a twofold covering of B; the absolute value, however, is well-defined on B.

Proof. The lemma follows from Theorem 4.3.8 of [40], coupled with certain observations. Theorem 5 of [18] is an alternative reference, but the notation of this paper more closely resembles that of [40]. Both the statement and the proof of Theorem 4.3.8 in [40] overlook the possibility that several non-conjugate Cartan subgroups of $M_\beta \cdot B_{\beta,-}$ can be G-conjugate. Hence, for each i, one must take the expression in Theorem 4.3.8 with $J = B_i$, transfer it to B via $\operatorname{Ad} g_i$, and sum over i. Since the formula to be proved is really meaningful only on the regular set, b may be assumed to be regular. In this case, the orbit of $g_i^{-1} b g_i$ under the normalizer of B_i in $M_\beta \cdot B_{\beta,-}$ has exactly as many elements as the Weyl group of B_i in $M_\beta \cdot B_{\beta,-}$, namely c_i (in the context of this paper, B is Abelian, as need not be true in [40]). In the beginning of § 4, ϕ was constructed from ϕ_0 in two steps.

---

[11] Strictly speaking, ψ is a well-defined function only on the regular set in $M_\beta \cdot B_{\beta,-}$.

The formula (5.1) already takes into account the passage from ϕ_0 to ϕ_1. As far as the step from ϕ_1 to ϕ is concerned, I claim that

(5.4) every coset in $M_\beta/M_\beta^\dagger$ has a representative g which normalizes B_i;

this will be shown below. Since $M_\beta^\dagger$ is normal in M_β, such a representative also normalizes $B_i \cap (M_\beta^\dagger \cdot B_{\beta,-})$. Moreover, $\mathrm{Ad}\, g$ preserves the absolute value of the expression (5.3), pulled back to B_i via $\mathrm{Ad}\, g_i^{-1}$. The contribution of B_i to the formula for $\Theta|_B$ in effect involves a summation over $W(G, B_i)$. Hence, in the formula for Θ, the passage from ϕ_1 to ϕ reflects itself by a multiplicative factor, namely the order of $M_\beta/M_\beta^\dagger$, which equals a, according to (2.58). Now only (5.4) remains to be verified. For any $g \in M_\beta$, $\mathrm{Ad}\, g$ maps $B_i \cap M_\beta^0$ onto another Cartan subgroup of M_β^0. According to (2.52), $B_i \cap M_\beta^0$ and its image under $\mathrm{Ad}\, g$ are conjugate to each other, relative to the adjoint group of $\mathfrak{m}_\beta^{\mathbb{C}}$. As Rothschild has shown[12] (Corollary 2.4 in [33]), these two Cartan subgroups must then also be conjugate under M_β^0. Thus, if g is modified by a suitable factor in M_β^0, it will normalize $B_i \cap M_\beta^0$, and hence B_i. This concludes the argument.

It will be necessary to have an explicit description of the B_i. For this purpose, I consider a Cartan subgroup $B = B_S$, where $S \subset \Phi^n(\mathfrak{h})$ is a strongly orthogonal set of noncompact roots. Let G_S be the subgroup of G defined in (2.42). Then $B_{S,+}^0 \cdot G_S$ is the centralizer of $B_{S,+}^0$, and $H \subset B_{S,+}^0 \cdot G_S$. I set $\Phi' = \Phi(B_{S,+}^0 \cdot G_S, H)$; equivalently,

(5.5) $$\Phi' = \{\alpha \in \Phi(G, H) | \alpha \in \mathbb{Q}\text{-linear span of } S\}.$$

According to (2.15), $Y_\beta + Y_{-\beta}$ spans $\mathfrak{b}_{\beta,-}$. Hence, for $g \in G$, $\mathrm{Ad}\, g^{-1}(B_S)$ lies in $M_\beta \cdot B_{\beta,-}$ if and only if $\mathrm{Ad}\, g(Y_\beta + Y_{-\beta}) \in \mathfrak{b}_S$. Now let $w \in W(G, H)$ be such that $w\beta \in \Phi'$. I choose $k \in K$ such that $\mathrm{Ad}\, k$ represents w. Then $\mathrm{Ad}\, k(Y_\beta + Y_{-\beta})$ lies in $\mathfrak{g}_S$; moreover, as a conjugate of $Y_\beta + Y_{-\beta}$, $\mathrm{Ad}\, k(Y_\beta + Y_{-\beta})$ is semisimple, with real eigenvalues. It follows that every split Cartan subalgebra of $\mathfrak{g}_S$, for example $\mathfrak{b}_{S,-}$, contains a G_S-conjugate of $\mathrm{Ad}\, k(Y_\beta + Y_{-\beta})$. Thus, if $g \in G_S$ is suitably chosen, $\mathrm{Ad}\, g k(Y_\beta + Y_{-\beta}) \in \mathfrak{b}_{S,-}$. As was pointed out before, $\mathrm{Ad}(gk)^{-1}(B_S)$ is therefore a Cartan subgroup of $M_\beta \cdot B_{\beta,-}$.

(5.6) **Lemma.** *The $M_\beta \cdot B_{\beta,-}$-conjugacy class of the Cartan subgroup $\mathrm{Ad}(gk)^{-1}(B_S)$ depends only on the root $w\beta \in \Phi'$; it is independent of the particular choices of w, k, and g. Thus, to each root $w\beta \in \Phi'$, with $w \in W(G, H)$, there corresponds an $M_\beta \cdot B_{\beta,-}$-conjugacy class of Cartan subgroups of $M_\beta \cdot B_{\beta,-}$. Every Cartan subgroup of $M_\beta \cdot B_{\beta,-}$, which is G-conjugate to B_S, belongs to one of these conjugacy classes. Two roots $w_1\beta$, $w_2\beta$, with $w_i \in W(G, H)$, determine the same conjugacy class if and only if $w_2\beta = \pm w w_1 \beta$, for some $w \in W(G, H)$, such that $w\Phi' = \Phi'$.*

Proof. A preliminary statement will be helpful. Let $F \in \mathfrak{b}_{S,-}$ be given; if $g \in G_S$ is such that $\mathrm{Ad}\, g(F) \in \mathfrak{b}_{S,-}$, then

(5.7) g can be factored as $g = g_1 g_2$, with $g_1 \in$ normalizer of $\mathfrak{b}_{S,-}$ in G_S and $g_2 \in$ centralizer of F in G_S.

Indeed, both $\mathfrak{b}_{S,-}$ and $\mathrm{Ad}\, g^{-1}(\mathfrak{b}_{S,-})$ are split Cartan subalgebras of the centralizer of F in G_S, and hence they are conjugate under $\mathrm{Ad}\, g_2^{-1}$, for some g_2 in the centralizer of F. But then $g_1 = g g_2^{-1}$ normalizes $\mathfrak{b}_{S,-}$. As for the first statement of the

[12] This is also implicit in the results of [38].

lemma, the ambiguity in the choices of w and k means that $\mathrm{Ad}\,k(Y_\beta+Y_{-\beta})$ is determined only up to the action of $\mathrm{Ad}\,h$, for some $h\in H\cap G_S$. The factor h can be absorbed into g, so that one only needs to worry about the ambiguity in the choice of $g\in G_S$, with k fixed. If g and g' are two possible choices, one can apply (5.7), with $g'g^{-1}$ taking the place of g, and with $F=\mathrm{Ad}\,gk(Y_\beta+Y_{-\beta})$:

$$g'g^{-1}=g_1\,gkmk^{-1}g^{-1},$$

for some g_1 in the normalizer of $\mathfrak{b}_S$, and some m in the centralizer of $Y_\beta+Y_{-\beta}$, i.e. $m\in M_\beta\cdot B_{\beta,-}$. Now $\mathrm{Ad}(g'k)^{-1}(B_S)=\mathrm{Ad}(g_1\,gkm)^{-1}(B_S)=\mathrm{Ad}\,m^{-1}(\mathrm{Ad}(gk)^{-1})(B_S)$, as desired.

To continue the proof, let $B\subset M_\beta\cdot B_{\beta,-}$ be a Cartan subgroup which is G-conjugate to B_S. Replacing B by one of its $M_\beta\cdot B_{\beta,-}$-conjugates, one can arrange that $B\cap M_\beta^0$ is obtained from $H\cap M_\beta^0$ by the Cayley transform construction: there exists a strongly orthogonal set $S'\subset\Phi(G,H)$, all of whose members are orthogonal to β and noncompact, viewed as roots of $(M_\beta^0, H\cap M_\beta^0)$, such that

$$\mathfrak{b}^{\mathbb{C}}\cap\mathfrak{m}_\beta^{\mathbb{C}}=c_{S'}(\mathfrak{h}^{\mathbb{C}}\cap\mathfrak{m}_\beta^{\mathbb{C}}); \tag{5.8}$$

the Cayley transform $c_{S'}$ is defined relative to the group M_β^0, rather than G. Just as in the proof of (2.61), one can show that β is strongly orthogonal to all roots in S', with at most one exception. I shall proceed assuming there is an exception; the other case may be treated similarly, but with fewer complications. Thus, for exactly one $\gamma\in S'$, $\beta\pm\gamma$ are roots. According to (2.61), γ must be compact, viewed as a root of (G,H), so that $\beta\pm\gamma$ is a pair of strongly orthogonal noncompact roots. Let $S''=S'-\{\gamma\}$; then every root in S'' is strongly orthogonal to β. The Cayley transforms corresponding to S'', relative to M_β and G, therefore coincide. Also, $c_{S'}=c_{S''}\circ c_\gamma$, where c_γ is the Cayley transform corresponding to γ, relative to the group M_β^0. Since $\mathfrak{b}_\beta^{\mathbb{C}}=c_\beta\mathfrak{h}^{\mathbb{C}}$, and in view of (5.8),

$$\mathfrak{b}^{\mathbb{C}}=c_{S''}\circ c_\gamma\circ c_\beta\,\mathfrak{h}^{\mathbb{C}}.$$

I claim that $c_\gamma\circ c_\beta\,\mathfrak{h}^{\mathbb{C}}=\mathrm{Ad}\,g\circ c_{\beta+\gamma}\circ c_{\beta-\gamma}\mathfrak{h}^{\mathbb{C}}$, for some $g\in G$ which commutes with $c_{S''}$. As far as the verification of this claim is concerned, one may as well assume that $\Phi(G,H)=\{\pm\beta,\pm\gamma,\pm(\beta\pm\gamma)\}$, and $G=Sp(2,\mathbb{R})$. But then the claim becomes essentially obvious, because both $c_\gamma\circ c_\beta\,\mathfrak{h}^{\mathbb{C}}$ and $c_{\beta+\gamma}\circ c_{\beta-\gamma}\mathfrak{h}^{\mathbb{C}}$ are complexifications of split Cartan subalgebras. Thus, up to conjugacy, B corresponds to the strongly orthogonal subset $S''\cup\{\beta\pm\gamma\}$ of $\Phi^n(G,H)$. According to (2.16), for some $w\in W(G,H)$, $w(S''\cup\{\beta\pm\gamma\})=S$; at least if some of the roots in $S''\cup\{\beta\pm\gamma\}$ are replaced by their negatives, which is legitimate. In particular, $w\beta\in\Phi'$. Without loss of generality, I now assume that $w=1$. If $g\in G_S$ is properly chosen, $\mathrm{Ad}\,g(Y_\beta+Y_{-\beta})$ lies in $\mathfrak{b}_{S,-}$. It must be shown that $\mathrm{Ad}\,g^{-1}(B_S)$ is $M_\beta\cdot B_{\beta,-}$-conjugate to B. For this assertion, only the sub-root system $\{\pm\beta,\pm\gamma,\pm(\beta\pm\gamma)\}$ really matters; the problem can again be reduced to the special case $G=Sp(2,\mathbb{R})$, where it presents no particular difficulty.

Only the final statement of the lemma remains to be proven. Let $w_1, w_2\in W(G,H)$ be such that $w_i\beta\in\Phi'$, $i=1,2$. For $i=1,2$, I choose a $k_i\in K$ which represents w_i, and $g_i\in G_S$, so that $\mathrm{Ad}(g_i k_i)(Y_\beta+Y_{-\beta})\in\mathfrak{b}_{S,-}$. I first suppose that $w_2\beta=\pm w w_1\beta$, for some $w\in W(G,H)$ with $w\Phi'=\Phi'$. The two strongly orthogonal subsets S and wS

of $\Phi''(G,H)$ span all of Φ'. Hence (2.16), applied to G_S, coupled with the fact that any two split Cartan subalgebras of $\mathfrak{g}_S$ must be conjugate, shows that $\pm S$ and $\pm wS$ are conjugate under $W(B^0_{S,+}\cdot G_S, H)$. Hence w can be factored into a product of an element of $W(B^0_{S,+}\cdot G_S, H)$ and an element of

(5.9) $$\{w\in W(G,H) \mid \pm wS = \pm S\};$$

these two cases may be treated separately. If w has a representation as $\mathrm{Ad}\,k$, with $k\in G_S$, the k_i and g_i can be chosen so that $k_2 = k k_1$, $g_2 = g_1$ (provided $w_2 = w w_1$, as may be assumed). An argument based on (5.7), similar to one near the beginning of this proof, now implies that the two Cartan subgroups $\mathrm{Ad}(g_i k_i)^{-1}(B_S)$ of $M_\beta\cdot B_{\beta,-}$ are conjugate. Next, if w belongs to the group (5.9), it can be represented as $\mathrm{Ad}\,k$, with k normalizing H, B_S, and G_S (cf. (2.20)). Again, if the g_i and k_i are suitably chosen, $k_2 = k k_1$, $g_2 = g_1$. Thus,

$$g_2 k_2 = g_1 k k_1 = g_1 k g_1^{-1} k^{-1} k g_1 k_1,$$

with $g_1 k g_1^{-1} k^{-1}\in G_S$. Applying (5.7) to $g = g_1 k g_1^{-1} k^{-1}$ and $F = \mathrm{Ad}(k g_1 k_1)(Y_\beta + Y_{-\beta})$, one can find $g_3\in$ normalizer of $\mathfrak{b}_{S,-}$ in G_S, and $m\in M_\beta\cdot B_{\beta,-}$, satisfying

$$g_1 k g_1^{-1} k^{-1} = g_3 k g_1 k_1 m k_1^{-1} g_1^{-1} k^{-1}.$$

Hence $g_2 k_2 = g_1 k k_1 = g_3 k g_1 k_1 m$. Since $g_3 k$ normalizes B_S, the two Cartan subalgebras $\mathrm{Ad}(g_i k_i)^{-1}(B_S)$ are related by $\mathrm{Ad}\,m$, and hence $M_\beta\cdot B_{\beta,-}$-conjugate.

Conversely, I suppose

$$\mathrm{Ad}(g_2 k_2)^{-1}(B_S) = \mathrm{Ad}\,m^{-1}\circ \mathrm{Ad}(g_1 k_1)^{-1}(B_S),$$

for some $m\in M_\beta\cdot B_{\beta,-}$. Then $g_2 k_2 = g_3 g_1 k_1 m$, with $g_3\in$ normalizer of B_S. Because of (2.36) and (2.20), $g_3 = k g_4$ for some k which normalizes H, B_S, and G_S, and such that $g_4\in G_S$. Since g_4 normalizes B_S, it can be absorbed into g_1; then

$$g_2 k_2 = k g_1 k_1 m = k g_1 k^{-1} k k_1 m.$$

Since G_S contains g_2 and $k g_1 k^{-1}$, $\mathrm{Ad}\,k_2(Y_\beta + Y_{-\beta})$ and $\mathrm{Ad}\,k k_1(Y_\beta + Y_{-\beta})$ are G_S-conjugate. Let $w\in W(G,H)$ be the element determined by $\mathrm{Ad}\,k$. Because of the properties of k, $w\Phi' = \Phi'$. The two noncompact roots $w_2\beta$ and $w w_1\beta$, which both lie in Φ', give rise to Cartan subalgebras of $\mathfrak{g}_S$ with one dimensional split parts. According to what was just said, these split parts are conjugate under G_S. Two Cartan subalgebras of $\mathfrak{g}_S$ are conjugate whenever their split parts are. Hence $w_2\beta$ and $w w_1\beta$ determine conjugate Cartan subalgebras of $\mathfrak{g}_S$. In view of (2.16), for some $w'\in W(B^0_{S,+}\cdot G_S, H)$, $w_2\beta = \pm w' w w_1\beta$. Thus $w_1\beta$ and $\pm w_2\beta$ are related by an element of $W(G,H)$ which preserves Φ'. This completes the proof of Lemma (5.6).

I shall now combine (5.2) and (5.6). The symbols β, ϕ_0, ζ, v, and Θ shall have the same meaning as in the beginning of this section; S is again a strongly orthogonal subset of $\Phi''(\mathfrak{h})$, and Φ' is defined by (5.6). I enumerate the set

$$\{w\in W(G,H) \mid w\beta\in\Phi'\}$$

as $\{w_1,\dots,w_n\}$; the integer n may not agree with that in Lemma (5.2), of course. For $1\leq i\leq n$, one can choose a representative $k_i\in K$ of w_i, such that

$$\mathrm{Ad}\,k_i(Y_\beta + Y_{-\beta}) = Y_{w_i\beta} + Y_{-w_i\beta},$$

in the notation of (2.7). Since $\operatorname{Ad} k_i(Y_\beta + Y_{-\beta}) \in \mathfrak{g}_S$, as was argued in preparation of (5.6), there exists $g_i \in G_S$ with $\operatorname{Ad}(g_i k_i)(Y_\beta + Y_{-\beta}) \in \mathfrak{b}_S$. Thus $\operatorname{Ad}(g_i k_i)^{-1}(B_S)$ is a Cartan subgroup of $M_\beta \cdot B_{\beta,-}$. I set

$$X = Y_\beta + Y_{-\beta}, \tag{5.10}$$
$$X_i = \operatorname{Ad} k_i(X) = Y_{w_i\beta} + Y_{-w_i\beta}.$$

According to (2.15), X spans $\mathfrak{b}_{\beta,-}$, and X_i spans $\mathfrak{b}_{w_i\beta,-}$, for $1 \leqq i \leqq n$. I recall the definition of $W(B_S)_-$: it is the Weyl group of B_S in $B^0_{S,+} \cdot G_S$ (cf. (2.40) and (2.42)).

(5.11) **Lemma.** *For* $b \in B_S$,

$$\Theta(b) = \sum_{i=1}^n C_i \cdot \sum_{w \in W(B_S)_-} \psi(\operatorname{Ad}(g_i k_i)^{-1}(wb)) \cdot \left|\prod_{\alpha \in \Phi(G,B_S),\, \langle \alpha, \operatorname{Ad} g_i(X_i)\rangle > 0} (e^{\alpha/2} - e^{-\alpha/2})(wb)\right|^{-1},$$

where

$$C_i = \frac{\#\{w \in W(B^0_{S,+} \cdot G_S, B_{w_i\beta}) \mid w X_i = X_i\}}{\#\{w \in W(B^0_{S,+} \cdot G_S, B_S) \mid w \operatorname{Ad} g_i(X_i) = \operatorname{Ad} g_i(X_i)\}} \cdot \frac{1}{\# W(M^0_{w_i\beta}, B^0_{w_i\beta,+}) \cdot \# W(B^0_{S,+} \cdot G_S, H)}.$$

Proof. According to (2.36), every $w \in W(G, B_S)$ can be factored as $w = w' w''$, with $w'' \in W(B_S)_-$ and $w' \in U(B_S)$. Moreover, as follows from (2.43),

$$U(B_S) \cap W(B_S)_- \simeq \{w \in W(B^0_{S,+} \cdot G_S, H) \mid wS \subset S \cup (-S)\}. \tag{5.12}$$

Thus,

(5.13) every element of $W(G, B_S)$ can be expressed as a product uw, with $u \in U(B_S)$ and $w \in W(B_S)_-$, in exactly as many ways as the cardinality of the group (5.12).

I enumerate the elements of $U(B_S)$ as $\{u_1, \dots, u_m\}$. In view of Lemma (2.20), with $S_1 = S_2 = S$, and in view of the definition of G_S, each $u_j \in U(B_S)$ can be represented as $\operatorname{Ad} v_j$ for some $v_j \in K$, such that

$$v_j \text{ normalizes } H,\ B_S, \text{ and } G_S. \tag{5.14}$$

As i runs from 1 to n, each conjugacy class of Cartan subgroups of $M_\beta \cdot B_{\beta,-}$, which contains a G-conjugate of B_S, is represented by some $\operatorname{Ad}(g_i k_i)^{-1}(B_S)$. The conjugacy class of $\operatorname{Ad}(g_i k_i)^{-1}(B_S)$ occurs exactly as many times as the integer

$$\frac{\#\{w \in W(G,H) \mid w w_i \beta = \pm w_i \beta\} \cdot \#\{w \in W(G,H) \mid w \Phi' = \Phi'\}}{\#\{w \in W(G,H) \mid w\Phi' = \Phi',\ w w_i \beta = \pm w_i \beta\}}. \tag{5.15}$$

All this follows from (5.6). Now let a' be the integer a of (5.2), divided by the order of the group (5.12), and c'_i the order of

$$\{w \in W(G, B_S) \mid w \operatorname{Ad} g_i(X_i) = \operatorname{Ad} g_i(X_i)\}, \tag{5.16}$$

times the integer (5.15). Putting together (5.2) and the various statements above, one finds

$$\Theta(b)=a'\sum_{i=1}^{n}\frac{1}{c_i'}\sum_{j=1}^{m}\sum_{w\in W(B_S)_-}\psi(\mathrm{Ad}(k_i^{-1}g_i^{-1}v_j)(wb)) \cdot \left|\prod_{\alpha\in\Phi(G,B_S),\langle\alpha,\mathrm{Ad}g_i(X_i)\rangle>0}(e^{\alpha/2}-e^{-\alpha/2})(\mathrm{Ad}\,v_j(wb))\right|^{-1}, \tag{5.17}$$

for $b\in B_S$.

Because of (5.14), for any given i and j, $\mathrm{Ad}(v_j^{-1}k_i)$ represents again an element of the set

$$\{w\in W(G,H)|w\beta\in\Phi'\},$$

say w_l, with l depending on i and j, of course. Also, $v_j^{-1}g_iv_j$ is an element of G_S, such that

$$\mathrm{Ad}(v_j^{-1}g_iv_jv_j^{-1}k_i)(X)\in\mathfrak{b}_S.$$

Hence,

$$\psi(\mathrm{Ad}(k_i^{-1}g_i^{-1}v_j)(wb))=\psi(\mathrm{Ad}(g_lk_l)^{-1}(wb)),$$

and similarly

$$\left|\prod_{\alpha\in\Phi(G,B_S),\langle\alpha,\mathrm{Ad}g_i(X_i)\rangle>0}(e^{\alpha/2}-e^{-\alpha/2})(\mathrm{Ad}\,v_j(wb))\right| =\left|\prod_{\alpha\in\Phi(G,B_S),\langle\alpha,\mathrm{Ad}g_l(X_l)\rangle>0}(e^{\alpha/2}-e^{-\alpha/2})(wb)\right|.$$

Since $\mathrm{Ad}\,v_j$ preserves S, B_S, G_S, etc., one obtains $c_i'=c_l'$. The summation over j in (5.17) can therefore be eliminated, if one multiplies the right hand side with the order of $U(B_S)$ and sets v_j equal to the identity.

It remains to be shown that the constant C_i in the statement of the lemma equals $\frac{a'}{c_i'}\#U(B_S)$. Since $\mathrm{Ad}\,k_i$ establishes an isomorphism between B_β and $B_{w_i\beta}$, one has

$$a'=\frac{\#\{w\in W(G,B_{w_i\beta})|wX_i=X_i\}}{\#W(M_{w_i\beta}^0,B_{w_i\beta,+}^0)\cdot\#\{w\in W(B_{S,+}^0\cdot G_S,H)|wS\subset S\cup(-S)\}} \tag{5.18}$$

for $i=1,\ldots,n$. Because of (2.36) and (2.43), there is an isomorphism

$$\{w\in W(G,B_{w_i\beta})|wX_i=X_i\}\simeq\{w\in W(G,H)|ww_i\beta=\pm w_i\beta\}. \tag{5.19}$$

Next, I shall investigate the group $\{w\in W(G,H)|w\Phi'=\Phi'\}$. For every w in this group, wS is a strongly orthogonal subset of $\Phi^n(G,H)$, which spans Φ'. Under the correspondence (2.16), applied to the group G_S, both S and wS determine the same conjugacy class of Cartan subgroups of G_S, namely the conjugacy class of split Cartan subgroups. Hence there must exist $w'\in W(B_{S,+}^0\cdot G_S,H)$, such that $w'wS\subset S\cup(-S)$. Taking into account (2.35), one may deduce

$$\#\{w\in W(G,H)|w\Phi'=\Phi'\} =\frac{\#W(B_{S,+}^0\cdot G_S,H)\cdot\#U(B_S)}{\#\{w\in W(B_{S,+}^0\cdot G_S,H)|wS\subset S\cup(-S)\}}\cdot \tag{5.20}$$

At this point, (5.18–20) give the identity

$$\frac{a'}{c'_i}\cdot \# U(B_S)=\frac{1}{\# W(M^0_{w_i\beta}, B^0_{w_i\beta,+})\cdot \# W(B^0_{S,+}\cdot G_S, H)} \cdot\frac{\#\{w\in W(G,H)| w\Phi'=\Phi', ww_i\beta=\pm w_i\beta\}}{\#\{w\in W(G,B_S)| w\,\mathrm{Ad}\, g_i(X_i)=\mathrm{Ad}\, g_i(X_i)\}}. \tag{5.21}$$

To complete the proof, the right hand side of (5.21) must be shown to be equal to C_i. Now let

$$\Phi'_i=\{\alpha\in\Phi'|\alpha\perp w_i\beta\};$$

via (2.60), Φ'_i can be identified with a sub-root system of $\Phi(G, B_{w_i\beta})$, namely

$$\Phi'_i\simeq\Phi(G_i, B_{w_i\beta}),\quad \text{where}\quad G_i=\text{centralizer of } B_{w_i\beta,-} \text{ in } B^0_{S,+}\cdot G_S. \tag{5.22}$$

In terms of the isomorphism (5.19), one finds

$$\{w\in W(G,H)| w\Phi'=\Phi', ww_i\beta=\pm w_i\beta\}\simeq\{w\in W(G,B_{w_i\beta})| w\Phi'_i=\Phi'_i, wX_i=X_i\}.$$

Since G_S has a split Cartan subgroup, so does the semisimple part of G_i. Consequently, Φ'_i must admit a strongly orthogonal spanning set S'_i, consisting of roots which are noncompact when viewed as roots of $(M^0_{w_i\beta}, B^0_{w_i\beta,+})$. In complete analogy to (5.20), with virtually the same arguments, one may conclude

$$\begin{aligned}&\#\{w\in W(G,B_{w_i\beta})| w\Phi'_i=\Phi'_i, wX_i=X_i\}\\ &=\frac{\# W(G_i,B_{w_i\beta})\cdot\#\{w\in W(G,B_{w_i\beta})| wS'_i\subset S'_i\cup(-S'_i), wX_i=X_i\}}{\#\{w\in W(G_i,B_{w_i\beta})| wS'_i\subset S'_i\cup(-S'_i)\}}.\end{aligned} \tag{5.24}$$

It should be also observed that

$$W(G_i,B_{w_i\beta})=\{w\in W(B^0_{S,+}\cdot G_S, B_{w_i\beta})| wX_i=X_i\}. \tag{5.25}$$

By construction, $\mathrm{Ad}\, g_i^{-1}(B_S)$ is a Cartan subgroup of G_i, whose toroidal part lies in the center of G_i. Thus, g_i can be modified, without destroying any of its required properties, such that $\mathrm{Ad}\, g_i^{-1}(B_S)$ becomes the Cartan subgroup corresponding to the strongly orthogonal set S'_i, whose elements are regarded as noncompact roots of G_i, or equivalently of $M_{w_i\beta}$. If one applies (2.36) and (2.43) in this context, with $M_{w_i\beta}$ taking the place[13] of G and S'_i the place of S, one is lead to the equality

$$\begin{aligned}&\#\{w\in W(G,B_S)| w\,\mathrm{Ad}\, g_i(X_i)=\mathrm{Ad}\, g_i(X_i)\}\\ &=\#\{w\in W(G,\mathrm{Ad}\, g_i^{-1}(B))| wX_i=X_i\}\\ &=\frac{\# W(G_i,\mathrm{Ad}\, g_i^{-1}(B_S))\cdot\#\{w\in W(G,B_{w_i\beta})| wS'_i\subset S'_i\cup(-S'_i), wX_i=X_i\}}{\#\{w\in W(G_i,B_{w_i\beta})| wS'_i\subset S'_i\cup(-S'_i)\}}.\end{aligned} \tag{5.26}$$

Furthermore, as is essentially clear from the definitions,

$$\begin{aligned}\# W(G_i,\mathrm{Ad}\, g_i^{-1}(B_S))&=\#\{w\in W(B^0_{S,+}\cdot G_S,\mathrm{Ad}\, g_i^{-1}(B_S))| wX_i=X_i\}\\ &=\#\{w\in W(B^0_{S,+}\cdot G_S, B_S)| w\,\mathrm{Ad}\, g_i(X_i)=\mathrm{Ad}\, g_i(X_i)\}.\end{aligned} \tag{5.27}$$

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[13] Properly interpreted, these statements remain correct, even though $M_{w_i\beta}$ need not be connected.

Combining (5.21) with (5.23–27), one finally obtains the desired identity $\frac{a'}{c'_i} \cdot \# U(B_S) = C_i$. This completes the proof of Lemma (5.11).

Stating the hypotheses of the main results of this section requires some care. I shall consider two alternative sets of inductive hypotheses. Either, if G/K is Hermitian symmetric, I assume that (4.15), (4.21) and (4.22) hold for all those groups which satisfy the requirements listed in the beginning of § 2, whose quotient by a maximal compact subgroup is Hermitian symmetric, and which are of lower dimension than G. Or – and now G/K need not be Hermitian symmetric – I shall assume that a modified version of (4.15) holds for all lower dimensional groups which satisfy the requirements of § 2: for any such group, invariant eigendistributions $\Theta(\Psi, \lambda)$ shall be defined in some definite manner, compatibly with inner automorphisms[14], so that (4.15b–e), (4.21), and (4.22) hold.

I now let $\Psi \subset \Phi(G, H)$ be a system of positive roots, β a noncompact root which is simple for Ψ, and Ψ_1 the system of positive roots obtained from Ψ by replacing β with $-\beta$. Further, an admissible (cf. (4.11)) linear functional $\lambda \in i\mathfrak{h}^*$ shall be given, so that the condition (*) in (4.15) holds. I consider a particular invariant eigendistribution on G – it will be convenient to denote it by $\Theta(\Psi, \lambda)$ –, and I define $\Theta(\Psi_1, \lambda)$ by the formula

$$\Theta(\Psi, \lambda) + \Theta(\Psi_1, \lambda) = \Theta, \tag{5.28}$$

with Θ having the same meaning as in (4.15c).

(5.29) **Lemma.** *If $\Theta(\Psi, \lambda)$ satisfies* (4.15b) *and* (4.15e), *then $\Theta(\Psi_1, \lambda)$ is an invariant eigendistribution, which also satisfies* (4.15b) *and* (4.15e).

Proof. The infinitesimal character of $\Theta(\Psi, \lambda)$ can be read off from (4.15e): it is χ_λ, in Harish-Chandra's notation [6]. Similarly, one can identify the infinitesimal character of the invariant eigendistribution ϕ_0 on M^0_β, which enters the construction of Θ. It is known how to compute the infinitesimal character of an induced invariant eigendistribution (e.g. Theorem 4.3.8 of [40]). Conclusion: both $\Theta(\Psi, \lambda)$ and Θ have infinitesimal character χ_λ; hence so does $\Theta(\Psi_1, \lambda)$. Because of (5.1), $\Theta(\Psi_1, \lambda)$ and the negative of $\Theta(\Psi, \lambda)$ agree on H, which implies (4.15e) for $\Theta(\Psi_1, \lambda)$. Because of (4.15b), applied to ϕ_0, and in view of the proof of (4.12), ϕ_0 transforms under $Z(M^0_\beta)$ according to the rule

$$\phi_0(zm) = e^{\lambda - \rho}(z)\, \phi_0(m),$$

for $m \in M^0_\beta$, $z \in Z(M^0_\beta) \subset H$. By definition, $\zeta_\lambda(f) = e^{\lambda - \rho}(f)$ for $f \in F_\beta$. This gives

$$\phi_1(zm) = e^{\lambda - \rho}(z)\, \phi_1(m),$$

whenever $m \in M^\dagger_\beta$, $z \in Z(M^\dagger_\beta) \subset H$ (cf. (2.54–56)).

