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# Closures of Conjugacy Classes of Matrices are Normal

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## 0. Introduction

*0.1.* The purpose of this paper is to prove the following theorem: Let $A$ be an $n\times n$ matrix over an algebraically closed field $K$ of characteristic zero, $C_A$ the conjugacy class of $A$ and $\overline{C_A}$ its (Zariski-) closure.

**Theorem.** *$\overline{C_A}$ is normal, Cohen-Macaulay with rational singularities.*

If a variety $X$ with a $G$-action ($G$ reductive) is the closure of an orbit $\mathcal{O}$ and $\dim(X\setminus\mathcal{O})\leqq \dim X-2$, it is a crucial question for the geometry of $X$ to decide whether the singularity (in $X\setminus\mathcal{O}$) is normal. In fact the normality of $X$ allows to identify the ring $K[X]$ of regular functions on $X$ with the functions on the orbit $\mathcal{O}$ and so, by Frobenius reciprocity, to analyse $K[X]$ as a representation of $G$ (cf. [11], [1]). In our case this is closely related to the "multiplicity conjecture" of Dixmier; we refer the reader to the paper [1] for a detailed description of this connection and some applications.

A different proof of this theorem will appear in [23].

*0.2.* The theorem has also another interesting application, shown to us by Th. Vust, in the spirit of the classical theory of Schur. If $U$ is a finite dimensional vector space one has the classical relation between the action of $GL(U)$ and of the symmetric group $\mathfrak{S}_m$ on the tensor space $U^{\otimes m}$. If we restrict to the subgroup $G_A$ of $GL(U)$ centralizing a fixed matrix $A$ ($\in$ End $U$), then one can still compute the centralizer of $G_A$ acting on $U^{\otimes m}$ and one obtains (see Sect. 6): $\mathrm{End}_{G_A}(U^{\otimes m})$ *is spanned by the endomorphisms*

$$\sigma\cdot A^{h_1}\otimes A^{h_2}\otimes\ldots\otimes A^{h_m}\qquad(\sigma\in\mathfrak{S}_m, h_1,\ldots,h_m\in\mathbb{N}).$$

We remark that the group $G_A$ is not reductive and the commuting algebra is not semisimple in general.

*0.3.* In many ways the motivation to study this problem came from a fundamental paper of B. Kostant [11] in which he studies in detail the adjoint action on a semisimple Lie algebra $\mathfrak{g}$. In the course of his analysis he shows the normality of

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the variety $\overline{C_A}$ in the case in which $A$ is a *regular nilpotent element* of $\mathfrak{g}$ (i.e. $\overline{C_A}$ is the *nilpotent cone* of $\mathfrak{g}$). His method depends on the fact that, in this case, $\overline{C_A}$ can be proved to be a *complete intersection* in $\mathfrak{g}$. This is no more true for the non regular classes in general, nevertheless some particular cases were treated by W. Hesselink [8]; we wish to thank him for his comments on an earlier version of the paper. Our method, on the other hand, consists in constructing an auxiliary variety $Z$ which is a complete intersection and of which $\overline{C_A}$ is a "quotient" (1.4).

*0.4. Remark.* It is known (see [8] proposition 1, or use the method of associated cones [1]) that it is sufficient for the proof of the theorem to treat the case of a *nilpotent matrix A* and so we restrict to this case. Then $\overline{C_A}$ has a resolution of singularities $\pi: X \to \overline{C_A}$ where $X$ is the cotangent bundle of $GL_n/P$, $P$ a parabolic subgroup of $GL_n$ ([4], or [1] Anhang). Then the canonical divisor of $X$ is 0 and so by the theorem of Grauert-Riemenschneider (cf. [9] p. 50) it follows that $\overline{C_A}$ has *rational singularities* and the normality of $\overline{C_A}$ is sufficient to insure also the Cohen-Macaulay property. So the main point of the paper is to prove that $\overline{C_A}$ is *normal.* The proof we give should be adaptable also to positive characteristic; it yields at least that the *normalisation of* $\overline{C_A}$ *is purely inseparable over* $\overline{C_A}$ (cf. remark 5.7).

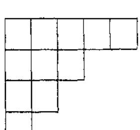
*0.5.* Let us remark finally that the methods developed here have analogues for all the classical groups. In this case, which will be treated in a subsequent paper, there occur different phenomena which are not yet fully understood. Of course the non connected conjugacy classes have non normal closure, but there are also *infinitely many* connected conjugacy classes $C_A$ for which $\overline{C_A}$ is *not normal*; the simplest known cases are: for the symplectic groups the one in $\mathfrak{sp}_8$ relative to the partition (3, 3, 1, 1), for the orthogonal groups the one in $\mathfrak{so}_{13}$ relative to the partition (4, 4, 2, 2, 1).

## 1. Notations, Some Known Results

*1.1.* Let us fix some notations. Any nilpotent matrix is conjugate to one in normal Jordan block form:

$$\begin{pmatrix} J_{p_1} & 0 & 0 & \dots & 0 \\ 0 & J_{p_2} & 0 & & \vdots \\ 0 & 0 & & & \vdots \\ 0 & 0 & \dots & \dots & J_{p_k} \end{pmatrix}, \quad J_t := \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \quad \text{a } t \times t\text{-block.} \tag{$*$}$$

We can assume $p_1 \geqq p_2 \geqq \dots \geqq p_k$; this decreasing sequence $\eta = (p_1, p_2, \dots, p_k)$ is a partition of $n$ and it is convenient to represent it geometrically as a *Young-diagram* with rows consisting of $p_1, p_2, \dots, p_k$ boxes respectively:

e.g. the diagram  corresponds to the partition (5, 3, 2, 1, 1).

The *dual partition* $\hat{\eta}=(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_m)$ is defined setting $\hat{p}_i$ equal to the length of the $i^{th}$ column of the diagram $\eta$; more formally $\hat{p}_i := \#\{j \mid p_j \geqq i\}$. In case of a partition $\eta$ associated to the normal Jordan block form of a nilpotent matrix $A$, the dual partition $\hat{\eta}$ has the following interpretation:

$$\dim \operatorname{Ker} A^j = \sum_{i=1}^{j} \hat{p}_i$$

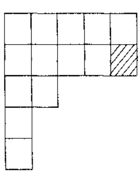
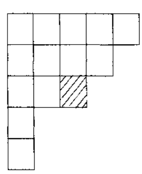
or equivalently

$$\operatorname{rk} A^j = \sum_{i>j} \hat{p}_i.$$

*1.2.* Given two partitions $\eta=(p_1, \ldots, p_s)$ and $\nu=(q_1, \ldots, q_t)$ of $n$, *we say* $\eta \geqq \nu$, *if we have* $\sum_{i=1}^{j} p_i \geqq \sum_{i=1}^{j} q_i$ *for all* $j$. This is equivalent to $\sum_{k>j} \hat{p}_k \geqq \sum_{k>j} \hat{q}_k$ for all $j$.

A simple property of this ordering, which expresses it geometrically, is the following (cf. [7] Proposition 3.9):

**Proposition.** *If* $\eta > \nu$ *and no other partition is in between them* (i.e. $\eta$ and $\nu$ are adjacent in the ordering), *then the diagram of* $\eta$ *is obtained from the one of* $\nu$ *raising a box from one row to the first allowable position.*

(e.g.  $=\eta$ and  $=\nu$)

*1.3.* From now on, if $\eta$ is a partition of $n$, we will indicate with $C_\eta$ the conjugacy class of the matrix $(*)$ in normal Jordan block form with partition $\eta$. The following is the basic theorem on degenerations of orbits (cf. [7] Theorem 3.10 and Corollary 3.8 (a)).

