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Classification Problems in Differential Topology. V

On Certain 6-Manifolds

C. T. C. Wall (Liverpool)

The object of this paper is, first to give the classification up to diffeomorphism of closed, smooth, simply-connected 6-manifolds; and then to use this to study other classifications and related questions. Most of our results are valid only for manifolds which satisfy the additional hypothesis.

(H) The homology of M^6 is torsion-free, and $w_2(M)=0$.

Since, by smoothing theory, it is known that any piecewise-linear 6-manifold admits a differential structure, unique up to concordance, it follows that our classifications apply equally to this case.

The problem was suggested to the author by P. E. Newstead, as one of the manifolds above arises in his classification of holomorphic vector bundles of rank 2 and degree 1 over a Riemann surface of genus 2. There is no dependence on the previous papers in the series – most of the problems investigated in them belonged to the "metastable" or quadratic range: here for the first time we consider cubic forms.

1. Splitting Theorem

Theorem 1. *Let M be a closed, smooth, 1-connected 6-manifold. Then we can write M as a connected sum $M_1 \# M_2$, where $H_3(M_1)$ is finite and M_2 is a connected sum of copies of $S^3 \times S^3$.*

Proof. Write $H_3^w(M)$ for the quotient of $H_3(M)$ by its torsion subgroup. Then $H_3^w(M)$ is a finitely generated free abelian group, and intersection numbers induce a skew-symmetric integer-valued bilinear form on it which, by the Poincaré duality theorem, is nonsingular. It follows by a standard result that $H_3^w(M)$ admits a symplectic basis $\{e_i, e_i' : 1 \leqq i \leqq r\}$, so that

$$e_i \cap e_j = e_i' \cap e_j' = 0, \qquad e_i \cap e_j' = \delta_{ij}.$$

Now $H_3(M)$ maps onto $H_3^w(M)$. Also, since M is simply-connected, the Hurewicz theorem implies that $\pi_3(M)$ maps onto $H_3(M)$. Choose elements of $\pi_3(M)$ with weak homology classes e_i, e_i', and represent them by maps $f_i, f_i' : S^3 \to M$. Since M is simply-connected, a theorem of Haefliger [3] shows that these maps can be taken to be embeddings.

Also we may suppose (by a general position argument) that the image spheres meet each other transversely in a finite set of points, none of which lies on more than two of the spheres.

The above-quoted result of HAEFLIGER depends on an argument of WHITNEY [23] which shows how to remove a pair of intersections of opposite sign. Using this argument (which applies here since M is simply-connected and our spheres have codimension 3) we find that we can remove all intersections except those forced on us: i.e. for each i a single transverse intersection of f_i and f_i'.

Let $*$ be a base point in S^3: we may suppose our intersections are $f_i(*)=f_i'(*)$. Also, since the intersection is transversal, we can find a neighbourhood $D^3 \times D^3$ of $* \times *$ in $S^3 \times S^3$ to which

$$(f_i \times *) \cup (* \times f_i'): \ (S^3 \times *) \cup (* \times S^3) \to M$$

extends as an embedding. Now the normal bundle of each sphere in M is trivial, as $\pi_2(SO_3)$ vanishes, so we can extend our imbedding to a (closed) neighbourhood N of $(S^3 \times *) \cup (* \times S^3)$ in $S^3 \times S^3$. We may suppose N chosen so that the closure of its complement is a disc D^6, and small enough for us to have embeddings (for each i) $F_i: N \to M$, which are disjoint.

If we remove from M the interior of each $F_i(N)$ and attach in its place the disc D_i^6, we obtain a closed manifold M_1. It is clear from the construction that M is diffeomorphic to the connected sum of M_1 and of a number of copies of $S^3 \times S^3$ – say of M_2. Also the original choice of the e_i and e_i' shows that the map

$$H_3(M_2) \to H_3(M) \to H_3^w(M)$$

is an isomorphism. Thus $H_3(M_1)$ is finite.

We observe that since the fundamental group of a connected sum (in dimension >2) is a free product, M_1 is simply-connected.

2. A Normal Form

The proof below could replace that of Theorem 1 in our case, but does not appear to generalise usefully.

Theorem 2. *Let M satisfy (H) and $H_3(M)=0$. Then M can be obtained from S^6 by surgery on a disjoint set of embeddings $g_i: S^3 \times D^3 \to S^6$.*

Proof. By duality, $H_2(M)$ is a free abelian group: choose a free basis $\{e_i\}$. By the Hurewicz theorem, we can represent the e_i by maps $\bar{f}_i: S^2 \to M$; by a general position argument, these may be supposed disjoint embeddings. Now $\pi_1(SO_4) \cong \mathbb{Z}_2$ classifies SO_4-bundles over S^2, and is detected by the second Stiefel class w_2. Since $w_2(M)=0$, the $\bar{f}_i(S^2)$ have trivial normal bundles, and $\bar{f}_i$ extends to an embedding $f_i: S^2 \times D^4 \to M$.

Form W' from $M \times I$ by using the f_i to attach copies of $D^3 \times D^4$ to $M \times 1$. Since, by construction, the map $H_3(W', M) \to H_2(M)$ is an isomorphism, W' is 3-connected. If V is the component other than $M \times 0$ of $\partial W'$, we see by duality that $H_i(W', V) \cong H^{7-i}(W', M)$ vanishes for $i \neq 4$, and for $i=4$ is the isomorphic image of $H_4(W') \cong H_4(M)$. Since V is clearly simply-connected, it is a homotopy 6-sphere. According to SMALE [17], V is diffeomorphic to S^6.

Reversing the construction above, we see that M can be obtained from $S^6 \cong V$ by surgery as stated. We also observe that we can attach D^7 to W' along V, to give a manifold W with boundary M. And W is a handlebody, formed by attaching handles h^4 to D^7. We shall use W later on to construct embeddings of M.

3. Invariants of Torsion-Free 6-Manifolds

We now consider closed, smooth, simply-connected 6-manifolds M with torsion-free homology. By Theorem 1, we may restrict attention to manifolds with $H_3(M)=0$. Then the only nonvanishing homology (or cohomology) groups are in dimensions 0, 2, 4 and 6. We orient M, so the groups in dimensions 0 and 6 have given isomorphisms with $\mathbb{Z}$. Write H for the free abelian group $H^2(M) \cong H_4(M)$, and $\hat{H}$ for its dual

$$\hat{H} = \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z}) \cong H_2(M) \cong H^4(M),$$

the isomorphisms being natural given the orientation, and induced by cup and cap products.