Every Ad m, with $m \in M_\beta$, certainly operates trivially on $Z(G)$, so that

$$\phi(zm) = e^{\lambda - \rho}(z)\, \phi(m),$$

[14] If Theorem (4.15) holds, the compatibility with inner automorphisms follows from the uniqueness statement.

for $m \in M_\beta$, $z \in Z(G)$ (note: because $Z(G)$ is compact, it must lie in M_β; cf. (2.52)). This last identity, together with the containment $Z(G) \subset M_\beta$, implies

$$\Theta(zg) = e^{\lambda-\rho}(z)\,\Theta(g),$$

if $z \in Z(G)$, $g \in G$, as follows from the definition of Θ in (4.5). Hence $\Theta(\Psi_1, \lambda)$ satisfies (4.15b).

The hypotheses which preceed the statement of Lemma (5.29) continue to be in force. I also assume that $\Theta(\Psi, \lambda)$ satisfies (4.15b) and (4.15e), so that the conclusions of Lemma (5.29) hold.

(5.30) **Proposition.** *If* $\Theta(\Psi, \lambda)$ *satisfies* (4.21), *then so does* $\Theta(\Psi_1, \lambda)$.

Proof. The identity which (4.21) asserts is invariant under inner automorphisms. It therefore suffices to verify it for one Cartan subgroup from each conjugacy class; without loss of generality, I assume $B = B_S$, for some strongly orthogonal subset $S \subset \Phi^n(G, H)$.

Next, I claim that it suffices to prove the identity only on B_S, rather than on all of G'. Loosely speaking, the statement (4.21) is transitive, in the following sense. Let B' be a Cartan subgroup of G' (hence also of G; cf. (2.40)), and $\tilde{B}'$ its inverse image in $\tilde{G}'$. In (4.21), one can reapply the statement (4.21) to the invariant eigendistributions

$$\Theta_{\tilde{G}'}(w\Psi \cap \Phi(G', H), w\lambda),$$

with $\tilde{G}'$ taking the place of G and $\tilde{B}'$ the place of B. This double application of (4.21) then becomes equivalent to a single application, with B' in place of B. The verification of the preceding remark is straightforward and will be left to the reader. Because of the transitivity of (4.21), and because of the inductive hypotheses, it is indeed enough to check the formula in (4.21) on B_S.

By assumption, (4.21) holds for $\Theta(\Psi, \lambda)$. I shall now rephrase this identity, restricted to B_S, in a slightly different form. First of all, instead of summing over the quotient of the two Weyl groups, one may sum over $W(G, H)$ and divide by the order of $W(G', H)$. Secondly, the positive root system Ψ_0, which was used to define $\Delta_{G,G'}$, can be chosen in any convenient manner; I shall assume $\Psi_0 = \Psi$. For any $\alpha \in \Phi(G, H)$, I define

$$\text{(5.31)} \qquad \operatorname{sgn}_\Psi \alpha = \begin{cases} +1 & \text{if } \alpha \in \Psi \\ -1 & \text{if } \alpha \notin \Psi. \end{cases}$$

Then, with $\Phi' = \Phi(G', H)$,

$$\text{(5.32)} \qquad \varepsilon_{G,G'}(w\Psi) = \textstyle\prod_{\alpha \in w\Psi, \alpha \notin \Phi'} \operatorname{sgn}_\Psi \alpha.$$

According to (4.19), $\Delta_{G,G'}$ is invariant under inner automorphisms of the complexification of $\tilde{G}'$. The Cayley transform c_S is such an automorphism. Via c_S, one can transfer Ψ to a system of positive roots Ψ_S in $\Phi(G, B_S)$, so that

$$\text{(5.33)} \qquad \Psi = c_S^* \Psi_S.$$

It should be recalled that

$$\text{(5.34)} \qquad \Phi(B_S)_- = \{\alpha \in \Phi(G, B_S) \mid \langle \alpha, \mathrm{b}_{S,+} \rangle = 0\} = \Phi(G', B_S).$$

Hence, on the inverse image $\tilde{B}_S$ of B_S in $\tilde{G}'$, one has

$$\Delta_{G,G'}|_{\tilde{B}_S} = \prod_{\alpha\in\Psi_S,\alpha\notin\Phi(B_S)_-}(e^{\alpha/2}-e^{-\alpha/2}). \tag{5.35}$$

With these preparations out of the way, the formula in (4.21), restricted to B_S, can be rewritten as follows:

$$\begin{aligned}\Theta(\Psi,\lambda)|_{B_S} = {} & (-1)^{q-q'}(\# W(G',H))^{-1}\Big(\prod_{\alpha\in\Psi_S,\alpha\notin\Phi(B_S)_-}(e^{\alpha/2}-e^{-\alpha/2})\Big)^{-1}\\ & \cdot\sum_{w\in W(G,H)}\Big(\prod_{\alpha\in w\Psi,\alpha\notin\Phi'}\operatorname{sgn}_\Psi\alpha\Big)\cdot\Theta_{\tilde{G}'}(w\Psi\cap\Phi',w\lambda).\end{aligned} \tag{5.36}$$

The analogous formula for $\Theta(\Psi_1,\lambda)$ is precisely what must be proven. Since Ψ_1 and Ψ coincide, except for β,

$$\Theta_{\tilde{G}'}(w\Psi_1\cap\Phi',w\lambda)=\Theta_{\tilde{G}'}(w\Psi\cap\Phi',w\lambda) \tag{5.37}$$

whenever $w\beta\notin\Phi'$. Similarly,

$$\prod_{\alpha\in w\Psi_1,\alpha\notin\Phi'}\operatorname{sgn}_\Psi\alpha=\pm\prod_{\alpha\in w\Psi,\alpha\notin\Phi'}\operatorname{sgn}_\Psi\alpha, \tag{5.38}$$

with the upper sign applying if $w\beta\in\Phi'$, and the lower sign if $w\beta\notin\Phi'$. The formula (5.36) is assumed to hold. In view of (5.28), and because of (5.37) and (5.38), the formula to be proven becomes equivalent to

$$\begin{aligned}\Theta|_{B_S} = {} & (-1)^{q-q'}(\# W(G,H))^{-1}\Big(\prod_{\alpha\in\Psi_S,\alpha\notin\Phi(B_S)_-}(e^{\alpha/2}-e^{-\alpha/2})\Big)^{-1}\\ & \cdot\sum_{w\in W(G,H),w\beta\in\Phi'}\Big(\prod_{\alpha\in w\Psi,\alpha\notin\Phi'}\operatorname{sgn}_\Psi\alpha\Big)\\ & \cdot\big(\Theta_{\tilde{G}'}(w\Psi\cap\Phi',w\lambda)+\Theta_{\tilde{G}'}(w\Psi_1\cap\Phi',w\lambda)\big).\end{aligned} \tag{5.39}$$

I shall deduce (5.39) from Lemma (5.11), plus the inductive hypotheses.

In order to apply (5.11) to the present situation, one must give ϕ_0, ζ, and v the same meaning as in (4.15c). For each noncompact root $\gamma\in\Phi(G,H)$, I set

$$\Phi_\gamma=\{\alpha\in\Phi(G,H)\,|\,\alpha\perp\gamma\}. \tag{5.40a}$$

According to (2.59) and (2.60), one can identify Φ_γ with the root system of $(M^0_\gamma, B^0_{\gamma,+})$, in a natural manner. To simplify the notation, I shall write this identification as an identity:

$$\Phi_\gamma=\Phi(M^0_\gamma,B^0_{\gamma,+}). \tag{5.40b}$$

In terms of the notation just established,

$$\phi_0=\Theta_{M^0_\beta}(\Psi\cap\Phi_\beta,\lambda|_{\mathfrak{b}_{\beta,+}}). \tag{5.41}$$

I let w_i, g_i, k_i, X_i, etc. be the same objects as in (5.11). However, whereas previously g_i was chosen subject only to the two conditions

$$g_i\in G_S,\qquad \operatorname{Ad}g_i(X_i)\in\mathfrak{b}_{S,-}, \tag{5.42}$$

the choice will now have to be made in a more specific manner. Relative to the complexification $G_S^{\mathbb{C}}$ of G_S, X_i is conjugate to $Z_{w_i\beta}$ (cf. (2.7) and (2.10)), and hence $\operatorname{Ad}g_i(X_i)$ and $c_S Z_{w_i\beta}$ are $G_S^{\mathbb{C}}$-conjugate whenever $g_i\in G_S$. If g_i satisfies (5.42), both $c_S Z_{w_i\beta}$ and $\operatorname{Ad}g_i(X_i)$ lie in $\mathfrak{b}^{\mathbb{C}}_{S,-}$, and must therefore be conjugate even under the action of the Weyl group of $\mathfrak{b}_{S,-}^{\mathbb{C}}$ in $\mathfrak{g}_S^{\mathbb{C}}$. This Weyl group coincides with the Weyl group of $B_{S,-}$ in G_S. Thus, if g_i is modified by a suitable element of the normalizer of $B_{S,-}$ in G_S, one can arrange that

(5.43) $$\operatorname{Ad} g_i(X_i) = \pm c_S Z_{w_i\beta}.$$

I have left the sign in (5.43) undetermined, in order to be able to impose a further condition. If $w_j\beta = \pm w_i\beta$, for $1 \leqq i, j \leqq n$, then $X_i = X_j$ and $Z_{w_i\beta} = \pm Z_{w_j\beta}$. Hence, without violating (5.43), I can and shall insist on

(5.44) $$w_j\beta = \pm w_i\beta \Rightarrow g_j = g_i.$$

To prevent the subscripts from sprouting too wildly, I shall introduce some notational conventions, which will be followed for the remainder of the proof:

(5.45) $$\begin{aligned} &G' = B_{S,+}^0 \cdot G_S, \qquad \Phi' = \Phi(G', H), \\ &\beta_i = w_i\beta, \qquad M_i = M_{w_i\beta}, \qquad B_i = B_{w_i\beta}, \\ &\Phi_i = \{\alpha \in \Phi(G, H) \mid \alpha \perp \beta_i\} = \Phi(M_i^0, B_{i,+}^0), \\ &W_i = W(M_i^0, B_{i,+}^0), \end{aligned}$$

for $1 \leqq i \leqq n$; cf. (5.40). Since $\operatorname{Ad} k_i$ preserves H and maps B_β onto B_i, and since the invariant eigendistributions $\Theta_{M_\beta^0}(\ldots, \ldots)$ on M_β^0 are defined compatibly with inner automorphisms, (5.41) is equivalent to

(5.46) $$\phi_0 \circ \operatorname{Ad} k_i^{-1} = \Theta_{M_i^0}(w_i\Psi \cap \Phi_i, w_i\lambda|_{\mathfrak{b}_{i,+}}).$$

By construction, B_S is a Cartan subgroup of G'. Because $g_i \in G_S$, $\operatorname{Ad} g_i^{-1}(B_S)$ is also a Cartan subgroup of G, and

(5.47) $$\operatorname{Ad} g_i^{-1}(B_{S,+}^0) = B_{S,+}^0.$$

At the same time, as follows from the choice of g_i,

(5.48) $\operatorname{Ad} g_i^{-1}(B_S)$ is a Cartan subgroup of $M_i \cdot B_{i,-}$.

I now define

(5.49) $$G_i' = G' \cap M_i^0;$$

equivalently, one can describe G_i' as the centralizer of $B_{S,+}^0$ in $M_{w_i\beta}^0$. In view of (2.15), the containment $w_i\beta \in \Phi'$ implies $B_{S,+}^0 \subset B_{i,+}^0$, and hence

(5.50) $$B_{S,+}^0 \subset M_i^0.$$

As can be inferred from (5.47–50), G_i' plays the same role with respect to M_i^0 and its Cartan subgroup

(5.51) $$\operatorname{Ad} g_i^{-1}(B_S) \cap M_i^0,$$

which G' plays for G and B_S. In particular,

(5.52) G_i' is connected.

Because of the inductive hypotheses, I may apply (4.21) to the invariant eigendistribution (5.46), with G_i' taking the place of G'. One thus has to consider $\Delta_{M_i^0, G_i'}$ and $\varepsilon_{M_i^0, G_i'}$, as described by (4.19) and (4.20). The role of the reference positive root system Ψ_0 will be played by the positive root system $\Psi \cap \Phi_i$. Then, on the appropriate finite covering of $B_{i,+}^0$,

$$\Delta_{M_i^0, G_i'} = \prod\nolimits_{\alpha \in \Psi \cap \Phi_i, \alpha \notin \Phi'} (e^{\alpha/2} - e^{-\alpha/2}). \tag{5.53}$$

Every element of the Weyl group W_i of $B_{i,+}^0$ in M_i^0, considered as an automorphism of $B_{i,+}^0$, can be uniquely extended to an element of $W(G, H)$; cf. (2.36) and (2.43). I may therefore regard W_i as a subgroup of

$$\{w \in W(G, H) | w\beta_i = \pm \beta_i\}. \tag{5.54}$$

In terms of this convention, for $w \in W_i$,

$$\varepsilon_{M_i^0, G_i'}(w(w_i \Psi \cap \Phi_i)) = \prod\nolimits_{\alpha \in w w_i \Psi, \alpha \notin \Phi', \alpha \perp \beta_i} \mathrm{sgn}_\Psi \alpha, \tag{5.55}$$

which is analogous to (5.32).

In order to get the formula for $\Delta_{M_i^0, G_i'}$ on the Cartan subgroup (5.51) – or on an appropriate finite covering thereof – with a definite choice of sign, one must transfer the positive root system $\Psi \cap \Phi_i$ from $B_{i,+}^0$ to the Cartan subgroup (5.51), via an inner automorphism of the complexification of G_i'. In a connected, reductive, complex Lie group, the centralizer of a subalgebra of a Cartan subalgebra is automatically connected. Hence, in the preceding sentence, I may replace the phrase "via an inner automorphism of the complexification of G_i'" by the phrase "via an inner automorphism of the complexification of G', which operates trivially on X_i". I let c_i be the Cayley transform corresponding to the noncompact root β_i. According to (2.10), $c_i^{\pm 1} \ Z_{\beta_i} = \pm X_i$. Combined with (5.43), this gives the identity

$$\mathrm{Ad}\, g_i^{-1} \circ c_S \circ c_i^{\pm 1}(X_i) = X_i;$$

the sign of the exponent depends on the sign in (5.43). Since $\mathrm{Ad}\, g_i$, c_S, and c_i all belong to the complexification of G', and since c_i operates trivially on $\mathfrak{b}_{i,+}^{\mathbb{C}}$, the positive root system $\Psi \cap \Phi_i$ corresponds to the following positive root system for the Cartan subgroup (5.51) of G_i': the image under $\mathrm{Ad}\, g_i^{-1}$ of

$$\{\alpha \in \Psi_S | \langle \alpha, \mathrm{Ad}\, g_i(X_i) \rangle = 0\};$$

Ψ_S, it should be recalled, is the positive root system Ψ, carried over into $\Phi(G, B_S)$ by c_S. As a consequence of those considerations, on a sitable finite covering of $\mathrm{Ad}\, g_i^{-1}(B_S) \cap M_i^0$,

$$\Delta_{M_i^0, G_i'} = \left(\prod\nolimits_{\alpha \in \Psi_S, \alpha \notin \Phi(B_S)_-, \langle \alpha, \mathrm{Ad}\, g_i(X_i) \rangle = 0} (e^{\alpha/2} - e^{-\alpha/2})\right) \circ \mathrm{Ad}\, g_i. \tag{5.56}$$

With (5.55) and (5.56) as ingredients, I now apply (4.21) to the invariant eigendistribution (5.46). As before, I consider W_i as a subgroup of the group (5.54). Then, on $\mathrm{Ad}\, g_i^{-1}(B_S) \cap M_i^0$,

$$\begin{aligned} \phi_0 \circ \mathrm{Ad}\, k_i^{-1} &= (-1)^{q_i - q_i'} \cdot (\# W(G_i', B_{i,+}^0))^{-1} \\ &\cdot \left(\prod\nolimits_{\alpha \in \Psi_S, \alpha \notin \Phi(B_S)_-, \langle \alpha, \mathrm{Ad}\, g_i(X_i) \rangle = 0} (e^{\alpha/2} - e^{-\alpha/2})\right)^{-1} \circ \mathrm{Ad}\, g_i \\ &\cdot \sum\nolimits_{w \in W_i} \left(\prod\nolimits_{\alpha \in w w_i \Psi, \alpha \notin \Phi', \alpha \perp \beta_i} \mathrm{sgn}_\Psi \alpha\right) \Theta_{\bar{G}_i'}(w w_i \Psi \cap \Phi_i', w w_i \lambda|_{\mathfrak{b}_{i,+}}), \end{aligned} \tag{5.57}$$

where $\Phi'_i = \Phi_i \cap \Phi' = \Phi(G'_i, B^0_{i,+})$, and $q_i = \frac{1}{2}\dim_{\mathbb{R}} M^0_i / M^0_i \cap K$, $q'_i = \frac{1}{2}\dim_{\mathbb{R}} G'_i/G'_i \cap K$ (note: both M^0_i and G'_i are θ-stable, so that their intersections with K are maximal compact subgroups).

For each i between 1 and n, I let I_i be the set of indices j, $1 \leq j \leq n$, such that $\beta_j = \pm\beta_i$. In other words, $\{w_j | j \in I_i\}$ is the w_i-coset of the subgroup (5.54) of $W(G, H)$. If $j \in I_i$, one has $X_j = X_i$, $M_j = M_i$, $G'_j = G'_i$, and $g_j = g_i$ (cf. (5.44)). One can therefore add the identities (5.57), corresponding to all the indices in I_i. Since W_i is really a subgroup of the group (5.54), the summation over W_i merely accounts to multiplication by the order of W_i. Thus, on $\mathrm{Ad}\, g_i^{-1}(B_S) \cap M^0_i$,

$$\begin{aligned}\sum_{j \in I_i} \phi_0 \circ \mathrm{Ad}\, k_j^{-1} = {} & (-1)^{q_i - q'_i} \cdot \# W_i \cdot (\# W(G'_i, B^0_{i,+}))^{-1} \\ & \cdot \Big(\prod_{\alpha \in \Psi_S, \alpha \notin \Phi(B_S)_-, \langle \alpha, \mathrm{Ad}\, g_i(X_i)\rangle = 0} (e^{\alpha/2} - e^{-\alpha/2})\Big)^{-1} \circ \mathrm{Ad}\, g_i \\ & \cdot \sum_{j \in I_i} \Big(\prod_{\alpha \in w_j \Psi, \alpha \notin \Phi', \alpha \perp \beta_j} \mathrm{sgn}_\Psi \alpha\Big)\, \Theta_{\tilde{G}'_j}(w_j \Psi \cap \Phi'_j, w_j \lambda|_{\mathfrak{b}_{j,+}}).\end{aligned} \tag{5.58}$$

The centralizer of $B_{i,-}$ in G' can be factored as

$$Z_{G'}(B_{i,-}) = (G' \cap M_i) \cdot B_{i,-};$$

this follows from, and is analogous to, the factorization

$$Z_G(B_{i,-}) = M_i \cdot B_{i,-}.$$

Since G'_i is the identity component of $G' \cap M_i$,

$$(G' \cap M_i)^\dagger = G'_i \cdot F_{w_i \beta} = G' \cap M_i^\dagger \tag{5.59}$$

(cf. (2.55); note: $F_{w_i\beta} \subset G'$). For $1 \leq i \leq n$, I define ψ_i on $Z_{G'}(B_{i,-})$ as follows:

$$\begin{aligned} &\psi_i(g) = 0 \quad \text{if } g \notin (G' \cap M_i)^\dagger \cdot B_{i,-}, \\ &\psi_i(mfb) = \Theta_{\tilde{G}'_i}(w_i \Psi \cap \Phi'_i, w_i \lambda|_{\mathfrak{b}_{i,+}})(m) \cdot \zeta_{w_i \lambda}(f)\, e^{\nu} \circ \mathrm{Ad}\, k_i^{-1}(b) \\ &\qquad\qquad \text{if } m \in G'_i,\ f \in F_{\beta_i}, \text{ and } b \in B_{i,-}. \end{aligned} \tag{5.60}$$

Strictly speaking, ψ_i makes sense only on a suitable finite covering of $Z_{G'}(B_{i,-})$. However, not ψ_i itself, only the product of ψ_i with Δ_{M_i, G'_i} will occur below, and the product does make sense on $Z_{G'}(B_{i,-})$. In (5.60), ν has the same meaning as in (4.15c), and the character $\zeta_{w_i\lambda}$ of F_{β_i} is defined by (4.13); equivalently, $\zeta_{w_i\lambda} = \zeta_\lambda \circ \mathrm{Ad}\, k_i^{-1}$.

The definition (5.1) of ψ, together with (5.58–60), allows one to rewrite the formula (5.11) in terms of the ψ_i: on B_S,

$$\begin{aligned}\Theta = {} & \sum_{i=1}^n (-1)^{q_i - q'_i} \cdot C'_i \cdot \Big(\prod_{\alpha \in w_i \Psi, \alpha \notin \Phi', \alpha \perp \beta_i} \mathrm{sgn}_\Psi \alpha\Big) \\ & \cdot \sum_{w \in W(B_S)_-} \Big\{ \Big|\prod_{\alpha \in \Phi(G, B_S), \alpha \notin \Phi(B_S)_-, \langle \alpha, \mathrm{Ad}\, g_i(X_i)\rangle > 0} (e^{\alpha/2} - e^{-\alpha/2}) \circ w\Big|^{-1} \\ & \cdot \Big(\prod_{\alpha \in \Psi_S, \alpha \notin \Phi(B_S)_-, \langle \alpha, \mathrm{Ad}\, g_i(X_i)\rangle = 0} (e^{\alpha/2} - e^{-\alpha/2}) \circ w\Big)^{-1} \\ & \cdot \Big|\prod_{\alpha \in \Phi(B_S), \langle \alpha, \mathrm{Ad}\, g_i(X_i)\rangle > 0} (e^{\alpha/2} - e^{-\alpha/2}) \circ w\Big|^{-1} \cdot \psi_i \circ \mathrm{Ad}\, g_i^{-1} \circ w \Big\},\end{aligned} \tag{5.61}$$

with C'_i given by

$$\begin{aligned}C'_i = {} & \frac{1}{\# W(G'_i, B^0_{\beta_i,+}) \cdot \# W(G', H)} \\ & \cdot \frac{\# \{w \in W(G, B_{\beta_i}) | w X_i = X_i\}}{\# \{w \in W(G', B_S) | w\, \mathrm{Ad}\, g_i(X_i) = \mathrm{Ad}\, g_i(X_i)\}}.\end{aligned} \tag{5.62}$$

If $\alpha\in\Phi(G,B_S)$ satisfies $\alpha\notin\Phi(B_S)_-$, $\langle\alpha,\operatorname{Ad}g_i(X_i)\rangle>0$, then so does its complex conjugate $\bar\alpha$; moreover, α and $\bar\alpha$ are distinct. Hence

$$(5.63)\qquad \begin{aligned}&\left|\prod_{\alpha\in\Phi(G,B_S),\alpha\notin\Phi(B_S)_-,\langle\alpha,\operatorname{Ad}g_i(X_i)\rangle>0}(e^{\alpha/2}-e^{-\alpha/2})\right|\\ &=\prod_{\alpha\in\Phi(G,B_S),\alpha\notin\Phi(B_S)_-,\langle\alpha,\operatorname{Ad}g_i(X_i)\rangle>0}(e^{\alpha/2}-e^{-\alpha/2})\\ &=(-1)^{m_i}\prod_{\alpha\in\Psi_S,\alpha\notin\Phi(B_S)_-,\langle\alpha,\operatorname{Ad}g_i(X_i)\rangle\neq 0}(e^{\alpha/2}-e^{-\alpha/2}),\end{aligned}$$

with $m_i=\#\{\alpha\in\Psi_S|\alpha\notin\Phi(B_S)_-,\langle\alpha,\operatorname{Ad}g_i(X_i)\rangle>0\}$. Under the Cayley transform c_S, Ψ_S corresponds to Ψ, $\Phi(B_S)_-$ corresponds to Φ', and $\operatorname{Ad}g_i(X_i)$ to either Z_{β_i} or $-Z_{\beta_i}$ (cf. (5.43)). In the derivation (5.63), the inequality $\langle\alpha,\operatorname{Ad}g_i(X_i)\rangle>0$ could just as well have been reversed, so that the sign in (5.43) is really irrelevant. As follows from (2.7),

$$\langle\lambda,Z_{\beta_i}\rangle=2(\lambda,\beta_i),\qquad\text{whenever }\lambda\in i\mathfrak{h}^*.$$

These remarks now imply the identity

$$(5.64)\qquad \begin{aligned}(-1)^{m_i}&=\prod_{\alpha\in\Psi,\alpha\notin\Phi',(\alpha,\beta_i)\neq 0}\operatorname{sgn}(\alpha,\beta_i)\\ &=\prod_{\alpha\in w_i\Psi,\alpha\notin\Phi',(\alpha,\beta_i)\neq 0}\operatorname{sgn}_\Psi\alpha\cdot\operatorname{sgn}(\alpha,\beta_i).\end{aligned}$$

One may consider $W(B_S)_-$ both as the Weyl group of B_S in G and as a subgroup of $W(G,B_S)$. The two expressions

$$\prod_{\alpha\in\Psi_S}(e^{\alpha/2}-e^{-\alpha/2})\quad\text{and}\quad\prod_{\alpha\in\Psi_S\cap\Phi(B_S)_-}(e^{\alpha/2}-e^{-\alpha/2})$$

are therefore $W(B_S)_-$-alternating, and their quotient is $W(B_S)_-$-invariant. Together with (5.63) and (5.64), the just mentioned fact makes (5.61) equivalent to

$$(5.65)\qquad \begin{aligned}\Theta=&\sum_{i=1}^n(-1)^{q_i-q_i'}\cdot C_i'\cdot\Big(\prod_{\alpha\in w_i\Psi,\alpha\notin\Phi',(\alpha,\beta_i)\neq 0}\operatorname{sgn}(\alpha,\beta_i)\Big)\\ &\cdot\Big(\prod_{\alpha\in w_i\Psi,\alpha\notin\Phi'}\operatorname{sgn}_\Psi\alpha\Big)\cdot\Big(\prod_{\alpha\in\Psi_S,\alpha\notin\Phi(B_S)_-}(e^{\alpha/2}-e^{-\alpha/2})\Big)^{-1}\\ &\cdot\sum_{w\in W(B_S)_-}\left|\prod_{\alpha\in\Phi(B_S)_-,\langle\alpha,\operatorname{Ad}g_i(X_i)\rangle>0}(e^{\alpha/2}-e^{-\alpha/2})\circ w\right|^{-1}\cdot\psi_i\circ\operatorname{Ad}g_i^{-1}\circ w,\end{aligned}$$

again on B_S.

The final sum on the right hand side in (5.65), except for a suitable multiplicative constant, represents an induced invariant eigendistribution on $\tilde G'$ (cf. (5.2); since B_S splits modulo the center of G', the integer n in (5.2), with $G=\tilde G'$, $B=\tilde B_S$, is equal to one). Thus let Θ_i be the induced invariant eigendistribution on $\tilde G'$, which corresponds to the following data: The noncompact root $\beta_i\in\Phi(\tilde G',\tilde H)$, the invariant eigendistribution

$$\Theta_{\tilde G_i'}(w_i\Psi\cap\Phi_i',w_i\lambda|_{\mathfrak{b}_{i,+}})$$

on $\tilde G_i'$, the character ζ_{w_i} of F_{β_i}, and the linear functional $\nu\circ\operatorname{Ad}k_i^{-1}$ on $\mathfrak{b}_{i,-}$. Then, on B_S,

$$(5.66)\qquad \begin{aligned}\Theta=&(\#W(G',H))^{-1}\cdot\Big(\prod_{\alpha\in\Psi_S,\alpha\notin\Phi(B_S)_-}(e^{\alpha/2}-e^{-\alpha/2})\Big)\\ &\cdot\sum_{i=1}^n\Big\{(-1)^{q_i-q_i'}\cdot\Big(\prod_{\alpha\in w_i\Psi,\alpha\notin\Phi',(\alpha,\beta_i)\neq 0}\operatorname{sgn}(\alpha,\beta_i)\Big)\Big(\prod_{\alpha\in w_i\Psi,\alpha\notin\Phi'}\operatorname{sgn}_\Psi\alpha\Big)\cdot\Theta_i\Big\}.\end{aligned}$$

On the other hand, because of the inductive hypotheses, one can apply (4.15c) on G', to conclude

$$(5.67)\qquad \Theta_i=\Theta_{\tilde G'}(w_i\Psi\cap\Phi',w_i\lambda)+\Theta_{\tilde G'}(w_i\Psi_1\cap\Phi',w_i\lambda).$$

I recall that $\{w_1, \ldots, w_n\}$ is an enumeration of those $w \in W(G, H)$ which map β into Φ', and such that $\beta_i = w_i \beta$. In order to complete the proof, I had to verify (5.39). In view of (5.66) and (5.67), the problem comes down to checking that

(5.68) $$(-1)^{q-q'} = (-1)^{q_i - q_i'} \cdot \prod_{\alpha \in w_i \Psi, \alpha \notin \Phi', (\alpha, \beta_i) \neq 0} \operatorname{sgn}(\alpha, \beta_i).$$

The root β_i is noncompact, it is simple, relative to the system of positive roots $w_i \Psi$, and it lies in the sub-root system Φ'. By a β_i-ladder, I shall mean a maximal string of roots of the form $\{\alpha + l\beta_i \mid l \in \mathbb{Z}\}$. Because of the properties of β_i which were just mentioned, each β_i-ladder lies either wholly in Φ' or wholly outside Φ'; also, with the exception of $\{\pm\beta_i\}$, it lies entirely in $w_i \Psi$ or entirely in $-w_i \Psi$; and finally, again with the exception of $\{\pm\beta_i\}$, the roots in a β_i-ladder are compact and noncompact in an alternating fashion. In particular, the complement of Φ' in $w_i \Psi$ is a disjoint union of β_i-ladders. Evidently $q - q'$ can be described as the number of noncompact roots in $w_i \Psi$, outside of Φ', whereas $q_i - q_i'$ is the number of roots in $w_i \Psi$, outside of Φ', perpendicular to β_i, which are noncompact when viewed as roots of $(M_i^0, B_{i,+}^0)$; cf. (2.61), with $S = \{\beta_i\}$. To demonstrate (5.68), one only needs to make sure that each β_i-ladder in $w_i \Psi$, outside of Φ', contributes equally to both sides of (5.68).

If such a β_i-ladder contains an even number of roots, say $2n$, then exactly n of these are noncompact, exactly n have a negative inner product with β_i, and none is perpendicular to β_i. Hence the ladder does contribute equally. If a β_i-ladder consists of a single root α, then α is strongly orthogonal to β_i. According to (2.61), the contributions of $\{\alpha\}$ to $(-1)^{q-q'}$ and to $(-1)^{q_i - q_i'}$ agree, and there is no contribution to the last factor on the right hand side of (5.68). As the only remaining possibility, I now consider a β_i-ladder of length 3, lying in $w_i \Psi$ and outside of Φ'. Exactly one of the three roots has a negative inner product with β_i. If two of the three roots are noncompact, then the middle one is compact, and hence noncompact relative to M_i^0 (cf. (2.61)). On the other hand, if only one root is noncompact, it must be the middle one, which is therefore compact relative to M_i^0. In both cases, the ladder again contributes equally to both sides of (5.68). At this point, (5.68) has been verified, and proof of Proposition (5.30) is now complete.

The hypotheses which were stated above Proposition (5.30) shall continue to be in force. In particular, $\Theta(\Psi, \lambda)$, $\Theta(\Psi_1, \lambda)$, and Θ shall have the same meaning as in (5.30). Moreover, I shall assume that $\Theta(\Psi, \lambda)$, and hence also $\Theta(\Psi_1, \lambda)$, satisfies (4.21).

(5.69) **Proposition.** *If* $\Theta(\Psi, \lambda)$ *satisfies* (4.22), *then so does* $\Theta(\Psi_1, \lambda)$.

The proof will be preceeded by several lemmas. However, one simplifying assumption can be made right away. The two statements (4.21) and (4.22) are consistent, in the following sense: if one applies (4.21), with the given Cartan subgroup B, and then (4.22), with G_B and $B \cap G_B$ (cf. (2.40)) taking the place of G and B, the result amounts to a single application of (4.22). Hence, without loss of generality, I shall assume that G is semisimple, and that B is a split Cartan subgroup. As in the statement of (4.22), I set $G' = G(B^j)$. The identity component B^0 of B lies in G'; according to the hypotheses, G' also contains H. Consequently, there exists a strongly orthogonal set $S \subset \Phi(G', H)$, such that B^0 is G'-conjugate to the identity component of B_S. Since $B^j = z \cdot B^0$, for some z in the center of G',

B^j is G'-conjugate to a component of B_S. It clearly suffices to verify (4.22) on this component of B_S, instead of B^j. Thus, in the proof of (5.69), it is legitimate to make the following assumptions:

(5.70)
G is semisimple;
B^j is a connected component of a split Cartan subgroup B;
the group $G' = G(B^j)$ contains H; and
$B = B_S$, for some strongly orthogonal subset $S \subset \Phi^n(G', H)$.