**Proposition.** a) *Given two partitions* $\eta$ *and* $\nu$ *of* $n$, *we have* $\eta \geqq \nu$ *if and only if* $\overline{C_\eta} \supseteq C_\nu$.

b) *If* $\eta=(p_1, \ldots, p_t)$ *is a partition of* $n$ *and* $\hat{\eta}=(\hat{p}_1, \ldots, \hat{p}_k)$ *the dual partition, we have:*

$$\dim C_\eta = n^2 - \sum_{i,j=1}^{t} \min(p_i, p_j) = n^2 - \sum_{i=1}^{k} \hat{p}_i^2 = 2 \sum_{i<j} \hat{p}_i \hat{p}_j.$$

*1.4.* We are working always with affine varieties and we will use the following terminology. If $X$ is an (affine) variety with the action of a reductive group $G$ and $\pi\colon X \to Y$ a morphism, we say that $\pi$ is a *quotient* (under $G$), if the coordinate ring of $Y$ is identified, via $\pi$, with the ring of $G$-*invariant functions* on $X$. We denote this quotient by $\pi\colon X \to X/G$. The following properties of quotient maps are well known ([20], Chap. 1, § 2):

a) *Let* $Z \subseteq X$ *be a* $G$*-stable closed subvariety. Then* $\pi(Z) \subseteq X/G$ *is closed and* $\pi|_Z\colon Z \to \pi(Z)$ *is a quotient.*

b) *Consider the following fibre product:*

$$\begin{array}{ccc} X' := Y \times_{X/G} X & \xrightarrow{\phi'} & X \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\phi} & X/G. \end{array}$$

*The action of $G$ on $X$ induces an action on the fibre product $X'$ in a natural way and $\pi'$ is a quotient with respect to this action.*

## 2. The Induction Lemma

*2.1.* If $U, V$ are vector spaces we will write $\mathrm{L}(U,V)$ for the space of linear maps from $U$ to $V$ and $\mathrm{L}(U)$ instead of $\mathrm{L}(U,U)$. If $V$ is $n$ dimensional and $\eta$ is a partition of $n$, we may consider the elements of $\mathrm{L}(V)$ as $n \times n$ matrices and so $C_\eta \subseteq \mathrm{L}(V)$.

*2.2.* Let $\eta = (p_1, \dots, p_k)$ be a partition of $n$. Erasing the *first column* in the Young diagram $\eta$ one obtains a partition $\eta' = (p'_1, p'_2, \dots, p'_k)$ of $m := n - \hat{p}_1 = n - k$, formally defined by $p'_i = p_i - 1$ for all $i$. In terms of dual partitions we have $\hat{\eta}' = (\hat{p}_2, \hat{p}_3, \dots)$.
Fix vector spaces $U, V$ of dimension $m$, $n$ respectively and consider the two maps

$$\begin{array}{ccc} \mathrm{L}(U,V) \times \mathrm{L}(V,U) & \xrightarrow{\pi} & \mathrm{L}(U) \\ \big\downarrow{\scriptstyle \rho} & & \\ \mathrm{L}(V) & & \end{array}$$

defined by $\pi(A,B) = BA$, $\rho(A,B) = AB$.

**Theorem** (First fundamental theorem of invariant theory): *$\pi$ and $\rho$ are quotient maps* (under $GL(V)$, $GL(U)$ respectively) *and the image of $\rho$ is the determinantel variety of matrices of rank $\leqq m$.* (cf. [22] § 3, Théorème 3 or [18] II.6, Theorem 2.6. A; for a characteristic free proof see [2] § 3.)

*2.3.* Consider finally the orbits $C_{\eta'} \subseteq \mathrm{L}(U)$, $C_\eta \subseteq \mathrm{L}(V)$ and the variety $N_\eta := \pi^{-1}(\overline{C_{\eta'}})$.

**Lemma.** $\rho(N_\eta) = \overline{C_\eta}$:

$$\begin{array}{ccc} \pi^{-1}(\overline{C_{\eta'}}) = N_\eta & \xrightarrow{\pi} & \overline{C_{\eta'}} \\ \big\downarrow{\scriptstyle \rho} & & \\ \overline{C_\eta} & & \end{array}$$

*Proof.* First of all we show that $\rho(N_\eta) \subseteq \overline{C_\eta}$, using repeatedly Proposition 1.3 a). Let $(A,B) \in N_\eta$, i.e. $BA \in \overline{C_{\eta'}}$. To prove that $AB \in \overline{C_\eta}$ we must verify that, for any

$i \geqq 1$, we have $\operatorname{rk}(AB)^i \leqq \sum_{j \geqq i+1} \hat{p}_j$. Now $(AB)^i = A(BA)^{i-1} B$, so $\operatorname{rk}(AB)^i \leqq \operatorname{rk}(BA)^{i-1} \leqq \sum_{j \geqq i} \hat{p}'_j = \sum_{j \geqq i+1} \hat{p}_j$ as desired.

To show that $\rho(N_\eta) = C_\eta$ it is sufficient to prove that $C_\eta \subseteq \rho(N_\eta)$ (since $\rho$ is a quotient map and so $\rho(N_\eta)$ is closed, cf. 1.4a). Let us then fix $D \in C_\eta$, $D\colon V \to V$. We can clearly identify $U$ with $D(V)$ since $\operatorname{rk} D = m$. It is immediate to verify that $D|_U$ has Young diagram $\eta'$ and clearly we have a factorization

$$\begin{tikzcd} V \arrow[rr, "D"] \arrow[dr, "B"'] & & V \\ & U \arrow[ur, "A"'] & \end{tikzcd}$$

where $A$ is the inclusion and $B$ coincides with $D$. On the other hand $BA$ is just $D|_U$ so that the pair $(A, B)$ is in $N_\eta$ and the claim is proved[1]. qed.

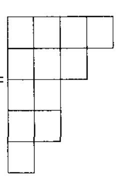
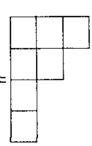
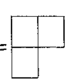
We will use this lemma to present the variety $\overline{C_\eta}$ as a quotient of a suitable variety $Z$ for which we will be able to prove normality (Theorem 3.3).

## 3. The Variety $Z$

*3.1.* Notations being as in section 2 we make the following construction. Starting with a fixed partition $\eta = (p_1, \ldots, p_k)$ and dual partition $\hat{\eta} = (\hat{p}_1, \ldots, \hat{p}_t)$, $t := p_1$, we define a sequence of partitions

$$\eta_t := \eta, \quad \eta_{t-1}, \quad \eta_{t-2}, \ldots, \eta_1$$

by $\eta_{i-1} := \eta'_i$ (i.e. by erasing successively the first column of the corresponding Young diagrams);

e.g. $\eta = \eta_4 =$  , $\eta_3 =$  , $\eta_2 =$  , $\eta_1 =$  .

Then $\eta_i$ is a partition of $n_i := \hat{p}_t + \hat{p}_{t-1} + \ldots + \hat{p}_{t-i+1}$ with dual partition $\hat{\eta}_i = (\hat{p}_{t-i+1}, \ldots, \hat{p}_{t-1}, \hat{p}_t)$.

Construct next vector spaces $U_1, U_2, \ldots, U_t$ of dimensions $n_1, n_2, \ldots, n_t$ respectively and consider the affine spaces:

$$\begin{aligned} M := {} & \mathrm{L}(U_1, U_2) \times \mathrm{L}(U_2, U_1) \times \mathrm{L}(U_2, U_3) \\ & \times \mathrm{L}(U_3, U_2) \times \ldots \times \mathrm{L}(U_{t-1}, U_t) \times \mathrm{L}(U_t, U_{t-1}) \end{aligned}$$

and

$$N := \mathrm{L}(U_1) \times \mathrm{L}(U_2) \times \ldots \times \mathrm{L}(U_{t-1}).$$

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[1] This argument due to W. Hesselink replaces a more direct matrix computation we had made.

We will indicate a point $\alpha$ of $M$ by

$$\alpha=(A_1,B_1,A_2,B_2,\dots,A_{t-1},B_{t-1})$$

where $A_i\colon U_i\to U_{i+1}$, $B_i\colon U_{i+1}\to U_i$.