There is also the cup product $H^2(M) \times H^2(M) \to H^4(M)$, or in the above notation $H \times H \to \hat{H}$. To give this is equivalent to giving the iterated product $\mu: H \times H \times H \to \mathbb{Z}$. Then μ is symmetric; H and μ determine the entire homology and cohomology structure. As to cohomology operations, μ already determines the 4-type of M, and the only operation from H^4 to H^6 is Sq^2 (with mod 2 coefficients).

We next come to characteristic classes. Modulo 2 we have the Wu class $1+v_2$ determining the Stiefel classes $w_2 = v_2$, $w_4 = v_2^2 = w_2^2$; other w_i vanish. And v_2 is determined as dual to

$$Sq^2:\ H^4(M; \mathbb{Z}_2) \to H^6(M; \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

Apart from the Euler class, which is determined by the homology, the only integral characteristic class is $p_1 \in H^4(M; \mathbb{Z}) \cong \hat{H}$.

There are no relations between these invariants over the integers, but certain congruence relations hold. Certainly w_2^2 is the mod 2 reduction of p_1. We also obtain a relation mod 2 by considering Sq^2 on decomposable elements $xy(x, y \in H)$. For by the Cartan formula,

$$Sq^2(xy) = Sq^2 x \cdot y + Sq^1 x \cdot Sq^1 y + x \cdot Sq^2 y$$
$$= x^2 y + 0 + x y^2 \quad (\text{mod } 2).$$

But, modulo 2 again, $Sq^2(xy) = v_2 xy = w_2 xy$. An analogous argument provides also

$$x^3 \equiv \mathscr{P}^1(x) \equiv p_1 \cdot x \quad (\text{mod } 3).$$

Finally, results of WU [24] lead to a congruence mod 4.
To sum up, we have

Theorem 3. *The invariants of a closed, smooth, simply-connected 6-manifold M with torsion-free homology can be described as:*

Two free abelian groups $H = H^2(M)$, $G = H^3(M)$.

A symmetric trilinear map $\mu: H \times H \times H \to \mathbb{Z}$.

A homomorphism $p_1: H \to \mathbb{Z}$.

An element $w_2 \in H \otimes \mathbb{Z}_2$, *the image say of* $W_2 \in H$.

These satisfy the relations:

For $x, y \in H$,

$$\mu(x, y, x + y + W_2) \equiv 0 \quad (\text{mod } 2).$$

For $x \in H$,

$$p_1(x) \equiv \mu(x, W_2, W_2) \quad (\text{mod } 4),$$
$$p_1(x) \equiv \mu(x, x, x) \quad (\text{mod } 3).$$

We observe that the reintroduction of $H_3(M)$ does not affect the remaining invariants.

We conjecture that the above invariants determine M up to diffeomorphism. We shall prove this below in the case when M satisfies (H) — which amounts here to the extra hypothesis $w_2(M) = 0$. After Theorem 1, it will be sufficient to consider only the case when $H_3(M) = 0$; and by Theorem 2, M is then obtainable by surgery on a framed link of 3-spheres in S^6.

4. The Classification of Framed Links of S^3 in S^6

First consider the case when we have only a single S^3, so we have just a framed knot. Fortunately, HAEFLIGER gives a detailed discussion at the end of § 5 of [7] of precisely the case which interests us. Write FC_3^3 for the group of isotopy classes of embeddings $g: S^3 \times D^3 \to S^6$, C_3^3

for the group of isotopy classes of embeddings $\bar{g}: S^3 \to S^6$. There is a natural homomorphism $\varphi: FC_3^3 \to C_3^3$, obtained by taking $\bar{g}$ as the restriction of g to $S^3 \times 0$. From [7] (or we can easily see directly) there is a short exact sequence

$$0 \to \pi_3(SO_3) \xrightarrow{\tau} FC_3^3 \xrightarrow{\varphi} C_3^3 \to 0,$$

where τ is the map which twists the tubular neighbourhood g of $\bar{g}$. Also, $\pi_3(SO_3)$ is infinite cyclic. Finally, C_3^3 is also infinite cyclic (so the sequence splits), and a generator is given by the explicit embedding $g_1: S^3 \to S^6$ of ([5], p. 463).

We note in passing that the gap in [6] for $k=1$ — where HAEFLIGER would like to perform surgery on a simply-connected framed manifold V with boundary S^3 and signature zero — can be filled by using a result of [20] which implies that by taking a connected sum of V with many copies of $S^2 \times S^2$ we obtain a connected sum of D^4 with copies of the same. This observation is due to B. STEER.

Next we consider the classification of framed links, i.e. of embeddings in S^6 of a disjoint union of copies of $S^3 \times D^3$. The group of such links was studied in [6]: we shall use the final theorem of that paper. Although the case $d=2$ is excluded by the statement of the theorem, the observation above (also a more recent argument by HAEFLIGER) justifies our repairing the omission. (The cases $d=3$, 7 are more truly exceptional, owing to the existence of maps of Hopf invariant one).

Proposition. *The class of a link as above is determined by the knot class of each component and the linking elements λ_j^i, λ_{jk}^i. These are subject to the sole relations $S\lambda_j^i = S\lambda_i^j$, and that λ_{jk}^i is symmetric in i, j and k.*

Here $\lambda_j^i \in \pi_3(S^2)$ is the homotopy class of the composite of the embedding g_i of S^3 in $S^6 - g_j(S^3)$, and a homotopy equivalence of the latter with S^2, chosen using an embedded copy of S^2 whose linking number with $g_j(S^3)$ is $+1$ (to fix signs). Write now $S_i^3 = g_i(S^3)$.

More complicated is λ_{jk}^i: the complement of $S_j^3 \cup S_k^3$ has the homotopy type of $S^2 \vee S^2$, and S_i^3 maps into this. But, using the easiest case of the Hilton-Milnor theorem (or the Blakers-Massey theorem),

$$\pi_3(S^2 \vee S^2) \cong \pi_3(S^2) \oplus \pi_3(S^2) \oplus \pi_3(S^3),$$

the third summand being injected by the Whitehead product of the inclusion maps of the two copies of S^2. Write $\lambda_{jk}^i \in \pi_3(S^3)$ for the projection on this summand of the class of S_i^3.