In order to state the first lemma, I let G be semisimple, subject to the usual hypotheses, $\beta \in \Phi(G, H)$ a noncompact root, B a split Cartan subgroup of G which is contained in $M_\beta^0 \cdot B_{\beta,-}$, and B^j a connected component of B. Since M_β^0 lies in the singular set of G, there exists a root $\gamma \in \Phi(G, B)$ which vanishes on $\mathfrak{b} \cap \mathfrak{m}_\beta$; it is uniquely determined, up to sign.

(5.71) **Lemma.** *Under the hypotheses which were just mentioned, B^j lies in $M_\beta^\dagger \cdot B_{\beta,-}$ if and only if e^γ assumes positive values on B^j. Moreover, if B^j lies in $M_\beta^\dagger \cdot B_{\beta,-}$, then*

$$B^j \subset (G(B^j) \cap M_\beta)^0 \cdot F_\beta \cdot B_{\beta,-}$$

(cf. (2.44); $(\ldots)^0$ *denotes the identity component of* $\ldots$).

Proof. First, suppose $B^j \subset M_\beta^\dagger \cdot B_{\beta,-}$, and let $b \in B^j$ be given. According to (2.55), $b = b_0 \cdot b_- \cdot f$, with $b_0 \in M_\beta^0$, $b_- \in B_{\beta,-}$, and $f \in F_\beta$. Since B contains both $B_{\beta,-}$ and F_β, b_0 must lie in $B \cap M_\beta^0$. Like any element of $B \cap M_\beta^0$, b_0 can be expressed as $\exp X$, with $X \in \mathfrak{m}_\beta^{\mathbb{C}} \cap \mathfrak{b}^{\mathbb{C}}$. The root γ vanishes on $\mathfrak{m}_\beta^{\mathbb{C}} \cap \mathfrak{b}^{\mathbb{C}}$, so that $e^\gamma(b_0) = 1$. On F_β, e^γ is identically equal to one (cf. the remark above (2.56)), and e^γ assumes only positive values on $B_{\beta,-}$. Hence $e^\gamma(b) > 0$.

Conversely, suppose that e^γ assumes positive values on B^j. Since $\exp \mathfrak{b}$ is the identity component of B, and in view of (2.38), one can pick an element $b \in B^j$, of the form $b = \exp(iX)$, with $X \in \mathfrak{b}$, and $\langle \lambda, X \rangle \in \pi\mathbb{Z}$, for all $\lambda \in \Lambda(\Phi(B))$, Now $X = X_0 + X_1$, with $X_0 \in \mathfrak{m}_\beta \cap \mathfrak{b}$, $X_1 \in \mathfrak{b}_{\beta,-}$. I shall show that $\exp(iX_0) \in M_\beta^0$, and that $\exp(iX_1) \in F_\beta$. Indeed, the root γ vanishes on $\mathfrak{m}_\beta \cap \mathfrak{b}$, and $\langle \gamma, X \rangle$ is an integral multiple of π; hence

$$\exp \langle \gamma, iX_1 \rangle = \exp \langle \gamma, iX \rangle = \pm 1.$$

On the other hand, $\exp \langle \gamma, iX \rangle = e^\gamma(b) > 0$, so that

$$\langle \gamma, X \rangle = \langle \gamma, X_1 \rangle \in 2\pi\mathbb{Z}.$$

As can be checked (cf. the remarks above (2.56)), this implies $\exp(iX_1) \in F_\beta$. Now let μ be an element of the weight lattice of the root system $\Phi(M_\beta^0, B \cap M_\beta^0)$. In analogy to the statement (2.41), there exists a $\lambda \in \Lambda(\Phi(B))$, whose restriction to $\mathfrak{m}_\beta \cap \mathfrak{b}$ coincides with μ. Since λ is a weight, λ has the same restriction to $\mathfrak{b}_{\beta,-}$ as a suitable half integral multiple $\frac{k}{2}\gamma$ of γ. Recall that $\langle \gamma, X \rangle \in 2\pi\mathbb{Z}$. Hence

$$\langle \mu, X_0 \rangle = \langle \lambda, X_0 \rangle = \left\langle \lambda - \frac{k}{2}\gamma, X_0 \right\rangle = \left\langle \lambda - \frac{k}{2}\gamma, X \right\rangle = \langle \lambda, X \rangle - \frac{k}{2} \langle \gamma, X \rangle \in \pi\mathbb{Z}.$$

Appealing to (2.38), one may conclude that $\exp(iX_0) \in M_\beta^0$. The containment $\exp(iX_1) \in F_\beta$ is already known. Thus $b \in M_\beta^\dagger$. Since B^j lies either entirely inside or entirely outside of $M_\beta^\dagger \cdot B_{\beta,-}$, this gives $B^j \subset M_\beta^\dagger \cdot B_{\beta,-}$.

It remains to be shown that $B^j \subset M_\beta^\dagger \cdot B_{\beta,-}$ implies

$$B^j \subset (G(B^j) \cap M_\beta)^0 \cdot F_\beta \cdot B_{\beta,-}.$$

Applying the statement (2.38) to the group $G(B^j)$, one can select a $b \in B^j$, of the form $b = \exp(iX)$, with $X \in \mathfrak{b}$ satisfying $\langle \lambda, X \rangle \in \pi \mathbb{Z}$, for all λ in the weight lattice of the root system $\Phi(G(B^j), B \cap G(B^j))$. I now proceed just as in the argument above, with $G(B^j)$ playing the role of G, and $(G(B^j) \cap M_\beta)^0$ the role of M_β^0. Conclusion: $b \in (G(B^j) \cap M_\beta)^0 \cdot F_\beta$. Since $B^j = b \cdot B^0$, one obtains the desired result.

(5.72) **Lemma.** *Let G be a simple Lie group, which contains a compact Cartan subgroup H, as well as a split Cartan subgroup. For any two noncompact roots $\beta_1, \beta_2 \in \Phi(G, H)$ of the same length, there exists some $w \in W(G, H)$, such that $w\beta_1 = \pm \beta_2$. Also, if $\Phi(G, H)$ contains roots of two different lengths, then so does $\Phi^n(G, H)$.*

Remark. For the first assertion, the assumption that G has a split Cartan subgroup is not really needed; however, it shortens the proof.

Proof. The noncompact roots, modulo sign and modulo conjugacy under $W(G, H)$, classify the conjugacy classes of Cartan subgroups with one-dimensional split part; this follows from (2.16). Also, if β is a noncompact root, the root system $\Phi(\mathfrak{b}_\beta)_-$ (cf. (2.21)) is spanned by a short root whenever β is short, and by a long root whenever β is long. Now let $A \subset G$ be a split Cartan subgroup. According to the results of [38], coupled with (2.18), the conjugacy classes of Cartan subgroups B with one-dimensional split parts also correspond to the set $\Phi(G, A)$, modulo the action of the Weyl group $W(G, A)$. If such a Cartan subgroup B corresponds to a root $\alpha \in \Phi(G, A)$, then $\Phi(B)_-$ is spanned by a short root whenever α is short, and by a long root whenever α is long. In an abstract, irreducible root system, the Weyl group operates transitively on the set of roots of any given length. Since $W(G, A)$ is the full Weyl group of the root system $\Phi(G, A)$, these statements imply the lemma.

I now impose the conditions (5.70), and I let the symbols β, ϕ, ζ, v, Θ, and ψ have the same meaning as in the very beginning of this section. Then

$$X = Y_\beta + Y_{-\beta} \tag{5.73}$$

spans $\mathfrak{b}_{\beta,-}$; for $g \in G$, $\operatorname{Ad} g^{-1}(B_S) \subset M_\beta \cdot B_{\beta,-}$ if and only if $\operatorname{Ad} g(X) \in \mathfrak{b}_S$. Since B_S is a split Cartan subgroup, any two conjugates of B_S which lie in $M_\beta \cdot B_{\beta,-}$ are already conjugate under M_β. Hence, according to (5.2), if $g \in G$ is chosen so that $\operatorname{Ad} g(X) \in \mathfrak{b}_S$, for $b \in B_S$,

$$\begin{aligned}\Theta(b) = C \cdot \textstyle\sum_{w \in W(G, B_S)} \psi(\operatorname{Ad} g^{-1}(wb)) \\ \cdot \left|\textstyle\prod_{\alpha \in \Phi(G, B_S),\, \langle \alpha, \operatorname{Ad} g(X) \rangle > 0} (e^{\alpha/2} - e^{-\alpha/2})(wb)\right|^{-1},\end{aligned} \tag{5.74a}$$

where

$$C = \frac{\#\{w \in W(G, B_\beta) \mid wX = X\}}{\# W(M_\beta^0, B_{\beta,+}^0) \cdot \#\{w \in W(G, B_S) \mid w \operatorname{Ad} g(X) = \operatorname{Ad} g(X)\}}. \tag{5.74b}$$

(5.75) **Lemma.** *If $g \in G$ satisfies $\operatorname{Ad} g(X) \in \mathfrak{b}_S$, the restriction of $\psi \circ \operatorname{Ad} g^{-1}$ to B_S depends only on $\operatorname{Ad} g(X)$, and otherwise not on the particular choice of g.*

Proof. Suppose $\operatorname{Ad}\tilde{g}(X)=\operatorname{Ad}g(X)$; then $\tilde{g}=g\cdot m$, for some $m\in M_\beta$. Both $\operatorname{Ad}g^{-1}(B_S)\cap M_\beta^0$ and $\operatorname{Ad}\tilde{g}^{-1}(B_S)\cap M_\beta^0$ are split Cartan subgroups of M_β^0, and hence M_β^0-conjugate. It follows that $m=m_1\cdot m_0$, for some $m_0\in M_\beta^0$, and some $m_1\in M_\beta$ which normalizes $\operatorname{Ad}g^{-1}(B_S)$. Since ψ is $\operatorname{Ad}M_\beta^0$-invariant, I may as well assume that $m=m_1$, i.e. that m normalizes $\operatorname{Ad}g^{-1}(B_S)$. According to (2.52), $\operatorname{Ad}m\colon \mathfrak{m}_\beta^{\mathbb{C}}\to\mathfrak{m}_\beta^{\mathbb{C}}$ is an inner automorphism. Hence $\operatorname{Ad}m$ operates on $\operatorname{Ad}g^{-1}(B_S)\cap M_\beta^0$ as an element of the Weyl group of the root system $\Phi(M_\beta^0, \operatorname{Ad}g^{-1}(B_S)\cap M_\beta^0)$; in other words, as an element of $W(M_\beta^0, \operatorname{Ad}g^{-1}(B_S)\cap M_\beta^0)$. Since ψ, restricted to $\operatorname{Ad}g^{-1}(B_S)$, is invariant under the action of this Weyl group, the lemma follows.

I continue with the hypotheses which were stated above (5.75); in particular, G' shall denote the group $G(B_S^j)$. I enumerate the set

$$\{w\in W(G,H)\,|\,w\beta\in\Phi(G',H)\}$$

as $\{w_1,\ldots,w_n\}$. For each i, I define

(5.76) $$\beta_i=w_i\beta,\qquad\text{and } X_i=Y_{\beta_i}+Y_{-\beta_i},$$

in the notation of (2.7). Next, I select representatives k_i for w_i, so that

(5.77a) $$\operatorname{Ad}k_i(X)=X_i,$$

with $X=Y_\beta+Y_{-\beta}$, as in (5.73). Because of the hypotheses (5.70), the Cayley transform c_S may be thought of as being defined relative to the group G'; also, each β_i is a root of (G',H). Hence, in complete analogy to (5.43), with G' playing the role of G, one can pick $g_i\in G'$, such that

(5.77b) $$\operatorname{Ad}g_i(X_i)=\pm c_S Z_{\beta_i}.$$

(5.78) **Lemma.** *For* $b\in B_S^j$,

$$\Theta(b)=\sum_{i=1}^n C_i\cdot\sum_{w\in W(G',B_S\cap G')}\psi(\operatorname{Ad}(g_ik_i)^{-1}(wb))\cdot\Big|\prod_{\alpha\in\Phi(G,B_S),\,\langle\alpha,\operatorname{Ad}g_i(X_i)\rangle>0}(e^{\alpha/2}-e^{-\alpha/2})(wb)\Big|^{-1},$$

where

$$C_i=\frac{\#\{w\in W(G',H)\,|\,w\beta_i=\pm\beta_i\}}{\#W(G',H)\cdot\#W(M_{\beta_i}^0,B_{\beta_i,+}^0)\cdot\#\{w\in W(G',B_S\cap G')\,|\,w\operatorname{Ad}g_i(X_i)=\operatorname{Ad}g_i(X_i)\}}.$$

Proof. The lemma is a consequence of (5.74) and (5.75), together with the following two statements:

(5.79) if $g\in G$ satisfies $\operatorname{Ad}g(X)\in\mathfrak{b}_S$, and if $\operatorname{Ad}g^{-1}(B_S^j)\subset M_\beta^\dagger\cdot B_{\beta,-}$, then $\operatorname{Ad}g(X)=w\circ\operatorname{Ad}g_i(X_i)$, for some i and some $w\in W(G',B_S\cap G')$;

and

(5.80) the number of times which any particular $w\circ\operatorname{Ad}g_i(X_i)$ comes up, as i runs from 1 to n and w over $W(G',B_S\cap G')$, is equal to

$$\#\{w\in W(G',B_S\cap G')\,|\,w\circ\operatorname{Ad}g_i(X_i)=\operatorname{Ad}g_i(X_i)\}\cdot\frac{\#\{w\in W(G,B_\beta)\,|\,wX=X\}\cdot\#W(G',H)}{\#\{w\in W(G',H)\,|\,w\beta_i=\pm\beta_i\}}.$$

It therefore suffices to prove (5.79) and (5.80).

To begin with, if $\mathrm{Ad}\,g(X)\in\mathfrak{b}_S$, there exists a unique root $\gamma\in\Phi(\mathrm{G}, \mathrm{Ad}\,g^{-1}(B_S))$ which assumes the value 2 on X, and which vanishes on the orthogonal complement of X in $\mathrm{Ad}\,g^{-1}(\mathfrak{b}_S)$; also, γ belongs to the same simple factor of G to which β belongs, and it has the same length as β. This is true because $\mathrm{Ad}\,g^{-1}(\mathfrak{b}_S^{\mathbb{C}})$ and $\mathfrak{b}_\beta^{\mathbb{C}}$ are conjugate under the action of $M_\beta^{\mathbb{C}}$, and in view of (2.7) and (2.10). Since $\mathrm{Ad}\,g(X)$ determines g up to a right factor in M_β, one can transfer γ back to B_S via $\mathrm{Ad}\,g$ without ambiguity. The Weyl group of B_S in G operates transitively on the roots in $\Phi(G, B_S)$ which have the same length as β, and which belong to the same simple factor as β. Hence the G-conjugates of X in $\mathfrak{b}_S$ are in one-to-one correspondence with those roots in $\Phi(G, B_S)$ which belong to the same simple factor as β, and which have the same length as β; the correspondence is given as follows: $\mathrm{Ad}\,g(X)$ corresponds to the unique root $\alpha\in\Phi(G, B_S)$ such that $\langle\alpha, \mathrm{Ad}\,g(X)\rangle=2$, and such that α vanishes on the orthogonal complement of $\mathrm{Ad}\,g(X)$ in $\mathfrak{b}_S$. Moreover, in this situation, $\mathrm{Ad}\,g^{-1}(B_S^{j})$ lies in $M_\beta^{\dagger}\cdot B_{\beta,-}$ if and only if $\alpha\in\Phi(G', B_S\cap G')$, as can be deduced from (5.71).

Now let $\alpha\in\Phi(G', B_S\cap G')$ be given, of the same length as β, and belonging to the same simple factor of G. According to (5.72), applied to the appropriate simple factor of G, there exists a noncompact root $\beta'\in\Phi(G', H)$, which has the same length as α, and hence the same length as β, and which belongs to the same simple factor of G' as α, and hence to the same simple factor of G as β. Replacing β' by its negative, if necessary, one can arrange that $\beta'=\beta_i$, for some i; this follows again from (5.72). Under the correspondence set up above, the root β_i, transferred to B_S via c_S, corresponds to $c_S Z_{\beta_i}=\pm\mathrm{Ad}\,g_i(X_i)$. Because of the particular way in which $\beta'=\beta_i$ was chosen, the root β_i, transferred to B_S via c_S, is conjugate to $\pm\alpha$ under the action of $W(G', B_S\cap G')$. Hence, for suitable element w of this Weyl group, α corresponds to $w\circ\mathrm{Ad}\,g_i(X_i)$, which verifies the statement (5.79).

As for the multiplicity of $w\circ\mathrm{Ad}\,g_i(X_i)$, if i is kept fixed, the number of w's in $W(G', B_S\cap G')$ which give the same $w\circ\mathrm{Ad}\,g_i(X_i)$ is precisely

(5.81a) $$\#\{w\in W(G', B_S\cap G')\,|\,w\circ\mathrm{Ad}\,g_i(X_i)=\mathrm{Ad}\,g_i(X_i)\}.$$

The number of indices i for which $\mathrm{Ad}\,g_i(X_i)=\pm c_S Z_{\beta_i}$ takes any given value is given by
$$\#\{w\in W(G', H)\,|\,w\beta=\pm\beta\},$$
at least if the $g_i\in G'$ are chosen so that $g_i=g_j$ whenever $\beta_i=\pm\beta_j$; for the purpose of verifying (5.80), I shall make this legitimate assumption. As follows from (2.36) and (2.43), the integer above coincides with

(5.81b) $$\#\{w\in W(G, B_\beta)\,|\,wX=X\}.$$

The $W(G', B_S\cap G')$-orbits of $\mathrm{Ad}\,g_i(X_i)$ and of $\mathrm{Ad}\,g_j(X_j)$ agree if and only if the corresponding roots β_i, β_j, transferred to B_S via c_S, are conjugate under the action of the Weyl group of B_S in G'; in other words, if and only if β_i and β_j have the same length and belong to the same simple factor of G'. According to (5.72), this happens precisely when $\pm\beta_i$ and $\pm\beta_j$ are $W(G', H)$-conjugate. The number of noncompact roots in $\Phi(G', H)$ which are $W(G', H)$-conjugate to any given β_i or its negative is

(5.81c) $$\frac{\#W(G', H)}{\#\{w\in W(G', H)\,|\,w\beta_i=\pm\beta_i\}}.$$

The multiplicity of $w \circ \operatorname{Ad} g_i(X_i)$ is given by the product of the three integers (5.81a–c). At this point, both (5.79) and (5.80), and hence the lemma itself, have been verified.

The preceeding lemma is the main ingredient of the proof of Proposition (5.69), which closely parallels the proof of (5.30).

Proof of (5.69). As was pointed out already, I may and shall impose the conditions (5.70). In particular, the hypotheses of Lemma (5.78) are fulfilled.

By assumption, $\Theta(\Psi, \lambda)$ satisfies the identity (4.22): on B_S^j,

$$\begin{aligned}\Theta(\Psi, \lambda) = &(-1)^{q-q'} \cdot (W(G', H))^{-1} \cdot (\Delta_{G,G'})^{-1} \\ &\cdot \textstyle\sum_{w \in W(G,H)} \varepsilon_{G,G'}(w\Psi) \cdot \Theta_{\tilde{G}'}(w\Psi \cap \Phi(G', H), w\lambda).\end{aligned} \tag{5.82}$$

In order to simplify the notation in what follows, I set

$$\Phi(B_S)' = \Phi(G', B_S \cap G'), \qquad W(B_S)' = W(G', B_S \cap G'); \tag{5.83}$$

$\Phi(B_S)'$ is a sub-root system of $\Phi(G, B_S)$, and $W(B_S)'$ is a subgroup of $W(G, B_S)$. As in (5.33), I let Ψ_S be the system of positive roots Ψ, transferred to $\Phi(G, B_S)$ via c_S. In the definitions of $\Delta_{G,G'}$ and $\varepsilon_{G,G'}$, the role of Ψ_0 shall be played be Ψ. Then

$$\varepsilon_{G,G'}(w\Psi) = \textstyle\prod_{\alpha \in w\Psi, \alpha \notin \Phi(G', H)} \operatorname{sgn}_\Psi \alpha, \tag{5.84}$$

and on the appropriate covering of B_S,

$$\Delta_{G,G'} = \textstyle\prod_{\alpha \in \Psi_S, \alpha \notin \Phi(B_S)'} (e^{\alpha/2} - e^{-\alpha/2}); \tag{5.85}$$

these identities are analogous to (5.32) and (5.35). Thus, on B_S^j,

$$\begin{aligned}\Theta(\Psi, \lambda) = &(-1)^{q-q'} \cdot (\# W(G', H))^{-1} \cdot \left(\textstyle\prod_{\alpha \in \Psi_S, \alpha \notin \Phi(B_S)'} (e^{\alpha/2} - e^{-\alpha/2})\right)^{-1} \\ &\cdot \textstyle\sum_{w \in W(G,H)} \left(\prod_{\alpha \in w\Psi, \alpha \notin \Phi(G',H)} \operatorname{sgn}_\Psi \alpha\right) \cdot \Theta_{\tilde{G}'}(w\Psi \cap \Phi(G', H), w\lambda).\end{aligned} \tag{5.86}$$

The analogous formula for $\Theta(\Psi_1, \lambda)$ on B_S^j is precisely what must be proven. In view of (5.28), this formula which must be proven can be written as follows: on B_S^j,

$$\begin{aligned}\Theta = &(-1)^{q-q'} \cdot (\# W(G', H))^{-1} \cdot \left(\textstyle\prod_{\alpha \in \Psi_S, \alpha \notin \Phi(B_S)'} (e^{\alpha/2} - e^{-\alpha/2})\right)^{-1} \\ &\cdot \textstyle\sum_{w \in W(G,H),\, w\beta \in \Phi(G',H)} \left\{\left(\prod_{\alpha \in w\Psi, \alpha \notin \Phi(G',H)} \operatorname{sgn}_\Psi \alpha\right)\right. \\ &\cdot \left.\left(\Theta_{\tilde{G}'}(w\Psi \cap \Phi(G', H), w\lambda) + \Theta_{\tilde{G}'}(w\Psi_1 \cap \Phi(G', H), w\lambda)\right)\right\}.\end{aligned} \tag{5.87}$$

The derivation of (5.87) is very similar to that of (5.39). I shall now deduce (5.87) from (5.78), plus the inductive hypotheses.

Without further mention, I shall use the notation of (5.78). Since the sign in (5.77b) was left undetermined, I may insist that

$$\beta_j = \pm \beta_i \Rightarrow g_j = g_i. \tag{5.88}$$

Some further notational conventions:

$$\begin{aligned}&M_i = M_{\beta_i}, \quad B_i = B_{\beta_i}, \quad W_i = W(M_i^0, B_{i,+}^0), \\ &\Phi_i = \{\alpha \in \Phi(G, H) \mid \alpha \perp \beta_i\} = \Phi(M_i^0, B_{i,+}^0), \quad \Phi' = \Phi(G', H).\end{aligned} \tag{5.89}$$

In terms of these conventions, one has

$$\phi_0 \circ \operatorname{Ad} k_i^{-1} = \Theta_{M_i^0}(w_i \Psi \cap \Phi_i, w_i \lambda|_{\mathfrak{b}_{i,+}}), \tag{5.90}$$

which is entirely analogous to (5.46).

For each i, $\operatorname{Ad} g_i^{-1}(B_S)$ is a Cartan subgroup of $M_i \cdot B_{i,-}$, and its identity component is the identity component of a split Cartan subgroup of G' (recall: $g_i \in G'$). I now set

$$G_i' = \text{identity component of } G' \cap M_i^0. \tag{5.91}$$

According to (5.71), and because of the manner in which the g_i and k_i were chosen (cf. the proof of (5.78)),

$$\operatorname{Ad} g_i^{-1}(B_S^j) \subset G_i' \cdot F_{\beta_i} \cdot B_{i,-}. \tag{5.92}$$

Hence, for each i, one can select an $f_i \in F_{\beta_i}$, such that

$$f_i \cdot \operatorname{Ad} g_i^{-1}(B_S^j) \subset G_i' \cdot B_{i,-}. \tag{5.93}$$

Because of (5.88), I may also assume that

$$\beta_j = \pm \beta_i \Rightarrow f_j = f_i. \tag{5.94}$$

In this situation,

(5.95) $(f_i \cdot \operatorname{Ad} g_i^{-1}(B_S^j)) \cap G_i'$ is a connected component of the Cartan subgroup $\operatorname{Ad} g_i^{-1}(B_S) \cap M_i^0$ of M_i^0.

For every $\alpha \in \Phi(G, \operatorname{Ad} g_i^{-1}(B_S))$ which vanishes on X_i, e^α is identically equal to 1 on F_{β_i}. Hence $\Phi(G_i', \operatorname{Ad} g_i^{-1}(B_S) \cap G_i')$ consists precisely of those roots $\alpha \in \Phi(M_i^0, \operatorname{Ad} g_i^{-1}(B_S) \cap M_i^0)$, such that e^α assumes positive values on $(f_i \cdot \operatorname{Ad} g_i^{-1}(B_S^j)) \cap G_i'$. In other words, G_i' plays the same role with respect to M_i^0 and $(f_i \cdot \operatorname{Ad} g_i^{-1}(B_S^j)) \cap G_i'$, which G' plays with respect to G and B_S^j.

Because of the inductive hypotheses, I may apply the identity (4.22), with M_i^0 in place of G and $(f_i \cdot \operatorname{Ad} g_i^{-1}(B_S^j)) \cap G_i'$ in place of B^j. In order to get an explicit formula for $\Delta_{M_i^0, G_i'}$ on $\operatorname{Ad} g_i^{-1}(B_S)$, I let $\Psi \cap \Phi_i$ play the role of Ψ_0 in (4.19). As was remarked in the proof of (5.30), every $w \in W_i$ can be uniquely extended to an element of $W(G, H)$. In this manner, one obtains an inclusion

$$W_i \subset \{w \in W(G, H) | w\beta_i = \pm \beta_i\}. \tag{5.96}$$

For each $w \in W_i$, one has

$$\varepsilon_{M_i^0, G_i'}(w(w_i \Psi \cap \Phi_i)) = \prod\nolimits_{\alpha \in w w_i \Psi,\, \alpha \notin \Phi',\, \alpha \perp \beta_i} \operatorname{sgn}_\Psi \alpha. \tag{5.97}$$

The same argument which was used to verify (5.56) can be used in the present context: on the appropriate finite covering of $\operatorname{Ad} g_i^{-1}(B_S) \cap M_i^0$,

$$\Delta_{M_i^0, G_i'} = \left(\prod\nolimits_{\alpha \in \Psi_S,\, \alpha \notin \Phi(B_S)',\, \langle \alpha, \operatorname{Ad} g_i(X_i) \rangle = 0} (e^{\alpha/2} - e^{-\alpha/2})\right) \circ \operatorname{Ad} g_i. \tag{5.98}$$

Keeping in mind the inclusion (5.96), and using (5.97) and (5.98), I now inductively apply (4.22). On $(f_i \cdot \operatorname{Ad} g_i^{-1}(B_S^j)) \cap G_i'$,

$$\begin{aligned} \phi_0 \circ \operatorname{Ad} k_i^{-1} = & (-1)^{q_i - q_i'} \cdot (\# W(G_i', B_{i,+}^0))^{-1} \\ (5.99) \quad & \cdot \left(\prod\nolimits_{\alpha \in \Psi_S,\, \alpha \notin \Phi(B_S)',\, \langle \alpha, \operatorname{Ad} g_i(X_i) \rangle = 0} (e^{\alpha/2} - e^{-\alpha/2})\right)^{-1} \circ \operatorname{Ad} g_i \\ & \cdot \sum\nolimits_{w \in W_i} \left(\prod\nolimits_{\alpha \in w w_i \Psi,\, \alpha \notin \Phi',\, \alpha \perp \beta_i} \operatorname{sgn}_\Psi \alpha\right) \cdot \Theta_{\tilde{G}_i'}(w w_i \Psi \cap \Phi_i', w w_i \lambda|_{\mathfrak{b}_{i,+}}) \end{aligned}$$

(recall (5.90)!). Here $\Phi_i' = \Phi_i \cap \Phi' = \Phi(G_i', B_{i,+}^0)$, and

$$q_i = \tfrac{1}{2} \dim_{\mathbb{R}} M_i^0 / M_i \cap K, \qquad q_i' = \tfrac{1}{2} \dim G_i'/G_i' \cap K.$$

This step is completely analogous to the derivation of (5.57). Again just as in the proof of (5.30), I let I_i be the set of indices j, $1 \leqq j \leqq n$, such that $\beta_j = \pm \beta_i$. For $j \in I_i$, one has $X_j = X_i$, $M_j = M_i$, $f_j = f_i$, etc. The arguments which preceed (5.58) can now be applied to (5.99). On $(f_i \cdot \operatorname{Ad} g_i^{-1}(B_S^j)) \cap G'$,

$$\begin{aligned} \sum\nolimits_{j \in I_i} \phi_0 \circ \operatorname{Ad} k_j^{-1} = & (-1)^{q_i - q_i'} \cdot \# W_i \cdot (\# W(G_i', B_{i,+}^0))^{-1} \\ (5.100) \quad & \cdot \left(\prod\nolimits_{\alpha \in \Psi_S,\, \alpha \notin \Phi(B_S)',\, \langle \alpha, \operatorname{Ad} g_i(X_i) \rangle = 0} (e^{\alpha/2} - e^{-\alpha/2}) \circ \operatorname{Ad} g_i\right)^{-1} \\ & \cdot \sum\nolimits_{j \in I_i} \left(\prod\nolimits_{\alpha \in w_j \Psi,\, \alpha \notin \Phi',\, \alpha \perp \beta_j} \operatorname{sgn}_\Psi \alpha\right) \cdot \Theta_{\tilde{G}_j'}(w_j \Psi \cap \Phi_j', w_j \lambda|_{\mathfrak{b}_{i,+}}) \end{aligned}$$

I define ψ_i on $Z_{G'}(B_{i,-})$, for $1 \leqq i \leqq m$, as follows:

$$\begin{aligned} & \psi_i(g) = 0 \quad \text{if } g \notin G_i' \cdot F_{\beta_i} \cdot B_i; \\ (5.101) \quad & \psi_i(mfb) = \Theta_{\tilde{G}_i'}(w_i \Psi \cap \Phi_i', w_i \lambda|_{\mathfrak{b}_{i,+}})(m) \cdot \zeta_{w_i \lambda}(f) \cdot e^\nu \circ \operatorname{Ad} k_i^{-1}(b) \\ & \text{if } m \in G_i', \quad f \in F_{\beta_i}, \quad b \in B_{i,-}. \end{aligned}$$

Just as in the proof of (5.3), ψ_i may make sense only on a suitable finite covering of $Z_{G'}(B_{i,-})$; however, only the product of ψ_i with $\Delta_{M_i^0, G_i'}$ occurs below, and this product does make sense on $Z_{G'}(B_{i,-})$. Now let $g \in B_S^j$ be given. Because of (5.93),

$$(5.102) \qquad \operatorname{Ad} g_i^{-1}(g) = f_i^{-1} \cdot b_0 \cdot b_-, \qquad \text{with } b_0 \in G_i' \quad b_- \in B_{i,-}.$$

Since $\operatorname{Ad} g_i(X_i) \in \mathfrak{b}_S$, and since X_i spans $\mathfrak{b}_{i,-}$, $\operatorname{Ad} g_i^{-1}(B_S)$ contains both f_i and b_-. Hence

$$(5.103) \qquad b_0 \in (f_i \cdot \operatorname{Ad} g_i^{-1}(B_S^j)) \cap G_i'.$$

Also, $\operatorname{Ad} k_i^{-1}(f_i) \in F_\beta$, $\operatorname{Ad} k_i^{-1}(b_0) \in M_\beta^0$, and $\operatorname{Ad} k_i^{-1}(b_-) \in B_{\beta,-}$. Thus

$$\begin{aligned} (5.104) \quad \psi(\operatorname{Ad}(g_i k_i)^{-1}(g)) & = \psi \circ \operatorname{Ad} k_i^{-1}(f_i^{-1} \cdot b_0 \cdot b_-) \\ & = \phi_0 \circ \operatorname{Ad} k_i^{-1}(b_0) \cdot \zeta_{w_i \lambda}(f_i^{-1}) \cdot e^\nu \circ \operatorname{Ad} k_i^{-1}(b_-); \end{aligned}$$

similarly one obtains the corresponding identity for ψ_i, on B_S^j, lifted to the appropriate finite covering of B_S. Combining (5.100–5.104), and taking into account (5.88) and (5.94), one finds

$$\begin{aligned} \sum\nolimits_{j \in I_i} \psi \circ \operatorname{Ad}(g_j k_j)^{-1} = & (-1)^{q_i - q_i'} \cdot \# W_i \cdot (\# W(G_i', B_{i,+}^0))^{-1} \\ (5.105) \quad & \cdot \left(\prod\nolimits_{\alpha \in \Psi_S,\, \alpha \notin \Phi(B_S)',\, \langle \alpha, \operatorname{Ad} g_i(X_i) \rangle = 0} (e^{\alpha/2} - e^{-\alpha/2})\right)^{-1} \\ & \cdot \sum\nolimits_{j \in I_i} \left(\prod\nolimits_{\alpha \in w_j \Psi,\, \alpha \notin \Phi',\, \alpha \perp \beta_j} \operatorname{sgn}_\Psi \alpha\right) \cdot \psi_j \circ \operatorname{Ad} g_j^{-1}; \end{aligned}$$

this formula holds on B_S^j, but it may be quite false on other components of B_S.