*3.2.* We are now ready to define the variety $Z$. *It is the subvariety of $M$ defined by the equations*

$$\begin{aligned}
&B_1A_1=0\\
&B_2A_2=A_1B_1\\
&B_3A_3=A_2B_2\\
&\quad\vdots\\
&B_{t-1}A_{t-1}=A_{t-2}B_{t-2}.
\end{aligned}\tag{**}$$

In more suggestive notation we write

$$\alpha\colon U_0=0\underset{B_0=0}{\overset{A_0=0}{\rightleftarrows}}U_1\underset{B_1}{\overset{A_1}{\rightleftarrows}}U_2\underset{B_2}{\overset{A_2}{\rightleftarrows}}U_3\dots U_{t-1}\underset{B_{t-1}}{\overset{A_{t-1}}{\rightleftarrows}}U_t$$

for the elements of $Z$. The equations just require that for each $i=1\dots t-1$ the two compositions $U_{i-1}\rightleftarrows U_i\rightleftarrows U_{i+1}$ yield the same endomorphism of $U_i$. (See also [19] 5.3, where this objects occur as representations of a certain Lie algebra.)

In due time we will prove that the equations we have given actually define a reduced variety; for the moment we think of $Z$ as a scheme, possibly not reduced, and we indicate by $Z_{\mathrm{red}}$ the reduced variety associated.

The best way to understand the equations is to construct a map $\Phi\colon M\to N$ given by the formula:

$$\begin{aligned}
&\Phi(A_1,B_1,A_2,B_2,\dots,A_{t-1},B_{t-1})\\
&\quad=(B_1A_1,B_2A_2-A_1B_1,B_3A_3-A_2B_2,\dots,B_{t-1}A_{t-1}-A_{t-2}B_{t-2}).
\end{aligned}$$

Then $Z$, as a scheme, is the fiber $\Phi^{-1}(0)$ of the 0 point in $N$.

*3.3.* Consider the group $G:=GL(U_1)\times GL(U_2)\times\dots\times GL(U_t)$ and its normal subgroup $H:=GL(U_1)\times GL(U_2)\times\dots\times GL(U_{t-1})$. The group $G$ acts on $M$ and $N$ in a natural way:

On

$$\begin{aligned}
&M\colon (g_1,g_2,\dots,g_t)(A_1,B_1,A_2,B_2,\dots,A_{t-1},B_{t-1})\\
&\quad:=(g_2A_1g_1^{-1},g_1B_1g_2^{-1},\dots,g_tA_{t-1}g_{t-1}^{-1},g_{t-1}B_{t-1}g_t^{-1})
\end{aligned}$$

and on

$$\begin{aligned}
&N\colon (g_1,g_2,\dots,g_t)(E_1,E_2,\dots,E_{t-1})\\
&\quad:=(g_1E_1g_1^{-1},\dots,g_{t-1}E_{t-1}g_{t-1}^{-1}).
\end{aligned}$$

It is easy to verify that the map $\Phi$: $M \to N$ is equivariant under $G$ and so $Z$ is invariant under $G$. Now the main theorem is an immediate consequence of the following more precise result (use the fact that a quotient of a normal variety is also normal):

**Theorem.** i) *$Z$ is a complete intersection in $M$; the equations (**) give a regular sequence.*

ii) *$Z$ is non singular in codimension* 1.

iii) *$Z$ is reduced, irreducible and normal.*

iv) *There is an isomorphism $Z/H \xrightarrow{\sim} \overline{C_\eta}$ being compatible with the actions of $GL(U_t) = G/H$.*

The rest of the paper is devoted to the proof of this theorem. We first reduce it to a lemma (3.7) whose proof will be given in Sect. 5 (5.4, 5.5, 5.6) using the theory of nilpotent pairs (section 4) and a dimension formula for nilpotent pair orbits (5.3).

*3.4.* First of all we settle part iv) which gives the connection between $Z$ and $\overline{C_\eta}$. We consider the map

$$\Theta: M \to \mathrm{L}(U_t)$$

given by $(A_1, B_1, \ldots, A_{t-1}, B_{t-1}) \mapsto A_{t-1} B_{t-1}$, which is clearly $GL(U_t)$ equivariant.

**Proposition.** *$\Theta(Z_{\mathrm{red}}) = \overline{C_\eta}$ and the induced map $\Theta'$: $Z_{\mathrm{red}} \to \overline{C_\eta}$ is a quotient map under $H$* (i.e. $Z_{\mathrm{red}}/H \xrightarrow{\sim} \overline{C_\eta}$).

*Proof.* We use repeatedly lemma 2.3. Since $B_1 A_1 = 0$ the pair $(A_1, B_1)$ is in the variety $N_{\eta_2}$ and so $A_1 B_1 \in \overline{C_{\eta_2}}$. By induction we may assume $A_{i-1} B_{i-1} \in \overline{C_{\eta_i}}$. Since $\underline{B_i A_i} = A_{i-1} B_{i-1}$ we have that $(A_i, B_i) = N_{\eta_{i+1}}$ and so again by 2.3, $A_i B_i \in \overline{C_{\eta_{i+1}}}$. Thus finally $\Theta$ maps $Z_{\mathrm{red}}$ into $\overline{C_\eta} = \overline{C_{\eta_t}}$ and the same lemma 2.3 applied inductively shows that $Z_{\mathrm{red}}$ is mapped onto $\overline{C_\eta}$. To see that the map is a quotient under $H$ we perform the quotients in succession. First under $GL(U_1)$, we have the quotient map (Theorem 2.2)

$$\begin{aligned}\Theta_1: M \to\ & \mathrm{L}(U_2) \times \mathrm{L}(U_2, U_3) \times \mathrm{L}(U_3, U_2) \times \ldots \\ & \times \mathrm{L}(U_{t-1}, U_t) \times \mathrm{L}(U_{t-1}\, U_{t-1})\end{aligned}$$

given by

$$(A_1, B_1, A_2, B_2, \ldots, A_{t-1}, B_{t-1}) \mapsto (A_1 B_1, A_2, B_2, \ldots, A_{t-1}, B_{t-1}).$$

If we restrict this map to $Z_{\mathrm{red}}$ we have again a quotient map (since we are in characteristic zero cf. 1.4a). Now on $Z_{\mathrm{red}}$ we have $A_1 B_1 = B_2 A_2$ (if $t > 2$); thus we see that $\Theta_1$ maps $Z_{\mathrm{red}}$ into the graph of the map

$$\begin{aligned}\gamma: M_1 := &\ \mathrm{L}(U_2, U_3) \times \mathrm{L}(U_3, U_2) \times \ldots \\ & \times \mathrm{L}(U_{t-1}, U_t) \times \mathrm{L}(U_t, U_{t-1}) \to \mathrm{L}(U_2) \\ & (A_2, B_2, A_3, B_3, \ldots, A_{t-1}, B_{t-1}) \mapsto B_2 A_2.\end{aligned}$$

Thus we may replace the graph of $\gamma$ with its domain and drop the first coordinate $A_1 B_1$. Hence on $Z_{\mathrm{red}}$ the mapping

$$(A_1, B_1, \ldots, A_{t-1}, B_{t-1}) \mapsto (A_2, B_2, \ldots, A_{t-1}, B_{t-1}) \in M_1$$

is a quotient under $GL(U_1)$. Its image $Z_1 \subseteq M_1$ is easily seen to be defined by the "equations":

$$\begin{aligned} &B_2 A_2 \in \overline{C_{\eta_2}} \\ &A_2 B_2 = B_3 A_3 \\ &A_3 B_3 = B_4 A_4 \\ &\quad \vdots \\ &A_{t-2} B_{t-2} = B_{t-1} A_{t-1}. \end{aligned}$$

Similarly (if $t > 3$) $Z_1/GL(U_2) = Z_{\mathrm{red}}/GL(U_1) \times GL(U_2)$ is naturally contained in

$$M_2 := \mathrm{L}(U_3, U_4) \times \mathrm{L}(U_4, U_3) \times \ldots \times \mathrm{L}(U_{t-1}, U_t) \times \mathrm{L}(U_t, U_{t-1})$$

and given by the "equations":

$$\begin{aligned} &B_3 A_3 \in \overline{C_{\eta_3}} \\ &A_3 B_3 = B_4 A_4 \\ &\quad \vdots \\ &A_{t-2} B_{t-2} = B_{t-1} A_{t-1}. \end{aligned}$$

Then finally by induction $Z_{\mathrm{red}}/GL(U_1) \times GL(U_2) \times \ldots \times GL(U_{t-2})$ is given by

$$\{(A_{t-1}, B_{t-1}) \mid B_{t-1} A_{t-1} \in \overline{C_{\eta_{t-1}}}\} \subseteq \mathrm{L}(U_{t-1}, U_t) \times \mathrm{L}(U_t, U_{t-1}),$$

i.e. it is the variety $N_\eta = N_{\eta_t}$, and hence

$$Z_{\mathrm{red}}/H \cong N_\eta/GL(U_{t-1}) \cong \overline{C_\eta}$$

(the isomorphism being induced by $\Theta$). qed.