Since $\pi_3(S^2)$ and $\pi_3(S^3)$ are infinite cyclic, our links can be characterised by a set of integers. To fix these, we choose generators: the Hopf map for $\pi_3(S^2)$ and the identity map for $\pi_3(S^3)$. We must also choose a left inverse to τ. Now if $g: S^3 \times D^3 \to S^6$ is an embedding; we can use it to attach a 4-handle to D^7, and the stable tangent bundle of the result determines an element of $\pi_3(SO)$, which can be shown to be an even multiple of the generator, and hence to lie in the (monomorphic) image of $\pi_3(SO_3)$. (In fact it suffices to check this with some framing of g_1). Thus we have defined a map $\beta': FC_3^3 \to \pi_3(SO_3)$ which gives stably $\beta: FC_3^3 \to \pi_3(SO)$. As $\beta\tau$ is clearly induced by suspension, $\beta'\tau$ is the identity.

To sum up, our framed link is determined by a set of integers, which we can write as $\beta'_i, \varphi_i, \lambda^i_j, \lambda^i_{jk}$; which are subject to the sole relations that λ^i_{jk} be symmetric and that $\lambda^i_j \equiv \lambda^j_i$ (mod 2).

5. Identification of the Invariants

Suppose $g_i: S^3 \times D^3 \to S^6$ a disjoint set of embeddings. Use them to attach 4-handles to D^7 to obtain W; set $M = \partial W$. The handles have homology classes in $H_4(W, D^7) \cong H_4(W) \cong H_4(M) \cong H^2(M)$; denote these classes in $H^2(M)$ by e_i. The two previous sections are now tied together by

Theorem 4. *We have, for $i<j<k$,*

$$\mu(e_i, e_j, e_k) = \lambda^i_{jk} \qquad \mu(e_i, e_i, e_j) = \lambda^j_i$$

$$\mu(e_i, e_i, e_i) = 6\varphi_i + \beta'_i \qquad p_1(e_i) = 4\beta'_i.$$

Proof. Form X from S^6 by deleting the interiors of the images of the g_i. Then X has the homotopy type of a wedge of copies of S^2 (plus various 5-cells), and we can choose S_i^2 to link S_i^3 once and the other S_j^3 not at all. M is formed by attaching copies of $D^4 \times S^2$ to X; hence by attaching 4-cells and 6-cells. The attaching maps of the 4-cells are null-homologous in X so they represent homology classes in M; in fact the classes corresponding to e_i. The above choice of linking numbers shows that the S_i^2 represent the dual base of $H_2(M)$.

We now see that (up to homotopy) M is obtained from $\vee S_i^2$ by attaching 4-cells D_k^4 and a 6-cell. We calculate the attaching maps of the D_k^4. By the Hilton-Milnor theorem again,

$$\pi_3(\vee S_i^2) \cong \oplus_i \pi_3(S_i^2) \oplus_{i<j} \pi_3(S_{ij}^3).$$

Thus the class of ∂D_k^4 is determined by integers, which in turn determine products in cohomology. Also, by definition, the component of this class in $\pi_3(S_i^2)$ for $i \neq k$ is λ^k_i and in $\pi_3(S_{ij}^3)$ for $k \neq i, j$ is λ^k_{ij}.

Now the isomorphism of $\pi_3(S_i^2)$ with $\mathbb{Z}$ is defined by the Hopf invariant, which is a functional square. Hence the value on the homology class of D_k^4 of the square of e_i (the dual base to S_i^2) is λ_i^k. As this homology class is dual to e_k, we have $e_i^2 e_k = \lambda_i^k$. Also, the functional product takes the value 1 on a Whitehead product element, so $e_i e_j$ has value λ_{ij}^k on the class of D_k^4, and $e_i e_j e_k = \lambda_{ij}^k$.

The remaining two equations are related only to our classification of framed knots. By definition, the restriction to D_k^4 of the stable tangent bundle of W (hence also of M) gives $2\beta_k'$ times the generator of $\pi_3(SO)$. Thus the Pontrjagin class takes the value $4\beta_k'$ on this homology class, so $p_1(e_k) = 4\beta_k'$. Finally, e_i^3 is (as above) determined by a class in $\pi_3(S^2)$, thus we must evaluate the map $FC_3^3 \to \pi_3(S^2)$. On the subgroup $\pi_3(SO_3)$ it is easy to see that the map is induced by the fibre projection $SO_3 \to S^2$, hence is an isomorphism. Also, HAEFLIGER has shown ([7], end of § 5) that the framed knot g_1 determines 6 times the generator of $\pi_3(S^2)$. It can now be checked without difficulty that the signs in the stated result are correct.

It is now a simple matter to prove our main result.

Theorem 5. *Diffeomorphism classes of oriented manifolds satisfying* (H) *correspond bijectively to isomorphism classes of systems of invariants*:

two free abelian groups H, G,

a symmetric trilinear map $\mu: H \times H \times H \to \mathbb{Z}$,

a homomorphism $p_1: H \to \mathbb{Z}$

subject to: *for* $x, y \in H$,

$$\mu(x,x,y) \equiv \mu(x,y,y) \pmod 2,$$

for $x \in H$,

$$p_1(x) \equiv 4\mu(x,x,x) \pmod{24}.$$

Proof. By Theorem 3, these are indeed invariants of diffeomorphism class. To demonstrate existence and uniqueness of a manifold with given invariants, it suffices (by Theorem 1) to consider only the case $G=0$. Choose a base $\{e_i\}$ for H. By Theorem 2, there exists a framed link of 3-spheres in S^6 on which surgery can be performed to give M, with the given base $\{e_i\}$ for $H^2(M)$. By the Proposition and Theorem 4, μ and p_1 determine the framed link, and satisfy the given congruence relation. Hence they also determine M. To prove existence of M with the given invariants, we only have to choose a framed link as specified by Theorem 4 and perform surgery.

We note that the final congruence condition is stronger than that obtained in Theorem 3. A direct proof can be given by applying ADEM'S secondary cohomology operation to the Thom class of the tangent (or normal) bundle of M.

6. Homeomorphism and Homotopy Classifications

Since cup products are certainly homotopy type invariants, it is only necessary to investigate the Pontrjagin class p_1. Now NOVIKOV has shown [*16*] that the image of p_1 in rational cohomology is a topological invariant of M [parts of his original proof are unclear, but see Izvestia Akad. Nauk SSSR. 30 (1966) 208–246]. Thus we have

Theorem 6. *Two manifolds satisfying* (H) *are homeomorphic if and only if they are diffeomorphic.*

We leave unsolved the problem whether a topological 6-manifold satisfying (H) is homeomorphic to a smooth manifold. (For homotopy spheres this has recently been shown by M.H.A. Newman). We next come to homotopy classification.

