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Dehn's Lemma for Certain 4-Manifolds

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Unless otherwise stated all manifolds and maps are smooth.

Let M be a 4-manifold and K a knot M (i.e. an embedded circle). Suppose K is homotopic to zero in M. We wish to construct an embedded 2-disc in M with boundary K. We can map a 2-disc into M so that its boundary is embedded as K and its interior is mapped into $\operatorname{int} M$. We can deform the map (rel. boundary) slightly into general position, when it becomes an immersion whose self-intersections are at a finite number, n say, of points where exactly two sheets cross transversely (we shall call this an *n-point immersion;* a 0-point immersion is an embedding).

Lemma 1. *Let D be an n-point immersed 2-disc in $M (n>0)$. Suppose there exists in* $\operatorname{int} M$ *an embedded 2-sphere S^2 which intersects D transversely and in a single point q. Then there exists in M an $(n-1)$-point immersed 2-disc with boundary ∂D. If further S^2 has trivial normal bundle then there exists in M an embedded 2-disc with boundary ∂D.*

Proof. Let p be a self-intersection of D and let D_1 be a small 2-disc neighbourhood of p in one sheet of D. Let D_2 be a small 2-disc neighbourhood of q in S^2. Take an arc pq in D disjoint from the other self-intersections of D, and from $D_1 \setminus \{p\}$. We can construct a thin tube $D^1 \times S^1$ (with its axis along pq) joining ∂D_1 to ∂D_2 with its interior disjoint from D and S^2. The $(n-1)$-immersed disc is

$$(D \setminus \operatorname{int} D_1) \cup (D^1 \times S^1) \cup (S^2 \setminus \operatorname{int} D_2)$$

with the corners smoothed (see Fig. 1).

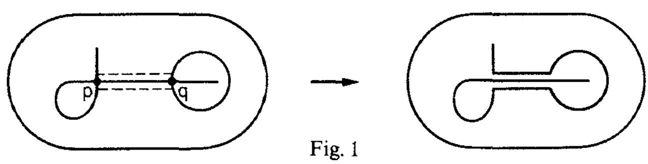

Fig. 1

If S^2 has trivial normal bundle we can use the above construction, with a sequence of cross-sections as the 2-spheres, to eliminate all the self-intersections.

* R. A. Norman died in an accident on the Welsh mountains in the spring of 1967. This paper has been compiled from some of his results.

It is well known (and, in fact, an easy consequence of Lemma 1) that if the knot K contains points of int M then it does indeed bound a 2-disc in M. We consider, then, the case where $K \subset \partial M$.

Theorem 2. *Let the 4-manifold M be a connected sum $N \# (S^2 \times S^2)$. Let K be any knot in ∂M. Suppose that K is homotopic to zero in M. Then K bounds an embedded 2-disc in M.*

Proof. Clearly K homotopic to zero in M implies K homotopic to zero in N. So K bounds in N an n-point immersed 2-disc, D say, for some n. Choose a generator of the first factor of $S^2 \times S^2$ and take the connected sum of pairs $(N, D) \# (S^2 \times S^2, \text{generator})$. Now $D \#$ generator is an n-point immersed 2-disc, and any generator of the second factor of $S^2 \times S^2$ is a sphere with trivial normal bundle. We can apply Lemma 1 for the result.

Corollary 3. *Any knot in S^3 bounds a 2-disc in M, where*

$$M = (S^2 \times S^2) \backslash \operatorname{int} D^4 \quad \textit{and} \quad S^3 = \partial M.$$

Remark. Theorem 2 remains true with $S^2 \times S^2$ replaced by T, the other 2-sphere bundle over S^2 (see [2]). For take $D \#$ (cross-section) and cancel its self-intersections with fibres.

If M is as in Corollary 3, and α and β are the obvious generators of $H_2(M, \partial M)$, then the embedded 2-disc constructed by the method of Theorem 2 represents $\alpha + m\beta$ for some integer m. In fact it is not hard to see that we can arrange for $m = 0$. (Cf. the proof of Theorem 4 below.) A natural question that arises is "Given a knot K in S^3, is there a manifold M with $\partial M = S^3$, such that K bounds in M a disc which represents $0 \in H_2(M, \partial M)$?" The answer is in the affirmative. A suitable M is $rT \backslash \operatorname{int} D^4$ (where rT denotes the connected sum of r copies of the manifold T of the above remark, and the integer r depends on K).

Theorem 4. *For any knot K in S^3, there is an r such that K bounds a 2-disc D in $M = rT \backslash \operatorname{int} D^4$, where $S^3 = \partial M$, and D represents $0 \in H_2(M, \partial M)$.*

Proof. Consider any projection of K. It is well known that on changing certain undercrossings of K to overcrossings the new knot K^* obtained is trivial. This implies that there is in $S^3 \times [0, 1]$ a properly embedded

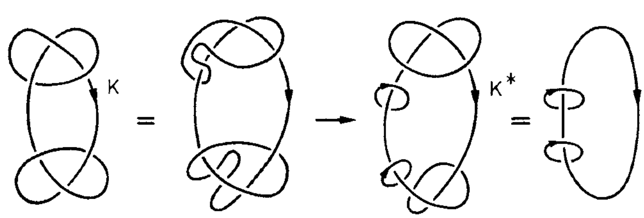

Fig. 2

surface of genus 0 with part of its boundary, namely K, in $S^3 \times \{0\}$, and the rest (in $S^3 \times \{1\}$) consisting of K^* together with some circles which link K^* once and each other not at all (see Fig. 2).