According to (2.44), $B_S^i \subset B_S^0 \cdot Z(G')$. Since every $w \in W(G', B_S \cap G')$ can be realized as $\operatorname{Ad} g$, with $g \in G'$,

$$W(G', B_S \cap G') \text{ preserves } B_S^i. \tag{5.106}$$

In view of the two statements (2.36) and (2.43), there is an isomorphism

$$\{w \in W(G', H) \mid w\beta_i = \pm\beta_i\} \simeq \{w \in W(G', B_i) \mid wX_i = X_i\}. \tag{5.107}$$

At this point, (5.78) and (5.105–5.107) lead to the following identity on B_S^i:

$$\begin{aligned} \Theta = {} & \textstyle\sum_{i=1}^n (-1)^{q_i - q_i'} \cdot C_i' \cdot \big(\prod_{\alpha \in w_i\Psi,\, \alpha \notin \Phi',\, \alpha \perp \beta_i} \operatorname{sgn}_\Psi \alpha\big) \\ & \cdot \textstyle\sum_{w \in W(B_S)'} \{ |\prod_{\alpha \in \Phi(G, B_S),\, \alpha \notin \Phi(B_S)',\, \langle \alpha, \operatorname{Ad} g_i(X_i) \rangle > 0} (e^{\alpha/2} - e^{-\alpha/2}) \circ w|^{-1} \\ & \cdot \big(\textstyle\prod_{\alpha \in \Psi_S,\, \alpha \notin \Phi(B_S)',\, \langle \alpha, \operatorname{Ad} g_i(X_i) \rangle = 0} (e^{\alpha/2} - e^{-\alpha/2}) \circ w\big)^{-1} \\ & \cdot |\textstyle\prod_{\alpha \in \Phi(B_S)',\, \langle \alpha, \operatorname{Ad} g_i(X_i) \rangle > 0} (e^{\alpha/2} - e^{-\alpha/2}) \circ w|^{-1} \cdot \psi_i \circ \operatorname{Ad} g_i^{-1} \circ w \} \end{aligned} \tag{5.108}$$

with C_i' given by a formula which looks identical to (5.62). Just as (5.61) implies (5.39), with practically the same arguments, (5.87) can be deduced from (5.108). The detailed verification will be left to the reader. The proof of Proposition (5.69) is now complete.

§ 6. The Proofs of the Main Theorems

Using the preparatory work of the last section, I shall verify Theorems (4.15), (4.21), and (4.22). In addition to the usual hypotheses about G, I now require that G/K be Hermitian symmetric. Inductively, I may and shall assume that the theorems in question hold for all groups of lower dimension than G. For the purpose of constructing the invariant eigendistributions $\Theta(\Psi, \lambda)$ on G, I keep fixed a particular system of positive roots Ψ in $\Phi = \Phi(G, H)$. Since G/K is Hermitian symmetric, one can choose another system of positive roots Ψ_0 in Φ, such that

(6.1)
a) Ψ_0 has the property (3.1), and
b) $\Psi_0 \cap \Phi^c = \Psi \cap \Phi^c$

(Φ^c = set of compact roots in Φ). Indeed, any system of positive roots with the property (3.1) is $W(G, H)$-conjugate to one which satisfies b). As was already remarked in (4.16), some special considerations become necessary when one of the simple factors of G is isomorphic to or a covering of $SO(2, 2n+1)$, $n \geq 2$.

(6.2) **Lemma.** *If $G = SO(2, 2n+1)$, any given system of positive roots $\Psi \subset \Phi = \Phi(G, H)$ contains a single short, noncompact root γ. The requirement $\gamma \in \Psi_0$, imposed in addition to (6.1), specifies Ψ_0 uniquely. In this situation, there exists a chain*

$$\Psi_0, \Psi_1, \ldots, \Psi_m = \Psi$$

of systems of positive roots in Φ, such that each Ψ_i is obtained from Ψ_{i-1} by reflection about a simple (relative to Ψ_{i-1}), long, noncompact root.

Proof. For $G = SO(2, 2n+1)$, $i\mathfrak{h}^*$ has an orthogonal basis $\{e_1, \ldots, e_{n+1}\}$, so that

$$\Phi^c = \{\pm(e_i \pm e_j),\ 2 \leqq i < j \leqq n+1;\ \pm e_i,\ 2 \leqq i \leqq n+1\},$$

$$\Phi^n = \{e_1;\ \pm(e_1 \pm e_i),\ 2 \leqq i \leqq n+1\}.$$

Replacing some of the e_i by their negatives, if necessary, and permuting $e_2, \ldots, e_{n+1}$, one can arrange that $e_1 \in \Psi$, and that

$$\Psi \cap \Phi^c = \{e_i - e_j,\ 2 \leqq i < j \leqq n+1;\ e_i,\ 2 \leqq i \leqq n+1\}.$$

If Ψ_0 is to satisfy the conditions (3.1), $\Psi \cap \Phi^c = \Psi_0 \cap \Phi^c$, and $e_1 \in \Psi_0$, one must set

$$\Psi_0 = \{e_i \pm e_j,\ 1 \leqq i < j \leqq n+1;\ e_i,\ 1 \leqq i \leqq n+1\}.$$

By successive reflections about simple (at the respective stage), long, noncompact roots, one can generate $n+1$ distinct positive root systems, all of which contain e_1, and have the same intersection with Φ^c as Ψ_0. This exhausts all such root systems, since the Weyl groups of Φ and Φ^c have order $2^{n+1}(n+1)!$ and $2^n n!$, respectively. Hence the lemma.

Now back to a general G! Because of the lemma, in addition to (6.1), I can make the following assumption about Ψ_0:

(6.3) For each simple factor of G which is locally isomorphic to $SO(2, 2n+1)$, $n \geqq 2$, Ψ_0 contains the same short, noncompact root as Ψ.

Let C and C_0 be the positive Weyl chambers of the root system Φ, corresponding to Ψ and Ψ_0, respectively. Because of (6.1b), C and C_0 lie in the same Weyl chamber of the root system Φ^c. Hence C can be connected to C_0 by a chain of Weyl chambers of Φ, all of which lie in the same Weyl chamber of Φ^c, such that any successive two have a face in common. Equivalently,

(6.4) there exists a chain $\Psi_0, \Psi_1, \ldots, \Psi_m = \Psi$ of positive root systems, such that each Ψ_i is obtained from Ψ_{i-1} by reflection about a simple (relative to Ψ_{i-1}), noncompact root.

This chain, of course, is not uniquely determined. As follows from (6.2) and (6.3), one can arrange that

(6.5) the noncompact root whose sign is reversed in the passage from Ψ_{i-1} to Ψ_i is long, whenever it belongs to a simple factor locally isomorphic to $SO(2, 2n+1)$, with $n \geqq 2$.

For the time being, I keep fixed a particular chain with the properties (6.4) and (6.5).

Next, I consider an admissible $\lambda \in i\mathfrak{h}^*$ (cf. (4.11)), such that

$$(6.6) \qquad (\lambda, \alpha) > 0 \quad \text{for all } \alpha \in \Psi \cap \Phi^c.$$

Since all of the Ψ_i have the same intersection with Φ^c, the condition (6.6) also holds with Ψ_i in place of Ψ. The positive root system Ψ_0 satisfies (3.1), so that the invariant eigendistribution $\Theta(\Psi_0, \lambda)$, as described in § 3, is defined. If Theorem (4.15), in particular (4.15c), is to hold, $\Theta(\Psi_{i-1}, \lambda) + \Theta(\Psi_i, \lambda)$ must be equal to a certain induced invariant eigendistribution, which I shall denote by Θ_i; here i ranges between 1 and n. I set

$$(6.7) \qquad \Theta(\Psi, \lambda) = (-1)^m\, \Theta(\Psi_0, \lambda) + \textstyle\sum_{i=1}^m (-1)^{m-i}\, \Theta_i.$$

It remains to be shown that $\Theta(\Psi, \lambda)$ does not depend on the particular choice of the Ψ_i, and that it has all the desired properties.

(6.8) **Lemma.** *If the positive root system* Ψ_0 *has the property* (3.1), *then* $\Theta(\Psi_0, \lambda)$ *satisfies the statements of Theorems* (4.21) *and* (4.22).

Proof. In the case of both (4.21) and (4.22), one may as well assume that the Cartan subgroup in question is of the form B_S, for some strongly orthogonal subset $S \subset \Phi^n$. I claim in both situations

(6.9) $$S \subset \Phi(G', H).$$

Indeed, if $G' = B_{S,+} \cdot G_S$, this follows directly from the description of G_S. In the situation of (4.22), on the other hand, G' equals $B_{S,+} \cdot G(B_S^j)$, for some connected component B_S^j of B_S. Every simple factor of G_S is locally isomorphic to $Sp(n, \mathbb{R})$, for some n (cf. (2.40)). The root system C_n contains a strongly orthogonal spanning set which is uniquely determined, except for the signs of its members, all of which are long roots. When C_n is identified with the root system of $Sp(n, \mathbb{R})$, relative to a split Cartan subgroup A, every long root α exponentiates to a character e^α with positive values everywhere on A. The Cayley transform c_S maps S onto a strongly orthogonal spanning set of $\Phi(G_S, B_S \cap G_S)$. As follows from the preceeding remarks, this strongly orthogonal spanning set must lie in the root system of $G(B_S^j)$, which implies (6.9) also in the situation of (4.22). In both cases, a root $\alpha \in \Phi(G, H)$ is compact if and only if it is compact, viewed as a root of G. Hence

(6.10) $\Psi_0 \cap \Phi(G', H)$ has the property (3.1), relative to the root system $\Phi(G', H)$.

Using (6.9) and (6.10) one can read off the statement of the lemma from Hecht's explicit formula (3.11).

Combining the Lemma with (5.29), (5.30), and (5.69), one obtains

(6.11) **Corollary.** *The formula* (6.7) *describes an invariant eigendistribution, which satisfies the statements* (4.15b), (4.15e), (4.21) *and* (4.22).

Now let $B \subset G$ be an arbitrary Cartan subgroup, with identity component B^0, and with Lie algebra $\mathfrak{b}$. I choose and keep fixed an inner automorphism d of $\mathfrak{g}^{\mathbb{C}}$, which can be realized as the composition of an inverse Cayley transform and an inner automorphism of G, and which induces an isomorphism $d\colon \mathfrak{b}^{\mathbb{C}} \xrightarrow{\sim} \mathfrak{h}^{\mathbb{C}}$. The dual isomorphism d^* maps Ψ to a system of positive roots $d^*\Psi$ in $\Phi(G, B)$. The root system $\Phi(\mathfrak{b})_-$, which was defined in (2.21), is a sub-root system of $\Phi(G, B)$; it consists precisely of those $\alpha \in \Phi(G, B)$ which assume real values on $\mathfrak{b}$. By restriction, $d^*\Psi$ determines a system of positive roots in $\Phi(\mathfrak{b})_-$. I let $\{\gamma_1, \ldots, \gamma_S\}$ be the corresponding set of simple roots in $\Phi(\mathfrak{b})_-$, and I define

(6.12) $$C = \{b \in B^0 \mid e^{\gamma_i}(b) > 1 \quad \text{for } 1 \leqq i \leqq s\}.$$

Since $W(G, B)$ contains the group generated by the reflections about the roots in $\Phi(\mathfrak{b})_-$, the conjugates of C under the action of $W(G, B)$ cover a dense open subset of B^0, which includes all regular elements of B^0. Thus knowing $\Phi(\Psi, \lambda)$ on C amounts to knowing it on all of B^0.

Under d^*, λ corresponds to a complex valued, linear functional $d^*\lambda$ on $\mathfrak{b}$. The Weyl group of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ will be denoted by $W_{\mathbb{C}}$. As w runs over $W_{\mathbb{C}}$, $d^* w\lambda$ ranges over all linear functionals on $\mathfrak{b}$ which are conjugate to $d^*\lambda$ under the action of the Weyl group of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$.

(6.13) **Lemma.** *For every $w \in W_{\mathbb{C}}$ there exists an integer $n(w)$, such that*

$$\Theta(\Psi, \lambda)|_C = \left(\prod_{\alpha \in \Psi} (e^{d^*\alpha/2} - e^{-d^*\alpha/2})\right)^{-1} \cdot \sum_{w \in W_{\mathbb{C}}} n(w) \cdot e^{d^* w \lambda}.$$

The integers $n(w)$ do not depend on the particular choice of λ.

It will turn out, of course, that the integers $n(w)$ do not depend on the particular choice of the Ψ_i, either, but this fact is not yet known. As should also be pointed out, the numerator and the denominator individually may make sense only on a suitable finite covering of B; the quotient, however, is well defined on B itself.

Proof. For $\Psi = \Psi_0$, the statement of the lemma follows directly from the explicit formula (3.11). It therefore suffices to prove the analogous assertion about each Θ_i. Inductively, one may assume that the lemma is correct in the case of all groups of lower dimension than G. This inductive assumption allows one to deduce the desired statement about the Θ_i from Lemma (5.2). The following observations about (5.2) are crucial for this purpose. First of all, the expression[15]

$$\prod_{\alpha \in \Phi(G,B),\ \langle \alpha, \operatorname{Ad} g_i(X) \rangle > 0} (e^{\alpha/2} - e^{-\alpha/2})$$

is real on B: if $\alpha \in \Phi(G,B)$ satisfies the inequality $\langle \alpha, \operatorname{Ad} g_i(X) \rangle > 0$, then so does its complex conjugate $\bar{\alpha}$, which therefore also contributes in the product. Secondly if g lies in B^0, then $\operatorname{Ad} g_i(g) = m\,b$, for some $b \in B_{\beta,-}$ and some $m \in (M_\beta \cap \operatorname{Ad} g_i^{-1}(B))^0$; thus $\psi(g) = \phi_0(m) \cdot e^\nu(b)$, and the inductive hypothesis can be applied to $\phi_0(m)$. Thirdly, since $\operatorname{Ad} g_i(M_\beta) \cap B^0$ is the kernel in B of e^α, for some $\alpha \in \Phi(G,B)$, every w in the Weyl group of the root system

$$\Phi(\operatorname{Ad} g_i(M_\beta^0), B \cap \operatorname{Ad} g_i(M_\beta^0))$$

can be continued to an element of $d^{-1} \circ W_{\mathbb{C}} \circ d$, which operates trivially on $\operatorname{Ad} g_i(X)$. As the reader can verify, these remarks imply the desired statements about the Θ_i, except for the integrality of the coefficients of the exponential terms.

To deduce the integrality of the coefficients from the inductive hypotheses, one may argue as follows. In the formula (5.2), ψ can be replaced by

$$\sum_{g \in M_\beta^\dagger \backslash M_\beta} \psi \circ \operatorname{Ad} g,$$

provided the multiplication by a $= \# M_\beta / M_\beta^\dagger$ is omitted. With this new version of the formula, instead of summing over $W(G,B)$, one may sum, in a well defined manner, over the quotient

$$\{w \in W(G,B) | w \operatorname{Ad} g_i(X) = \operatorname{Ad} g_i(X)\} \backslash W(G,B),$$

provided the constant $\dfrac{1}{c_i}$ is dropped. In other words, the division by c_i in (5.2) does not introduce any denominators. This concludes the proof.

According to a criterion of Harish-Chandra (Theorem 7 of [13]),

(6.14) an invariant eigendistribution Θ on G is tempered if and only if, for every Cartan subgroup B,

$$\left|\prod_{\alpha \in \Phi(G,B)} (e^{\alpha/2} - e^{-\alpha/2})(b)\right|^{\frac{1}{2}} \cdot |\Theta(b)| \leq a(1 + \sigma(b))^m$$

for all $b \in B$, with suitable positive constants a, m.

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[15] One may have to go to a finite covering of B; cf. the remark below (6.13).

As usual, I have identified Θ with the function that represents it. The function σ is defined by

$$\sigma(b_+ \cdot \exp X) = \|X\|,$$

whenever $b_+ \in B_+$ and $X \in \mathfrak{b}_-$. Hence, using the notation of (6.12) and (6.13), one finds

(6.15) if $\Theta(\Psi, \lambda)$ is tempered, $n(w) = 0$ unless $e^{d^* w \lambda}$ remains bounded on C.

Conversely,

(6.16) $\Theta(\Psi, \lambda)$ satisfies the temperedness condition on B^0, provided $n(w) = 0$ whenever $e^{d^* w \lambda}$ fails to the bounded on C.

Remark. In Harish-Chandra's criterion (6.14), the sufficiency is more elementary than the necessity. For the purposes of this paper, if the inductive hypotheses are suitably modified, only the sufficiency is needed.

(6.17) **Lemma.** *Suppose G is not isomorphic to $Sp(n, \mathbb{R})$ or its adjoint group, for any n. Then the invariant eigendistribution $\Theta(\Psi, \lambda)$ of* (6.7) *does not depend on the particular choice of the Ψ_i. Moreover, $\Theta(\Psi, \lambda)$ is tempered whenever λ satisfies the condition* (**) *of* (4.15).

Proof. In the context of Theorems (4.21) and (4.22), every simple factor of G' is locally isomorphic to $Sp(n, \mathbb{R})$ (cf. (2.40) and (2.44)). Because of the assumption about G, every simple factor of G' therefore has strictly lower dimension than G. In particular, Theorem (4.15) holds for any G' which can occur in an application of (4.21) and (4.22) (it is possible to reduce this question from the case of G' to that of its simple factors, in the manner described by (4.17)). According to (6.11), it is legitimate to apply (4.21) and (4.22) to G. It follows immediately that $\Theta(\Psi, \lambda)$ cannot depend on arbitrary choices. In view of (6.15) and (6.16), the statement (4.15f) for the various groups G' which can occur gives the temperedness of $\Theta(\Psi, \lambda)$, provided λ satisfies the condition (**).

(6.18) **Lemma.** *Suppose $G \simeq Sp(n, \mathbb{R})$ for some n, and suppose B is a Cartan subgroup of G which is not split. Then $\Theta(\Psi, \lambda)|_B$ does not depend on the choice of the Ψ_i. Also, $\Theta(\Psi, \lambda)$ satisfies the condition for temperedness at least on B, provided $(\lambda, \alpha) \geqq 0$ for all $\alpha \in \Psi^n$. Similarly, if B^j is a connected component of a split Cartan subgroup B, and if $B^j \not\subset Z(G) \cdot B^0$, the restriction of $\Theta(\Psi, \lambda)$ to B^j does not depend on the Ψ_i; moreover, $\Theta(\Psi, \lambda)$ has the temperedness property on B^j, whenever* (**) *in* (4.15) *holds.*

Proof. Via the statements (4.21) and (4.22), the problems can be reduced to the case of lower dimensional subgroups. The arguments are very similar to those in the proof of (6.17).

If G is the adjoint group of $Sp(n, \mathbb{R})$, the distribution $\Theta(\Psi, \lambda)$ can be pulled back to $Sp(n, \mathbb{R})$, and one can check its various properties there, rather than on G itself. For $G = Sp(n, \mathbb{R})$, if B^j is a connected component of $Z(G) \cdot B^0$, for some split Cartan subgroup B, one completely understands the behaviour of $\Theta(\Psi, \lambda)$ on B^j, as soon as one knows it on B^0 – thanks to (4.15b), which was verified already. As follows from these remarks, the question of whether $\Theta(\Psi, \lambda)$ is well defined, and the problem of proving (4.15f), at this point need to be taken up only for the identity component of a split Cartan subgroup of $Sp(n, \mathbb{R})$.

I now turn to this special case. Thus G will denote the group $Sp(n,\mathbb{R})$, and A a split Cartan subgroup of G. The other symbols, like Ψ, λ, $\Theta(\Psi,\lambda)$, etc., will retain their previous meanings. I choose an inner automorphism d of $G^{\mathbb{C}}$, which can be realized as the composition of an inverse Cayley transform and an inner automorphism of G, such that

(6.19) $$d\colon \mathfrak{a}^{\mathbb{C}} \xrightarrow{\sim} \mathfrak{h}^{\mathbb{C}}$$

($\mathfrak{a}$ = Lie algebra of A). Let $\alpha_1, \ldots, \alpha_n$ be the simple roots for Ψ. Then $d^*\alpha_i$, $1 \leqq i \leqq n$, is the set of simple roots in $\Phi(G,A)$, relative to the system of positive roots $d^*\Psi$. As in (6.12), I define

(6.20) $$C = \{a \in A^0 \mid e^{d^*\alpha_i}(a) > 1, \quad 1 \leqq i \leqq n\}.$$

Thus C is the image under exp of a Weyl chamber in $\mathfrak{a}$. According to Lemma (6.13), for suitable integers $n(w)$,

(6.21) $$\Theta(\Psi,\lambda)|_C = \left(\prod\nolimits_{\alpha\in\Psi}(e^{d^*\alpha/2} - e^{-d^*\alpha/2})\right)^{-1} \cdot \sum\nolimits_{w\in W_{\mathbb{C}}} n(w)\cdot e^{d^* w\lambda}.$$

The remarks which were made below (6.13) also apply here.

It will be necessary to consider the Cartan subgroups of G which have a one dimensional toroidal part. For the present purposes, the enumeration given by Proposition (3.16) is not so helpful. Instead, I shall use Konstant's and Sugiura's classification directly. For each root $\alpha \in \Phi(G,A)$, one can choose a generator $E_\alpha \in \mathfrak{g}$ of the α-root space, and an element $F_\alpha \in \mathfrak{a}$, subject to the conditions

(6.22) $$[F_\alpha, E_\alpha] = 2E_\alpha, \quad [F_\alpha, E_{-\alpha}] = -2E_{-\alpha}, \quad [E_\alpha, E_{-\alpha}] = F_\alpha.$$

These conditions determine the F_α's completely: F_α spans the bracket product of the root spaces corresponding to α and $-\alpha$, and $\langle\alpha, F_\alpha\rangle = 2$. As follows from this remark,

(6.23) $$d(F_\alpha) = Z_\gamma, \quad \text{if } \alpha = d^*\gamma$$

(cf. (2.7)). Each triple $\{F_\alpha, E_\alpha, E_{-\alpha}\}$ spans a copy of $\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{g}$, which centralizes

(6.24) $$\{X \in \mathfrak{a} \mid \langle\alpha, X\rangle = 0\}.$$

The inner automorphism

(6.25) $$e_\alpha = \operatorname{Ad}\exp\frac{i\pi}{4}(E_\alpha + E_{-\alpha})$$

of $\mathfrak{g}^{\mathbb{C}}$ acts as the identity on the subspace (6.24) of $\mathfrak{a}$, and it maps F_α to $iE_{-\alpha} + iE_\alpha$; e_α is a type of Cayley transform. In particular,

(6.26) $$\{X \in \mathfrak{a} \mid \langle\alpha, X\rangle = 0\} \oplus \mathbb{R}(E_\alpha - E_{-\alpha})$$

is a Cartan subalgebra of $\mathfrak{g}$, with complexification $e_\alpha(\mathfrak{a}^{\mathbb{C}})$. For any given root $\gamma \in \Phi(G,H)$, I let $\mathfrak{a}_\gamma$ denote the Cartan subalgebra (6.26), with $\alpha = d^*\gamma$. The corresponding Cartan subgroup will be referred to as A_γ. I define $d_\gamma = d \circ e_\alpha^{-1}$, with $\alpha = d^*\gamma$; then

(6.27) $$d_\gamma\colon \mathfrak{a}_\gamma^{\mathbb{C}} \xrightarrow{\sim} \mathfrak{h}^{\mathbb{C}}.$$

It may seem strange that I have enumerated the Cartan subalgebras $\mathfrak{a}_\gamma$ in terms of roots of (G, H), rather than roots of (G, A); however, this enumeration will turn out to be convenient.

The toroidal direction in $\mathfrak{a}_\gamma$ is spanned by $d_\gamma^{-1}(Z_\gamma)$. Hence, in the notation of (2.21),

$$\Phi(\mathfrak{a}_\gamma)_- = \{d_\gamma^* \alpha | \alpha \in \Phi(G, H), \alpha \perp \gamma\}.$$

By restriction, Ψ determines a system of positive roots in

$$\{\alpha \in \Phi(G, H) | \alpha \perp \gamma\}.$$

I let $\beta_1, \ldots, \beta_{n-1}$ be the resulting set of simple roots. In analogy to (6.12), I set

(6.28) $$C_\gamma = \{a \in A_\gamma^0 | e^{d_\gamma^* \beta_i}(a) > 1, \ 1 \leqq i \leqq n-1\}.$$

According to Lemma (6.13), for suitably chosen integers $n_\gamma(w)$,

(6.29) $$\Theta(\Psi, \lambda)|_{C_\gamma} = \left(\prod_{\alpha \in \Psi} (e^{d_\gamma^* \alpha/2} - e^{-d_\gamma^* \alpha/2})\right)^{-1} \cdot \sum_{w \in W_{\mathbb{C}}} n_\gamma(w) \cdot e^{d_\gamma^* w \lambda}.$$

For each $\gamma \in \Phi(G, H)$, $s_\gamma \in W_{\mathbb{C}}$ shall denote the reflection about the root γ.

(6.30) **Lemma.** *If $\gamma \in \Phi(G, H)$ is simple, relative to the system of positive roots Ψ, then*

$$n(w) - n(s_\gamma w) = n_\gamma(w) - n_\gamma(s_\gamma w),$$

for all $w \in W_{\mathbb{C}}$.

The identity stated in the lemma follows from general results of Harish-Chandra about invariant eigendistributions [11] (see also Theorem 1 of [20]). However, in the case at hand, the lemma can be proven by essentially combinatorial arguments. Since such arguments are more in the spirit of this paper, I shall give a direct proof of lemma. The proof, which depends on some sublemmas, will be postponed to the end of this section. I now continue with the main arguments, assuming the statement of the lemma.

Because of (6.18), the integers $n_\gamma(w)$ do not depend on the particular choice of the Ψ_i, which were used to construct $\Theta(\Psi, \lambda)$. Since the reflections about the simple roots generate all of $W_{\mathbb{C}}$, the lemma implies the following:

(6.31) **Corollary.** *For any two elements w_1, w_2 of $W_{\mathbb{C}}$, the difference $n(w_1) - n(w_2)$ does not depend on the particular choice of the Ψ_i.*

Two definitions are needed for the statement of the next lemma:

(6.32)
$$\Lambda(\Psi) = \{\lambda \in \Lambda | (\lambda, \alpha) \geqq 0 \quad \text{for all } \alpha \in \Psi\},$$
$$R(\Psi) = \{w \in W_{\mathbb{C}} | e^{d^* w \lambda} \text{ is bounded on } C, \text{ for all } \lambda \in \Lambda(\Psi)\}$$

(cf. (6.20)).

(6.33) **Lemma.** *There exists an integer n, such that $n = n(w)$ whenever $w \notin R(\Psi)$.*

Proof[16]. Let $\{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots in $\Phi(G, H)$, relative to the system of positive roots Ψ, and let $\{\lambda_1, \ldots, \lambda_n\}$ be the set of fundamental highest weights. Thus λ_i is the unique element of Λ such that $2(\lambda_i, \alpha_i) = (\alpha_i, \alpha_i)$, and $(\lambda_i, \alpha_j) = 0$ if $i \neq j$. Apparently $\Lambda(\Psi)$ consists precisely of the integral linear combinations,

[16] The argument below is similar to Harish-Chandra's proof of the uniqueness of the Θ_λ, in [12].

with nonnegative coefficients, of $\lambda_1, \ldots, \lambda_n$. Similarly, for any given $\lambda \in \Lambda$, $e^{d^*\lambda}$ is bounded on C if and only if λ is a linear combination, with nonpositive coefficients, of $\alpha_1, \ldots, \alpha_n$; equivalently, if and only if $(\lambda, \lambda_i) \leqq 0$, $1 \leqq i \leqq n$. Hence

(6.34) $w \in W_{\mathbb{C}}$ belongs to $R(\Psi)$ if and only if $(w\lambda_i, \lambda_j) \leqq 0$, *for* $1 \leqq i, j \leqq n$.

To simplify the notation, I shall write s_i for the reflection about the root α_i, rather than s_{α_i}. I now claim

(6.35) if $(w\lambda_i, \lambda_j) > 0$ for some integers i, j, then $n(s_k w) = n(w)$ whenever $k \neq j$.

To prove the claim, I let w, i, j be given, so that $(w\lambda_i, \lambda_j) > 0$, and I let k be an integer between 1 and n, $k \neq j$. It is then possible to pick an admissible $\lambda \in \Lambda(\Psi)$, which satisfies $(w\lambda, \lambda_j) > 0$. The invariant eigendistribution $\Theta(\Psi, \lambda)$ satisfies the temperednes condition on A_{α_k}, as follows from (6.18). By restriction, Ψ determines a system of positive roots in

$$\{\alpha \in \Phi(G, H) | \alpha \perp \alpha_k\}.$$

I enumerate the set of simple roots in this root system as $\beta_1, \ldots, \beta_{n-1}$. One can express $w\lambda$ as a linear combination of these, plus α_k:

$$w\lambda = \textstyle\sum_i b_i \beta_i + c\alpha_k, \qquad b_i, c \in \mathbb{R}. \tag{6.36}$$

Each β_i lies in Ψ; hence

$$\beta_i = \textstyle\sum_l m_{i,l} \alpha_l, \qquad \text{with } m_{i,l} \geqq 0 \tag{6.37}$$

Combining these identities, one finds

$$w\lambda = \textstyle\sum_{l \neq k} \left(\sum_i b_i m_{i,l}\right) \alpha_l + c' \alpha_k,$$

for some $c' \in \mathbb{R}$. The inequality $(w\lambda, \lambda_j) > 0$ implies

$$\textstyle\sum_i b_i m_{i,j} > 0.$$

In view of (6.37), not all of the coefficients b_i in (6.36) can be negative. Thus, for $\gamma = \alpha_k$, $e^{d_\gamma^* w\lambda}$ does not remain bounded on C_γ. On the other hand, $\Theta(\Psi, \lambda)$ is known to satisfy the temperedness condition on A_{α_k}. Hence $n_{\alpha_k}(w) = 0$. Similarly, one may conclude $n_{\alpha_k}(s_k w) = 0$, because s_k acts as the identity on $\beta_1, \ldots, \beta_{n-1}$. The statement (6.35) now follows from Lemma (6.30).

To complete the proof of the lemma, I shall show that $n(w) = n(1)$, whenever $w \in W_{\mathbb{C}}$ does not belong to $R(\Psi)$. It will simplify matters to use the specific properties of the root system $\Phi(G, H)$, which is of type C_n. If an orthogonal basis $\{e_1, \ldots, e_n\}$ of $i\mathfrak{h}^*$ is suitably chosen,

$$\begin{aligned} &\alpha_i = e_i - e_{i+1}, \quad 1 \leqq i \leqq n-1; \quad \alpha_n = 2e_n; \\ &\lambda_i = e_1 + e_2 + \cdots + e_i, \quad 1 \leqq i \leqq n. \end{aligned} \tag{6.38a}$$

Moreover,

(6.38b) $W_{\mathbb{C}}$ is the group generated by the permutations of the e_i, and the reflections about the e_i.