Theorem 7. *Two manifolds satisfying* (H) *admit an orientation-preserving homotopy equivalence if and only if the corresponding groups G have the same rank, and there is an isomorphism between the groups H which preserves* μ, *and* p_1 (mod 48).

Only the final condition calls for comment: invariance of p_1 (mod 24) was previously known, but follows trivially in our case by the relation with μ. In fact, after Theorem 4, if μ is given we know p_1 (mod 24): to determine p_1 (mod 48) is equivalent to determining the numbers φ_i (mod 2).

Proof. We first give an indirect proof of the mod 48 invariance of p_1, which indicates a reason for the fact; then we give a computational proof which will establish also the sufficiency clause.

We follow MILNOR and KERVAIRE [*10*]. Since $w_2(M)=0$, the tangent bundle of M is trivial on the 3-skeleton; if ϑ is the obstruction to trivialising on the 4-skeleton we have $p_1=2\vartheta$. Now extend the structure monoid from SO_6 to SG_6 (=self-maps of degree 1 of S^5). The resulting fibration is (at least stably) a homotopy type invariant of M, by a result of SPIVAK (Princeton thesis, 1964). Hence so is the image of ϑ under

$$H^4(M;\mathbb{Z})\approx H^4(M;\pi_3(SO_6))\to H^4(M;\pi_3(SG_6)).$$

But we can identify $\pi_3(SG_6)$ with $\pi_9(S^6)$, and the coefficient map with the J-homomorphism, which reduces mod 24. Hence ϑ is a homotopy invariant mod 24, hence also $2\vartheta=p_1$ mod 48.

Our second proof is valid only when $G=0$; by Theorem 1, we only need prove the sufficiency clause in this case. Choose a base $\{e_i\}$ of $H^2(M)$ giving rise by duality to bases of $H_2(M)$ and of $H_4(M)$. Since M is 1-connected and $H_*(M)$ is torsion-free, M is (homotopy equivalent to) a cell complex whose cells determine these basis elements for homology: this follows from arguments of SMALE [17] up to homeomorphism – a simpler argument of MILNOR (see [21]) provides homotopy equivalence. We have already mentioned that functional cup products detect the third homotopy group of a wedge K^2 of 2-spheres, so the homotopy type of the 4-skeleton K^4 of M is determined by μ. Now the sequence

$$\pi_5(K^2)\to\pi_5(K^4)\to\pi_5(K^4,K^2)$$

is exact, and a simple computation shows that $\pi_5(K^4, K^2)$ is detected by functional cup-product and functional Sq^2, hence in our case the image of the attaching map of the 6-cell of M in $\pi_5(K^4, K^2)$ is well-determined. Unfortunately, $\pi_5(K^2)$ is exceedingly large (if n is the rank of H, it has rank $\binom{n}{4}$)) and most of its summands map injectively to $\pi_5(K^4)$. But the class in $\pi_5(K^4)$ of the attaching map is not a homotopy invariant of M, as K^4 has many self-homotopy equivalences homologous to the identity.

We must thus argue differently. Now consider M as obtained from S^6 by surgery on a framed link, and take the piecewise linear point of view. In this sense, S^3 unknots in S^6. If M_1 and M_2 have corresponding forms μ, the two framed links must arise from the same unframed link; also we may suppose the regular neighbourhoods which are the images of the $g_i: S^3\times D^3\to S^6$ are identical: only the actual maps differ. To form M_i we delete the interior of the image of g_i and attach $D^4\times S^2$: a 4-cell and a 6-cell. Only the attaching maps distinguish M_1 from M_2.

The 4-cell is attached by a map $S^3\to S^3\times S^2$: on the first component, the degree is 1; on the second, if we use the standard framing, the Hopf invariant is $\mu(e_i, e_i, e_i)$ (c.f. proof of Theorem 4). Thus its homotopy class is uniquely determined. Twist $S^3\times S^2$ by the corresponding element of $\pi_3(SO_3)\approx\pi_3(S^2)$. Then the attaching map becomes the class of $S^3\times *$. Adding the 4-cell to $S^3\times S^2$ now changes its homotopy type to that of $S^5\vee S^2$.

It remains, for each i, to attach a 6-cell to $S^5\vee S^2$ by a map of degree 1 on S^5. But we have

$$\pi_5(S^5\vee S^2)\approx\pi_5(S^5)\oplus\pi_5(S^2),$$

so only the class in $\pi_5(S^2)\approx\mathbb{Z}_2$ remains to be considered. That it is equivalent to φ_i (mod 2) is now clear from the first argument, but we may see it directly as follows.

The fibration $\Omega^2 S^2\to SG_3\to S^2$ (here SG_3 is the space of self-maps of S^2 of degree 1, and the projection is the natural one) induces a short exact sequence

$$\xrightarrow{0}\pi_5 S^2\longrightarrow\pi_3 SG_3\longrightarrow\pi_3 S^2\xrightarrow{0}$$

which necessarily splits. As $\pi_3(G, SO)=0$, the exact sequence of HAEFLIGER [7, 5. 11] shows that FC_3^3 maps onto $\pi_3(SG_3)$. We thus have

$$\begin{array}{ccccccc} 0\rightarrow\pi_3(SO_3) & \xrightarrow{\tau} & FC_3^3 & \xrightarrow{\varphi} & C_3^3 & \rightarrow 0 \\ & & \downarrow & & & \\ 0\rightarrow \pi_5(S^2) & \longrightarrow & \pi_3(SG_3) & \longrightarrow & \pi_3(S^2) & \rightarrow 0 \end{array}$$

and the composite $\pi_3(SO_3)\rightarrow\pi_3(S^2)$ is evidently the isomorphism induced by the natural projection. This splits the second sequence, and our split of the first induces a surjection $C_3^3\rightarrow\pi_5(S^2)$, so that the class in $\pi_5(S^2)$ is just the class φ_i in C_3^3 reduced mod 2.