Moreover we may suppose that the same number, r say, of the circles link K^* left-handedly as link it right-handedly, for we may introduce new ones at will (Fig. 3).

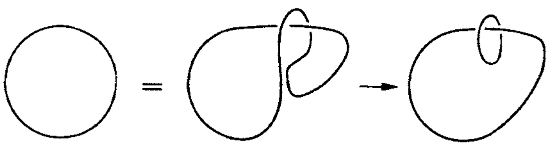

Fig. 3

In $S^3 \times [1, 2]$ we properly embed another surface of genus 0 such that its intersection with $S^3 \times \{1\}$ is K^* and its linking circles, and with $S^3 \times \{2\}$ is $2r$ links of the most elementary kind (Fig. 4).

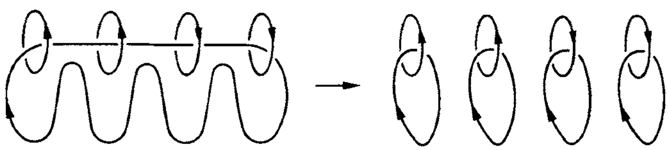

Fig. 4

Now attach to a 3-disc in $S^3 \times \{2\}$ containing exactly one such link a copy of $CP^2 \setminus \operatorname{int} D^4$ by a 3-disc in its boundary. The circles of the link bound disjoint 2-discs in $CP^2 \setminus \operatorname{int} D^4$, and (if CP^2 is suitably oriented) the union of the 2 discs represents zero in the relative homology group. Do this for each link. The boundary of $S^3 \times [0, 2]$ plus attachments is $S^3 \times \{0\}$ together with another 3-sphere. Attach to the latter a final 4-disc and smooth all corners. The resulting manifold is $rT \setminus \operatorname{int} D^4$ (see Lemma 1 of [2]), and the 2-disc bounding K which we have constructed in it has the required property.

Any 3-dimensional 2-handlebody-boundary (using the notation of Smale [1], any boundary of a manifold in $\mathscr{H}(4, k, 2)$) is formed from S^3 by spherical modifications. That is to say, one removes from S^3 the interiors of a set of disjoint embedded solid tori $(D^2 \times \partial D^2)_i$ and attaches solid tori $(\partial D^2 \times D^2)_i$ along the boundaries. We denote by S_i the embedded circle $(\partial D^2 \times 0)_i$ in the resulting 3-manifold.

Lemma 5. *Let M be a 4-manifold with ∂M a 2-handlebody boundary. If the circles S_i in ∂M bound disjoint locally flatly embedded 2-discs in M, then any PL knot in ∂M bounds a PL embedded 2-disc in M.*

Proof. We may assume that the discs bounded by the circles S_i are properly embedded (i.e. with interiors in int M) for all i. We ambient isotope the given knot off the handles $(\partial D^2 \times D^2)_i$ to obtain a knot K which in S^3 possibly links the solid tori $(D^2 \times \partial D^2)_i$. We can decompose K as the sum of a knot K^* lying in a 3-disc D^3 in S^3 disjoint from the solid tori and a finite set of circles in S^3 each linking exactly one solid torus exactly once (see Fig. 5).

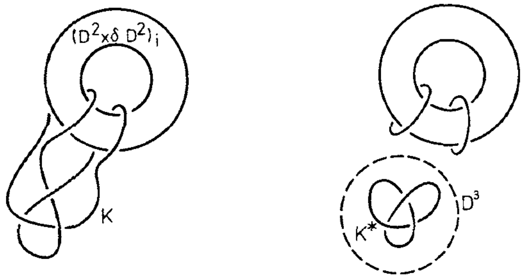

Fig. 5

These circles are essentially the S_i above and thus bound 2-discs. If exactly one circle links each solid torus then the discs can be chosen disjoint by hypothesis. If several link the same torus, choosing cross-sections of the normal bundle of the 2-disc bounded by the relevant S_i enables us to span all the circles with disjoint discs (in this case it may be necessary to have the circles linked with one another in S^3, but this is easy to arrange). Now we put a cone on K^* (the cone is a PL 2-disc). We can do this in a small neighbourhood of D^3 disjoint from the previously constructed 2-discs. Finally we join all the discs together to obtain a PL disc bounded by K, and then ambient isotope K back to its original position.

An immediate consequence of this Lemma is

Corollary 6. *If* $M \in \mathscr{H}(4, k, 2)$ *then any PL knot in* ∂M *bounds a PL embedded 2-disc in* M.

Theorem 7. *Let* M *be a* 4*-manifold with* $\partial M = (S^1 \times S^2) \# \cdots \# (S^1 \times S^2)$. *Suppose that the inclusion* $\partial M \subset M$ *induces the zero map on fundamental groups. Then every PL knot in* ∂M *bounds a PL embedded* 2*-disc in* M.

Proof. Let S_i be a generator of the first factor of the i-th copy of $S^1 \times S^2$. Then S_i is homotopic to zero in M by hypothesis. Hence it bounds an n-point immersed 2-disc D_i for some n. By isotoping a generator of the second factor of $S^1 \times S^2$ into int M we construct a 2-sphere in int M which intersects D_i transversely and in a single point. This 2-sphere has trivial normal bundle. By Lemma 1, then, we can assume that D_i is an embedded disc. Moreover using the cancelling technique of the proof of Lemma 1 we can ensure that D_i and D_j are disjoint for $i \neq j$. Hence we may apply Lemma 5 for the result.

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