I now suppose that $w \notin R(\Psi)$. According to (6.34), there exist integers i, j, such that $(w\lambda_i, \lambda_j) > 0$; I choose j minimal with respect to this property. In view of

(6.38),

(6.39) $$w\lambda_i = \sum_k a_k e_k, \quad \text{with } a_k = \pm 1 \text{ or } 0.$$

Because of the choice of j, $a_j = 1$. Let W_j be the subgroup of $W_{\mathbb{C}}$ generated by all the reflections about the simple roots, except for the reflection s_j. As can be checked, W_j keeps λ_j fixed. Hence, and because of (6.35), $n(w'w) = n(w)$, for any $w' \in W_j$. Thus, without loss of generality, I may replace w by $w'w$, for some convenient $w' \in W_j$. By doing so, one can arrange that, for some integer $l \geqq j$,

(6.40) $$a_1 = 1; \quad a_k = 1 \text{ if } j < k \leqq l; \quad a_k = 0 \text{ if } k > l.$$

In this situation, $(w\lambda_i, \lambda_1) > 0$. Now the very same argument can be repeated, with j replaced by 1; the condition (6.40) then becomes equivalent to the identity $w\lambda_i = \lambda_i$. This is possible only if $we_1 = e_k$, for some $k \leqq i$. Hence $(w\lambda_1, \lambda_i) > 0$, and the entire argument can be repeated, with 1 taking the place of the integer i. Arguing as before, I may assume $w\lambda_1 = \lambda_1$; moreover, $n(w'w) = n(w)$, whenever $w' \in W_1$. As can be checked, $w\lambda_1 = \lambda_1$ implies $w \in W_1$. Hence $n(w) = n(1)$, for any $w \in W_{\mathbb{C}}$ which lies outside of $R(\Psi)$. The lemma follows.

(6.41) **Lemma.** *If λ satisfies $(\lambda, \alpha) \geqq 0$, for all $\alpha \in \Psi \cap \Phi^n$, the invariant eigendistribution $\Theta(\Psi, \lambda)$ is tempered.*

Proof. According to (6.18), $\Theta(\Psi, \lambda)$ satisfies the temperedness condition on every Cartan subgroup which is not split, as well as on every component of A outside of $A^0 \cdot Z(G)$. Since $\Theta(\Psi, \lambda)$ is known to have the property (4.15b), only the temperedness on A^0 needs to be checked. Because of (6.33), it suffices to show that $n(1) = 0$; I shall do so by induction on the length m of the chain $\Psi_0, \ldots, \Psi_m$. If $m = 0$, $\Psi = \Psi_0$ has the property (3.1), and $\Theta(\Psi, \lambda)$ is explicitly given by (3.11). As was remarked in § 3, $\Theta(\Psi, \lambda)$ is then tempered, provided λ satisfies the condition (**). Now let m be arbitrary. There exists a simple, noncompact root $\beta \in \Psi$, such that

(6.42) $$\Psi_{m-1} = s_\beta \Psi$$

($s_\beta \in W_{\mathbb{C}}$ is the reflection about β). According to (6.7),

(6.43) $$\begin{aligned} &\Theta(\Psi, \lambda) = \Theta_m - \Theta(\Psi_{m-1}, \lambda), \quad \text{where} \\ &\Theta(\Psi_{m-1}, \lambda) = (-1)^{m-1}\, \Theta(\Psi_0, \lambda) + \textstyle\sum_{i=1}^{m-1} (-1)^{m-i-1}\, \Theta_i. \end{aligned}$$

As follows from the induction hypothesis,

(6.44) $$\Theta(\Psi_{m-1}, \lambda) \text{ is tempered, provided } (\lambda, \alpha) > 0 \text{ for all } \alpha \in \Psi_{m-1}.$$

With suitably chosen integers $n'(w)$, $n''(w)$, one can express the restrictions of Θ_m and of $\Theta(\Psi_{m-1}, \lambda)$ to C as follows:

(6.45) $$\begin{aligned} \Theta_m|_C &= \Big(\textstyle\prod_{\alpha\in\Psi}(e^{d^*\alpha/2} - e^{-d^*\alpha/2})\Big)^{-1} \cdot \textstyle\sum_{w\in W_{\mathbb{C}}} n'(w)\, e^{d^* w\lambda} \\ \Theta(\Psi_{m-1}, \lambda)|_C &= \Big(\textstyle\prod_{\alpha\in\Psi}(e^{d^*\alpha/2} - e^{-d^*\alpha/2})\Big)^{-1} \cdot \textstyle\sum_{w\in W_{\mathbb{C}}} n''(w)\, e^{d^* w\lambda}. \end{aligned}$$

I shall show that $n'(1) = n''(1) = 0$, except when $n = 1$. The case of $Sp(1, \mathbb{R})$ is well understood, and may be excluded.

Let $\lambda \in i\mathfrak{h}^*$ be admissible, and such that $(\lambda, \alpha) > 0$, for every $\alpha \in \Psi$. Then, according to (6.42), $(s_\beta \lambda, \alpha) > 0$ whenever $\alpha \in \Psi_{m-1}$. In view of (6.44), this implies the temperedness of $\Theta(\Psi_{m-1}, s_\beta \lambda)$. As follows from formula (6.45), $n''(w s_\beta) = 0$ unless $w \in R(\Psi)$. It was assumed that $n \neq 1$; hence there exist simple roots (relative to Ψ) other than β. I consider the fundamental highest weight corresponding to one of the other simple roots, say μ. The reflection about β preserves μ; this is a consequence of the definition of the fundamental highest weights. Thus $(s_\beta \mu, \mu) = (\mu, \mu) > 0$, and hence s_β cannot belong to $R(\Psi)$; cf. (6.34). For the reasons which were given a above, one may conclude that $n''(1) = 0$.

It remains to be shown that $n'(1) = 0$. The induced invariant eigendistribution Θ_m is equal to Θ, as defined in (4.15c). I shall use the notation established here. Let $\lambda \in i\mathfrak{h}^*$ be admissible, and such that $(\lambda, \alpha) > 0$, for all $\alpha \in \Psi$. The restriction μ of λ to $\mathfrak{b}_{\beta,+}$ then satisfies $(\mu, \alpha) > 0$, for all $\alpha \in \Psi_\beta$. According to the inductive hypotheses stated at the beginning of this section, Theorem (4.15) holds for the group M_β^0. Thus, subject to the assumption about λ,

(6.46) $$\phi_0 \text{ is tempered}.$$

I set $X = Y_\beta + Y_{-\beta}$. The argument which preceeds (5.43) shows that $g \in G$ can be chosen, such that

(6.47) $$\operatorname{Ad} g(X) = d^{-1}(Z_\beta).$$

In particular, $\operatorname{Ad} g^{-1}(A)$ is then a Cartan subgroup of $M_\beta \cdot B_{\beta,-}$. Let $\gamma_1, \ldots, \gamma_{n-1}$ be the set of simple roots, relative to the system of positive roots which Ψ cuts out in the root system

(6.48) $$\{\alpha \in \Phi(G, H) \mid \alpha \perp \beta\}.$$

I define

$$C' = \{a \in A^0 \mid e^{d^* \gamma_i}(a) > 1, 1 \leqq i \leqq n-1\};$$

then $C' \cap \operatorname{Ad} g(M_\beta^0)$ plays the same role with respect to the groups $\operatorname{Ad} g(M_\beta^0)$ and $A^0 \cap \operatorname{Ad} g(M_\beta^0)$ which C plays with respect to G and A^0. Let ψ be defined by (5.1), and let $W'_{\mathbb{C}}$ be the subgroup of $W_{\mathbb{C}}$ generated by the reflections about the roots of the root system (6.48). If one applies Lemma (6.13) to ϕ_0, one obtains a formula

(6.49) $$\psi \circ \operatorname{Ad} g^{-1}|_{C'} = \left(\prod\nolimits_{\alpha \in \Psi, \alpha \perp \beta} (e^{d^* \alpha/2} - e^{-d^* \alpha/2})\right)^{-1} \cdot \sum\nolimits_{w \in W'_{\mathbb{C}}} m(w) \cdot e^{d^* w \lambda},$$

for suitable integers $m(w)$, $w \in W'_{\mathbb{C}}$; here one must take into account (6.47). Moreover, as follows from (6.46),

(6.50) $$m(1) = 0.$$

I now use (5.2), with A taking the place of B. Since A is split, in the notation of (5.2), one has $n = 1$. For every element u of $W(G, A) = d^{-1} \circ W_{\mathbb{C}} \circ d$, there exists a unique $u' \in d^{-1} \circ W'_{\mathbb{C}} \circ d$, such that $u C \subset C'$. Hence, in (5.2), instead of summing over $W(G, A)$, one may sum over the set

(6.51) $$\{u \in W(G, A) \mid u C \subset C'\}.$$

Via d, the set (6.51) corresponds to the set

$$\{w \in W_{\mathbb{C}} | w^{-1}\alpha \in \Psi \text{ whenever } \alpha \in \Psi, \alpha \perp \beta\},$$

whose members I enumerate as $w_1, \ldots, w_N$. Lemma (5.2), as reformulated above, coupled with (6.45) and (6.49), allows one to compute the integers $n'(w)$ in terms of the $m(w)$. For this purpose, I extend the definition of the integers $m(w)$ to all of $W_{\mathbb{C}}$, by setting $m(w)=0$, if $w \notin W'_{\mathbb{C}}$. Then, for every $w \in W_{\mathbb{C}}$,

$$n'(w) = \textstyle\sum_{i=1}^{N} a_i \cdot m(w_i w),$$

with $a_i = \pm a$. Of the w_i^{-1}, only the identity belongs to $W'_{\mathbb{C}}$. Hence $n'(1)=m(1)=0$; cf. (6.50). As was pointed out before, the equality $n(1)=0$, which has now been verified, implies the statement of the lemma.

In particular, Lemma (6.41) implies the identity $n(1)=0$. Therefore, and because of (6.31), the integers $n(w)$ do not depend on the particular choice of the Ψ_i. The restriction of $\Theta(\Psi, \lambda)$ to any non-split Cartan subgroup, and to any component of A outside of $A^0 \cdot Z(G)$, is already known to be independent of the choice of the Ψ_i. Thus,

(6.52) **Corollary.** *The invariant eigendistribution $\Theta(\Psi, \lambda)$ does not depend on the particular choice of the Ψ_i.*

The proofs of Theorems (4.15), (4.21), and (4.22) are now almost complete. I drop the assumption that $G = Sp(n, \mathbb{R})$. I define the invariant eigendistribution $\Theta(\Psi, \lambda)$ by the prescription (6.7). In all possible cases, the definition does not depend on arbitrary choices and is therefore meaningful; cf. (6.17), (6.18), and (6.52). If the system of positive roots Ψ has the property (3.1), in the definition of $\Theta(\Psi, \lambda)$, one may set $\Psi = \Psi_0$, $m=0$. Thus (4.15a) is automatic. The transformation rule (4.15b) has already been verified in (6.11). As a direct consequence of their construction, the invariant eigendistributions $\Theta(\Psi, \lambda)$ satisfy (4.15c). The Definition (6.7) is forced by (4.15c); hence the uniqueness of the $\Theta(\Psi, \lambda)$. The identity

$$\Theta(w\Psi, w\lambda) = \Theta(\Psi, \lambda),$$

for $w \in W(G, H)$, is compatible with (4.15a–c). Thus the uniqueness implies (4.15d). According to (6.11), the $\Theta(\Psi, \lambda)$ satisfy (4.15e). In the various possible cases, (4.15f) follows from (6.17), (6.18), and (6.41). Finally, Theorems (4.21) and (4.22) hold, thanks to Lemma (6.11).

At this point, only Lemma (6.30) remains to be proven. In the statement of the lemma, the choice of the system of positive roots Ψ enters in various ways, at least implicitly: in the definitions of C and of C_γ, in determining the notion of a simple root, in fixing the signs of the denominator in formulas (6.21) and (6.29), as well as in the original definition of $\Theta(\Psi, \lambda)$. I claim that the choice of the system of positive roots is really irrelevant, as long as the same one is used to define C, C_γ, the notion of a simple root, and the signs of the denominators. Indeed, any two systems of positive roots are related by some $w \in W_{\mathbb{C}}$. Under d, w corresponds to an element of $W(G, A)$, which can be represented by $\mathrm{Ad}\, g$, for some $g \in G$. If g is suitably chosen, $\mathrm{Ad}\, g$ maps A_γ onto $C_{w\gamma}$, it maps C_γ, defined relative to Ψ, onto $C_{w\gamma}$, defined relative to $w\Psi$, and similarly for C. These facts imply what was claimed above.

The identity asserted by Lemma (6.30) is linear. It therefore suffices to verify it individually for $\Theta(\Psi_0, \lambda)$ and the Θ_i.

(6.53) **Lemma.** *The invariant eigendistributions* $\Theta(\Psi, \lambda)$ *of Theorem* (3.11) *satisfy the statement of Lemma* (6.30).

Proof. Let Ψ be a system of positive roots with the property (3.1). Because of the remarks above, in verifying (6.30) for the $\Theta(\Psi, \lambda)$ of Theorem (3.11), I may use Ψ to define C, C_γ, etc. In $\Phi(G, H)$, there exists a unique strongly orthogonal spanning set S, all of whose members lie in Ψ. As follows from (3.11), on the set

(6.54) $$\{a \in A^0 \mid e^{d^*\alpha}(a) < 1 \text{ for all } \alpha \in S\},$$

$\Theta(\Psi, \lambda)$ is given by the formula

$$(-1)^q \left(\prod\nolimits_{\alpha \in \Psi} (e^{d^*\alpha/2} - e^{-d^*\alpha/2})\right)^{-1} \cdot \sum\nolimits_{w \in W(G,H)} \varepsilon(w)\, e^{d^* w \lambda}.$$

If w_0 is the unique element of $W_{\mathbb{C}}$ which maps Ψ to $-\Psi$, and if the automorphism $d^{-1} \circ w \circ d$ of $\mathfrak{a}$ is represented by $\operatorname{Ad} g$, with $g \in G$, then $\operatorname{Ad} g$ maps C into the set (6.54). Since $\Theta(\Psi, \lambda)$ is invariant under $\operatorname{Ad} g$, one finds that $n(w) = (-1)^q \varepsilon(w)$ if $w_0 w \in W(G, H)$, and that $n(w) = 0$ if $w_0 w \notin W(G, H)$. With slight modifications, the same argument can be used to compute the integers $n_\gamma(w)$: it turns out that $n_\gamma(w) = n(w)$, for all $w \in W_{\mathbb{C}}$. The lemma follows.

As far as the assertion of Lemma (6.30) for the Θ_i is concerned, the specific situation does not simplify the argument. I therefore consider a more general setting. Let G be a connected, linear, semisimple Lie group, which contains both a compact Cartan subgroup H and a split Cartan subgroup A. One can then choose an inner automorphism d of $\mathfrak{g}^{\mathbb{C}}$, which can be expressed as the composition of an inner automorphism of $\mathfrak{g}$ and the inverse of a Cayley transform, such that d: $\mathfrak{a}^{\mathbb{C}} \xrightarrow{\sim} \mathfrak{h}^{\mathbb{C}}$. I keep fixed a particular system of positive roots $\Psi \subset \Phi(G, H)$. In this situation, d_γ, A_γ, C, C_γ can be defined exactly as before.

Now let $\beta \in \Phi(G, H)$ be a noncompact root. As follows from the argument above (5.43), for a suitably chosen $g \in G$,

(6.55) $$\operatorname{Ad} g(Y_\beta + Y_{-\beta}) = d^{-1} Z_\beta.$$

Since $Y_\beta + Y_{-\beta}$ spans $\mathfrak{b}_{\beta,-}$, $\operatorname{Ad} g^{-1}(A)$ is a Cartan subgroup of $M_\beta \cdot B_{\beta,-}$. If $\gamma \in \Phi(G, H)$ is perpendicular to β, the inner automorphism e_α of (6.25), with $\alpha = d^*\gamma$, acts as the identity on $d^{-1} Z_\beta$. Hence,

(6.56) $$\text{if } \gamma \perp \beta, \quad \operatorname{Ad} g(Y_\beta + Y_{-\beta}) = d_\gamma^{-1} Z_\beta.$$

In particular, $\operatorname{Ad} g^{-1}(A_\gamma)$ is a Cartan subgroup of $M_\beta \cdot B_{\beta,-}$, whenever $\gamma \perp \beta$. In the root system

$$\{\alpha \in \Phi(G, H) \mid \alpha \perp \beta\},$$

which can be naturally identified with $\Phi(M_\beta^0, B_{\beta,+}^0)$, Ψ cuts out a system of positive roots. I enumerate the corresponding simple roots as $\{\gamma_i\}$, and I define

(6.57) $$C' = \{a \in A^0 \mid e^{d^*\gamma_i}(a) > 1, \text{ for all } i\}.$$

With respect to $\operatorname{Ad} g(M_\beta)$ and $A \cap \operatorname{Ad} g(M_\beta)$, the intersection of C' with $\operatorname{Ad} g(M_\beta)$ plays the same role which C plays with respect to G and A. For any given $\gamma \in \Phi(G, H)$, with $\gamma \perp \beta$, I now let $\{\eta_i\}$ be the simple roots in

$$\{\alpha \in \Phi(G, H) | \alpha \perp \beta, \alpha \perp \gamma\},$$

relative to the system of positive roots induced by Ψ. In analogy to C_γ, I set

$$C'_\gamma = \{a \in A^0_\gamma | e^{d^*_\gamma \eta_i}(a) > 1, \text{ for all } i\}. \tag{6.58}$$

I consider an invariant eigendistribution ϕ_0 on M^0_β, and a character ζ of F_β, subject to (4.7). Moreover, a linear functional $\nu \in \mathfrak{b}^*_{\beta,-}$ shall be given. As described in §4, the data β, ϕ_0, ζ, ν can be used to construct an induced invariant eigendistribution Θ. I define ψ as in (5.1). Now the crucial hypothesis: I suppose that there exists a $\lambda \in i\mathfrak{h}^*$, and constants $n'(w)$, $n'_\gamma(w)$, such that

$$\psi \circ \operatorname{Ad} g^{-1}|_{C'} = \left(\prod_{\alpha \in \Psi, \alpha \perp \beta} (e^{d^*\alpha/2} - e^{-d^*\alpha/2})\right)^{-1} \cdot \sum_{w \in W_{\mathbb{C}}} n'(w) \cdot e^{d^* w \lambda} \tag{6.59}$$

($W_{\mathbb{C}}$ = Weyl group of $\Phi(G, H)$), and, for every $\gamma \in \Phi(G, H)$ which is perpendicular to β,

$$\psi \circ \operatorname{Ad} g^{-1}|_{C'_\gamma} = \left(\prod_{\alpha \in \Psi, \alpha \perp \beta} (e^{d^*_\gamma \alpha/2} - e^{-d^*_\gamma \alpha/2})\right)^{-1} \cdot \sum_{w \in W_{\mathbb{C}}} n'_\gamma(w) \cdot e^{d^*_\gamma w \lambda}. \tag{6.60}$$

Finally, if γ is simple with respect to the system of positive roots determined by Ψ in

$$\{\alpha \in \Phi(G, H) | \alpha \perp \beta\}, \tag{6.61}$$

the identity

$$n'(w) - n'(s_\gamma w) = n'_\gamma(w) - n'_\gamma(s_\gamma w) \tag{6.62}$$

is to hold; here $s_\gamma \in W_{\mathbb{C}}$ denotes the reflection about γ.

(6.63) *Remark.* If Θ is one of the eigendistributions Θ_i of (6.7), these hypotheses are satisfied.

Indeed, an application of Lemma (6.13) to ϕ_0, plus the formula which defines ψ, assure that the restrictions of $\psi \circ \operatorname{Ad} g^{-1}$ to C' and C'_γ can be expressed in the manner of (6.59) and (6.60). Actually, in both cases, one only needs to sum over the subgroup of $W_{\mathbb{C}}$ generated by the reflections about the roots in the root system (6.61). By induction on the dimension of G, one may assume that the analogue of Lemma (6.30) holds for the group M^0_β. The identity (6.62) is then a direct consequence. This verifies the remark.

Back to the general situation! For suitably chosen constants $n(w)$, $n_\gamma(w)$,

$$\begin{aligned} \Theta|_C &= \left(\prod_{\alpha \in \Psi} (e^{d^*\alpha/2} - e^{-d^*\alpha/2})\right)^{-1} \cdot \sum_{w \in W_{\mathbb{C}}} n(w) \cdot e^{d^* w \lambda}, \\ \Theta|_{C_\gamma} &= \left(\prod_{\alpha \in \Psi} (e^{d^*_\gamma \alpha/2} - e^{-d^*_\gamma \alpha/2})\right)^{-1} \cdot \sum_{w \in W_{\mathbb{C}}} n_\gamma(w)\, e^{d^*_\gamma w \lambda}; \end{aligned} \tag{6.64}$$

this can be shown just as in the proof of Lemma (6.13).

(6.65) **Lemma.** *Let $u_i, \ldots, u_N$ be an enumeration of the set*

$$\{w \in W_{\mathbb{C}} | w^{-1} \alpha \in \Psi \text{ if } \alpha \in \Psi, \alpha \perp \beta\},$$

then $n(w) = \# M_\beta / M^\dagger_\beta \cdot \sum_{i=1}^N n'(u_i w)$.

Proof. I use Lemma (5.2), with A playing the role of B. Since A is split, in the notation of (5.2), one has $n = 1$. The automorphism d induces an isomorphism

$$W(G, A) \simeq W_{\mathbb{C}}. \tag{6.66}$$

Under this isomorphism, the group

$$(6.67)\qquad \{w\in W(G,A)\,|\,w\circ \mathrm{Ad}\, g(Y_\beta+Y_{-\beta})=\mathrm{Ad}\, g(Y_\beta+Y_{-\beta})\}$$

corresponds to $\{w\in W_{\mathbb{C}}|w\beta=\beta\}$ = subgroup of $W_{\mathbb{C}}$ generated by the reflections about the roots in the root system (6.61).

For every $v\in W(G,A)$, there exists a unique element v' of the group (6.67), such that $vC\subset v'C'$. Hence, in (5.2), instead of summing over $W(G,A)$, one may sum over the set

$$(6.68)\qquad \{v\in W(G,A)\,|\,vC\subset C'\},$$

provided the factor $1/c_i$ is dropped. The isomorphism (6.66) maps the set (6.68) onto the set $\{u_i|1\leqq i\leqq N\}$. If an element v of the set (6.68) corresponds to u_i,

$$\begin{aligned}\prod\nolimits_{\alpha\in\Psi,\alpha\perp\beta}&(e^{d^*\alpha/2}-e^{-d^*\alpha/2})\circ v\\ &=\prod\nolimits_{\alpha\in\Psi,\alpha\perp\beta}(e^{d^*u_i^{-1}\alpha/2}-e^{-d^*u_i^{-1}\alpha/2})\\ &=\prod\nolimits_{\alpha\in\Psi,\alpha\perp u_i^{-1}\beta}(e^{d^*\alpha/2}-e^{-d^*\alpha/2}),\end{aligned}$$

because $u_i^{-1}\alpha\in\Psi$ whenever $\alpha\in\Psi$, $\alpha\perp\beta$. With v and u_i as above,

$$\prod\nolimits_{\alpha\in\Psi,(\alpha,u_i^{-1}\beta)\neq 0}(e^{d^*\alpha/2}-e^{-d^*\alpha/2})$$

is positive on C, and hence agrees on C with

$$\left|\prod\nolimits_{\alpha\in\Phi(G,H),\langle\alpha,\mathrm{Ad}\, g(Y_\beta+Y_{-\beta})\rangle>0}(e^{d^*\alpha/2}-e^{-d^*\alpha/2})\circ v\right|$$

(cf. (6.55)). As a consequence of these remarks,

$$\Theta|_C=a\cdot\left(\prod\nolimits_{\alpha\in\Psi}(e^{d^*\alpha/2}-e^{-d^*\alpha/2})\right)^{-1}\cdot\sum\nolimits_{i=1}^N\sum\nolimits_{w\in W_{\mathbb{C}}}n'(w)\,e^{d^*u_i^{-1}w\lambda}.$$

According to (2.58), $a=\#M_\beta/M_\beta^\dagger$. The lemma follows.

It will be necessary to have an analogous formula for the $n_\gamma(w)$. Instead of expressing the $n_\gamma(w)$ in terms of the $n'_\gamma(w)$, it is easier to work with certain intermediate quantities $n''_\gamma(w)$, which I shall now describe. For any root $\gamma\in\Phi(G,H)$, d_γ identifies $W(G,A_\gamma)$ with a subgroup of $W_{\mathbb{C}}$. I distinguish two possible cases:

$$(6.69\,\mathrm{a})\qquad d_\gamma^{-1}\circ s_\gamma\circ d_\gamma\in W(G,A_\gamma),$$

and

$$(6.69\,\mathrm{b})\qquad d_\gamma^{-1}\circ s_\gamma\circ d_\gamma\notin W(G,A_\gamma)$$

($s_\gamma\in W_{\mathbb{C}}$ is the reflection about γ). I now suppose that $\gamma\perp\beta$. In terms of the constants $n'_\gamma(w)$ of (6.60), I define

(6.70) $n''_\gamma(w)=n'_\gamma(w)$ in the situation (6.69a), and $n''_\gamma(w)=\frac{1}{2}(n'_\gamma(w)-n'_\gamma(s_\gamma w))$ in the situation (6.69b).

(6.71) **Lemma.** *Let $\gamma\in\Psi$ be a simple root, let $u_1,\ldots,u_M$ be an enumeration of the set*

$$\{w\in W_{\mathbb{C}}|w\gamma\perp\beta;\ w^{-1}\alpha\in\Psi \text{ whenever } \alpha\in\Psi,\alpha\perp\beta\},$$

and let $\gamma_i = u_i \gamma$. *Then, for every* $w \in W_{\mathbb{C}}$,

$$n_\gamma(w) = \# M_\beta / M_\beta^\dagger \cdot \sum_{i=1}^{M} n''_{\gamma_i}(u_i w).$$

Proof. The group $\{w \in W_{\mathbb{C}} | w\beta = \beta\}$ operates on the set of roots which are perpendicular to β and $W_{\mathbb{C}}$-conjugate to γ. I choose a complete set of representatives $\eta_1, \ldots, \eta_k$ of the orbits. For each η_i, there exists a unique $w_i \in W_{\mathbb{C}}$ such that

(6.72) $$w_i \gamma = \eta_i, \text{ and } w_i \alpha \in \Psi \text{ whenever } \alpha \in \Psi, \alpha \perp \gamma.$$

Indeed, the Weyl group of the root system

$$\{\alpha \in \Phi(G, H) | \alpha \perp \gamma\},$$

namely the isotropy subgroup of $W_{\mathbb{C}}$ at γ, operates simply transitively on the set of Weyl chambers of this root system. Hence the existence and uniqueness of w_i.

Every Cartan subgroup of G, with one dimensional toroidal part, is conjugate to one of the form A_η, for some $\eta \in \Phi(G, H)$. Two such Cartan subgroups A_η, A_ζ are conjugate precisely when η and ζ lie in the same $W_{\mathbb{C}}$-orbit. These statements follow directly from Kostant's and Sugiura's classification [24, 38]. The analogous assertion about $M_\beta \cdot B_{\beta,-}$ is also correct. Each η_i is perpendicular to β, so that $\mathrm{Ad}\, g^{-1}(A_{\eta_i})$ lies in $M_\beta \cdot B_{\beta,-}$ (cf. (6.56); $g \in G$ has the same meaning as in (6.55)). Thus: every Cartan subgroup of $M_\beta \cdot B_{\beta,-}$ which is G-conjugate to A_γ, is M_β-conjugate to exactly one among the $\mathrm{Ad}\, g^{-1}(A_{\eta_i})$, $1 \leqq i \leqq k$. In the notation of (5.2), with $B = A_\gamma$, the integer n is now equal to k.

Via d, $w_i \in W_{\mathbb{C}}$ corresponds to an element of $W(G, A)$, which can be represented as $\mathrm{Ad}\, h_i$, for some $h_i \in G$. If h_i is suitably chosen,

$$\mathrm{Ad}\, h_i \colon A_\gamma \xrightarrow{\sim} A_{\eta_i}.$$

On $\mathfrak{a}_\gamma$, one then obtains the identity

(6.73) $$d_{\eta_i} \circ \mathrm{Ad}\, h_i = w_i \circ d_\gamma.$$

Because of the conditions (6.72),

(6.74) $$\mathrm{Ad}\, h_i^{-1}(C'_{\eta_i}) = \{a \in A_\gamma^0 | e^{d_\gamma^* \alpha}(a) > 1 \text{ if } \alpha \in \Psi, \alpha \perp \gamma, \alpha \perp w_i^{-1}\beta\}.$$

In Lemma (5.2), with A_γ playing the role of B, I may set $g_i = h_i^{-1} g$, with g as in (6.55). According to (6.60) and (6.72), and because $w_i^{-1}\alpha \in \Psi$ whenever $\alpha \in \Psi$, $\alpha \perp \eta_i$,

(6.75) $$\begin{aligned} &\psi \circ \mathrm{Ad}\,(h_i^{-1} g)^{-1}|_{\mathrm{Ad}\, h_i^{-1}(C'_{\eta_i})} \\ &= \left(\textstyle\prod_{\alpha \in \Psi, \alpha \perp \beta, (\alpha, \eta_i) \neq 0} \mathrm{sgn}_\Psi(w_i^{-1}\alpha)\right) \\ &\quad \cdot \left(\textstyle\prod_{\alpha \in \Psi, \alpha \perp w_i^{-1}\beta} (e^{d_\gamma^* \alpha/2} - e^{-d_\gamma^* \alpha/2})\right)^{-1} \cdot \textstyle\sum_{w \in W_{\mathbb{C}}} n'_{\eta_i}(w_i w)\, e^{d_\gamma^* w \lambda}; \end{aligned}$$

here, as before, $\mathrm{sgn}_\Psi \alpha = 1$ or -1, depending on whether $\alpha \in \Psi$ or $-\alpha \in \Psi$. Under d_γ, the Weyl group $W(G, A_\gamma)$ corresponds to a subgroup of $W_{\mathbb{C}}$. Since

$$\Phi(\mathfrak{a}_\gamma)_- = \{d_\gamma^* \alpha | \alpha \in \Phi(G, H), \alpha \perp \gamma\}$$

(cf. (2.21)),

(6.76)
$$d_\gamma \circ W(G, A_\gamma) \circ d_\gamma^{-1} = \{w \in W_{\mathbb{C}} | w\gamma = \gamma\}$$
$$d_\gamma \circ W(G, A_\gamma) \circ d_\gamma^{-1} = \{w \in W_{\mathbb{C}} | w\gamma = \pm\gamma\},$$

depending on whether (6.69a) or (6.69b) holds.

For every $w \in W(G, A_\gamma)$, there exists a unique element w' of the group

(6.77)
$$\{d_\gamma^{-1} \circ v \circ d_\gamma | v \in W_{\mathbb{C}},\ v\gamma = \gamma,\ v w_i^{-1}\beta = w_i^{-1}\beta\},$$

such that $wC_\gamma \subset w' \circ \mathrm{Ad}\, h_i^{-1}(C'_{\eta_i})$; moreover, the restriction of $\psi \circ \mathrm{Ad}(g^{-1}h_i)$ to A_γ is invariant under the group (6.77). Hence, in (5.2), instead of summing over $W(G, A_\gamma)$, one may sum over the set

(6.78)
$$\{w \in W(G, A_\gamma) | wC_\gamma \subset \mathrm{Ad}\, h_i^{-1}(C'_{\eta_i})\},$$

provided one multiplies the sum by the order of the group (6.77).