Finally, the image of $g_i\in FC_3^3$ in $\pi_3(SG_3)$ induces up to homotopy an automorphism of the fibration $S^3\times S^2\rightarrow S^3$: this is the attaching map for $D^4\times S^2$ to obtain M. Perform the automorphism in two stages: first by $\pi_3(S^2)\approx\pi_3(SO_3)$ as a bundle automorphism. What remains is just the element of $\pi_5(S^2)$ needed above.

Theorem 8. *Let X be a simply-connected CW-complex, with $H^3(X;\mathbb{Z}_2)$ zero, satisfying Poincaré duality with a fundamental class $z\in H_6(X)$. Then X is homotopy equivalent to a closed smooth manifold.*

Proof. By SPIVAK's thesis, for N large there is a unique (up to homotopy) fibration with base X, with fibre homotopy equivalent to S^N, the fundamental homology class of whose Thom space is spherical. By a result of STASHEFF [18], this is classified by a map $X\rightarrow BG$. The obstructions to factorising this map through BSO lie in the groups $H^i(X;\pi_{i-1}(G, SO))$. Now the homotopy group vanishes for $i=2, 4, 6$; the cohomology group vanishes for $i=1, 5$ since X is simply-connected and satisfies duality, and for $i=3$ by hypothesis. Thus the map can be factorised through BSO; we see that a homotopy equivalent fibration is a bundle with group SO_{N+1}.

We now apply the technique of surgery (due to NOVIKOV and BROWDER [0] for this situation) as follows: let T be the Thom space, $f: S^{N+6}\rightarrow T$ the map of degree 1. We identify X with the zero cross-section $\subset T$, let $M=f^{-1}(X)$, and use a transversality argument to make M a manifold, and then surgery to make the map $f\mid M: M\rightarrow X$ a homotopy equivalence. The only obstruction (at the last step of the argument) is an Arf invariant, but since any 3-sphere in M has trivial normal bundle ($\pi_2(SO_3)=0$), we can still do surgery, provided we are willing to change the bundle defined by the map $X\rightarrow BSO$. This proves the result.

If the hypothesis $H^3(X;\mathbb{Z}_2)=0$ be dropped, then there is an obstruction in this group to the first step in the argument. This is the "exotic characteristic class" computed by GITLER and STASHEFF [2].

7. Almost Complex Structures

Since our problem arose by considering a complex manifold, it is now natural to turn to the question of complex structures. The obstructions to reducing the group of the tangent bundle from SO_6 to U_3 (i.e. to finding an almost complex structure) lie in the groups $H^i(M; \pi_{i-1}(SO_6, U_3))$. Now we have

Lemma. $\pi_2(SO_6, U_3) \cong \mathbb{Z}$. For $i=0, 1, 3, 4, 5, 6$: $\pi_i(SO_6, U_3)=0$. For $i<6$, these are given by MASSEY [*14*]: the case $i=6$ follows by a similar elementary computation. (In fact, $SO_6/U_3 = P_3(\mathbb{C})$.) We thus have

Theorem 9. *Let M be a smooth oriented 6-manifold. Then M has an almost complex structure if and only if $W_3(M)=0$. When this is so, there is just one (homotopy class of) almost complex structure for each $c_1 \in H^2(M)$ whose* mod 2 *reduction is $W_2(M)$.*

The other Chern classes are determined by the usual relations: $p_1 = c_1^2 - 2c_2$ and c_3 is the Euler class (when $H_2(M)$ has 2-torsion, more care is necessary with c_2).

When we wish to consider complex structures we meet the disturbing fact that there is no known necessary condition for a homotopy class of almost complex structures to contain a complex structure: also no known sufficient condition (except by listing manifolds). We must leave these problems open.

However, it is well known (see. e.g. [*25*]) that a necessary condition for M to admit a Kähler complex structure is that there exist $\omega \in H^2(M; \mathbb{R})$, with $\omega^3 \neq 0$, cup product with which induces an isomorphism

$$H^2(M; \mathbb{R}) \to H^4(M; \mathbb{R}).$$

Moreover, if the structure is projective algebraic, then ω comes from a well defined integral class (which we also denote by ω), which is (up to sign) dual to the homology class of a hyperplane section. We now rewrite this result in the notation introduced in § 3.

Proposition. *Let M be a closed oriented 6-manifold satisfying (H). Suppose M homeomorphic to a nonsingular projective algebraic variety. Then for some $\omega \in H$ we have $\mu(\omega, \omega, \omega) \neq 0$, and the quadratic form of the symmetric bilinear map $H \times H \to \mathbb{Z}$ defined by $(x, y) \mapsto \mu(\omega, x, y)$ is nondegenerate.*

We observe finally that the manifold of NEWSTEAD (which is projective algebraic) has the following invariants:

G has rank 4;

H has rank 1, generator e (say): let $\hat{e}$ be the dual generator of $\hat{H}$; $e^2 = 4\hat{e}$, $c_1 = 2e$, $c_2 = 12\hat{e}$, so $p_1 = -8\hat{e}$.

8. Immersions

We next determine the least dimensions of Euclidean spaces into which our manifolds can be immersed. We first consider smooth immersions.

Theorem 10. *Let M satisfy (H). Then for smooth immersions we have*

(i) M does not immerse in $\mathbb{R}^6$.

(ii) M immerses in $\mathbb{R}^7 \Leftrightarrow p_1(M)=0$.

(iii) M immerses in $\mathbb{R}^8 \Leftrightarrow$ for some $X \in 2H$, $p_1(M)+X^2=0$.

(iv) M immerses in $\mathbb{R}^9$.

Proof. Since $M=M^6$ is closed, it cannot immerse in $\mathbb{R}^6$. For $q>0$, there is by [9] an obstruction theory for immersing M in $\mathbb{R}^{6+q}$, with obstruction groups

$$H^i(M;\pi_{i-1}(SO, SO_q)).$$

If $q=1$, the nonzero groups are those with $i=2$ or 4: the first obstruction is easily identified with $w_2(M)=0$. The second maps to p_1 under the map of coefficient groups

$$\pi_3(SO, SO_1)=\pi_3(SO)\cong\pi_4(BSO)\to H_4(BSO)\cong\mathbb{Z}$$

and hence vanishes if and only if p_1 does.