This reasoning can be also displayed in a more suggestive way constructing a diagram, e.g. $t=5$:

$$\begin{array}{ccccccccc}
Z(1,5) & \to & Z(1,4) & \to & Z(1,3) & \to & N_{\eta_1} & \to & \overline{C_{\eta_1}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
Z(2,5) & \to & Z(2,4) & \to & N_{\eta_2} & \to & \overline{C_{\eta_2}} & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
Z(3,5) & \to & N_{\eta_3} & \to & \overline{C_{\eta_3}} & & & & \\
\downarrow & & \downarrow & & & & & & \\
N_{\eta_4} & \to & \overline{C_{\eta_4}} & & & & & & \\
\downarrow & & & & & & & & \\
\overline{C_{\eta_5}} & & & & & & & &
\end{array}$$

where each $Z(i,j)$ is constructed inductively forming a fiber product. If we proceed on a column we see that

$$Z(k,n) \xleftarrow{\sim} Z(1,n)/GL(U_1)\times GL(U_2)\times \ldots \times GL(U_{k-1}) \quad \text{for } k<n-1$$

(cf. 1.4b).

3.5. We now make a simple remark on the basic map $\Phi$: $M \to N$ (3.2) for which $Z=\Phi^{-1}(0)$.

Let $M^0$ be the open subset of $M$ of those elements $(A_1,B_1,A_2,B_2,\ldots,A_{t-1},B_{t-1})$ such that *for each* $i=1,2,\ldots,t-1$ *either* $A_i$ *or* $B_i$ *has maximal rank*. Then we have:

**Proposition.** *The differential* $d\Phi$ *of* $\Phi$ *is onto at every point* $\alpha$ *of* $M^0$.

*Proof.* Let $\alpha=(A_1,B_1,A_2,B_2,\ldots,A_{t-1},B_{t-1})\in M^0$. We can identify the tangent space of $M$ in $\alpha$ with $M$ itself and take a point $T=(X_1,Y_1,X_2,Y_2,\ldots,X_{t-1},Y_{t-1})$ in it. Then the tangent map gives

$$\begin{aligned} d\Phi_\alpha(T)=(&Y_1A_1+B_1X_1, Y_2A_2+B_2X_2-X_1B_1-A_1Y_1,\ldots,Y_{t-1}A_{t-1}\\ &+B_{t-1}X_{t-1}-X_{t-2}B_{t-2}-A_{t-2}Y_{t-2}). \end{aligned}$$

If $W=(W_1,W_2,\ldots,W_{t-1})$ is any tangent vector in $\Phi(\alpha)\in N$ we can solve inductively the equations

$$\begin{aligned} &Y_1A_1+B_1X_1=W_1\\ &Y_2A_2+B_2X_2-X_1B_1-A_1Y_1=W_2\\ &\vdots\\ &Y_{t-1}A_{t-1}+B_{t-1}X_{t-1}-X_{t-2}B_{t-2}-A_{t-2}Y_{t-2}=W_{t-1} \end{aligned}$$

provided that for each $i$ either $A_i$ or $B_i$ has maximal rank. In fact if $A_i$ has maximal rank then there is an $\bar{A}_i$: $U_{i+1}\to U_i$ with $\bar{A}_iA_i=Id_{U_i}$, so the equation $Y_iA_i=R_i$ is solved by $Y_i:=R_i\bar{A}_i$. Similarly if $B_i$ has maximal rank then there is an element $\bar{B}_i$: $U_i\to U_{i+1}$ with $B_i\bar{B}_i=Id_{U_i}$ and the equation $B_iX_i=S_i$ is solved by $X_i:=\bar{B}_iS_i$. qed.

3.6. The net result of this proposition is this:

**Corollary.** *The open subvariety* $Z^0:=Z\cap M^0$ *of* $Z$ *is smooth and of codimension* $\sum_{i=1}^{t-1} n_i^2$ in $M$.

*Proof.* The only thing to prove is that $Z^0\neq\emptyset$ since then the statement is a consequence of 3.5 $\left(\dim N=\sum_{i=1}^{t-1} n_i^2 \text{ by definition 3.1}\right)$. Now if we recall the proof of 3.4 we see that we have constructed an element in $Z$ such that for all $i$ both $A_i$ and $B_i$ have maximal rank (see also the construction in 3.1). More precisely if $D$: $U\to U$ is any element of $C_\eta$ we may assume $U_t=U$, $U_{t-1}=D(U)$, $U_{t-2}=D^2(U),\ldots,U_1=D^{t-1}(U)$, $U_0=0$. Setting $A_i$: $U_i\to U_{i+1}$ the inclusion, and $B_i$: $U_{i+1}\to U_i$ the map $D$ itself, we have the required element. qed.

*Remark.* If we insist that for each $i$ both $A_i$ and $B_i$ have maximal rank we still get a non empty open set $Z'$ of $Z$. One can easily show that $Z'$ is an orbit under $G$ (cf. proof above). It will be proved in fact that this is the unique open orbit of $G$ in $Z$ (5.4).

*3.7.* To complete the proof of the theorem it is enough to show the following result:

**Lemma.** $\dim(Z \setminus Z^0) \leqq \dim Z - 2$.

In fact using 3.6 this lemma implies $\dim Z = \dim Z^0$ and that $Z$ is non singular in codimension one. Thus again by 3.6 we have that the codimension of $Z$ in $M$ is exactly the number $\sum_{i=1}^{t-1} n_i^2$ of equations defining $Z$ (3.2). This implies that these equations form a regular sequence and hence $Z$ is a complete intersection. Since $Z$ is a cone it is also connected. But then by Serre's criterion ([6] IV, Théorème 5.8.6) $Z$ is normal reduced and so also irreducible. This completes the proof of the theorem 3.3 modulo the lemma above.

For this basic statement we will need to stratify the complement of $Z^0$ in $Z$ with strata of which we can compute the dimension (5.1, 5.3) and this will lead us to the theory of nilpotent pairs (cf. the following section).

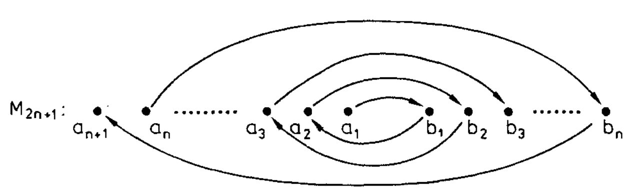
## 4. Nilpotent Pairs

Given two vector spaces $U, V$ we consider the space $L := \mathrm{L}(U, V) \times \mathrm{L}(V, U)$ of pairs of maps $U \underset{B}{\overset{A}{\rightleftarrows}} V$ as a representation of $GL(U) \times GL(V)$ in a canonical way:

$$(X, Y)(A, B) = (YAX^{-1}, XBY^{-1}).$$

The theory of orbits for this representation is known (cf. [3], [14], or [5]) and it is in fact a special case of the theory of vector space crowns. One can naturally think of such pairs as of a category of modules, and the classification is (like in the case of Jordan blocks) through indecomposable modules. Also the "invariant theory" of this representation is well known (see [12] and also [17]). In our case we are interested in a special class of pairs, those $U \underset{B}{\overset{A}{\rightleftarrows}} V$ for which $BA$ (or equivalently $AB$) is *nilpotent*. We will call such pairs "*nilpotent pairs*". They can be easily seen to be exactly the unstable vectors (in the sense of geometric invariant theory) of the representation $L$.

