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On Bloch's Conjecture

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In this paper everything is defined over **C**. Let A be an abelian variety, let X be a closed reduced irreducible subset of A and let $\alpha: X \to A$ be the inclusion. Let T be the space of 1-forms on A. We shall prove the following

Theorem 1. *If X is an algebraic variety of general type of dimension n, then there exists a system $\{\omega_1, \dots, \omega_{n+1}\}$ in T such that $\{\alpha^*(\omega_1 \wedge \dots \wedge \omega_{i-1} \wedge \omega_{i+1} \wedge \dots \wedge \omega_{n+1})\}_{i=1,\dots,n+1}$ are linearly independent.*

T. Ochiai proved in [3] that from Theorem 1 we can deduce

Theorem 2 (Bloch's conjecture). *Let X be a projective algebraic manifold of dimension n with irregularity $>n$. Then, any holomorphic curve $f: \mathbf{C} \to X$ is degenerate.*

For some related topics see [1] and [2].

1. *Proof of Theorem* 1. We may assume that $\alpha^*\omega \neq 0$ for every $\omega \in T$. The dual space T^* can be considered as a universal cover of A. Let $\tilde{X}$ be the pull back of X in T^*. Put $N = \{\omega \in \bigwedge^n T;\ \alpha^*\omega = 0\}$. For each $H \in \mathrm{Gr}_{n+1}(T)$, put $d(H) = \dim(\bigwedge^n H \cap N)$. We should prove that $\min_H d(H) = 0$. Assume the contrary and put $\min_H d(H) = d > 0$.

$U = \{H;\ d(H) \leqq n\}$ and $U_d = \{H;\ d(H) = d\}$ are Zariski open dense subsets of $\mathrm{Gr}_{n+1}(T)$. Each $H \in U$ defines a projection $T^* \to T^*/H^\perp$ and the image of $\tilde{X}$ by this is a divisor $X_H \subset T^*/H^\perp$. Let $\{dx_1, \dots, dx_{n+1}\}$ be a basis of H. We have $d(H)$-relations among

$$\omega_i = \alpha^*(dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \dots \wedge dx_{n+1}) \quad \text{for } i = 1, \dots, n+1.$$

After a suitable base change they are $\omega_1 = \dots = \omega_{d(H)} = 0$. Since $\omega_{n+1} \neq 0$, $\{x_1, \dots, x_n\}$ gives a local coordinate system of X_H near a certain smooth point p. The equation of X_H near p is such like $x_{n+1} = F(x_1, \dots, x_n)$. Then $\frac{\partial F}{\partial x_1} = \dots = \frac{\partial F}{\partial x_{d(H)}} = 0$. Thus $\alpha^*(dx_{d(H)+1} \wedge \dots \wedge dx_{n+1}) = 0$, which means that $\dim \alpha^*(\bigwedge^{n-d(H)+1} H) = \binom{n+1}{n-d(H)+1} - 1$, for other relations among $(n-d(H)+1)$-forms yield new relations among n-forms. Put $I = I(H) = \langle dx_{d(H)+1}, \dots, dx_{n+1} \rangle$, where $\langle\ \rangle$ denotes a linear subspace generated by the elements inside. Put $N_d=\{\omega\in\bigwedge^{n-d+1}T;\ \alpha^*\omega=0\}$. Then $\dim(\bigwedge^{n-d+1}H\cap N_d)=1$ for $H\in U_d$. Thus, $I\in\mathrm{Gr}_{n-d-1}(T)$ depends algebraically on $H\in U_d\subset\mathrm{Gr}_{n+1}(T)$.

Fix an $H\in U_d$ and $I=I(H)$.

Claim 1. For each $\theta\in T$ there is an element $\theta_H\in T$ such that $\theta-\theta_H\in H$ and $\alpha^*(\bigwedge^{n-d+1}\langle\theta_H, I\rangle)=0$.