If $q=2$, the nonzero groups have $i=3$ or 4. The first gives the obstruction $W_3(M)=0$. When we come to evaluate the second, we have already made a choice of immersion, hence of normal 2-plane bundle γ, over the 3-skeleton. Let X be the Euler class of γ. Then X^2 is the Pontrjagin class of the extension of γ over the 4-skeleton K (which exists and is unique: $\pi_3(BSO_2)$ and $\pi_4(BSO_2)$ vanish). Since we wish this extension to be inverse to the tangent bundle, we need $X^2+p_1(M)=0$. As $\pi_3(SO, SO_2)\cong \pi_3(SO)$ is infinite cyclic, it is clear that conversely, this condition suffices for our obstruction to vanish. It remains only to note that X defines γ, and γ is in fact inverse to the tangent bundle (on the 3-skeleton) $\Leftrightarrow w_2(M)+w_2(\gamma)=0$, i.e. $0=w_2(\gamma)$, the mod 2 reduction of X.

Finally consider $q=3$: here the nonzero groups are for $i=4$ and 6. The first obstruction is $w_4(M)$, which is zero as we saw in § 3. For the final obstruction, the coefficient group is

$$\pi_5(SO, SO_3)\cong\pi_4(SO_3).$$

We can interpret the obstruction as follows. M can be obtained from a 4-complex by attaching a 6-cell: call the attaching map $\alpha: S^5\to K$. We have already immersed a neighbourhood of the 4-complex, and so defined a normal bundle γ over K. The obstruction is the same as that to trivialising $\alpha^*(\gamma)$. But $w_2(\gamma)=0$, so γ is trivial on the 3-skeleton of K, and is induced by a map β of K to a wedge of 4-spheres. But $\beta\circ\alpha$ is detected by the functional cohomology operation Sq^2, i.e. by Sq^2 in M, which is zero. Thus $\beta\circ\alpha$ is nullhomotopic and $\alpha^*\gamma$ trivial, so our obstruction vanishes and we obtain the required immersion.

We next consider piecewise linear and topological immersions, restricted as follows. A map $f: M\to V$ of topological manifolds is a (locally flat) embedding if each $P\in f(M)$ has a coordinate neighbourhood $\varphi: U\to\mathbb{R}^v$ in V with $\varphi(U\cap f(M))=\mathbb{R}^m$. It is a (locally flat) immersion if each point of M has a neighbourhood embedded (locally flat) by f.

Theorem 11. *Let M satisfy (H). Then for topological immersions,*

(i) M does not immerse in $\mathbb{R}^6$.

(ii) M immerses in $\mathbb{R}^7 \Leftrightarrow p_1(M)=0$.

(iii) If M immerses in $\mathbb{R}^8$ with a normal (micro-) bundle; in particular, if there is a (locally flat) PL-immersion, we have an $X\in 2H$ with $p_1(M)+X^2=0$.

(iv) M immerses in $\mathbb{R}^9$.

We have not succeeded in deriving the condition $p_1(M)+X^2=0$ without the extra hypothesis in (iii).

Proof. (i) The image of such an immersion would be open (by invariance of domain) and compact, hence closed; contradicting connectedness of $\mathbb{R}^6$.

(iv) and the other sufficiency statements follow from theorem 10.

(ii) By a result of HAEFLIGER and POENARU ([8], Proposition 1), such an immersion induces a "neighbourhood" N of M. Cut N along M: the local flatness shows that we obtain a manifold with boundary. By a theorem of BROWN [1], the boundary has a collar neighbourhood. Hence M has a normal line bundle: let E be the total space. Since E^7 immerses in $\mathbb{R}^7$, it has trivial tangent bundle; in particular $p_1(E)=0$. But $E\simeq M$, and its tangent bundle is that of M plus a trivial line bundle. Hence $p_1(M)=0$.

(iii) First note that by the main result of [22], any locally flat PL embedding (hence, as above, also immersion) with codimension 2 has a normal PL-bundle. Also, by [12] microbundles are equivalent to bundles. So we assume M immersed in $\mathbb{R}^8$ with a topological normal bundle γ, with total space E: E has trivial tangent bundle.

By a result of MILNOR [15], the tangent bundle of E is the sum of the tangent bundle of M and the normal bundle of M in E – i.e. γ itself. Now the structure group of γ is the group orientation-preserving homeomorphisms of $\mathbb{R}^2$ leaving the origin fixed. But by [13] this is homotopy equivalent to SO_2. Thus we can give γ the structure of a vector bundle. Then, as before, γ has Euler class X, and $X^2=p_1(\gamma)$, $X^2+p_1(M)=p_1(E)=0$, and X reduces mod 2 to $w_2(\gamma)=w_2(M)=0$.

9. Embeddings

Now we consider the more difficult problem of embeddings: we first give necessary conditions.

Theorem 12. *Let M satisfy (H). Necessary conditions for topological embeddings of M are:*

(i) M embeds in $\mathbb{R}^7 \Rightarrow p_1(M)=0$, $\mu=0 \Leftrightarrow M$ is a connected sum of copies of $S^2\times S^4$ and $S^3\times S^3$.

(ii) If M embeds in $\mathbb{R}^8$ with a normal (micro-) bundle; in particular, if there is a (locally flat) PL-embedding, we have $p_1(M)=0$.

(iii) M embeds in $\mathbb{R}^9 \Rightarrow$ for some $X\in 2H$, $p_1(M)+X^2\equiv 0$ (mod 8).

Proof. (i) By Alexander duality, M separates $\mathbb{R}^7$; let B be the closure of the bounded complementary component, A of the unbounded. The Mayer-Vietoris exact sequence of the triad $(\mathbb{R}^7; A, B)$ yields isomorphisms

$$H=H^2(M)\approx H^2(A)\oplus H^2(B)=H_1\oplus H_2,$$

say

$$\hat{H}=H^4(M)\approx H^4(A)\oplus H^4(B).$$

And by Alexander duality, $H^4(A)\approx H_2(B)=\hat{H}_2$. The inclusion $i:A\subset M$ is compatible with the cup product, so if $x\in H$, $y\in H_2$ we have $0=i^*y$, hence $0=i^*(x\,y)$, so for $z\in H_2$, $0=\mu(x,y,z)$. Hence μ vanishes on $H_1\times H_2\times H_2$ and on $H_2\times H_2\times H_2$, and similarly with H_1 and H_2 interchanged. Since μ is symmetric, and $H_1+H_2=H$, μ vanishes identically. The conclusion $p_1(M)=0$ follows from Theorem 11 (ii). The final equivalence follows at once from Theorem 5.

(ii) Follows by combining (ii) of Theorem 11 with a standard argument which shows that the Euler class of the normal bundle of an embedding always vanishes.