*4.2.* The classification of the *indecomposable nilpotent pairs* is rather simple and resembles the theory of Jordan blocks. The indecomposables are of the following types:



i.e. the space $U$ is spanned by the basis $a_1, a_2, \dots, a_{n+1}$, $V$ has basis $b_1, b_2, \dots, b_n$ and

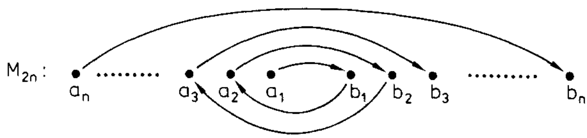
$$A\,a_i = b_i, \qquad B\,b_j = a_{j+1}.$$

This type will be in short indicated by a string

*abababab…ba*

with $n+1$ $a$'s and $n$ $b$'s.

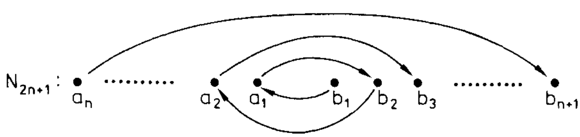
The type

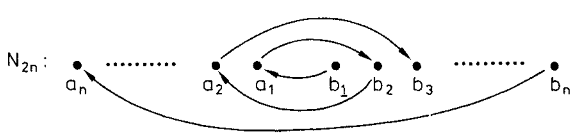



is defined in a similar way and is shortly indicated by the string

*abab…ab*

with $n$ $a$'s and $n$ $b$'s.
We have two other types starting with $b$ instead of $a$:



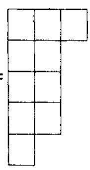

shortly indicated by *baba…bab* and *baba…ba* respectively.

*4.3.* In general a nilpotent pair $U \underset{B}{\overset{A}{\rightleftarrows}} V$ is a direct sum of indecomposables and so it will be determined by a finite set of such strings (*ab-strings*). We will refer to such a set of strings as the *ab-diagram of the pair.* It is easy to see that two distinct *ab*-diagrams give rise to non isomorphic pairs, since one can easily recover the *ab*-diagram from the knowledge of the ranks of all the compositions $BABA\dots$.

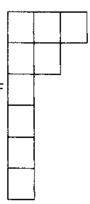
Given a nilpotent pair $U \underset{B}{\overset{A}{\rightleftarrows}} V$ through its *ab*-diagram $\delta$, it is simple to recognise the Young diagrams of the nilpotent matrices $BA\colon U \to U$ and $AB\colon V \to V$: For the diagram of $BA$ *suppress all the b's in the ab-strings of* $\delta$. In this way every *ab*-string gives rise to a string of $a$'s which can be interpreted as a row in a Young diagram. Similarly for $AB$ one has to suppress all the $a$'s. We call these diagrams the *associated a-diagram* and the *associated b-diagram* of $\delta$ and denote them as in 2.2 by $\pi(\delta)$ and $\rho(\delta)$.

$$\text{e.g. } \delta := \begin{array}{l} a\,b\,a\,b\,a\,b \\ b\,a\,b\,a \\ a\,b\,a \\ a\,b\,a \\ b\,a \\ b \end{array}$$

then

$$\pi(\delta) = \begin{array}{l} a\,a\,a \\ a\,a \\ a\,a \\ a\,a \\ a \end{array} =$$

$$= (3, 2, 2, 2, 1)$$

is the Young diagram of $BA \in \mathrm{L}(U)$ and

$$\rho(\delta) = \begin{array}{l} b\,b\,b \\ b\,b \\ b \\ b \\ b \\ b \end{array} =$$

$$= (3, 2, 1, 1, 1, 1)$$

is that of $AB \in \mathrm{L}(V)$.

*4.4.* One should make three remarks:

*Remark 1.* Not all pairs of Young diagrams describing a nilpotent orbit in $\mathrm{L}(U)$ and one in $\mathrm{L}(V)$ are associated to some nilpotent pair. Furthermore, there can be different nilpotent pairs giving rise to the same pair of Young diagrams.

*Remark 2.* For a nilpotent pair $(A, B)$ with *ab*-diagram $\delta$ one can immediately verify the following:

i) *$A$ is injective if and only if every ab-string in $\delta$ ends with b.*
ii) *$A$ is surjective if and only if every ab-string in $\delta$ starts with a.*
iii) *$B$ is injective if and only if every ab-string in $\delta$ ends with a.*
iv) *$B$ is surjective if and only if every ab-string in $\delta$ starts with b.*

*Remark 3.* For any $ab$-diagram $\delta$ we denote by $X_\delta$ its orbit in $L$ (under the group $GL(U) \times GL(V)$). We have the two maps

$$\begin{tikzcd} X_\delta \arrow[r, "\pi'"] \arrow[d, "\rho'"'] & C_{\pi(\delta)} \\ C_{\rho(\delta)} & \end{tikzcd}$$

induced by $\pi$ and $\rho$ (2.2) which are *fibrations* (being of the form $G/H \to G/H'$ with closed subgroups $H \subseteq H' \subseteq G := GL(U) \times GL(V)$). In particular $\pi'$ and $\rho'$ are *smooth*.

## 5. Nilpotent Strings, Proof of Lemma 3.7

*5.1.* We want to go back to the basic variety $Z$ (3.2) formed by strings

$$\alpha\colon\ U_0 = 0 \underset{B_0=0}{\overset{A_0=0}{\rightleftarrows}} U_1 \underset{B_1}{\overset{A_1}{\rightleftarrows}} U_2 \underset{B_2}{\overset{A_2}{\rightleftarrows}} U_3 \dots U_{t-1} \underset{B_{t-1}}{\overset{A_{t-1}}{\rightleftarrows}} U_t$$

with the conditions $B_{i+1}A_{i+1} = A_i B_i$ for $i = 0, 1, \dots, t-2$. Let us indicate by $Y_0 = \{\emptyset\}$, $Y_1, Y_2, \dots, Y_t$ the (finite) sets of diagrams indexing nilpotent conjugacy classes in $\mathrm{L}(U_0), \mathrm{L}(U_1), \dots, \mathrm{L}(U_t)$ respectively and by $\Psi_0, \Psi_1, \dots, \Psi_{t-1}$ the (finite) sets of *ab*-diagrams indexing conjugacy classes of nilpotent pairs in

$$\mathrm{L}(U_0, U_1) \times \mathrm{L}(U_1, U_0), \mathrm{L}(U_1, U_2) \times \mathrm{L}(U_2, U_1), \dots,$$
$$\mathrm{L}(U_{t-1}, U_t) \times \mathrm{L}(U_t, U_{t-1})$$

respectively. We have the already described maps associated to the two compositions (cf. 4.3):

$$Y_0 \xleftarrow{\pi_0} \Psi_0 \xrightarrow{\rho_0} Y_1 \xleftarrow{\pi_1} \Psi_1 \xrightarrow{\rho_1} Y_2$$
$$\xleftarrow{\pi_2} \Psi_2 \xrightarrow{\rho_2} Y_3 \leftarrow \dots \to \Psi_{t-1} \xrightarrow{\rho_{t-1}} Y_t.$$

We can form the iterated fiber products and construct the finite set $\Lambda$ of strings $\lambda = (\delta_0, \delta_1, \delta_2, \dots, \delta_{t-1})$ of *ab*-diagrams $\delta_i \in \Psi_i$ with

$$\rho_i(\delta_i) = \pi_{i+1}(\delta_{i+1}), \qquad i = 0, 1, \dots, t-2.$$

For each $\lambda \in \Lambda$ we have a stratum $Z_\lambda$ of the variety $Z$: *$Z_\lambda$ is the set of all points*

$$\alpha\colon\ 0 = U_0 \underset{B_0}{\overset{A_0}{\rightleftarrows}} U_1 \underset{B_1}{\overset{A_1}{\rightleftarrows}} U_2 \rightleftarrows \dots \rightleftarrows U_{t-1} \underset{B_{t-1}}{\overset{A_{t-1}}{\rightleftarrows}} U_t$$

*of $Z$ such that for each $i$ the nilpotent pair $(A_i, B_i)$ has ab-diagram $\delta_i$.*