For each η_i, I enumerate the elements of the set

$$\{w \in W_{\mathbb{C}} | w\gamma = \gamma;\ w^{-1}\alpha \in \Psi \text{ if } \alpha \in \Psi,\ \alpha \perp \gamma,\ \alpha \perp w_i^{-1}\beta\}$$

as $\{v_{i,j} | 1 \leqq j \leqq N_i\}$. Via d_γ, the set (6.78) corresponds to a subset of $W_{\mathbb{C}}$, namely

$\{v_{i,j} | 1 \leqq j \leqq N_i\}$ in the situation (6.69a), and

$\{v_{i,j},\ s_\gamma v_{i,j} | 1 \leqq j \leqq N_i\}$ in the situation (6.69b);

this follows from (6.74) and (6.76). For every $\alpha \in \Phi(G, H)$, the two roots $d_\gamma^*\alpha$ and $d_\gamma^* s_\gamma \alpha$ in $\Phi(G, A)$ are complex conjugates. Also, since γ is simple, α and $s_\gamma\alpha$ have the same sign with respect to Ψ, unless $\alpha = \pm\gamma$. Hence, if $w \in W(G, A_\gamma)$ is equal to $d_\gamma^{-1} \circ v_{i,j} \circ d_\gamma$ or $d_\gamma^{-1} \circ s_\gamma v_{i,j} \circ d_\gamma$,

$$\left|\prod\nolimits_{\alpha \in \Phi(G, A_\gamma),\ \langle \alpha, \mathrm{Ad}(h_i^{-1}g)(Y_\beta + Y_{-\beta})\rangle > 0} (e^{\alpha/2} - e^{-\alpha/2}) \circ w\right| |_{C_\gamma}$$
$$= \prod\nolimits_{\alpha \in \Psi,\ (\alpha, v_{i,j}^{-1} w_i^{-1}\beta) \neq 0} (e^{d_\gamma^*\alpha/2} - e^{-d_\gamma^*\alpha/2})|_{C_\gamma}$$

(note: $\gamma \perp w_i^{-1}\beta$!). Again if $w = d_\gamma^{-1} \circ v_{i,j} \circ d_\gamma$,

$$\prod\nolimits_{\alpha \in \Psi,\ \alpha \perp w_i^{-1}\beta} (e^{d_\gamma^*\alpha/2} - e^{-d_\gamma^*\alpha/2}) \circ w$$
$$= \left(\prod\nolimits_{\alpha \in \Psi,\ \alpha \perp w_i^{-1}\beta,\ (\alpha, \gamma) \neq 0} \mathrm{sgn}_\Psi(v_{i,j}^{-1}\alpha)\right) \cdot \prod\nolimits_{\alpha \in \Psi,\ \alpha \perp v_{i,j}^{-1} w_i^{-1}\beta} (e^{d_\gamma^*\alpha/2} - e^{-d_\gamma^*\alpha/2})$$

(note: $v_{i,j}^{-1}\alpha \in \Psi$ if $\alpha \in \Psi$, $\alpha \perp w_i^{-1}\beta$, $\alpha \perp \gamma$). Because γ is simple, if $w = d_\gamma^{-1} \circ s_\gamma v_{i,j} \circ d_\gamma$, the formula remains correct, except for a change in sign. In view of (6.76), the order of the group

$$\{w \in W(G, A_\gamma) | w \circ \mathrm{Ad}(h_i^{-1}g)(Y_\beta + Y_{-\beta}) = \mathrm{Ad}(h_i^{-1}g)(Y_\beta + Y_{-\beta})\}$$

is equal to the order of the group (6.77), or to twice the order of this group, depending on whether (6.69a) or (6.69b) holds. According to (6.75) and the preceeding remarks, an application of (5.2) shows that

(6.79)
$$n_\gamma(w) = \# M_\beta / M_\beta^\dagger \cdot \sum\nolimits_{i,j} \left(\prod\nolimits_{\alpha \in \Psi,\ \alpha \perp \beta,\ (\alpha, \eta_i) \neq 0} \mathrm{sgn}_\Psi(w_i^{-1})\right)$$
$$\cdot \left(\prod\nolimits_{\alpha \in \Psi,\ \alpha \perp w_i^{-1}\beta,\ (\alpha, \gamma) \neq 0} \mathrm{sgn}_\Psi(v_{i,j}^{-1}\alpha)\right) \cdot n'_{\eta_i}(w_i v_{i,j} w),$$

provided (6.69a) holds. In the situation (6.69b), the last factor $n'_{\eta_i}(\ldots)$ must be replaced by

$$\tfrac{1}{2}\left(n'_{\eta_i}(w_i v_{i,j} w) - n'_{\eta_i}(w_i s_\gamma v_{i,j} w)\right) = n''_{\eta_i}(w_i v_{i,j} w)$$

($w_i s_\gamma w_i^{-1} = s_{\eta_i}$; also, since A_γ and A_{η_i} are conjugate, if (6.69 b) holds, the analogous statement holds for A_{η_i}). Thus, if n'_{η_i} is replaced by n''_{η_i}, (6.79) becomes correct in both situations.

The two products of the sign factors in (6.79) can be combined into

(6.80) $$\prod_{\alpha\in\Psi,\ \alpha\perp\beta,\ (\alpha,\eta_i)\neq 0} \operatorname{sgn}_\Psi(v_{i,j}^{-1} w_i^{-1}\alpha).$$

I shall now assume, as I may, that the η_i have been chosen subject to the condition

(6.81) each η_i is simple, relative to the positive root system which Ψ induces in $\{\alpha\in\Phi(G,H)|\alpha\perp\beta\}$.

In this situation, s_{η_i} establishes a bijection between the two sets

$$\{\alpha\in\Psi|\alpha\perp\beta, (\alpha,\eta_i)<0\},$$

and

$$\{\alpha\in\Psi|\alpha\perp\beta, (\alpha,\eta_i)>0, \alpha\neq\eta_i\}.$$

Moreover, for any α in the first of these two sets,

$$\operatorname{sgn}_\Psi(v_{i,j}^{-1} w_i^{-1} s_{\eta_i}\alpha) = \operatorname{sgn}_\Psi(s_\gamma v_{i,j}^{-1} w_i^{-1}\alpha) = \operatorname{sgn}_\Psi(v_{i,j}^{-1} w_i^{-1}\alpha),$$

because γ is simple. Hence the product (6.80) is equal to one. For $1\leqq i\leqq k$, $1\leqq j\leqq N_i$, I set

$$w_{i,j} = w_i v_{i,j}.$$

Then, as follows from the preceeding remarks,

(6.82) $$n_\gamma(w) = \# M_\beta/M_\beta^\dagger \cdot \sum_{i=1}^{k} \sum_{j=1}^{N_i} n''_{\eta_i}(w_{i,j} w).$$

Now let η, ζ be two conjugate roots in $\{\alpha\in\Phi(G,H)|\alpha\perp\beta\}$ which are simple with respect to the system of positive roots cut out by Ψ. There exists a unique

$$v\in\{w\in W_{\mathbb{C}}|w\beta=\beta\},$$

such that $v\eta=\zeta$, and such that $v\alpha\in\Psi$ whenever $\alpha\in\Psi$, $\alpha\perp\beta$, $\alpha\perp\eta$. Via d, v corresponds to an element of $W(G,A)$, which can be realized as $\operatorname{Ad} h$, with $h\in G$. For a suitable choice of h, one can arrange that

$$\operatorname{Ad} h\colon\ A_\eta \xrightarrow{\sim} A_\zeta,$$

and that

(6.83) $$d_\eta\circ\operatorname{Ad} h = v\circ d_\zeta.$$

Because of the particular choice of v, $\operatorname{Ad} h$ maps C'_η to C'_ζ. As can be read off from (6.60) and (6.83),

(6.84) $$n'_\eta(w) = \Big(\prod_{\alpha\in\Psi,\ \alpha\perp\beta,\ (\alpha,\zeta)\neq 0} \operatorname{sgn}_\Psi(v^{-1})\Big)\cdot n'_\zeta(v w).$$

The same argument which shows that the expression (6.80) equals one can be applied to the product appearing in (6.84). Thus $n'_\eta(w) = n'_\zeta(v w)$, which implies the identity

(6.85) $$n''_\eta(w) = n''_\zeta(v w).$$

To complete the proof of the lemma, I shall construct a bijective mapping

(6.86) $$\tau\colon \{w_{i,j} \mid 1 \leqq i \leqq k,\ 1 \leqq j \leqq N_i\} \to \{u_l \mid 1 \leqq l \leqq M\},$$

such that $n''_{\gamma_l}(u_l w) = n''_{\eta_i}(w_{i,j} w)$, whenever $u_l = \tau(w_{i,j})$. The lemma will then follow directly from (6.82). I consider a particular $w_{i,j}$. For reasons which have been explained before, there exists a unique $\tilde{w}_{i,j} \in W_{\mathbb{C}}$, such that

(6.87) $$\tilde{w}_{i,j}\beta = \beta, \quad \text{and} \quad (\tilde{w}_{i,j} w_{i,j})^{-1}\alpha \in \Psi \quad \text{whenever } \alpha \in \Psi,\ \alpha \perp \beta.$$

I claim that $\tilde{w}_{i,j} w_{i,j}$ is one of the u_l. Indeed, because $\eta_i \perp \beta$ and $\tilde{w}_{i,j}\beta = \beta$,

$$\tilde{w}_{i,j} w_{i,j} \gamma = \tilde{w}_{i,j} w_i v_{i,j} \gamma = \tilde{w}_{i,j} \eta_i \perp \beta;$$

also, $(\tilde{w}_{i,j} w_{i,j})^{-1}$ maps $\{\alpha \in \Psi \mid \alpha \perp \beta\}$ into Ψ. Hence $\tilde{w}_{i,j} w_{i,j} = u_l$, for some l, $1 \leqq l \leqq M$. I define

(6.88) $$\tau(w_{i,j}) = u_l.$$

Because of the defining properties of u_l, $\gamma_l = u_l \gamma$ is a simple root, relative to the system of positive roots which Ψ determines in

$$\{\alpha \in \Phi(G, H) \mid \alpha \perp \beta\}.$$

According to (6.81), η_i is simple in the same sense. I want to show that $\tilde{w}_{i,j}$ can be used as v in (6.85), with $\eta = \eta_i$ and $\zeta = \gamma_l$. First of all,

$$\tilde{w}_{i,j}\eta_i = \tilde{w}_{i,j} w_{i,j}\gamma = u_l \gamma = \gamma_l.$$

Secondly, if $\alpha \in \Psi$, $\alpha \perp \beta$, $\alpha \perp \eta_i$, one must have $\tilde{w}_{i,j}\alpha \in \Psi$, for otherwise

$$\begin{aligned} & -\tilde{w}_{i,j}\alpha \in \Psi, \quad \alpha \perp \beta, \quad \alpha \perp \eta_i \\ \Rightarrow\ & -w_{i,j}^{-1}\alpha \in \Psi, \quad \alpha \perp \beta, \quad \alpha \perp \eta_i \\ \Rightarrow\ & -v_{i,j}^{-1} w_i^{-1}\alpha \in \Psi, \quad w_i^{-1}\alpha \perp w_i^{-1}\beta, \quad w_i^{-1}\alpha \perp \gamma \\ \Rightarrow\ & -w_i^{-1}\alpha \in \Psi, \quad w_i^{-1}\alpha \perp \gamma \ \Rightarrow\ -\alpha \in \Psi, \end{aligned}$$

which is a contradiction. The first implication holds because of (6.87), the second because of the definition of $w_{i,j}$, the third because of the properties of $v_{i,j}$, and the fourth because of the properties of w_i. This proves:

(6.89) $$n''_{\gamma_l}(u_l w) = n''_{\eta_i}(w_{i,j} w), \quad \text{if } u_l = \tau(w_{i,j}).$$

It remains to be shown that τ is bijective. If $u_l = \tau(w_{i,j})$, η_i is conjugate to γ_l under

(6.90) $$\{w \in W_{\mathbb{C}} \mid w\beta = \beta\}.$$

Hence $\tau(w_{i,j})$ uniquely determines the index i. On the other hand, if $u_l = \tau(w_{i,j})$, $w_{i,j}^{-1}$ maps β to $u_l^{-1}\beta$, η_i to γ, and

$$\{\alpha \in \Psi \mid \alpha \perp \beta,\ \alpha \perp \eta_i\}$$

into Ψ, as follows from the properties of w_i and $v_{i,j}$. These conditions describe $w_{i,j}$ uniquely. In other words, τ is injective. Now let u_l be given. Under the action of the group (6.90), γ_l is conjugate to η_i for some i. Hence there exists $\tilde{w} \in W_{\mathbb{C}}$, such that

$$\tilde{w}\beta = \beta; \quad \tilde{w}\eta_i = \gamma_l; \quad \tilde{w}\alpha \in \Psi \quad \text{if } \alpha \in \Psi, \quad \alpha \perp \beta, \quad \alpha \perp \eta_i.$$

In terms of $\tilde{w}$ and the index i, I define

$$v = w_i^{-1} \tilde{w}^{-1} u_l.$$

Then $v\gamma = \gamma$. Because of the properties of u_l, $\tilde{w}$, and w_i, $v^{-1}\alpha \in \Psi$ whenever $\alpha \in \Psi$, $\alpha \perp \gamma$, $\alpha \perp w_i^{-1}\beta$; thus v is one of the $v_{i,j}$. Since $u_l = \tilde{w} w_{i,j}$, one checks easily that $\tilde{w} = \tilde{w}_{i,j}$, and that $u_l = \tau(w_{i,j})$. Hence τ is bijective. The assertion of the lemma now follows from (6.82) and (6.89).

(6.91) **Lemma.** *If $\gamma \in \Phi(G, H)$ is simple with respect to Ψ, the constants $n(w)$ and $n_\gamma(w)$ of* (6.64) *satisfy the identity*

$$n(w) - n(s_\gamma w) = n_\gamma(w) - n_\gamma(s_\gamma w).$$

Proof. I enumerate the elements of the set

$$\{w \in W_{\mathbb{C}} \mid w^{-1}\alpha \in \Psi \text{ if } \alpha \in \Psi, \alpha \perp \beta\}$$

as $u_i, \ldots, u_N$, in a manner such that

$$u_i\gamma \perp \beta \quad \text{if and only if } 1 \leqq i \leqq M,$$

for some $M \leqq N$. The notation is then consistent with that of (6.65) and (6.67). Since γ is simple, if $u_i\gamma$ is not perpendicular to β, $u_i s_\gamma$ is again one of the u_i. Hence

(6.92) $$u_i \mapsto u_i s_\gamma \quad \text{establishes a bijection of } \{u_i \mid M < i \leqq N\}.$$

As follows from the defining property of the u_i, if $i \leqq M$, $\gamma_i = u_i\gamma$ is simple with respect to the system of positive roots which Ψ induces in

$$\{\alpha \in \Phi(G, H) \mid \alpha \perp \beta\}.$$

Hence, according to the hypothesis (6.62),

$$n'(w) - n'(s_{\gamma_i} w) = n'_{\gamma_i}(w) - n'_{\gamma_i}(s_{\gamma_i} w).$$

In view of the definition of the n''_{γ_i}, this implies

(6.93) $$n'(w) - n'(s_{\gamma_i} w) = n''_{\gamma_i}(w) - n''_{\gamma_i}(s_{\gamma_i} w),$$

provided $i \leqq M$. Hence

$$\begin{aligned}
&n(w) - n(s_\gamma w)\\
&\quad = \# M_\beta/M_\beta^\dagger \cdot \textstyle\sum_{i=1}^N \left(n'(u_i w) - n'(u_i s_\gamma w)\right) \qquad \text{(by (6.65))}\\
&\quad = \# M_\beta/M_\beta^\dagger \cdot \textstyle\sum_{i=1}^M \left(n'(u_i w) - n'(u_i s_\gamma w)\right) \qquad \text{(by (6.92))}\\
&\quad = \# M_\beta/M_\beta^\dagger \cdot \textstyle\sum_{i=1}^M \left(n'(u_i w) - n'(s_{\gamma_i} u_i w)\right) \qquad (u_i\gamma = \gamma_i)\\
&\quad = \# M_\beta/M_\beta^\dagger \cdot \textstyle\sum_{i=1}^M \left(n''_{\gamma_i}(u_i w) - n''_{\gamma_i}(s_{\gamma_i} u_i w)\right) \qquad \text{(by (6.93))}\\
&\quad = \# M_\beta/M_\beta^\dagger \cdot \textstyle\sum_{i=1}^M \left(n''_{\gamma_i}(u_i w) - n''_{\gamma_i}(u_i s_\gamma w)\right) \qquad (u_i\gamma = \gamma_i)\\
&\quad = n_\gamma(w) - n_\gamma(s_\gamma w) \qquad \text{(by (6.71))}.
\end{aligned}$$

This is the desired identity.

In view of (6.53), (6.63), and (6.91), Lemma (6.30) has been proven. This now completes the proofs of the main theorems.

§ 7. Blattner's Conjecture

With the help of the results stated in §4, it becomes a relatively simple matter to prove Blattner's conjecture for those linear, semisimple Lie groups, whose quotient by a maximal compact subgroup has a Hermitian symmetric structure. I begin with some general comments about the conjecture.

I consider a connected, semisimple Lie group G, with maximal compact subgroup K, such that K has the same rank as G. The set of equivalence classes of finite dimensional, irreducible K-modules will be denoted by $\hat{K}$. To each $j\in\hat{K}$, one can associate the character χ_j. Now let π be an irreducible, unitary representation of G. It is known that the restriction of π to K breaks up discretely, with finite multiplicities; moreover, the multiplicity of any given $j\in\hat{K}$ is bounded by the degree of j [7]. I set

$$n_j(\pi)=\text{multiplicity of } j.$$

Each χ_j can be viewed as a distribution on K, via integration against Haar measure. As follows from the bound on the $n_j(\pi)$, the series

$$\tau_\pi=\sum_{j\in\hat{K}} n_j(\pi)\,\chi_j \tag{7.1}$$

converges to a distribution on K [10]. I shall call τ_π the K-character of π. Harish-Chandra has also shown that τ_π, on K intersected with the regular set in G, is a real analytic function. Moreover, on this subset of K, τ_π, viewed as a function, agrees with the global character of π [10].

Because of the assumptions on the ranks, K contains a compact Cartan subgroup H of G. According to Harish-Chandra's enumeration of the representations of the discrete series, corresponding to each admissible $\lambda\in i\mathfrak{h}^*$ (cf. (4.11)), such that $(\lambda,\alpha)\neq 0$ for all $\alpha\in\Phi=\Phi(G,H)$, there exists a unique tempered, invariant eigendistribution Θ_λ on G, whose restriction to H, intersected with the regular set in G, is given by

$$\Theta_\lambda|_H=(-1)^q\Big(\prod_{\alpha\in\Phi,(\alpha,\lambda)>0}(e^{\alpha/2}-e^{-\alpha/2})\Big)^{-1}\sum_{w\in W(G,H)}\varepsilon(w)\,e^{w\lambda} \tag{7.2}$$

($q=\frac{1}{2}\dim G/K$; $\varepsilon(w)=$ sign of w). Moreover, Θ_λ is the character of a discrete series representation, and every discrete series character arises in this fashion [13]. In terms of λ, one can define a system of positive roots Ψ_λ in Φ:

$$\Psi_\lambda=\{\alpha\in\Phi(G,H)\,|\,(\lambda,\alpha)>0\}. \tag{7.3}$$

As in §2, I let Φ^c and Φ^n denote the sets of, respectively, compact and noncompact roots in Φ. I enumerate the roots in $\Phi^n\cap\Psi_\lambda$ as $\beta_1,\ldots,\beta_q$.

(7.4) **Blattner's Conjecture.** *Let π be a discrete series representation, with character Θ_λ, as in* (7.2). *Then*

$$\tau_\pi=\sum_{0\leq n_1,\ldots,n_q<\infty}\Big(\prod_{\alpha\in\Phi^c\cap\Psi_\lambda}(e^{\alpha/2}-e^{-\alpha/2})\Big)^{-1}\cdot\sum_{w\in W(G,H)}\varepsilon(w)\exp\Big[w\Big(\lambda+\sum_{i=1}^{q}(n_i+\tfrac{1}{2})\beta_i\Big)\Big]$$

is the K-character of π.

Remark. The formula is merely symbolic, in the following sense: formally, each summand is defined only on H; however, in view of Weyl's character formula, the summands extend to class functions on all of K, and it is these class functions on K which must be summed.

On the intersection of K with the regular set in G, the K-character which the conjecture predicts can be summed explicitly, and it agrees there with Θ_λ, as it must. This observation originally motivated the conjecture. In a formal sense, the conjecture is analogous to Kostant's formula for the multiplicity of a weight [25]. In order to see this, one should consider the partition function Q on $i\mathfrak{h}^*$, which is defined as follows: for each $\mu \in i\mathfrak{h}^*$, $Q(\mu)$ is the number of distinct ways in which μ can be expresses as a sum

$$\mu = n_1\,\beta_1 + n_2\,\beta_2 + \cdots + n_q\,\beta_q,$$

with nonnegative, integral coefficients n_i. Since Ψ spans a cone in $i\mathfrak{h}^*$, lying entirely in a half space, $Q(\mu)$ is well defined. I set

$$\rho_c = \tfrac{1}{2}\textstyle\sum_{\alpha\in\Phi^c\cap\Psi_\lambda}\alpha;$$

the conjecture (7.4) then becomes equivalent to

(7.4′) **Blattner's Conjecture.** *Let $\mu \in i\mathfrak{h}^*$ be a weight for K, which is dominant with respect to the system of positive roots $\Phi^c \cap \Psi_\lambda$ in Φ^c, and let π be a discrete series representation with character Θ_λ. In $\pi|_K$, the irreducible K-module of highest weight μ occurs with multiplicity*

$$\textstyle\sum_{w\in W(G,H)} \varepsilon(w)\, Q\big(w(\mu+\rho_c) - \lambda - \tfrac{1}{2}(\beta_1 + \cdots + \beta_q)\big).$$

The conjecture is usually stated as in (7.4′), although (7.4) seems more suggestive. One feature of the conjecture deserves special mention. According to (7.4′), the irreducible K-module with highest weight

$$\mu = \lambda + \tfrac{1}{2}\textstyle\sum_i \beta_i - \rho_c$$

occurs with multiplicity one in $\pi|_K$, and no irreducible consituent of $\pi|_K$ has highest weight $\mu - \beta_i$, $1 \leqq i \leqq q$. If λ is "sufficiently nonsingular", this property characterizes the discrete series representation π up to infinitesimal equivalence, among all irreducible, quasisimple representations of G (cf. Theorem 2 of [36] and Theorem II of [22]). It seems likely that the preceding statement holds for any nonsingular λ. It would then be a formal analogue of the "theorem of the highest weight" for finite dimensional representations.

The following theorem constitutes the main result of the present section:

(7.5) **Theorem.** *Blattner's conjecture holds for all connected, semisimple matrix groups G, whose quotient by a maximal compact subgroup admits a Hermitian symmetric structure.*

The previous status of Blattner's conjecture was already discussed in the introduction. In order to apply the results of §4 to the proof of (7.5), one must define K-multiplicities for all of the invariant eigendistributions $\Theta(\Psi, \lambda)$ occurring in Theorem (4.15). Actually, one can do so even for an arbitrary invariant eigendistribution, but this degree of generality is unnecessary. For the remainder of the present section, the phrase "irreducible representation" shall always refer to a quasi-simple, irreducible representation on a Hilbert space; two irreducible representations will be identified if they are infinitesimally equivalent. By a virtual representation, I shall mean a finite, formal linear combination of irreducible representations, with integral coefficients. In the obvious manner, each virtual representation of G has a global character and a K-character. Because the global characters of distinct irreducible representations are linearly independent [7], a virtual representation is fully determined by its global character.

(7.6) **Lemma.** *The invariant eigendistributions $\Theta(\Psi,\lambda)$ of Theorem* (4.15) *are characters of virtual representations.*

Proof. Every system of positive roots Ψ can be connected to a Ψ_0 with the property (3.1), by means of a chain $\Psi_0, \Psi_1, \ldots, \Psi_m = \Psi$, such that each Ψ_{i+1} is obtained from Ψ_i by reflection about a simple, noncompact root. It therefore suffices to show that the $\Theta(\Psi,\lambda)$ of Theorem (3.11), as well as the induced invariant eigendistributions Θ of (4.15c), are characters of virtual representations. The character of a quasi-simple, but not necessarily irreducible, representation on a Hilbert space is also the character of a virtual representation, namely of the sum of the irreducible quotients in a composition series. By construction, the $\Theta(\Psi,\lambda)$ of Theorem (3.11) are characters of quasi-simple representations, which can be realized, up to infinitesimal equivalence, on a Hilbert space [9, 16]. If one induces an irreducible representation from a parabolic subgroup, one obtains a quasi-simple representation. Hence, by induction on the dimension of G, one finds that the Θ's occurring in (4.15c) are characters of virtual representations. This completes the proof.

In view of the lemma, it makes sense to talk of the K-multiplicities for each of the $\Theta(\Psi,\lambda)$. In the obvious manner, one can extend Blattner's conjecture to cover all of the $\Theta(\Psi,\lambda)$. I shall prove this extended version of the conjecture, which then clearly implies Theorem (7.5). Thus let G be a group satisfying the hypotheses of Theorem (4.15), Ψ a system of positive roots in Φ, and $\lambda \in i\mathfrak{h}^*$ an admissible linear functional, such that condition (*) in (4.15) is satisfied. According to the preceding lemma, $\Theta(\Psi,\lambda)$ is the character of a virtual representation π, whose K-character I denote by τ_π. Let $\beta_1, \ldots, \beta_q$ be the roots in $\Psi \cap \Phi^n$.

(7.7) **Proposition.** *Under the hypotheses stated above,*

$$\tau_\pi = \sum_{0 \leqq n_1, \ldots, n_q < \infty} \Big(\prod_{\alpha \in \Phi^c \cap \Psi} (e^{\alpha/2} - e^{-\alpha/2})\Big)^{-1} \cdot \sum_{w \in W(G,H)} \varepsilon(w) \exp\big[w(\lambda + \textstyle\sum_{i=1}^{q} (n_i + \frac{1}{2})\beta_i)\big].$$

The remark below (7.4) also applies to the statement (7.7). The proof of the proposition, which proceeds by induction on the dimension of G, depends on two lemmas.

(7.8) **Lemma.** *The statement of Proposition* (7.7) *holds, provided the system of positive roots Ψ has the property* (3.1).

Proof. When Ψ satisfies condition (3.1), the Weyl group $W(G,H)$ permutes the positive, noncompact roots among each other. In this situation, the formula for τ_π in (7.7) can be rewritten as follows:

$$\tau_\pi = \Big(\prod_{\alpha \in \Phi^c \cap \Psi} (e^{\alpha/2} - e^{-\alpha/2})\Big)^{-1} \cdot \sum_{w \in W(G,H)} \varepsilon(w)\, e^{w(\lambda + \rho_n)} \cdot \sum_{0 \leqq n_1, \ldots, n_q < \infty} \exp\Big(\sum_{i=1}^{q} n_i \beta_i\Big)$$

(ρ_n = one half of the sum of the positive, noncompact roots). The product of the first two factors is the character of the irreducible K-module V_μ, with highest weight

$$\mu = \lambda - \rho_c + \rho_n$$

(cf. (3.7)), whereas the final factor is the (formal) character of the symmetric algebra of $\mathfrak{p}_+$ (cf. (3.2)). The assertion of the lemma is therefore equivalent to saying that the Frechét G-module F_λ, which was described in § 3, breaks up under K like the irreducible K-module V_μ, tensored with the symmetric algebra of $\mathfrak{p}_+$. This fact is proven in Harish-Chandra's original construction of the "holomorphic discrete series" [9]. I shall briefly sketch the argument. Via the Harish-Chandra embedding, G/K can be realized as a bounded domain in $\mathfrak{p}_-$. Let S be the Shilov boundary of G/K in $\mathfrak{p}_-$, and $H^2(S)$ the space of square integrable functions on S, which are boundary values of holomorphic functions on G/K. As is shown in [16], F_λ is infinitesimally equivalent to a representation on the Hilbert space $V_\mu \otimes H^2(S)$, in such a manner that the natural actions of K on the various spaces are preserved. The polynomial functions are dense in $H^2(S)$. Hence $H^2(S)$ breaks up under the action of K in the same way as the algebra of polynomial functions on $\mathfrak{p}_-$. Since $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are mutually dual, the polynomial algebra is K-isomorphic to the symmetric algebra of $\mathfrak{p}_+$. This implies the formula for τ_π.

The next lemma states that Proposition (7.7) is compatible with (4.15c). To be more precise, I consider a particular system of positive roots Ψ in $\Phi = \Phi(G, H)$, and a noncompact root β, which is simple with respect to Ψ. I make the following inductive hypothesis: Proposition (7.7) is correct for the group M_β^0. An admissible $\lambda \in i\mathfrak{h}^*$ shall be given, subject to condition (*) of (4.15). In terms of Ψ, β, and λ, I define the invariant eigendistribution Θ, exactly as in (4.15c). According to the arguments in the proof of Lemma (7.6), Θ is the character of a virtual representation π. Finally, let $\beta_1, \ldots, \beta_q$ be an enumeration of the noncompact roots in Ψ, such that $\beta = \beta_1$.

(7.9) **Lemma.** *In the situation which was just described, the K-character of π is given by*

$$\tau_\pi = \sum_{n_1=-\infty}^{\infty} \sum_{0 \leqq n_2, \ldots, n_q < \infty} \Big(\prod_{\alpha \in \Phi^c \cap \Psi} (e^{\alpha/2} - e^{-\alpha/2})\Big)^{-1} \cdot \sum_{w \in W(G,H)} \varepsilon(w) \exp\Big[w\Big(\lambda + \sum_{i=1}^{q} (n_i + \tfrac{1}{2}) \beta_i\Big)\Big];$$

this formula should be interpreted in the light of the remark below (7.4).

Proof. According to a standard integration formula, for any integrable function f on K,

$$\begin{aligned}
&\int_K f(k)\, dk \\
&= (\# W(G, H))^{-1} \int_H \int_K \Big|\prod_{\alpha \in \Phi^c \cap \Psi} (e^{\alpha/2} - e^{-\alpha/2})(h)\Big|^2 f(khk^{-1})\, dk\, dh \\
&= (\# W(G, H))^{-1} \int_H \int_K \Big\{\prod_{\alpha \in \Phi^c \cap \Psi} (e^{\alpha/2} - e^{-\alpha/2})(e^{-\alpha/2} - e^{\alpha/2})(h)\Big\} f(khk^{-1})\, dk\, dh
\end{aligned}$$

(note: $W(G, H)$ is also the Weyl group of H in K); the Haar measures dh and dk are normalized to have total mass one. The denominator of Weyl's character formula for K,

$$\prod_{\alpha \in \Phi^c \cap \Psi} (e^{\alpha/2} - e^{-\alpha/2}),$$

is $W(G, H)$-alternating. Hence, in (7.9), if the denominator is pulled inside the summation over $W(G, H)$ and composed with w, the factor $\varepsilon(w)$ must be dropped. Also, for $f \in C^\infty(K)$, the function

$$h \mapsto \int_K f(khk^{-1})\,dk$$

on H is $W(G, H)$-symmetric. These observations allow one to rewrite the formula for the distribution τ_π, which appears in the statement of the lemma, as follows: for $f \in C^\infty(K)$,

$$(7.10)\quad \begin{aligned} \tau_\pi(f) = \textstyle\sum_{n_1=-\infty}^{\infty} \sum_{0 \leq n_2, \ldots, n_q < \infty} \int_K \int_H \left(\prod_{\alpha \in \Phi^c \cap \Psi} (1-e^\alpha)(h)\right) \\ \cdot \exp\left(\lambda + \rho - \textstyle\sum_{\alpha \in \Phi^c \cap \Psi} \alpha + \sum_{i=1}^{q} n_i \beta_i\right)(h)\, f(khk^{-1})\,dh\,dk. \end{aligned}$$

The invariant eigendistribution φ_0 on M_β^0, which enters the construction of Θ, is the character of a virtual representation π_0 of M_β^0. Since the Cartan involution preserves M_β^0, $K \cap M_\beta^0$ is maximal compact in M_β. I denote the $K \cap M_\beta^0$-character of π_0 by σ_0; it is a distribution on $K \cap M_\beta^0$. According to (2.56), the group F_β is either trivial or of order two and is contained in H. I let d denote the nontrivial element of F_β, if there is one; otherwise, I set $d=1$. The virtual representation π_0 can be extended to a virtual representation π_1 of $M_\beta^\dagger$, by "letting d operate as multiplication by $\zeta_\lambda(d)$". The character of π_1 is then φ_1, as defined in (4.8). Since $K \cap M_\beta^\dagger$ consists of the union of $K \cap M_\beta^0$ and its translate by d, the $K \cap M_\beta^\dagger$-character σ_1 of π_1 can be easily computed in terms of σ_0:

$$(7.11)\quad \sigma_1(f) = \tfrac{1}{2}\left(\sigma_0(f|_{K \cap M_\beta^0}) + \zeta_\lambda(d)\,\sigma_0(f \circ l_d|_{K \cap M_\beta^0})\right), \qquad f \in C^\infty(K \cap M_\beta^\dagger);$$

here l_d denotes left translation by d. The factor $\frac{1}{2}$ insures that $K \cap M_\beta^\dagger$ is still assigned total Haar measure one. The passage from φ_1 to φ in (4.9) corresponds to inducing the virtual representation π_1 from $M_\beta^\dagger$ up to M_β. By extending the virtual representation in the appropriate manner to the parabolic subgroup (4.2), with $S = \{\beta\}$, and inducing up to G, one obtains the virtual representation π, whose character is Θ. Inducing a representation from the parabolic subgroup (4.2) to G and restricting it to K, amounts to first restricting the representation to K, intersected with the group (4.2), and then inducing it to K. As these remarks imply,

$$(7.12)\quad \tau_\pi(f) = \int_K \sigma_1(f \circ \mathrm{Ad}\,k|_{K \cap M_\beta^\dagger})\,dk,$$

for every $f \in C^\infty(K)$.