Proof. Let us choose a base $\{\omega_1,\ldots,\omega_{n+1}\}$ of H such that $I=\langle\omega_1,\ldots,\omega_{n-d+1}\rangle$. For $t\in\mathbf{C}$ near 0, $H_t=\langle\omega_1+t\theta,\omega_2,\ldots,\omega_{n+1}\rangle$ defines a 1-parameter family in U_d. Put

$$I_t=I(H_t)$$
$$=\left\langle\omega_1+t\theta+\sum_{j=1}^{n+1}f_{1j}(t)\,\omega'_j,\ \omega_2+\sum_{j=1}^{n+1}f_{2j}(t)\,\omega'_j,\ \ldots,\ \omega_{n-d+1}+\sum_{j=1}^{n+1}f_{n-d+1,j}(t)\,\omega'_j\right\rangle,$$

where $\omega'_j=\omega_1+t\theta$ (for $j=1$) or ω_j (for $j\neq 1$). We know that $f_{ij}(t)=tg_{ij}(t)$ for some holomorphic functions g_{ij} for $i=1,\ldots,n-d+1$ and $j=1,\ldots,n+1$. Compare the coefficients of t in the equation $\alpha^*(\bigwedge^{n-d+1}I_t)=0$, and we get

$$\alpha^*\Bigg(\theta\wedge\omega_2\wedge\ldots\wedge\omega_{n-d+1}$$
$$+\sum_{i=1}^{n-d+1}\sum_{j=n-d+2}^{n+1}a_{ij}\omega_j\wedge\omega_1\wedge\ldots\wedge\omega_{i-1}\wedge\omega_{i+1}\wedge\ldots\wedge\omega_{n-d+1}\Bigg)=0,$$

where $a_{ij}=\pm g_{ij}(0)$. After a suitable coordinate change of H we get an equation

$$\alpha^*\Bigg(\theta\wedge\omega_2\wedge\ldots\wedge\omega_{n-d+1}+a_{n-d+2}\,\omega_{n-d+2}\wedge\omega_2\wedge\ldots\wedge\omega_{n-d+1}$$
$$+\sum_{i=1}^{n-d+1}\sum_{j=n-d+3}^{n+1}b_{ij}\omega_j\wedge\omega_1\wedge\ldots\wedge\omega_{i-1}\wedge\omega_{i+1}\wedge\ldots\wedge\omega_{n-d+1}\Bigg)=0.$$

Put $\theta'=\theta+a_{n-d+2}\omega_{n-d+2}$. Then we have

$$(*)\quad\alpha^*\Bigg(\theta'\wedge\omega_2\wedge\ldots\wedge\omega_{n-d+1}$$
$$+\sum_{i=1}^{n-d+1}\sum_{j=n-d+3}^{n+1}b_{ij}\omega_j\wedge\omega_1\wedge\ldots\wedge\omega_{i-1}\wedge\omega_{i+1}\wedge\ldots\wedge\omega_{n-d+1}\Bigg)$$
$$=0.$$

Since $\omega_2\wedge\ldots\wedge\omega_{n+1}\neq 0$, there is an open dense subset X_{n-d+2} of X on which we can write

$$\alpha^*\omega_1=\sum_{i=2}^{n+1}f_i(x)\,\alpha^*\omega_i\quad\text{and}\quad\alpha^*\theta'=\sum_{i=2}^{n+1}g_i(x)\,\alpha^*\omega_i,$$

where the f_i and the g_i are holomorphic functions on X_{n+d+2}. Since $\alpha^*(\omega_1\wedge\ldots\wedge\omega_{n-d+1})=0$, we have $f_{n-d+2}=\ldots=f_{n+1}=0$. Therefore, $g_{n-d+2}=0$. Similarly, for each j $(n-d+2\leqq j\leqq n+1)$, we can find a complex number a_j as above. Put $\theta_H = \theta' + \sum_{j=n-d+2}^{n+1} a_j \omega_j$. Then, we can write on $\bigcap_{j=n-d+2}^{n+1} X_j$, $\alpha^* \theta_H = \sum_{i=2}^{n-d+1} h_i(x)\alpha^*\omega_i$, where the h_i are holomorphic functions. Thus,

$$\alpha^*(\bigwedge^{n-d+1}\langle\theta_H, I\rangle) = 0. \quad \text{Q.E.D.}$$

Claim 2. Let L be the subspace of T generated by I and all the θ_H for $\theta \in T$. Then, $\alpha^*(\bigwedge^{n-d+1} L) = 0$.

Proof. We shall show $\alpha^*(\bigwedge^{n-d-k+1} I \wedge (\theta_1)_H \wedge \ldots \wedge (\theta_k)_H) = 0$ for $k = 1, \ldots, n-d+1$ by induction on k. The case $k=1$ was already proved in claim 1. Suppose $k \geqq 2$ and that the assertion is proved for $k-1$. Choose a base $\{\omega_1, \ldots, \omega_{n-d+1}\}$ of I and suppose $\alpha^*(\omega_1 \wedge \ldots \wedge \omega_{n-d-k+1} \wedge (\theta_1)_H \wedge \ldots \wedge (\theta_k)_H) \neq 0$.