Put $\sigma_i := \rho_{i-1}(\delta_{i-1})$, $i = 1, 2, \dots, t$ and let us indicate as usual by $C_{\sigma_i}$ the conjugacy class of diagram $\sigma_i$ and by $X_{\delta_i}$ the nilpotent pair orbit of diagram $\delta_i$. The definition of $\Lambda$ implies that we have a fiber product diagram subordinate to the basic diagram constructing $Z$:

$$
\begin{tikzcd}
Z_\lambda = Z_\lambda(1,t) \arrow[r] \arrow[d] & \dots \arrow[r] & Z_\lambda(1,4) \arrow[r] \arrow[d] & Z_\lambda(1,3) \arrow[r] \arrow[d] & X_{\delta_1} \arrow[r] \arrow[d] & C_{\sigma_1} = (0) \\
Z_\lambda(2,t) \arrow[r] \arrow[d] & \dots \arrow[r] & Z_\lambda(2,4) \arrow[r] \arrow[d] & X_{\delta_2} \arrow[r] \arrow[d] & C_{\sigma_2} & \\
\vdots \arrow[dd] & & X_{\delta_3} \arrow[r] \arrow[d] & C_{\sigma_3} & & \\
 & & C_{\sigma_4} \arrow[dl, dotted, no head] & & & \\
X_{\delta_t} \arrow[r] \arrow[d] & C_{\sigma_{t-1}} & & & & \\
C_{\sigma_t} & & & & &
\end{tikzcd}
$$

in which each map is smooth (4.4 remark 3). Hence we get the following proposition:

**Proposition.** (i) $Z_\lambda$ *Is a locally closed, G-stable, smooth and irreducible subvariety of Z.*

(ii) *The set $\Lambda$ indexes a stratification of Z into smooth G-stable strata.*

*5.2.* The following result now clearly implies lemma 3.7.

**Lemma.** *For all $\lambda \in \Lambda$ either $Z_\lambda \subseteq Z^0$ or* $\dim Z_\lambda \leqq \dim Z - 2$.

The proof will be given in 5.4, 5.5, 5.6 using the following dimension formula for nilpotent pair orbits.

**5.3. Proposition.** *Let $X = X_\delta \subset \mathrm{L}(U, V) \times \mathrm{L}(V, U)$ be a nilpotent pair orbit projecting to the nilpotent conjugacy classes $C_1 \subset \mathrm{L}(U)$ and $C_2 \subset \mathrm{L}(V)$. Then*

$$\dim X_\delta = \tfrac{1}{2}(\dim C_1 + \dim C_2) + \dim U \cdot \dim V - \Delta,$$

$$\Delta := \sum_{i\,\mathrm{odd}} a_i b_i$$

*where $a_i$* (resp. $b_i$) *denotes the number of ab-strings of length i starting with a* (resp. *with b*).

*Proof.* The representation of $GL(U) \times GL(V)$ on $\mathrm{L} := \mathrm{L}(U, V) \times \mathrm{L}(V, U)$ is a $\Theta$-group in the sence of Vinberg [17] (cf. also [12]): Consider the automorphism $\Theta$ of $\mathrm{End}(U \oplus V)$ (and of $GL(U \oplus V)$) given by conjugation with $J = \begin{pmatrix} Id_U & 0 \\ 0 & -Id_V \end{pmatrix}$; then $GL(U) \times GL(V)$ is the fixed point group and $\mathrm{L} \subset \mathrm{End}(U \oplus V)$ the $(-1)$-

eigenspace of $\Theta$. Furthermore we have the following relation between the dimension of $X$ and the dimension of the conjugacy class $C \subset \mathrm{End}(U \oplus V)$ generated by $X$:

$$\dim X = \tfrac{1}{2} \dim C$$

(cf. [17] §2.5 Proposition 5, or [12] 1.3 Proposition 5). In order to calculate $\dim C$ denote by $r_i$ resp. $s_i$ the number of $a$'s resp. $b$'s in the $i^{\text{th}}$ row of the $ab$-diagram $\delta$ associated to $X$. Then the partition of the nilpotent conjugacy class $C$ is given by $(p_1, p_2, \ldots)$, $p_i := r_i + s_i$, hence

$$\dim C = (n+m)^2 - \sum_{i,j} \min(p_i, p_j)$$

(1.3 Proposition b), $n := \dim V$, $m := \dim U$). By definition we have $|r_i - s_i| \leqq 1$ and therefore $\min(p_i, p_j) = \min(r_i, r_j) + \min(s_i, s_j)$ except in the case $p_i = p_j$ odd, $r_i = s_j$ and $r_j = s_i$, where $\min(p_i, p_j) = \min(r_i, r_j) + \min(s_i, s_j) + 1$. This implies

$$\begin{aligned} \dim C &= (n+m)^2 - \sum_{i,j} \min(r_i, r_j) + \sum_{i,j} \min(s_i, s_j) + 2 \cdot \sum_{i \,\mathrm{odd}} a_i b_i \\ &= \dim C_1 + \dim C_2 + 2nm - 2\Delta, \end{aligned}$$

hence the required dimension formula.[2] qed.

Now let $\lambda \in \Delta$, $\lambda = (\delta_0, \ldots, \delta_{t-1})$ and $\lambda' = (\delta_0, \ldots, \delta_{t-2})$. Set $\sigma = \rho_{t-1}(\delta_{t-1})$, $\sigma' = \rho_{t-2}(\delta_{t-2})$, and $\Delta_\lambda = \sum_{i=0}^{t-1} \Delta_i$, $\Delta_i$ the $\Delta$ associated to $\delta_i$.

*Corollary.* $\dim Z_\lambda = \sum_{i=1}^{t-1} n_i n_{i+1} + \frac{1}{2} \dim C_\sigma - \Delta_\lambda$.

*Proof.* We have the fibre product diagram

$$\begin{tikzcd} Z_\lambda \arrow[r] \arrow[d] & Z_{\lambda'} \arrow[d] \\ X_{\delta_{t-1}} \arrow[r] \arrow[d] & C_{\sigma'} \\ C_\sigma & \end{tikzcd}$$

Now the proposition implies

$$\dim X_{\delta_{t-1}} = \tfrac{1}{2}(\dim C_\sigma + \dim C_{\sigma'}) + \dim U_t \cdot \dim U_{t-1} - \Delta_{t-1},$$

so we have by induction:

$$\begin{aligned} \dim Z_\lambda &= \dim Z_{\lambda'} + \dim X_{\delta_{t-1}} - \dim C_{\sigma'} \\ &= \sum_{i=1}^{t-2} n_i \cdot n_{i+1} + \tfrac{1}{2} \dim C_{\sigma'} + \tfrac{1}{2}(\dim C_{\sigma'} + \dim C_\sigma) \end{aligned}$$

[2] This proof was suggested by G. Kempken; it replaces a explicit but lengthy calculation of stabilizers we have made.

$$+n_{t-1}n_t-\dim C_{\sigma'}-\Delta_{t-1}$$
$$=\sum_{i=1}^{t-1} n_i n_{i+1}+\tfrac{1}{2}\dim C_\sigma-\Delta_\lambda. \quad \text{qed.}$$

5.4. We now look, in view of corollary 5.3, at the projection $\Theta\colon Z\to\overline{C_\eta}$ (3.4) and try to study the various strata $Z_\lambda$ which lie on top of a given orbit $C_\sigma$ in $\overline{C_\eta}$. First of all analyze the open orbit $C_\eta$: we have to describe the strings $\lambda=(\delta_0,\delta_1,\dots,\delta_{t-1})$ which lead to $\rho_{t-1}(\delta_{t-1})=\eta$. We claim that *there is only one such string.*

Let us take in general a Young diagram $\sigma$ and let $\sigma'$ be the diagram obtained from $\sigma$ erasing the first column. We want to find an $ab$-diagram such that the $a$-diagram and $b$-diagram associated are $\sigma'$ and $\sigma$.