The root system $\Phi(M_\beta^0, B_{\beta,+}^0)$ can be naturally identified with

$$\Phi_\beta = \{\alpha \in \Phi \mid \alpha \perp \beta\}.$$

In Φ_β, Ψ induces the system of positive roots

$$\Psi_\beta = \Phi_\beta \cap \Psi.$$

It should be recalled that a root $\alpha \in \Phi_\beta$ may be compact with respect to G, and noncompact with respect to M_β^0, or vice versa; cf. (2.61). I enumerate the compact (relative to M_β^0) roots in Ψ_β as $\gamma_1, \ldots, \gamma_s$, and the noncompact ones as $\eta_1, \ldots, \eta_t$. In the language of Theorem (4.15), the invariant eigendistribution φ_0 on M_β^0 corresponds to Ψ_β and the restriction of λ to $\mathfrak{b}_{\beta,+}^0$; this restriction is admissible with respect to M_β^0. The induction hypothesis implies an explicit formula for the distribution σ_0 on $K \cap M^0_\beta$. If one applies the same reasoning which leads to (7.10), one finds that

$$\text{(7.13)}\quad \begin{aligned}\sigma_0(g) = &\sum_{0 \leq m_1, \ldots, m_t < \infty} \int_{K \cap M^0_\beta} \int_{B^0_{\beta,+}} \left(\prod_{i=1}^s (1 - e^{\gamma_i})(b)\right) \\ &\cdot \exp\left(\lambda - \tfrac{1}{2}\sum_{i=1}^s \gamma_i + \tfrac{1}{2}\sum_{j=1}^t \eta_j + \sum_{j=1}^t m_j \eta_j\right)(b)\, g(m b m^{-1})\, db\, dm,\end{aligned}$$

for every $g \in C^\infty(K \cap M^0_\beta)$. Here $e^{\lambda + \cdots}$ is viewed as a function on $B^0_{\beta,+}$, by restriction.

The triple Y_β, $Y_{-\beta}$, Z_β of (2.7) spans a subalgebra of $\mathfrak{g}^{\mathbb{C}}$, which is the complexification of a copy of $\mathfrak{sl}(2, \mathbb{R})$ in $\mathfrak{g}$. This copy of $\mathfrak{sl}(2, \mathbb{R})$ intersects $\mathfrak{h}$ in

$$\mathfrak{t} = i\mathbb{R} Z_\beta;$$

$\mathfrak{t}$ is the Lie algebra of a one dimensional subtorus $T \subset H$. Since $\mathfrak{h}$ is the direct sum of $\mathfrak{b}^0_{\beta,+}$ and $\mathfrak{t}$,

$$H = B^0_{\beta,+} \cdot T.$$

Let db and dt be the Haar measures on $B^0_{\beta,+}$ and T, respectively, normalized to have total mass one. Then, whether or not T and $B^0_{\beta,+}$ have a trivial intersection,

$$\text{(7.14)}\quad \int_H g(h)\, dh = \int_{B^0_{\beta,+}} \int_T g(bt)\, dt\, db, \qquad \text{for all } g \in C^\infty(H).$$

I define $\delta = 0$ or $\delta = \frac{1}{2}$, so that

$$\sum_{\alpha \in \Phi,\, (\alpha, \beta) > 0} (\alpha, \beta)(\beta, \beta)^{-1} \equiv 2\delta \bmod 2$$

(note: $\frac{1}{2}\sum_{\alpha \in \Phi,\, (\alpha,\beta) > 0} (\alpha, \beta)$ is the inner product of β with the half-sum of the positive roots, relative to a suitably chosen system of positive roots). Since λ is admissible, the restriction of $\lambda - \delta\beta$ to $\mathfrak{t}$ exponentiates to a well-defined character of T. To keep the notation simple, I shall refer to this character simply as $e^{\lambda - \delta\beta}$. As can be checked, the generator d of F_β lies in T. Hence, and in view of the definition of ζ_λ,

$$\text{(7.15)}\quad \zeta_\lambda(d) = e^{\lambda - \delta\beta}(d).$$

I now claim that

$$\text{(7.16)}\quad \tfrac{1}{2}\left(g(h) + \zeta_\lambda(d)\, g(hd)\right) = \sum_{n=-\infty}^{\infty} \int_H e^{\lambda + (n - \delta)\beta}(t)\, g(ht)\, dt,$$

for all $g \in C^\infty(H)$ and $h \in H$.

In order to verify (7.16), I distinguish the two cases $d = 1$ and $d \neq 1$. In the first case, e^β generates $\hat{T}$, the group of characters of T. Also, because of (7.15), $e^{\lambda - \delta\beta}$ is an integral power of e^β, so that $e^{\lambda + (n - \delta)\beta}$ runs over $\hat{T}$, as n runs over $\mathbb{Z}$. If d fails to be the identity, it must be the unique element of order two in T. In this situation, e^β is the square of a generator of $\hat{T}$. In view of (7.15), depending on whether $\zeta_\lambda(d) = 1$ or $\zeta_\lambda(d) = -1$, $e^{\lambda + (n - \delta)\beta}$ ranges over all even powers or over all odd powers of a generator of $\hat{T}$. In all cases, the identity (7.16) comes down to a simple identity about Fourier series in one variable.

I now combine (7.11), (7.12), (7.13), and (7.16). In (7.13), if g is the restriction to $K \cap M^0_\beta$ of an Ad K-invariant function, the integration over $K \cap M^0_\beta$ becomes superfluous. In view of (7.14), the integrations over T and $B^0_{\beta,+}$ can be written as a single integration over H. The characters e^{γ_i} and e^{η_j} of H restrict trivially to T, whereas e^{β} restricts trivially to $B^0_{\beta,+}$. It should also be remembered that d commutes with M^0_β. Hence, for any $f \in C^\infty(K)$,

$$\begin{aligned}\tau_\pi(f) = &\textstyle\sum_{0 \leqq m_1, \ldots, m_t < \infty} \sum_{n=-\infty}^{\infty} \int_K \int_H \left(\prod_{i=1}^s (1-e^{\gamma_i})(h)\right) \\ &\cdot \exp\left(\lambda - \tfrac{1}{2}\textstyle\sum_{i=1}^s \gamma_i + \tfrac{1}{2}\sum_{j=1}^t \eta_j + \delta\beta + \sum_{j=1}^t m_j \eta_j + n\beta\right)(h)\, f(khk^{-1})\, dh\, dk.\end{aligned} \tag{7.17}$$

It remains to be shown that Eqs. (7.10) and (7.17) define the same distribution τ_π. I shall reduce this problem to the verification of three statements. First of all,

(7.18) for every $g \in C^\infty(H)$, the two series

$$\textstyle\sum_{n_1=-\infty}^{\infty} \sum_{0 \leqq n_2, \ldots, n_q < \infty} \int_H \exp\left(\sum_i n_i \beta_i\right)(h)\, g(h)\, dh$$

and

$$\textstyle\sum_{0 \leqq m_1, \ldots, m_t < \infty} \sum_{n=-\infty}^{\infty} \int_H \exp\left(n\beta + \sum_j m_j \eta_j\right)(h)\, g(h)\, dh$$

converge absolutely.

In particular, then, the series appearing in (7.10) and (7.17) can be rearranged at will. Secondly,

(7.19) there exists an integer l, such that

$$\lambda - \rho - \textstyle\sum_{\alpha \in \Phi^c \cap \Psi} \alpha = \lambda - \tfrac{1}{2}\sum_{i=1}^s \gamma_i + \tfrac{1}{2}\sum_{j=1}^t \eta_j + \delta\beta + l\beta.$$

Finally, let $\mathbb{Z}[\hat{H}]$ be the character ring of H, and M the $\mathbb{Z}$-module of all formal, possibly infinite, integral linear combinations of characters of H. In a natural manner, M becomes a $\mathbb{Z}[\hat{H}]$-module. Then

(7.20) in M, one has the identity

$$\begin{aligned}&\textstyle\prod_{\alpha \in \Phi^c \cap \Psi} (1-e^{\alpha}) \sum_{n_1=-\infty}^{\infty} \sum_{0 \leqq n_2, \ldots, n_q < \infty} \exp\left(\sum_i n_i \beta_i\right) \\ &\quad = \textstyle\prod_{i=1}^s (1-e^{\gamma_i}) \sum_{0 \leqq m_1, \ldots, m_t < \infty} \sum_{n=-\infty}^{\infty} \exp\left(n\beta + \sum m_j \eta_j\right).\end{aligned}$$

Clearly (7.18–7.20) imply the equality of the two distributions defined by Eqs. (7.10) and (7.17).

Both Ψ and $s_\beta \Psi$ (s_β = reflection about β) are systems of positive roots in Φ. It follows that the number of ways in which any given $\mu \in \Lambda$ can be expressed as a sum

$$\mu = \textstyle\sum_i n_i \beta_i, \qquad \text{with } n_i \in \mathbb{Z} \quad \text{and} \quad n_2, \ldots, n_q \geqq 0,$$

is bounded by some polynomial function of $\|\mu\|$. By standard arguments in Fourier analysis on a torus, the first of the two series in (7.18) converges absolutely. The second series is treated similarly. This verifies (7.18).

If $\alpha \in \Phi$ has a positive inner product with β, then so does $-s_\beta \alpha$, and $\alpha - s_\beta \alpha$ is proportional to β. Consequently,

$$\delta\beta - \tfrac{1}{2}\textstyle\sum_{\alpha \in \Phi, (\alpha,\beta) > 0} \alpha$$

is an integral multiple of β (recall the definition of δ!). This fact makes (7.19) equivalent to

$$\textstyle\sum_{\alpha \in \Phi^n \cap \Psi} \alpha - \sum_{\alpha \in \Phi^c \cap \Psi} \alpha + \sum_{i=1}^s \gamma_i - \sum_{j=1}^t \eta_j - \sum_{\alpha \in \Phi, (\alpha,\beta) > 0} \alpha \tag{7.21}$$

is an even integral multiple of β.

In order to demonstrate (7.21), I again consider the various β-ladders, i.e. the various maximal strings

$$\{\alpha+l\beta \mid l\in\mathbb{Z}\}$$

in $\Phi\cup\{0\}$. As was pointed out in the proof of (5.30), with the exception of $\{0, \pm\beta\}$, every β-ladder lies either wholly inside or wholly outside of Ψ. The exceptional β-ladder $\{0, \pm\beta\}$ contributes zero to the sum occurring in (7.21). Now let L be a β-ladder in Ψ, of even length. Then L contains none of the γ_i or η_j, and the contribution of $L\cup(-L)$ to the sum

$$\sum_{\alpha\in\Phi,\,(\alpha,\beta)>0}\alpha$$

is β or 4β, depending on whether L has length two or length four[17]. Since the roots in L are compact and noncompact in an alternating fashion,

$$\sum_{\alpha\in\Phi^c\cap L}\alpha-\sum_{\alpha\in\Phi^n\cap L}\alpha$$

equals $\pm\beta$ or $\pm 2\beta$, again depending on whether L has length two or four. In any event, every β-ladder of even length accounts for an even multiple of β in the sum appearing in (7.21). If the β-ladder $L\subset\Psi$ consists of a single root α, then the notions of compactness and noncompactness for α with respect to G and with respect to M_β^0 coincide (cf. (2.61)); also $\alpha\perp\beta$. It follows that $L\cup(-L)$ does not contribute to the sum in (7.21). The only possible remaining case is that of a β-ladder L in Ψ of length three, say

$$L=\{\alpha, \alpha\pm\beta\}.$$

According to (2.61), α is one of the η_j or one of the γ_i, depending on whether α is compact or not. It is now easy to check that the contribution of $L\cup(-L)$ to the sum in (7.21) vanishes. This completes the demonstration of (7.21), and thus of (7.19).

The assertion (7.20) still remains to be verified. The left hand side of the desired identity can be formally factored as follows:

$$\begin{aligned}(7.22)\qquad &\prod_{\alpha\in\Phi^c\cap\Psi}(1-e^{\alpha})\sum_{n_1=-\infty}^{\infty}\sum_{0\leqq n_2,\ldots,n_q}\exp\left(\sum_i n_i\beta_i\right)\\ &=\left(\sum_{n=-\infty}^{\infty}e^{n\beta}\right)\times\prod_{\alpha\in\Phi^c\cap\Psi}(1-e^{\alpha})\times\prod_{i=2}^{q}\left(\sum_{k=0}^{\infty}e^{k\beta_i}\right).\end{aligned}$$

Similarly, the right hand side has a formal factorization

$$\begin{aligned}(7.23)\qquad &\prod_{i=1}^{s}(1-e^{\gamma_i})\sum_{0\leqq m_1,\ldots,m_t<\infty}\sum_{n=-\infty}^{\infty}\exp\left(n\beta+\sum_j m_j\eta_j\right)\\ &=\left(\sum_{n=-\infty}^{\infty}e^{n\beta}\right)\times\prod_{i=1}^{s}(1-e^{\gamma_i})\times\prod_{j=1}^{t}\left(\sum_{k=0}^{\infty}e^{k\eta_j}\right).\end{aligned}$$

In both cases, the formal products do make sense, and they can be rearranged at will. As the reader can check, this is so because β is simple for Ψ, and because all of the other roots which appear are positive, but distinct from β.

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[17] Since G/K is Hermitian symmetric, a β-ladder of length four cannot occur. However, since the Hermitian symmetric structure of G/K is really irrelevant for the proof of Lemma (7.9), I do not want to use it at all.

Let $L_1, \dots, L_N$ be an enumeration of all those β-ladders which lie in, and make up, $\Psi - \{\beta\}$. Rearranging the right hand side of (7.22), one finds

$$\text{(7.24)} \quad \begin{aligned} &\prod_{\alpha \in \Phi^c \cap \Psi}(1-e^\alpha) \sum_{n_1=-\infty}^{\infty} \sum_{0 \leq n_2, \dots, n_q} \exp\left(\sum_i n_i \beta_i\right) \\ &\quad = \left(\sum_{n=-\infty}^{\infty} e^{n\beta}\right) \times \prod_{l=1}^{N} \left\{\prod_{\alpha \in L_l \cap \Phi^c}(1-e^\alpha) \prod_{\gamma \in L_l \cap \Phi^n}\left(\sum_{k=0}^{\infty} e^{k\gamma}\right)\right\}. \end{aligned}$$

Similarly,

$$\text{(7.25)} \quad \begin{aligned} &\prod_{i=1}^{s}(1-e^{\gamma_i}) \sum_{0 \leq m_1, \dots, m_t < \infty} \sum_{n=-\infty}^{\infty} \exp\left(n\beta + \sum_j m_j \eta_j\right) \\ &\quad = \left(\sum_{n=-\infty}^{\infty} e^{n\beta}\right) \times \prod_{l=1}^{N} \left\{\prod_{\gamma_i \in L_l}(1-e^{\gamma_i}) \prod_{\eta_j \in L_l}\left(\sum_{k=0}^{\infty} e^{k\eta_j}\right)\right\}. \end{aligned}$$

In order to establish (7.20), it now suffices to show that

$$\text{(7.26)} \quad \begin{aligned} &\left(\sum_{n=-\infty}^{\infty} e^{n\beta}\right) \prod_{\alpha \in L \cap \Phi^c}(1-e^\alpha) \prod_{\gamma \in L \cap \Phi^n}\left(\sum_{k=0}^{\infty} e^{k\gamma}\right) \\ &\quad = \left(\sum_{n=-\infty}^{\infty} e^{n\beta}\right) \prod_{\gamma_i \in L}(1-e^{\gamma_i}) \prod_{\eta_j \in L}\left(\sum_{k=0}^{\infty} e^{k\eta_j}\right), \end{aligned}$$

for every β-ladder $L \subset \Psi - \{\beta\}$.

If L has length one, say $L = \{\alpha\}$, then α is one of the γ_i or one of the η_j, depending on whether α is compact or noncompact. In this case, the identity (7.26) holds term-by-term. Next, I suppose L has length two, so that $L = \{\alpha, \alpha+\beta\}$, or $L = \{\alpha, \alpha-\beta\}$, for some positive, compact root α. The products on the right hand side of (7.6) are then trivial, whereas the left hand side can be rewritten as

$$\begin{aligned} &\left(\sum_{n=-\infty}^{\infty} e^{n\beta}\right)\left(\sum_{k=0}^{\infty} e^{k(\alpha \pm \beta)}\right)(1-e^\alpha) \\ &\quad = \left(\sum_{n=-\infty}^{\infty} e^{n\beta}\right)\left(\sum_{k=0}^{\infty} e^{k\alpha}\right)(1-e^\alpha) \\ &\quad = \sum_{n=-\infty}^{\infty} e^{n\beta}. \end{aligned}$$

Thus, for any L of length two, (7.26) holds. The cases of β-ladders of lengths three and four can be treated analogously. This completes the verification of (7.20). As was pointed out already, (7.18–7.20) together imply the lemma.

According to Lemma (7.8), the statement of Proposition (7.7) is correct whenever Ψ has the property (3.1). On the other hand, Lemma (7.9) asserts that the proposition is compatible with (4.15c). Conclusion: Proposition (7.7), and hence Theorem (7.5), have been proven.

§ 8. Explicit Realization of the Discrete Series

Of the various attempts to give an explicit realization of the discrete series, Parthasarathy's construction involving the Dirac operator [32] is probably the simplest and most satisfactory. In this section, I shall demonstrate that Blattner's conjecture, in the case of any group for which it is known to hold, implies the precise version of Parthasarathy's theorem, giving all of the discrete series, not just "most" of it, as in [32]. I shall also use this opportunity to show how Parthasarathy's arguments can be simplified, by using some of the ideas of [35, 36].

I consider a connected, semisimple Lie group G, containing a maximal compact subgroup K, of the same rank as G. It will not be necessary to insist that G is a linear group. I shall use the notation of § 2; in particular H denotes a compact Cartan subgroup of G, which lies inside K. For the remainder of this section, I fix an admissible (cf. (4.11)) $\lambda \in i\mathfrak{h}^*$, such that

$$(\lambda, \alpha) \neq 0, \qquad \text{for all } \alpha \in \Phi^c$$

(Φ^c = set of compact roots in $\Phi = \Phi(G, H)$). One can then choose a system of positive roots Ψ in Φ, subject to the conditions

$$(8.1) \qquad \begin{aligned} (\lambda, \alpha) &> 0 \quad \text{if } \alpha \in \Psi \cap \Phi^c \\ (\lambda, \alpha) &\geqq 0 \quad \text{if } \alpha \in \Psi \cap \Phi^n. \end{aligned}$$

If λ is nonsingular, i.e. $(\lambda, \alpha) \neq 0$ for all $\alpha \in \Phi$, these conditions determine Ψ uniquely.

As in § 3, I set ρ_c and ρ_n equal to one half the sum of all compact, resp. noncompact, roots in Ψ, and

$$(8.2) \qquad \rho = \rho_c + \rho_n = \tfrac{1}{2} \textstyle\sum_{\alpha \in \Psi} \alpha.$$

Since λ is admissible, and since

$$2(\rho, \alpha)(\alpha, \alpha)^{-1} \in \mathbb{Z}, \qquad 2(\rho_c, \alpha)(\alpha, \alpha)^{-1} \in \mathbb{Z}$$

for all $\alpha \in \Phi^c$, the linear functional

$$(8.3) \qquad \mu = \lambda - \rho_c$$

satisfies $2(\mu, \alpha)(\alpha, \alpha)^{-1} \in \mathbb{Z}$, for all $\alpha \in \Phi^c$, and

$$(8.4) \qquad (\mu, \alpha) \geqq 0 \quad \text{whenever } \alpha \in \Psi \cap \Phi^c.$$

Consequently, there exists an irreducible representation

$$(8.5) \qquad \tau: \mathfrak{k}^{\mathbb{C}} \to \operatorname{End}(V_\mu),$$

whose highest weight, relative to the system of positive roots $\Psi \cap \Phi^c$, is μ. Via ad, $\mathfrak{k}^{\mathbb{C}}$ operates on $\mathfrak{p}^{\mathbb{C}}$. When $\mathfrak{p}^{\mathbb{C}}$ is endowed with the Killing form, this action becomes skew-symmetric:

$$(8.6) \qquad \operatorname{ad}: \mathfrak{k}^{\mathbb{C}} \to \mathfrak{so}(\mathfrak{p}^{\mathbb{C}}).$$

Because K has the same rank as G, $\mathfrak{p}^{\mathbb{C}}$ is even-dimensional. One can thus consider the two half-spin modules of $\mathfrak{so}(\mathfrak{p}^{\mathbb{C}})$, which shall be referred to as S_+, S_-. By composing the actions of $\mathfrak{so}(\mathfrak{p}^{\mathbb{C}})$ on S_+ and S_- with the homomorphism (8.6), one obtains representations

$$(8.7) \qquad \begin{aligned} \sigma_+ &: \mathfrak{k}^{\mathbb{C}} \to \operatorname{End}(S_+) \\ \sigma_- &: \mathfrak{k}^{\mathbb{C}} \to \operatorname{End}(S_-). \end{aligned}$$

In order to pin down the labelling of S_+, S_-, it should be remarked that ρ_n is a weight of multiplicity one in $S_+ \oplus S_-$. Hence S_+ and S_- are completely determined by requiring that ρ_n be a weight of the $\mathfrak{k}^{\mathbb{C}}$-module S_+.

When the half-spin modules of $\mathfrak{so}(\mathfrak{p}^{\mathbb{C}})$ are constructed in terms of the Clifford algebra, as for example in [1], one obtains (nontrivial) $\mathfrak{so}(\mathfrak{p}^{\mathbb{C}})$-maps

$$(8.8) \qquad S_+ \otimes \mathfrak{p}^{\mathbb{C}} \to S_-, \qquad S_- \otimes \mathfrak{p}^{\mathbb{C}} \to S_+,$$

which are unique up to constant multiples. It is convenient to denote these mappings as follows:

(8.9) $$s\otimes X\mapsto c(X)s, \quad X\in\mathfrak{p}^{\mathbb{C}}, \quad s\in S_{\pm};$$

thus $c(X)$, for $X\in\mathfrak{p}^{\mathbb{C}}$, is a linear map from S_+ to S_- and from S_- to S_+. If the mappings (8.8) are appropriately normalized,

(8.10) $$c(X)^2=-B(X,X)\cdot 1,$$

for all $X\in\mathfrak{p}^{\mathbb{C}}$ (B = Killing form).

As another consequence of the usual constructions of the half-spin modules, one finds

(8.11) the set of weights of the $\mathfrak{k}^{\mathbb{C}}$-module S_+ (resp. of S_-) is $\{\rho_n-\beta_1-\cdots-\beta_l\}$, with $\beta_1,\ldots,\beta_l$ ranging over all l-tuples of distinct, positive, noncompact roots, and with l even (resp. l odd).

In particular, all weights of $V_\mu\otimes S_+$ and of $V_\mu\otimes S_-$ lift to characters of the torus H. Consequently,

(8.12) $\tau\otimes\sigma_+$ and $\tau\otimes\sigma_-$ lift to representations of K.

To keep the notation simple, I shall refer to the resulting representations of K by the same symbols.

Like any finite-dimensional K-module, the K-modules $V_\mu\otimes S_+$ and $V_\mu\otimes S_-$ associate homogeneous vector bundles $\mathscr{V}_\mu\otimes\mathscr{S}_+$ and $\mathscr{V}_\mu\otimes\mathscr{S}_-$ to the principal bundle

$$K\to G\to G/K.$$

I let p denote the projection $G\to G/K$, and for each open set $U\subset G/K$, I set $\Gamma_U(\ldots)$ = space of C^∞ sections of ... over U. One then obtains a canonical isomorphism

(8.13) $$\Gamma_U(\mathscr{V}_\mu\otimes\mathscr{S}_\pm)\cong\{F\in C^\infty(p^{-1}(U))\otimes V_\mu\otimes S_\pm \mid r\otimes\tau\otimes\sigma_\pm(k)F=F, \text{ for all } k\in K\}$$

($r(k)$ = right translation by k). Via infinitesimal right translation, the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ operates on $C^\infty(G)$, and more generally, on the space of C^∞ functions on any open subset of G. For $X\in\mathfrak{g}^{\mathbb{C}}$, $r(X)$ shall denote infinitesimal right translation by X. Now let $\{X_i \mid 1\leq i\leq 2q\}$ be a basis of $\mathfrak{p}^{\mathbb{C}}$, which is orthonormal with respect to the Killing form of $\mathfrak{g}^{\mathbb{C}}$. The operator

(8.14) $$\textstyle\sum_i r(X_i)\otimes 1\otimes c(X_i)\colon C^\infty(p^{-1}(u))\otimes V_\mu\otimes S_\pm\to C^\infty(p^{-1}(u))\otimes V_\mu\otimes S_\mp$$

is clearly independent of the particular choice of the basis $\{X_i\}$. Moreover, as can be verified by a simple direct computation, it maps the subspace of $r\otimes\tau\otimes\sigma_\pm(K)$-invariant elements to the subspace of $r\otimes\tau\otimes\sigma_\mp(K)$-invariants. Thus, composing the operators (8.14) with the isomorphism (8.13), one obtains homogeneous, first order differential operators

(8.15) $$D_\pm\colon\Gamma_u(\mathscr{V}_\mu\otimes\mathscr{S}_\pm)\to\Gamma_u(\mathscr{V}_\mu\otimes\mathscr{S}_\mp).$$

These are the Dirac operators.

The fibres of $\mathscr{V}_\mu \otimes \mathscr{S}_+$ and of $\mathscr{V}_\mu \otimes \mathscr{S}_-$ at the identity coset can be naturally identified with $V_\mu \otimes S_+$ and $V_\mu \otimes S_-$, respectively. Similarly, the tangent space of G/K at the identity coset is canonically isomorphic to $\mathfrak{p}$. The Killing form induces an isomorphism between $\mathfrak{p}$ and its dual space; thus the cotangent space of G/K at the identity coset also becomes isomorphic to $\mathfrak{p}$. When these identifications are made, the symbol at the identity coset of D_+, respectively D_-, evaluated on an $X \in \mathfrak{p}$, becomes

$$1 \otimes c(X) \colon V_\mu \otimes S_\pm \to V_\mu \otimes S_\mp .$$

Because the Killing form is positive definite on $\mathfrak{p}$, and in view of (8.10), this mapping is injective whenever $X \in \mathfrak{p}$ is nonzero. The injectivity, coupled with the homogeneous nature of D_+ and D_-, implies that the Dirac operators are elliptic.

As irreducible Spin($\mathfrak{p}$)-modules, S_+ and S_- carry essentially unique Hermitian structures, such that the action of Spin($\mathfrak{p}$) becomes unitary. With proper normalization of the inner products on $S_\pm$,

(8.16) $\qquad -c(\bar{X}) \quad$ is the adjoint of $c(X)$,

for any $X \in \mathfrak{p}^{\mathbb{C}}$ ($\bar{X}$ = complex conjugate of X, relative to the real form $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$). Also, V_μ has an essentially unique inner product, which makes the action of $\mathfrak{k}^{\mathbb{C}}$ skew-Hermitian. The resulting K-invariant inner products on $V_\mu \otimes S_\pm$ determine homogeneous, Hermitian structures on the vector bundles $\mathscr{V} \otimes \mathscr{S}_\pm$. With respect to these Hermitian structures,

(8.17) $\qquad D_-$ is the formal adjoint of D_+, and vice versa,

as follows from (8.16). Now let $L^2(\mathscr{V}_\mu \otimes \mathscr{S}_\pm)$ be the Hilbert spaces of L^2-sections of $\mathscr{V}_\mu \otimes \mathscr{S}_\pm$. By translation, G acts on these spaces unitarily. Because of the ellipticity and the homogeneity of the Dirac operations,

(8.18)
$$\begin{aligned} \mathbf{H}_{\lambda,+} &= L^2(\mathscr{V}_\mu \otimes \mathscr{S}_+) \cap \operatorname{Ker}\{D_+ : \Gamma(\mathscr{V}_\mu \otimes \mathscr{S}_+) \to \Gamma(\mathscr{V}_\mu \otimes \mathscr{S}_-)\}, \\ \mathbf{H}_{\lambda,-} &= L^2(\mathscr{V}_\mu \otimes \mathscr{S}_-) \cap \operatorname{Ker}\{D_- : \Gamma(\mathscr{V}_\mu \otimes \mathscr{S}_-) \to \Gamma(\mathscr{V}_\mu \otimes \mathscr{S}_+)\} \end{aligned}$$

are G-invariant Hilbert subspaces of $L^2(\mathscr{V}_\mu \otimes \mathscr{S}_\pm)$. Here, as always in this section, λ and μ are related by (8.3).

(8.19) **Lemma** (Parthasarathy). *Under the isomorphisms* (8.13), *the operators* $D_+ D_-$ *and* $D_- D_+$ *can be identified with*

$$-r(\Omega) \otimes 1 \otimes 1 + (\lambda + \rho, \lambda - \rho) \cdot 1$$

(Ω = *Casimir operator of* G).

This is proposition (3.1) of [32]. The proof comes down to the computation of the square of the operator (8.14). The basic algebraic fact is the identity

$$\sigma_\pm(Y) = \tfrac{1}{4} \textstyle\sum_{i,j} B(Y, [X_i, X_j])\, c(X_i)\, c(X_j),$$

which holds for any $Y \in \mathfrak{k}^{\mathbb{C}}$, and which is stated as Lemma 2.1 of [32].

In complete analogy to (8.13), there are natural isomorphisms

(8.20) $\quad L^2(\mathscr{V}_\mu \otimes \mathscr{S}_\pm) \cong \{F \in L^2(G) \otimes V_\mu \otimes S_\pm \mid r \otimes \tau \otimes \sigma_\pm(k) F = F, \text{ for all } k \in K\}.$

These isomorphisms provide embeddings of $\mathbf{H}_{\lambda,+}$ in $L^2(G)\otimes V_\mu\otimes S_+$, and of $\mathbf{H}_{\lambda,-}$ in $L^2(G)\otimes V_\mu\otimes S_-$:

(8.21) $$\mathbf{H}_{\lambda,\pm}\hookrightarrow\{F\in L^2(G)\otimes V_\mu\otimes S_\pm \mid F \text{ is } r\otimes\tau\otimes\sigma_\pm(K)\text{-invariant}\}.$$

If a section of $\mathscr{V}_\mu\otimes\mathscr{S}_+$ is annihilated by D_+, it must certainly lie in the kernel of D_-D_+; similarly $\mathbf{H}_{\lambda,-}$ is contained in the kernel of D_+D_-. Hence, and because of (8.19), on the right hand side of (8.21), $L^2(G)$ can be replaced by the $(\lambda+\rho,\lambda-\rho)$-eigenspace of Ω, acting on $L^2(G)$.

According to Harish-Chandra's work on the Plancherel formula [15], the set of classes in the dual $\hat{G}$, outside of the discrete series, on which the Casimir operator Ω acts as multiplication by any particular constant, has Plancherel measure zero. By $\hat{G}_d$, I shall denote the discrete series, and by $\hat{G}_d(\lambda)$, the subset consisting of those $i\in\hat{G}_d$ whose infinitesimal character assumes the value $(\lambda+\rho,\lambda-\rho)$ on Ω. For any $i\in\hat{G}_d$, I choose a particular representation π_i on a Hilbert space $\mathbf{H}_i$, which realizes the class i. Let π_i^* be the contragredient representation, on the dual space $\mathbf{H}_i^*$. To keep the notation simple, I shall refer to the completed tensor product of $\mathbf{H}_i$ and $\mathbf{H}_i^*$ as $\mathbf{H}_i\otimes\mathbf{H}_i^*$. When the natural Hilbert space structure of this tensor product is suitably renormalized,

$$\bigoplus_{i\in\hat{G}_d}\mathbf{H}_i\otimes\mathbf{H}_i^*$$

becomes the contribution of the discrete series to the Plancherel decomposition of $L^2(G)$, and

$$\bigoplus_{i\in\hat{G}_d(\lambda)}\mathbf{H}_i\otimes\mathbf{H}_i^*$$

becomes isomorphic to the $(\lambda+\rho,\lambda-\rho)$-eigenspace of Ω on $L^2(G)$. In this manner, (8.21) leads to injections

(8.22) $$\mathbf{H}_{\lambda,\pm}\hookrightarrow\bigoplus_{i\in\hat{G}_d(\lambda)}\mathbf{H}_i\otimes(\mathbf{H}_i^*\otimes V_\mu\otimes S_\pm)^K,$$

with $(\mathbf{H}_i^*\otimes V_\mu\otimes S_\pm)^K=$ space of K-invariants in $H_i^*\otimes V_\mu\otimes S_\pm$. Via the injection (8.22), the action of G on $\mathbf{H}_{\lambda,\pm}$ corresponds to the direct sum over i of π_i, tensored with the trivial action on $(\mathbf{H}_i^*\otimes V_\mu\otimes S_\pm)^K$.

Using (8.17), one can actually show that (8.22) is an isomorphism. When Blattner's conjecture is available for all discrete series representations in $\hat{G}_d(\lambda)$, by means of this isomorphism, it is possible to identify $\mathbf{H}_{\lambda,+}$ and $\mathbf{H}_{\lambda,-}$ directly; the argument comes down to a slight variant of the proof of Proposition (8.34) below. Instead of using this approach, I shall sketch an alternating sum argument, which is similar in spirit to Parthasarathy's argument, and which gives some information even if Blattner's conjecture is not known.