(iii) If $f:M\to\mathbb{R}^9$ is continuous and injective, then $(x,y)\to f(x)-f(y)$ defines a map $M\times M\to\mathbb{R}^9$ in which the inverse image of the origin is the diagonal ΔM. Removing this, and retracting $\mathbb{R}^9-0$ radially (by r) onto S^8 we have a map $F:M\times M-\Delta M\to S^8$ which is equivariant for the $\mathbb{Z}_2$-actions by interchange of factors on the left, and by the antipodal map on the right. We will show that our condition is necessary for the existence of such a map F; it will follow from our arguments that the condition is also sufficient.

Form the manifold M_0 from M by deleting the interior of an embedded disc. Then M_0 is a smooth regular neighbourhood of the 4-skeleton K^4 for a suitable C.W.-structure on M. By a theorem of HAEFLIGER [4] to the map $F|(M_0 \times M_0 - \Delta M_0)$ corresponds a smooth embedding $f_0: M_0 \to \mathbb{R}^9$, unique up to isotopy, with F equivariantly homotopic to $r \circ (f_0 \times f_0)$. Form M_1 from M_0 by removing a collar neighbourhood of ∂M_0 (or equivalently, from M by removing a larger concentric disc). Then $M_1 \simeq K$, and $\partial M_0 \approx S^5$ is embedded in $\mathbb{R}^9 - M_1 = C$, say. We assert that the inclusion map $j: S^5 \to C$ is nullhomotopic.

To prove this, note that $\mathbb{R}^9$ may be replaced by S^9 in the above. Then C is S-dual to K. Since $\dim K = 4$, C is 3-connected, and so it is sufficient to prove j stably nullhomotopic. But Map (M_1, S^8) is also S-dual to K, and j corresponds to $F|(S^5 \times M_1): S^5 \times M_1 \to S^8$, or rather to its adjoint $S^5 \to \text{Map}\,(M_1, S^8)$. This is nullhomotopic, and j is stably so.

The obstruction to finding a cross-section of the normal bundle γ of $f_0(M_1)$ is an element

$$q \in H^4(M_1; \pi_3(S^2)) \approx H^4(M; \mathbb{Z}) = \hat{H}.$$

(The first obstruction is zero, as we see easily that γ is trivial on the 3-skeleton). We first compute q. The cross-section can first be chosen on the 2-skeleton, and will than split $\gamma = \varepsilon \oplus \gamma_0$, where ε is a trivial line bundle and γ_0 a plane bundle. The Euler class $X(\gamma_0)$ can be anything reducing mod 2 to $w_2(\gamma)$, hence $X \in 2H$. Now X determines a bundle γ_0' over M_1, and the desired obstruction is that to extending the isomorphism $\gamma \to \varepsilon \oplus \gamma_0'$ from the 2-skeleton to the 4-skeleton. Now $p_1(\gamma) = -p_1(M)$ and $p_1(\varepsilon \oplus \gamma_0') = X^2$; also, the generator of $\pi_3(SO_3)$ (which is isomorphic by projection to $\pi_3(S^2)$) has Pontrjagin class 4. Hence $4q = p_1(M) + X^2$ (up to sign). It thus remains to show that $j \simeq 0$ implies $q = 0$ (mod 2); in fact we will see that these two conditions are equivalent.

Since C is connected, and the only nonzero reduced homology groups are $H_4(C) \approx H^4(K) \approx \hat{H}$, $H_5(C) \approx H^3(K) \approx G$, and $H_6(C) \approx H^2(K) = H$, we see that C has the homotopy type of a C.W. complex with only 4-cells, 5-cells and 6-cells, and the attaching maps all homologically trivial. The attaching maps of the 6-cells are determined by Sq^2 in C, hence (by S-duality) by Sq^2 in K, which amounts to cup square $H \to \hat{H}$, reduced mod 2. Then as $\pi_5(S^4) \approx \mathbb{Z}_2$, we have $\pi_5(C) \approx H_4(C; \mathbb{Z}_2) \oplus H_5(C)$, modulo the attaching maps of the 6-cells, i.e.

$$\pi_5(C) \approx H^4(M; \mathbb{Z}_2)/Sq^2\, H^2(M; \mathbb{Z}_2) \oplus H^3(M).$$

Note that if $q = \frac{1}{4} p_1 + Y^2$ (for $Y \in H$) is reduced mod 2, we find

$$q\,(\text{mod}\ 2) = \tfrac{1}{4}\, p_1\,(\text{mod}\ 2) + Sq^2\, H^2(M; \mathbb{Z}_2)$$

determines an element of the first summand. With this identification, we assert that q (mod 2) is the homotopy class of j.

First let $q=0$. Then we can find a nonzero section σ of γ, and hence an embedding g_0 of M_0 disjoint from f_0 except on ∂M_0 (where they agree). The image by g_0 of the 2-skeleton L of M_0 is nullhomotopic in C, so j extends to a map $M_0/L \to C$. But $S^5 \to M_0 \to M_0/L$ is stably nullhomotopic, hence so is j, and thus $j \simeq 0$. For M_0/L is homotopy equivalent to the 4-skeleton of M modulo the 2-skeleton, and hence to a wedge of 3-spheres and 4-spheres, and $\pi_5(M_0/L)$ splits correspondingly as a direct sum. We have zero in the component involving the 4-spheres, since

$$Sq^2: H^4(M;\mathbb{Z}_2) \to H^6(M;\mathbb{Z}_2)$$

vanishes. In the other, by Theorem 1, we have the attaching map of the top cell for a connected sum of copies of $S^3 \times S^3$. But this is a sum of Whitehead products, so its suspension is nullhomotopic.

In the general case, we construct a section which is nonzero on the 3-skeleton of K, and such that the intersection invariant of the resulting $g_0(M_0)$ with a 4-cell E of K can be identified with $q(E)$ (mod 2). The intersection invariant is that of *[19]*: the reduction mod 2 comes about since the map $(E, \partial E) \to (D^3, S^2) \to (S^3, *)$, defining it has homotopy class $\varepsilon\pi_4 S^3$ the suspension of $q(E) \in \pi_3(S^2)$. By adding a suitable tangential component at the zeroes, we can ensure (since $\dim K=4$) that $g_0(K)$ is disjoint from $f_0(K)$; and hence also (since M_1 is a regular neighbourhood of K in Int M_0) that $g_0(M_1)$ is disjoint from $f_0(M_1)$. It then follows as above that $g_0(\partial M_1)$ is nullhomotopic in C. Now let $A = M_0 - \operatorname{Int} M_1$. Then $g_0|A$ gives an isotopy of j to $g_0|\partial M_1$, which is nullhomotopic, so the linking of j and $f_0(K)$ is measured by the intersection of A with $f_0(K)$ and hence, by the above, by q (mod 2), as asserted.