Given $\sigma$ (as $b$-diagram) and a certain number $m$ of $a$'s, to construct an $ab$-diagram over $\sigma$ one has to proceed as follows: First of all every $b$-string of $\sigma$ has to be filled internally with $a$'s. This requires altogether as many $a$'s as the number of boxes $m'$ in $\sigma'$. If $m$ is equal to $m'$ the $ab$-diagram $\delta$ over $\sigma$ is unique, its associated $a$-diagram is $\sigma'$, and every $ab$-string of $\delta$ starts and ends with $b$. If $m<m'$ there is no $ab$-diagram over $\sigma$. If $m>m'$, after having used the $m'$ $a$'s, one can utilize the remaining $m-m'$ $a$'s in many ways: add an $a$ at the beginning or the end of an $ab$-string or create a row with single $a$.

$$\text{Example:}\quad \sigma=\begin{matrix} b\,b\,b\,b\,b\\ b\,b\,b\\ b\,b\,b\\ b\,b\\ b\end{matrix}\quad=(5,3,3,2,1),\ m'=9;$$

the unique $ab$-diagram with $m=9$ $a$'s is

$$\delta=\begin{matrix} b\,a\,b\,a\,b\,a\,b\,a\,b\\ b\,a\,b\,a\,b\\ b\,a\,b\,a\,b\\ b\,a\,b\\ b\end{matrix}$$

associated to $\sigma$ and

$$\pi(\delta)=\begin{matrix} a\,a\,a\,a\\ a\,a\\ a\,a\\ a\end{matrix}\quad=\sigma'$$

If we give $m=10$ $a$'s (for instance) one easily sees that it is possible to construct 11 $ab$-diagrams over $\sigma$, if $m=11$, we can construct 56 $ab$-diagrams etc.

Summing up the result we see by induction that *there is a unique string* $\lambda^0=(\delta_1^0,\delta_2^0,\dots,\delta_{t-1}^0)$ *such that* $\rho_{t-1}(\delta_{t-1}^0)=\eta=\eta_t$. For this string we have $\rho_i(\delta_i^0)=\eta_{i+1}$ for all $i$. Since every $ab$-string in each $\delta_i^0$ starts and ends with $b$, it follows from 4.4 remark 2 that $Z_{\lambda^0} \subseteq Z^0$ ($Z_{\lambda^0} = Z'$ with the notations of remark 3.6). Hence we get the following result:

**Lemma.** *There is a unique string* $\lambda^0 = (\delta_1^0, \delta_2^0, \ldots, \delta_{t-1}^0)$ *such that* $\rho_{t-1}(\delta_{t-1}^0) = \eta = \eta_t$. *For this string we have* $Z_{\lambda^0} \subseteq Z^0$.

5.5. We now make a further simple remark. By 3.6 and 3.1 we have

$$\dim Z^0 = 2\sum_{i=1}^{t-1} n_i n_{i+1} - \sum_{i=1}^{t-1} n_i^2 = \sum_{i=1}^{t-1} n_i n_{i+1} + \sum_{i=1}^{t-1} n_i(n_{i+1} - n_i)$$

$$= \sum_{i=1}^{t-1} n_i n_{i+1} + \sum_{i<j} \hat{p}_i \hat{p}_j = \sum_{i=1}^{t-1} n_i n_{i+1} + \tfrac{1}{2}\dim C_\eta.$$

Now if $\lambda \neq \lambda^0$, $Z_\lambda$ projects to some orbit $C_\sigma$ with $\sigma < \eta$ and thus $\dim Z_\lambda < \dim Z^0$ (Corollary 5.3). This implies (as one can also verify directly) that $\dim Z_{\lambda^0} = \dim Z^0$ and $Z_{\lambda^0}$ is the unique open orbit of $G$ acting on $Z$. The same estimate shows that, if $\dim C_\sigma \leqq \dim C_\eta - 4$, then $\dim Z_\lambda \leqq \dim Z^0 - 2$.

To complete the proof of 5.2 we are thus restricted to analyze the strings $\lambda$ such that $Z_\lambda$ projects to some $C_\sigma$ with $\dim C_\sigma = \dim C_\eta - 2$. In each degeneration the dimension of a nilpotent orbit decreases by at least 2 (since the orbits are even dimensional, see proposition 1.3b). Thus the only case in which we may have $\dim C_\sigma = \dim C_\eta - 2$ is if the diagram $\sigma$ is obtained from $\eta$ moving down a single box (Proposition 1.2). The explicit dimension formula (Proposition 1.3b)) shows, in fact, that the only case is *to move a box down to the next row.* Given such a $\sigma$ we must study which strings $(\delta_1, \delta_2, \ldots, \delta_{t-1})$ lead to $\rho_{t-1}(\delta_{t-1}) = \sigma$.

5.6. We analyze inductively this problem as before and claim that the analysis restricts to the following problem:

*Given a diagram* $\eta$, *let* $\eta'$ *be obtained from* $\eta$ *removing the first column, and let* $\sigma$ *be a one step degeneration of* $\eta$ (obtained by moving down one box to the next row). *We must study the ab-diagrams* $\delta$ *such that* $\rho(\delta) = \sigma$ *and* $\pi(\delta) \leqq \eta'$ ($\pi(\delta)$ and $\eta'$ with the same number of boxes).

We follow the same analysis as before, letting $m$ be the number of boxes of $\eta'$. Let $\sigma'$ be obtained from $\sigma$ erasing the first column and $m'$ its number of boxes. *We have clearly the two possibilities* $m = m'$ *and* $m = m' + 1$.

*Case I:* $m = m'$; this case is obtained when *the box moved in the degeneration of* $\eta$ *to* $\sigma$ *is attached to a non empty row.*

In this case the previous analysis (5.4) shows that i) there is a unique *ab*-diagram $\delta$ over $\sigma$, ii) $\delta$ is associated to $\sigma, \sigma'$ and $\sigma'$ is a one step degeneration of $\eta'$, and iii) every *ab*-string of $\delta$ starts and ends with $b$.

*Case II:* $m = m' + 1$; this case is obtained when $\sigma$ *is gotten from* $\eta$ *splitting off one box from the last row* (to form a new "one box" row). In this case the previous analysis shows that, to form an *ab*-diagram $\delta$ over $\sigma$, we are forced to place $m'$ $a$'s; the remaining single $a$ can be placed only in the last two rows or by itself since we must preserve the condition $\pi(\delta) \leqq \eta'$.

Let us consider thus the last two rows; after filling with $m'$ of the $a$'s they are:

$b\,a\,b\,a\,b\ldots a\,b$

$b$

For the remaining $a$ we have 5 choices:

($\alpha$) and ($\alpha'$): *We attach $a$ to the row $bab\ldots ab$ either to the right or to the left.* In this case the $ab$-diagram $\delta$ represents a pair in which one of the two maps has maximal rank. The associated $a$-diagram is just $\eta'$.

($\beta$) and ($\beta'$): *We attach $a$ to the row $b$ left or right.* In this case the $ab$-diagram $\delta$ represents also a pair in which one of the two maps has maximal rank. The associated $a$-diagram is a one step degeneration of $\eta'$.

($\gamma$): *We create a new row consisting of the remaining $a$.* In this case the associated $a$-diagram is a one step degeneration of $\eta'$ but neither map in the nilpotent pair has maximal rank. On the other hand for this $ab$-diagram we have, in the notations of 5.3 $a_1=b_1=1$ and hence $\Delta=1$. For the corresponding nilpotent pair orbit we have thus the dimension

$$\dim X_\delta=\tfrac{1}{2}(\dim C_{\sigma'}+\dim C_\sigma)+\dim U\cdot\dim V-1.$$

Summing up all this analysis we see by an easy induction, that we have proved:

*If $Z_\lambda$ projects to $C_\sigma$, $\sigma$ a one step degeneration of $\eta$, then either*

$$Z_\lambda\subseteq Z^0$$

*or*

$$\dim Z_\lambda\leqq\sum_{i=1}^{t-1}n_i n_{i+1}+\tfrac{1}{2}\dim C_\sigma-1\leqq\dim Z^0-2.$$

This completes the proof of 5.2.

5.7. *Remark.* The only place in which we have used characteristic zero, apart from the implication normal $\Rightarrow$ Cohen Macaulay, was in the proof of proposition 3.4, where we used the following fact: If $V$ is a affine variety on which a reductive group $G$ acts and $W$ a closed subvariety invariant under $G$, then the induced map $W/G\to V/G$ is a *closed immersion* (cf. 1.4a)). In characteristic $p>0$ one can only say that this map is finite and injective, i.e. $W/G$ is purely inseparable over its image (cf. [21] § 4).