Then, there is an open dense subset X' of X on which $\{\alpha^*(\omega_1), \ldots, \alpha^*(\omega_{n-d-k+1}), \alpha^*((\theta_1)_H), \ldots, \alpha^*((\theta_k)_H)\}$ gives a part of a base of the sheaf of the 1-forms everywhere, with which we can write down any 1-forms on X' as a linear combination with coefficients of holomorphic functions on X'. Since

$$\alpha^*(\omega_1 \wedge \ldots \wedge \omega_{n-d-k+2} \wedge (\theta_1)_H \wedge \ldots \wedge (\theta_{i-1})_H \wedge (\theta_{i+1})_H \wedge \ldots \wedge (\theta_k)_H) = 0,$$

for $i = 1, \ldots, k$, we have

$$\alpha^*(\omega_{n-d-k+2}) = \sum_{i=1}^{n-d-k+1} f_i(x)\alpha^*\omega_i,$$

where the f_i are holomorphic functions on X'. Hence, $\alpha^*(\omega_1 \wedge \ldots \wedge \omega_{n-d-k+2}) = 0$, which is a contradiction. Q.E.D.

We have proved that the image of $\bar{X}$ under the projection $T^* \to T^*/L^\perp$ is of dimension smaller than $n-d+1$. Since the codimension of L in T is not greater than d, this projection gives a fibering of $\bar{X}$ with fibers of linear subspaces of dimension d, i.e., X is not of general type. Q.E.D.

2. As an easy consequence of Theorem 2 we get

Theorem 3. *Let X be a closed reduced irreducible subset of an Abelian variety A and let $f: \mathbf{C} \to X$ be a holomorphic curve. Then the image $f(\mathbf{C})$ is contained in a translation of an abelian subvariety of A.*

In connection with this we have

Theorem 4. *Let X and A be as in Theorem 3. Let $Y = \bigcup \{Z;\ Z$ is a translation of an abelian subvariety of A and $Z \subset X\}$. If X is of general type, then Y is a closed algebraic subset of X such that $Y \neq X$.*

Proof. Let $\bar{A}$ be the universal cover of A and let $\bar{X}$ be the pull back of X in $\bar{A}$. Let $M = \mathrm{Gr}_1(\bar{A}) = \mathbf{P}((\bar{A})^*)$ and $p;\ \bar{A} - \{0\} \to M$ be the canonical projection. We construct an analytic subset $\bar{Y}$ of $\bar{A} \times M$ as follows: Let U be an arbitrary small open subset of M. For each section $s: U \to \bar{A}$ of p we define $s(\bar{X}) = \bigcup_{x \in U} (\bar{X}$

$+s(x), x)$, which is an analytic subset of $\bar{A} \times U$. Put $\bar{Y}_U = \bigcap_s s(\bar{X})$. Then the $\bar{Y}_U$ are compatible with the restrictions of open subsets and glue together to give $\bar{Y}$. For each $x \in M$ the fiber $\bar{Y}_x$ is the sum of the lines $\bar{C}$ contained in $\bar{X}$ whose directions are x. Thus, the image of $\bar{Y}$ by the projection to A is just Y. Let C be the image of $\bar{C}$ in A. Then the algebraic closure of C is a translation of an abelian subvariety B of A. Let Y_x be the image of $\bar{Y}_x$ in A. The set $N = \{x \in M;\ Y_x \neq \emptyset\}$ is an analytic subset of M, and hence an algebraic subset by GAGA. For each $x \in N$, Y_x is an analytic fiber bundle with a fiber $B = B_x$. Let D_x be the base space. Since $Y_x \neq X$, we have $\dim B_x + \dim D_x < \dim X$. We know that the function $\dim B_x$ on N is lower semi-continuous. Let us stratify N as follows: $N = \sum_{i \in I} N_i$, where the N_i are irreducible locally closed algebraic subsets of M and on each N_i the function $\dim B_x$ is constant and I is a finite set. For $x, y \in N_i$, we have $B_x = B_y$, and hence $D_x = D_y$. Therefore $Y_x = Y_y$. Put $Y_i = Y_x$ for an $x \in Y$. Then $Y = \bigcup_{i \in I} Y_i$. Q.E.D.

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