As follows from Harish-Chandra's enumeration of the discrete series representations [13], $\hat{G}_d(\lambda)$ is a finite set. Moreover, because $V_\mu\otimes S_\pm$ are finite-dimensional K-modules,

$$\dim(\mathbf{H}_i^*\otimes V_\mu\otimes S_\pm)^K<\infty,\qquad \text{for every } i\in\hat{G}_d.$$

In view of (8.22), these two remarks imply:

(8.23) $\mathbf{H}_{\lambda,+}$ and $\mathbf{H}_{\lambda,-}$ are finite direct sums of discrete series representations, each occurring with finite multiplicity.

Consequently, both $\mathbf{H}_{\lambda,+}$ and $\mathbf{H}_{\lambda,-}$ have well-defined characters.

Let $\mathbf{H}^*_{i,\infty}$ be the subspace of K-finite vectors in the irreducible, unitary G-module $\mathbf{H}^*_i$. Every $v \in \mathbf{H}^*_{i,\infty}$ is then an analytic vector. When $\mathbf{H}_i \otimes \mathbf{H}^*_i$ is embedded in $L^2(G)$ in the usual manner, every element of $\mathbf{H}_i \otimes \mathbf{H}^*_{i,\infty}$ maps to a real-analytic, and hence C^∞, function. Under the resulting injection

$$\mathbf{H}_i \otimes \mathbf{H}^*_{i,\infty} \hookrightarrow L^2(G) \cap C^\infty(G),$$

the action of any given $X \in \mathfrak{g}^{\mathbb{C}}$, considered as a vector field by infinitesimal right translation, corresponds to the action of $1 \otimes \pi^*_i(X)$ on $\mathbf{H}_i \otimes \mathbf{H}^*_{i,\infty}$. Since $V_\mu \otimes S_\pm$ are finite-dimensional K-modules,

$$(\mathbf{H}^*_i \otimes V_\mu \otimes S_\pm)^K = (H^*_{i,\infty} \otimes V_\mu \otimes S_\pm)^K.$$

In view of the preceding remarks, (8.22) and the description of the Dirac operators in terms of the operators (8.14) lead to isomorphisms

$$\mathbf{H}_{\lambda,\pm} \cong \bigoplus_{i \in \hat{G}_d(\lambda)} \mathbf{H}_i \otimes \operatorname{Ker} \{\textstyle\sum_j \pi^*_i(X_j) \otimes 1 \otimes c(X_j): \\ (\mathbf{H}^*_{i,\infty} \otimes V_\mu \otimes S_\pm)^K \to (\mathbf{H}^*_{i,\infty} \otimes V_\mu \otimes S_\mp)^K\}.$$

The two linear transformations

$$\textstyle\sum_j \pi^*_i(X_j) \otimes 1 \otimes c(X_j): (\mathbf{H}^*_{i,\infty} \otimes V_\mu \otimes S_\pm)^K \to (\mathbf{H}^*_{i,\infty} \otimes V_\mu \otimes S_\mp)^K,$$

corresponding to the two possible choices of sign, are adjoint to each other (cf. (8.16); also, $\mathfrak{g}^{\mathbb{C}}$ acts on $\mathbf{H}^*_{i,\infty}$ in a skew-Hermitian manner). The kernel of one is therefore isomorphic to the cokernel of the other. Consequently, the difference of the dimensions of the kernels of the two linear transformations equals the difference of the dimensions of the two finite-dimensional vector spaces in question. This proves:

(8.24)
$$\begin{aligned} &\text{multiplicity of } i \text{ in } \mathbf{H}_{\lambda,+} - \text{multiplicity of } i \text{ in } \mathbf{H}_{\lambda,-} \\ &= \dim (H^*_{i,\infty} \otimes V_\mu \otimes S_+)^K - \dim (H^*_{i,\infty} \otimes V_\mu \otimes S_-)^K, \end{aligned}$$

for every $i \in \hat{G}_d(\lambda)$.

The next lemma makes it possible to read off this difference of dimensions from Harish-Chandra's character formula. For the pupose of stating the lemma, I consider an irreducible unitary representation π of G, By τ_π, I shall denote the K-character of π, i.e. the distribution

$$\tau_\pi : f \mapsto \operatorname{trace} \textstyle\int_K f(k)\, \pi(k)\, dk, \quad f \in C^\infty(K).$$

Equivalently, τ_π is the sum, in the sense of distributions, of the characters of the K-irreducible constituents of π, each taken with the appropriate multiplicity. In a more or less obvious manner, τ_π can be pulled back to any finite covering of K. The infinitesimal representations σ_+ and σ_- of $\mathfrak{k}$ on S_+ and S_- determine global representations of some finite covering of K. On this covering,

(8.25)
$$\operatorname{trace} \sigma_+ - \operatorname{trace} \sigma_-$$

is then a well-defined, smooth class function. The product of τ_π, or rather of its pullback to the appropriate finite covering of K, with the function (8.25), can thus be defined; it is a distribution.

(8.26) **Lemma.** *The product τ_π(trace σ_+ − trace σ_-) is actually a function.*

Proof. Let $\mathbf{H}$ be the representation space of π, and $\mathbf{H}_\infty$ the subspace of K-finite vectors. As before, I choose an orthonormal basis $\{X_i\}$ of $\mathfrak{p}^{\mathbb{C}}$. The operators

(8.27) $$d_\pm = \sum_i \pi(X_i) \otimes c(X_i): \mathbf{H}_\infty \otimes S_\pm \to \mathbf{H}_\infty \otimes S_\mp$$

do not depend on the particular choice of the basis $\{X_i\}$, they commute with the actions of $\mathfrak{k}^{\mathbb{C}}$ via $\pi \otimes \sigma_\pm$, and they are mutually adjoint, relative to the natural structures of $\mathbf{H}_\infty \otimes S_\pm$ as pre-Hilbert spaces. By a computation, which is formally identical to the proof of Lemma (8.19), one finds that

(8.28) $$d_- d_+ = (\pi \otimes \sigma_+)(\Omega_K) - \pi(\Omega) \otimes 1 - [(\rho, \rho) - (\rho_c, \rho_c)] \cdot 1$$

(Ω = Casimir operator of G, Ω_K = Casimir operator of K); $d_+ d_-$ is given by the analogous formula. Since $\mathbf{H}_\infty$ is the algebraic direct sum of irreducible $\mathfrak{k}^{\mathbb{C}}$-modules, each occurring with finite multiplicity, the tensor product $\mathbf{H}_\infty \otimes S_+$ must also have this property. In particular, for a suitable algebraic basis of $\mathbf{H}_\infty \otimes S_+$, $d_- d_+$ becomes diagonal, with eigenvalues tending to $+\infty$. The kernel of d_+ is therefore finite-dimensional, as is the kernel of d_-, and thus also the cokernel of d_+. Because d_+ commutes with the action of $\mathfrak{k}^{\mathbb{C}}$, the kernel and the cokernel of d_+ are finite-dimensional $\mathfrak{k}^{\mathbb{C}}$-module; I shall denote their characters by χ_+ and χ_-. For purely formal reasons,

$$\tau_\pi(\text{trace}\,\sigma_+ - \text{trace}\,\sigma_-) = \chi_+ - \chi_-,$$

which proves the lemma.

In order to compute the integer appearing in (8.24), I consider a particular class $i \in \hat{G}_d(\lambda)$, whose K-character shall be denoted by τ_i. I set χ_μ = character of the K-module V_μ. Then

$$\begin{aligned}\dim(\mathbf{H}^*_{i,\infty} \otimes V_\mu \otimes S_+)^K &= \dim(\mathbf{H}_{i,\infty} \otimes V_\mu^* \otimes S_+^*)^K = \dim \operatorname{Hom}^K(V_\mu \otimes S_+, H_{i,\infty})\\ &= \tau_i(\bar{\chi}_\mu \cdot \text{trace}\,\sigma_+^*)\end{aligned}$$

($\bar{\chi}_\mu$ = complex conjugate of χ_μ). Subtracting the analogous identity for the dimension of $(\mathbf{H}^*_{i,\infty} \otimes V_\mu \otimes S_-)^K$, one finds

$$\dim(\mathbf{H}^*_{i,\infty} \otimes V_\mu \otimes S_+)^K - \dim(\mathbf{H}^*_{i,\infty} \otimes V_\mu \otimes S_-)^K = \tau_i\big(\bar{\chi}_\mu \cdot (\text{trace}\,\sigma_+^* - \text{trace}\,\sigma_-^*)\big).$$

As follows from (8.11),

(8.29) $$(\text{trace}\,\sigma_+ - \text{trace}\,\sigma_-)|_H = \prod_{\beta \in \Psi \cap \Phi^n} (e^{\beta/2} - e^{-\beta/2});$$

strictly speaking, both quantities may be well-defined only on a suitable finite covering of H. In any event, the difference of the characters of S_+ and S_- is real or purely imaginary, depending on whether there is an even or an odd number of positive, noncompact roots. Thus

$$\text{trace}\,\sigma_+^* - \text{trace}\,\sigma_-^* = (-1)^q(\text{trace}\,\sigma_+ - \text{trace}\,\sigma_-),$$

with $q = \frac{1}{2}\dim_{\mathbb{R}} G/K$, and hence

(8.30) $$\begin{aligned}&\dim(\mathbf{H}^*_{i,\infty} \otimes V_\mu \otimes S_+)^K - \dim(\mathbf{H}^*_{i,\infty} \otimes V_\mu \otimes S_-)^K\\ &= (-1)^q \tau_i\big((\text{trace}\,\sigma_+ - \text{trace}\,\sigma_-)\,\bar{\chi}_\mu\big).\end{aligned}$$

Because of Lemma (8.26), the product $\tau_i(\operatorname{trace}\sigma_+ - \operatorname{trace}\sigma_-)$ is completely determined by its restriction to the intersection of K with the regular set in G. According to a theorem of Harish-Chandra [10], on this intersection, τ_i is a real-analytic function, which coincides there with the restriction of the global character Θ_i of π_i. In view of Harish-Chandra's character formula [13], plus the identity (8.29),

$$(8.31) \qquad (-1)^q \Theta_i|_K \cdot (\operatorname{trace}\sigma_+ - \operatorname{trace}\sigma_-)$$

is, up to sign, the character of an irreducible $\mathfrak{k}^{\mathbb{C}}$-module[18]. As can be deduced from the preceding remarks, the integer

$$\dim(\mathbf{H}^*_{i,\infty} \otimes V_\mu \otimes S_+)^K - \dim(\mathbf{H}^*_{i,\infty} \otimes V_\mu \otimes S_-)^K$$

equals one (respectively minus one), if the expression (8.31) agrees with χ_μ (respectively, with $-\chi_\mu$), and it equals zero otherwise. When χ_μ is written out explicitly, in terms of Weyl's character formula, this assertion can be rephrased as follows:

$$\dim(\mathbf{H}^*_{i,\infty} \otimes V_\mu \otimes S_+)^K - \dim(\mathbf{H}^*_{i,\infty} \otimes V_\mu \otimes S_-)^K$$

$$(8.32) \qquad = \begin{cases} \pm 1 & \text{if } \Theta_i|_H = \pm(-1)^q \dfrac{\sum_{w \in W(K,H)} \varepsilon(w) e^{w\lambda}}{\Pi_{\alpha \in \Psi}(e^{\alpha/2} - e^{-\alpha/2})} \\ 0 & \text{otherwise} \end{cases}$$

(recall the relationship (8.3) between λ and μ!).

If λ is nonsingular, let Θ_λ be the discrete series character defined by Harish-Chandra in [13]. For any irreducible representation π with character Θ_λ, $\pi(\Omega)$ acts as multiplication by $(\lambda+\rho, \lambda-\rho)$. Thus Θ_λ is the character of a class in $\hat{G}_d(\lambda)$. At this point, Harish-Chandra's enumeration of the discrete series characters [13], coupled with (8.24) and (8.32), implies the following statement, which appears as theorem 1 in [32]:

(8.33) **Theorem** (Parthasarathy). *Both $\mathbf{H}_{\lambda,+}$ and $\mathbf{H}_{\lambda,-}$ are finite direct sums of discrete series representations. The character of $\mathbf{H}_{\lambda,+}$, minus the character of $\mathbf{H}_{\lambda,-}$, equals Θ_λ if λ is nonsingular, and it equals zero if λ is singular.*

Thus, as soon as $\mathbf{H}_{\lambda,-}$ is known to vanish, for any given nonsingular λ, $\mathbf{H}_{\lambda,+}$ must be an irreducible, unitary, square-integrable G-module, with character Θ_λ. In order to obtain explicit realizations of all of the discrete series representations of G, it suffices to show that $\mathbf{H}_{\lambda,-}=0$, whenever λ is nonsingular.

(8.34) **Proposition.** *Suppose Blattner's conjecture holds, for all classes in $\hat{G}_d(\lambda)$. Then $\mathbf{H}_{\lambda,+}=\mathbf{H}_{\lambda,-}=0$ if λ is singular, and $\mathbf{H}_{\lambda,-}=0$ if λ is nonsingular.*

Remark. In order to prove the proposition, one need not assume the full conjecture. Rather, it suffices to know that the actual multiplicities are less than or equal to the predicted multiplicities.

Proof. Because of (8.33), even if λ is singular, it suffices to show that $\mathbf{H}_{\lambda,-}=0$. If $\mathbf{H}_{\lambda,-} \neq 0$, there exists some $i \in \hat{G}_d(\lambda)$, such that

$$(8.35) \qquad (\mathbf{H}^*_i \otimes V_\mu \otimes S_-)^K \neq 0.$$

[18] Again, the quantity in question may make sense only on a suitable finite covering of K.

In Harish-Chandra's notation, the class i has character $\Theta_{\tilde{\lambda}}$, for some nonsingular $\tilde{\lambda}$. I can and shall pick $\tilde{\lambda}$ subject to the condition

$$\Phi^c \cap \Psi = \Phi^c \cap \tilde{\Psi}, \quad \text{where } \tilde{\Psi} = \{\alpha \in \Phi \mid (\alpha, \tilde{\lambda}) > 0\}. \tag{8.36}$$

Since i belongs to $\hat{G}_d(\lambda)$, $(\tilde{\lambda}+\tilde{\rho}, \tilde{\lambda}-\tilde{\rho}) = (\lambda+\rho, \lambda-\rho)$, or equivalently,

$$(\tilde{\lambda}, \tilde{\lambda}) = (\lambda, \lambda). \tag{8.37}$$

In a tensor product of two irreducible representations of $\mathfrak{k}^{\mathbb{C}}$, the highest weight of any irreducible subspace can be expressed as the sum of the highest weight of the first factor and some weight of the second factor. For every irreducible constituent of $V_\mu \otimes S_-$, the highest weight, relative to the system of positive roots $\Phi^c \cap \Psi$, is therefore of the form

$$\mu + \tilde{\rho}_n - \gamma_1 - \cdots - \gamma_t, \quad \gamma_i \in \Phi^n \cap \tilde{\Psi} \tag{8.38}$$

$(\tilde{\rho}_n =$ one half of the sum of the roots in $\Phi^n \cap \Psi)$.

According to Blattner's conjecture, the highest weight of any given irreducible $\mathfrak{k}^{\mathbb{C}}$-submodule of $\mathbf{H}_i$ can be written as

$$w(\tilde{\lambda} + \tilde{\rho}_n + \beta_1 + \cdots + \beta_s) - \rho_c, \quad \beta_i \in \Phi^n \cap \tilde{\Psi}, \; w \in W(K, H). \tag{8.39}$$

Because of (8.35), some irreducible summand of $V_\mu \otimes S_-$ does occur in H_i. Hence, for suitable choices of the β_i, γ_j, and w, the weights (8.38) and (8.39) agree. The relationship (8.3) between λ and μ now gives the identity

$$\tilde{\lambda} + \tilde{\rho}_n + \beta_1 + \cdots + \beta_s = w^{-1}(\lambda + \tilde{\rho}_n - \gamma_1 - \cdots - \gamma_t). \tag{8.40}$$

Being a highest weight, the weight (8.38) must be dominant with respect to $\Psi \cap \Phi^c$, as is ρ_c. Thus $\lambda + \tilde{\rho}_n - \gamma_1 - \cdots - \gamma_t$ is dominant. For every weight ν which is dominant with respect to $\Psi \cap \Phi^c$, and for every $w \in W(K, H)$, $\nu - w^{-1}\nu$ can be expressed as a sum of positive, compact roots. Hence there exist roots $\alpha_i \in \Phi^c \cap \Psi$, such that

$$w^{-1}(\lambda + \tilde{\rho}_n - \gamma_1 - \cdots - \gamma_t) = \lambda + \tilde{\rho}_n - \gamma_1 - \cdots - \gamma_t - \alpha_1 - \cdots - \alpha_r. \tag{8.41}$$

Putting together (8.40) with (8.41), one obtains an identity

$$\lambda = \tilde{\lambda} + \eta; \tag{8.42}$$

here η is a sum of positive roots, relative to $\tilde{\Psi}$. In particular,

$$(\tilde{\lambda}, \eta) \geqq 0. \tag{8.43}$$

In view of (8.37), one finds $\tilde{\lambda} = \lambda$, $\tilde{\Psi} = \Psi$, $t = 0$. But then (8.38) occurs as a highest weight in $V_\mu \oplus S_+$, not $V_\mu \oplus S_-$. Contradiction!

For the sake of completeness, I shall briefly discuss Parthasarathy's vanishing theorem (Theorem 2 of [32]). By a slight modification of Parthasarathy's argument, it is possible to avoid the assumption that G is a linear group, which was made in [32].

(8.44) **Proposition.** (Parthasarathy). *Suppose λ satisfies the conditions $(\lambda - \rho, \alpha) \geqq 0$, for all $\alpha \in \Phi^c \cap \Psi$, and $(\lambda - \rho, \alpha) > 0$ for all $\alpha \in \Phi^n \cap \Psi$. Then $\mathbf{H}_{\lambda,-} = 0$.*

Proof by contradiction. As in the proof of (8.34), if $\mathbf{H}_{\lambda,-} \neq 0$, there exists some $i \in \hat{G}_d(\lambda)$, such that

$$(8.45) \qquad (\mathbf{H}_i^* \otimes V_\mu \otimes S_-)^K \neq 0 .$$

By assumption, $\lambda-\rho$ is dominant; I let $V_{\lambda-\rho}$ be an irreducible $\mathfrak{k}^{\mathbb{C}}$-module with highest weight $\lambda-\rho$. Since an irreducible summand of S_+ has highest weight ρ_n, and since $\lambda-\rho+\rho_n=\mu$, there is a $\mathfrak{k}^{\mathbb{C}}$-invariant injection

$$(8.46) \qquad V_\mu \otimes S_- \hookrightarrow V_{\lambda-\rho} \otimes S_+ \otimes S_- .$$

It will be necessary to have a description of the highest weights, relative to the system of positive roots $\Phi^c \cap \Psi$, which occur in $S_+ \oplus S_-$. For this purpose, I enumerate those $w \in W(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ which preserve $\Phi^c \cap \Psi$ as $w_1, \ldots, w_N$:

$$(8.47) \qquad \{w_1, \ldots, w_N\} = \{w \in W(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}) \mid w(\Phi^c \cap \Psi) = \Phi^c \cap \Psi\} .$$

The set (8.47) is a complete set of representatives for the coset space

$$W(K,H) \setminus W(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}).$$

By standard arguments, one thus finds

$$(8.48) \qquad \begin{aligned}
&(\operatorname{trace}\sigma_+ - \operatorname{trace}\sigma_-)|_H = \textstyle\prod_{\beta\in\Phi^n\cap\Psi}(e^{\beta/2}-e^{-\beta/2})\\
&= \left(\textstyle\prod_{\alpha\in\Psi\cap\Phi^c}(e^{\alpha/2}-e^{-\alpha/2})\right)^{-1}\left(\textstyle\prod_{\alpha\in\Psi}(e^{\alpha/2}-e^{-\alpha/2})\right)\\
&= \left(\textstyle\prod_{\alpha\in\Psi\cap\Phi^c}(e^{\alpha/2}-e^{-\alpha/2})\right)^{-1}\left(\textstyle\sum_{w\in W(\mathfrak{g}^{\mathbb{C}},\mathfrak{h}^{\mathbb{C}})}\varepsilon(w)e^{w\rho}\right)\\
&= \textstyle\sum_{j=1}^N \varepsilon(w_j)\left\{\left(\prod_{\alpha\in\Psi\cap\Phi^c}(e^{\alpha/2}-e^{-\alpha/2})\right)^{-1}\left(\sum_{w\in W(K,H)}\varepsilon(w)e^{ww_j\rho}\right)\right\}.
\end{aligned}$$

In the difference $\operatorname{trace}\sigma_+ - \operatorname{trace}\sigma_-$, there can be no cancellation. Indeed, the $\mathfrak{k}^{\mathbb{C}}$-modules S_+ and S_- have no weights in common. This is implied by (8.11), plus the following observation: if some $\alpha \in \Phi^n$ is expressed as a linear combination of simple roots, the sum of the coefficients of the noncompact simple roots must be odd. For $1 \leqq j \leqq N$, $w_j\rho$ is dominant with respect to $\Phi^c \cap \Psi$ and nonsingular. Hence the term in curly parentheses appearing in (8.48) is the character of an irreducible $\mathfrak{k}^{\mathbb{C}}$-module, with highest weight $w_j\rho-\rho_c$. Thus:

(8.49) every $\mathfrak{k}^{\mathbb{C}}$-irreducible summand of S_+ (respectively, of S_-) has highest weight $w_j\rho-\rho_c$, with $1 \leqq j \leqq N$ and $\varepsilon(w_j)=+1$ (respectively, $\varepsilon(w_j)=-1$).

The preceding statement and the argument leading up to it appear as Lemma 2.2 in [32].

Because of (8.45) and (8.46), one can find an irreducible $\mathfrak{k}^{\mathbb{C}}$-module V_ν, with highest weight ν, such that

$$(8.50) \qquad (\mathbf{H}_i^* \otimes V_\nu \otimes S_+)^K \neq 0, \quad \text{and} \quad (V_\nu^* \otimes V_{\lambda-\rho} \otimes S_-)^K \neq 0 .$$

I now consider the positive semidefinite operator

$$d_- d_+ : \mathbf{H}_{i,\infty}^* \otimes S_+ \to \mathbf{H}_{i,\infty}^* \otimes S_+ ,$$

which was constructed in the proof of Lemma (8.26), with π_i^* playing the role of π. Because of (8.50), V_ν^* can be $\mathfrak{k}^{\mathbb{C}}$-equivariantly embedded in $\mathbf{H}_{i,\infty}^* \otimes S_+$. Let v be a nonzero vector in the image. Then

$$\pi_i^* \otimes \sigma_+ (\Omega_K)\, v = (\nu + 2\rho_c, \nu) v.$$

Since $i \in \hat{G}_d(\lambda)$, $\pi_i^*(\Omega)$ acts as multiplication by $(\lambda+\rho, \lambda-\rho)$. Applying $d_- d_+$ to v, taking into account the identity (8.28), as well as the fact that $d_- d_+ \geqq 0$, one arrives at the inequality

$$(8.51) \qquad (\lambda, \lambda) \leqq (\nu + \rho_c, \nu + \rho_c).$$

According to (8.49) and (8.50), for some j, $1 \leqq j \leqq N$, with $\varepsilon(w_j) = -1$, there exists a $\mathfrak{k}^{\mathbb{C}}$-equivariant injection

$$V_\nu \hookrightarrow V_{\lambda-\rho} \otimes V_{w_j\rho-\rho_c}$$

($V_{w_j\rho-\rho_c}$ is an irreducible $\mathfrak{k}^{\mathbb{C}}$-module with highest weight $w_j\rho - \rho_c$). On the tensor product of two irreducible $\mathfrak{k}^{\mathbb{C}}$-modules, with highest weight μ_1 and μ_2, the Casimir operator acts with eigenvalues not exceeding $\|\mu_1 + \mu_2 + \rho_c\|^2 - \|\rho_c\|^2$; this assertion follows, for example, from lemma 5.8 of [26]. Hence

$$(8.52) \qquad (\nu + \rho_c, \nu + \rho_c) \leqq (\lambda - \rho + w_j\rho, \lambda - \rho + w_j\rho).$$

In order to complete the proof, it suffices to derive a contradiction from the inequalities (8.51) and (8.52). Putting the inequalities together, one finds

$$\begin{aligned} 0 &\leqq (\lambda - \rho + w_j\rho, \lambda - \rho + w_j\rho) - (\lambda, \lambda) \\ &= (\rho - w_j\rho, \rho - w_j\rho) - 2(\lambda, \rho - w_j\rho) \\ &= -2(\rho, w_j\rho) + 2(\rho, \rho) - 2(\lambda, \rho - w_j\rho), \end{aligned}$$

so that $(\lambda - \rho, \rho - w_j\rho) \leqq 0$. Since w_j preserves the set $\Phi^c \cap \Psi$, and since $w_j \neq 1$ ($\varepsilon(w_j) = -1$!), $\rho - w_j\rho$ is a nonempty sum of positive, noncompact roots. As was assumed, every such root has a strictly positive inner product with $\lambda - \rho$. But then the inner product of $\lambda - \rho$ and $\rho - w_j\rho$ must be strictly positive. Contradiction!

§ 9. Some Postscripts

Most of the arguments of § 5 and § 6 work for an arbitrary semisimple matrix group G with a non-empty discrete series. The Hermitian symmetric structure of G/K was used mainly in order to provide the basic repertoire of characters of the "holomorphic discrete series"; the other discrete series characters were then built up from these. In this final section, I shall suggest that there always exists a distinguished class of discrete series representations, which can take the place of the "holomorphic discrete series" when the latter fails to exist. I shall also show, using results of Harish-Chandra, that the relationship between the various discrete series characters, which is implicit in (4.15c), holds whether or not G/K carries a Hermitian symmetric structure.

Let G be a connected, noncompact, simple matrix group, with a compact Cartan subgroup H. According to Borel-de Siebenthal [3], one can pick a system of positive roots in Φ, such that

(9.1) there exists a unique noncompact, simple root, and its multiplicity in the highest root is at most two.

The case of multiplicity one occurs precisely when G/K admits a Hermitian-symmetric structure. Now suppose that G is semisimple, rather than necessarily simple. A system of positive roots Ψ in Φ will be called *special* if its restriction to every noncompact simple factor has the property (9.1). I choose one such special positive root system Ψ, and I consider the discrete series characters Θ_λ parametrized by those λ's which are dominant with respect to Ψ. For a number of reasons, I believe that *these* discrete series characters can be expressed by a fairly simple, explicit, global formula. The formula presumably can be conjectured if it is worked out for some low dimensional groups, such as $SO(3,4)$ and $SO(4,4)$, which present essentially new features. A sufficiently explicit conjecture could then be verified, by checking that the formula in question satisfies the differential equations of an invariant eigendistribution (cf. Theorem 1 of [19]).

Once the invariant eigendistributions $\Theta(\Psi,\lambda)$, corresponding to every special system of positive roots Ψ, are known explicitly, it will be possible to use them to construct all of the discrete series characters, for every semisimple matrix group. One would have to show that these particular invariant eigendistributions have the properties[19] described by the statements (4.21), (4.22), (6.13), and (6.30). From this point on, the arguments of § 5 and § 6 apply, with only minor changes[20].

The proof of Blattner's conjecture in § 7 may not generalize as easily. However, there are some suggestive facts which should be mentioned in this connection. Again let Ψ be a special system of positive roots, and let π_λ be a discrete series representation, with character Θ_λ, such that λ is dominant with respect to Ψ. According to Proposition 1 of [36], Blattner's conjecture holds for π_λ, provided λ is "sufficiently nonsingular", i.e. provided $(\lambda,\alpha)>c$ whenever $\alpha\in\Psi$, for a certain positive constant c. It is at least conceivable that the methods of [34] and [36] can be pushed, to give Blattner's conjecture for all discrete series representations corresponding to every special system of positive roots. Unfortunately, to make the arguments of § 7 go through, one would need Blattner's conjecture for *all* invariant eigendistributions $\Theta(\Psi,\lambda)$ corresponding to a special positive root system Ψ, and not just for those which arise as discrete series characters.

As a final observation, I shall show that the relationship between the various discrete series characters, which is implicit in statement (4.15c), follows from the arguments of § 5 and § 6, even if G/K fails to have a Hermitian symmetric structure. Let $\Psi\subset\Phi$ be a particular system of positive roots. As a consequence of Harish-Chandra's construction of the discrete series characters [12], the characters Θ_λ, with λ ranging over the set

$$\{\lambda\in i\mathfrak{h}^* \mid \lambda \text{ is admissible, and } (\lambda,\alpha)>0 \text{ for all } \alpha\in\Psi\}, \tag{9.2}$$

satisfy the statement of Lemma (6.13). Hence, in the resulting description of Θ_λ, one can let λ wander over the larger set

$$\{\lambda\in i\mathfrak{h}^* \mid \lambda \text{ is admissible, and } (\lambda,\alpha)>0 \text{ for all } \alpha\in\Phi^c\cap\Psi\}; \tag{9.3}$$

in this fashion one obtains a family of invariant eigendistributions $\Theta(\Psi,\lambda)$, parameterized by the set (9.3), such that $\Theta(\Psi,\lambda)=\Theta_\lambda$ whenever λ lies in the set

[19] As they must, in view of Harish-Chandra's results on the discrete series.

[20] Cf. the discussion following Theorem (9.4) below.

(9.2), and such that the $\Theta(\Psi, \lambda)$ depend coherently on λ, in the sense of Lemma (6.13). The preceding remarks of course apply to any system of positive roots Ψ in Φ.

I now consider a system of positive roots $\Psi \subset \Phi$, and a noncompact root β which is simple for Ψ. Let $s_\beta \in W(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ be the reflection about β. For every λ in the set (9.3), the definition of the induced invariant eigendistribution Θ given in (4.15c) makes sense, whether or not G/K admits a Hermitian symmetric structure.

(9.4) **Theorem.** *Under the hypotheses stated above,*

$$\Theta(\Psi, \lambda) + \Theta(s_\beta \Psi, \lambda) = \Theta.$$

Proof. In view of the definition of Θ, the difference $\Theta - \Theta(\Psi, \lambda)$ must be an invariant eigendistribution. On any Cartan subgroup of G, the formula for this difference depends coherently on λ; this follows from the proof of Lemma (6.13). It therefore suffices to show that

$$\text{(9.5)} \qquad \Theta - \Theta(\Psi, \lambda) = \Theta_\lambda, \quad \text{provided } (\alpha, \lambda) > 0 \quad \text{if } \alpha \in s_\beta \Psi.$$

Since Θ was induced from a proper parabolic subgroup, it vanishes on the elliptic set. On the other hand, if $(\alpha, \lambda) > 0$ for $\alpha \in s_\beta \Psi$, $-\Theta_\lambda$ and $\Theta(\Psi, \lambda)$ have the same restriction to the elliptic set. Hence (9.5) becomes equivalent to

$$\text{(9.6)} \qquad \Theta - \Theta(\Psi, \lambda) \quad \text{is tempered, provided } (\alpha, \lambda) > 0 \quad \text{if } \alpha \in s_\beta \Psi.$$

Like any invariant eigendistribution, $\Theta(\Psi, \lambda)$ satisfies the statement of Lemma (6.30) [11]. From Harish-Chandra's construction of the discrete series characters, one can deduce that the Θ_λ, and hence also the $\Theta(\Psi, \lambda)$, have the properties described by Theorems (4.21) and (4.22). According to Propositions (5.30), (5.69), and the proof of Lemma (6.30), $\Theta - \Theta(\Psi, \lambda)$ must also have the properties described by Theorems (4.21) and (4.22), as well as by Lemma (6.30). Just as in the case of a group G with Hermitian symmetric quotient, an inductive argument allows one to assume that G has a split Cartan subgroup A, and that $\Theta - \Theta(\Psi, \lambda)$ satisfies the temperedness property, except possibly on the identity component A^0, provided $(\lambda, \alpha) > 0$ for all $\alpha \in s_\beta \Psi$. Up to covering, $Sp(n, \mathbb{R})$, $SO(2n, 2n)$, $SO(n, n+1)$, $E_{7(7)}$, $E_{8(8)}$, $F_{4(4)}$, and $G_{2(2)}$ are the only simple groups containing both a compact and a split Cartan subgroup. For the classical groups among these, the proofs of (6.33) and (6.41) go through essentially unchanged. In the other cases, minor modifications are necessary; however, I shall not go into details. These considerations imply (9.6), and hence the theorem.

Added in Proof. The derivation of Theroem 2 in [37] contains two minor errors. The two quantities appearing above formula (33) are added incorrectly, and the right hand side of (33) should read $\frac{1}{2}n(n+1) + \frac{1}{2}s(s+1) + sn + t$. This change affects the signs which occur in (36) and in Theorem 2. Also, the factor 2^t, which was computed in (32), should appear both in (36) and in Theorem 2.

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