## 6. An Application to Tensor Representation

6.1. We present here the application due to Th. Vust announced in the introduction. Let $A\in\mathrm{End}(U)$ be a matrix, $G_A$ the centralizer of $A$ in $GL(U)$. We consider the action of $G_A$ on the tensor space $U^{\otimes m}$ and wish to compute $\mathrm{End}_{G_A}(U^{\otimes m})$. One knows that $\mathrm{End}_{GL(U)}(U^{\otimes m})$ is spanned by the symmetric group $\mathfrak{S}_m$ acting on $U^{\otimes m}$ in the obvious way (cf. [2], [18]). Now clearly the endomorphisms $A^{h_1}\otimes A^{h_2}\otimes\ldots\otimes A^{h_m}\in\mathrm{End}(U^{\otimes m})$ also commute with $G_A$ and we have:

**Theorem (Th. Vust).** *The algebra* $\mathrm{End}_{G_A}(U^{\otimes m})$ *is spanned by the elements*

$$\sigma \cdot A^{h_1} \otimes \ldots \otimes A^{h_m}, \qquad \sigma \in \mathfrak{S}_m,\ h_1, \ldots, h_m \in \mathbb{N}.$$

The proof will require some lemmas (mostly well known).

*6.2.* Let $V$ be an affine variety, $G$ a reductive group acting on $V$, $W \subseteq V$ a $G$-stable closed subvariety, $M$ a linear representation of $G$ and $\varphi$: $W \to M$ a $G$-equivariant morphism.

**Lemma 1.** *There exists a G-equivariant morphism* $\Phi$: $V \to M$ *extending* $\varphi$.

*Proof.* Let $K[U]$, $K[W]$ be the coordinate rings of $V$, $W$. A $G$-equivariant morphism $\varphi$ from $W$ to $M$ is given by an element $u \in (K[W] \otimes M)^G$. To extend $\varphi$ to $V$ is equivalent to lift $u$ to $(K[V] \otimes M)^G$, and this is a simple consequence of linear reductivity. qed.

*6.3.* Let $V$ be as before, $G := GL(V)$. We take now $W$ to be the closure $\overline{GA}$ of an orbit $GA$ for an element $A \in V$. We assume:

i) $\dim(\overline{GA} \setminus GA) \leqq \dim GA - 2$,

ii) $\overline{GA}$ is a *normal* variety.

Let $G_A$ denote the stabilizer of $A$ in $G$ and $M$ be again a linear representation of $G$.

**Lemma 2.** *If* $B \in M$ *is invariant under* $G_A$, *then under the condition* i) *and* ii) *there exists a G-equivariant morphism* $\Phi$: $V \to M$ *such that* $\Phi(A) = B$.

*Proof.* First of all we construct a morphism $\varphi$: $GA \to M$ given by $gA \mapsto gB$; this is well defined since $B \in M^{G_A}$. The two hypotheses i) and ii) on $\overline{GA}$ imply that $\varphi$ extends (uniquely) to a $G$-equivariant map $\varphi'$: $\overline{GA} \to M$. Finally taking $W = \overline{GA}$ and applying lemma 1 we have the required conclusion. qed.

*6.4.* We now want to apply these lemmas to the following set up: $M := \mathrm{End}(U^{\otimes m}) = \mathrm{End}(U)^{\otimes m}$, $A \in \mathrm{End}(U)$ the given matrix, $B \in \mathrm{End}_{G_A}(U^{\otimes m})$, $V := \mathrm{End}(U)$. To finish our proof it only remains to explicit the set of $G$-equivariant maps $\Phi$: $\mathrm{End}(U) \to \mathrm{End}(U)^{\otimes m}$. This set can be easily computed (cf. [15]). We need two lemmas for which we refer to the literature.

**Lemma 3** ([10] Lemma 4.9, [15] Theorem 1.2). *Let* $\sigma \in \mathfrak{S}_m$ *be decomposed into cycles:* $\sigma = (i_1\, i_2 \ldots i_k)\,(j_1 j_2 \ldots j_e) \ldots (t_1\, t_2 \ldots t_s)$ (including cycles of length one), *and* $Y = X_1 \otimes X_2 \otimes \ldots \otimes X_m \in \mathrm{End}(U)^{\otimes m}$. *Then*

$$\mathrm{Tr}(\sigma \cdot Y) = \mathrm{Tr}(X_{i_1} X_{i_2} \ldots X_{i_k}) \cdot \mathrm{Tr}(X_{j_1} X_{j_2} \ldots X_{j_e}) \ldots \mathrm{Tr}(X_{t_1} X_{t_2} \ldots X_{t_s}).$$

**Lemma 4** ([16] Theorem 1, [15] Theorem 1.3). *The ring of invariants of the space of m-tuples of matrices* $(X_1, X_2, \ldots, X_m)$ *under simultaneous conjugation under* $GL(U)$ *is generated by the invariants*

$$\mathrm{Tr}(X_{v_1} X_{v_2} \ldots X_{v_k}), \qquad k \in \mathbb{N},\ v_1, \ldots, v_k \in \{1, 2, \ldots, m\}.$$

*6.5.* Let us now look at the space $L$ of $G$-equivariant maps $\Phi$: $\mathrm{End}(U) \to \mathrm{End}(U)^{\otimes m}$. Clearly $L$ is a module over the *ring* $R$ *of invariants of* $\mathit{End}(U)$.

**Proposition.** *$L$ is spanned, as an $R$-module, by the maps of type:*

$$X \mapsto \sigma \cdot X^{h_1} \otimes X^{h_2} \otimes \ldots \otimes X^{h_m}, \qquad \sigma \in \mathfrak{S}_m,\ h_i \in \mathbb{N}.$$

*Proof.* Let $\Phi$: $\mathrm{End}(U) \to \mathrm{End}(U)^{\otimes m}$ be a $G$-equivariant map. We introduce $m$ new variables $Y_1, Y_2, \ldots, Y_m$ in $\mathrm{End}(U)$ and construct a function $\Psi$ on $\mathrm{End}(U)^{m+1}$ by setting

$$\Psi(X, Y_1, Y_2, \ldots, Y_m) := \mathrm{Tr}(\Phi(X) \cdot Y_1 \otimes Y_2 \otimes \ldots \otimes Y_m).$$

By the non degeneracy of the trace form the mapping $\Phi \mapsto \Psi$ is an injection from $L$ to the space of invariants of $X, Y_1, Y_2, \ldots, Y_m$ which are linear in $Y_1, Y_2, \ldots, Y_m$. Now by lemma 4 such invariants are of type

$$\sum t(X) \cdot \mathrm{Tr}(X^{h_1} Y_{i_1} X^{h_2} Y_{i_2} \ldots X^{h_k} Y_{i_k}) \cdot \mathrm{Tr}(X^{p_1} Y_{j_1} X^{p_2} Y_{j_2} \ldots) \ldots$$
$$\ldots \mathrm{Tr}(X^{s_1} Y_{t_1} X^{s_2} Y_{t_2} \ldots) \qquad (t(X) \in R).$$

The previous lemma 3 shows then that any such invariant is of type $\mathrm{Tr}(\Phi(X) \cdot Y_1 \otimes Y_2 \otimes \ldots \otimes Y_m)$, where $\Phi(X)$ is a linear combination with coefficients in $R$ of the special maps $X \mapsto \sigma \cdot X^{h_1} \otimes X^{h_2} \otimes \ldots \otimes X^{h_m}$. The previous remark about the injectivity of $\Phi \mapsto \Psi$ completes the proof. qed.

6.6. We now sum all our work and prove the main theorem 6.1:

Let $B \in \mathrm{End}_{G_A}(U^{\otimes m}) = (\mathrm{End}(U)^{\otimes m})^{G_A}$; we have seen that there exists a $G$-equivariant map $\Phi$:$\mathrm{End}(U) \to \mathrm{End}(U^{\otimes m})$ such that $\Phi(A) = B$. By the previous proposition 6.5 we know all equivariant maps. The very formula given by the proposition implies immediately the theorem. qed.

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