We now come to sufficient conditions: here our object is to obtain smooth embeddings, and so prove that topological embeddability implies smooth embeddability.

Theorem 13. *(i) A connected sum of copies of* $S^2 \times S^4$ *and* $S^3 \times S^3$ *embeds smoothly in* $\mathbb{R}^7$.

Let M *satisfy* (H). *Sufficient conditions for smooth embeddability are:*

(ii) If $p_1(M)=0$, M *embeds in* $\mathbb{R}^8$.

(iii) M *embeds in* $\mathbb{R}^{10}$.

Proof. (i) is trivial, since embeddability of two manifolds easily implies that of their connected sum. In fact by Theorem 1 we can restrict ourselves throughout to manifolds M with $H_3(M)=0$.

By Theorem 2, and the remark at the end of § 2, we can now write $M = \partial W$, with W a handlebody.

(ii) Now $W \times I$ (with corners rounded) is again a handlebody, given by a framed link of 3-spheres in S^7, the suspension of the earlier link. Since all the 3-spheres lie in the equator S^6, they are in fact all unlinked. The framing is given by an element of $\pi_3(SO_4)$ which determines the Pontrjagin classes of W, and of M. Since $p_1(M)=0$, we have the standard framing. Hence c.f. [*19*] $W \times I$ is the standard handlebody, diffeomorphic to a boundary-connected sum of copies of $S^4 \times D^4$. Hence $W \times I$ embeds in $\mathbb{R}^8$; so does $M = \partial W$.

(iii) We again use W, but now construct a bundle E with fibre D^3 over W. The choice of basis of M gave an expression of W as a wedge of 4-spheres: we choose that bundle such that the characteristic class on the i^{th} sphere is given by $-\beta_i' \in \pi_3(SO_3)$. By the definition of β_i', it follows that E has trivial tangent bundle. But E (with corners rounded) is again a handlebody; since we are now in the stable range, E is a sum of copies of $S^4 \times D^6$. So E embeds in $\mathbb{R}^{10}$; so do the zero cross-section W, and $M = \partial W$.

Finally, we must consider embeddings in $\mathbb{R}^9$. These seem to be considerably more difficult to study. The cleanest result we can give concerns "almost smooth embeddings" – i.e. embeddings which are smooth except on a disc, where they are piecewise smooth. Here we have

Theorem 14. *A necessary and sufficient condition for the existence of an almost smooth embedding $M^6 \to \mathbb{R}^9$ is that there exist a class $X \in 2H$ with $X^2 + p_1(M) = 0$.*

It follows at once that our condition is necessary for smooth embeddings and sufficient for piecewise smooth, hence for piecewise linear ones. We conjecture that it is also necessary for a PL embedding.

Proof. Necessity. Let M_0 be the closure of the complement of the bad disc. Take a tubular neighbourhood of M_0 and extend to a smooth regular neighbourhood A of M. Extend the projection of the normal bundle of M_0 to the natural collapsing map of the neighbourhood onto M. The fibre over each point is collapsible, and meets ∂A in a homology 2-sphere. We have a spectral sequence

$$H^p(M; H^q(S^2)) \Rightarrow H^n(\partial A).$$

The sequence restricted to M_0 is that of the fibration, and so is induced from the universal example, the sequence of the fibration $S^2 \to BSO_2 \to BSO_3$. Work with rational coefficients: then this sequence is trivial; $E_2 = E_\infty$ is the tensor product of the exterior algebra on the generator s of $H^2(S^2; \mathbb{Z})$ in $E_2^{0,2}$ and the polynomial algebra on the universal Pontrjagin class $p_1 \in E_2^{4,0}$. But $H^*(BSO_2)$ is the polynomial algebra on the universal Euler class X; we have $X^2 = p_1$, and the restriction of X to the fibre S^2 gives the Euler class of the tangent bundle of S^2, i.e. $2s$.

We deduce in our spectral sequence, since there is no torsion, that the differentials vanish (only d_3 was in doubt), and that we can identify $H^*(\partial A)$ with $H^*(M)\otimes H^*(S^2)$ as groups, with ring structure given by $(2s)^2=p_1(M)$.

Now let B be the closure of the other complementary component of ∂A in S^9. The inclusions $\partial A\to A$ and $\partial A\to B$ induce ring homomorphisms $H^*(M)\approx H^*(A)\to H^*(\partial A)$ and $H^*(B)\to H^*(\partial A)$: in dimensions other than 0, 8, we know by duality that $H^i(\partial A)\approx H^i(M)\oplus H^i(B)$. The rest of the proof is now purely algebraic.

Let $e_1, \dots, e_n$ be a base of $H^2(M)$, z of $H^6(M)$, and $\hat{e}_1, \dots, \hat{e}_n$ the dual base of $H^4(M)$. Using the dummy suffix convention, write

$$s^2=\tfrac{1}{4}p_1=\beta'_i\,\hat{e}_i; \qquad e_i\,e_j=\lambda_{ijk}\,\hat{e}_k$$

as before. We only look at even dimensions; suppose bases of (the images of) the $H^i(B)$ are:

$$\begin{array}{ll} s+a_i\,e_i & \text{(dimension 2)} \\ s\,e_i+b_{ij}\,\hat{e}_j & \text{(dimension 4)} \\ s\,\hat{e}_i+c_i\,z & \text{(dimension 6)}. \end{array}$$

We now write down the conditions for these to define a subring. Products which lie in dimension 8 must vanish: this yields $a_i+c_i=0$ and $b_{ij}+b_{ji}=0$. Writing down the conditions for the product of $s+a_ie_i$ by itself, resp. by $H^4(B)$, to lie in $H^4(B)$, resp. $H^6(B)$ we find

$$2\,a_i\,b_{ij}=\beta'_j+a_i\,a_k\,\lambda_{ijk}; \qquad b_{ij}\,c_j+\lambda_{ijk}\,c_j\,a_k=\beta'_i+b_{ij}\,a_j$$

from which we deduce $b_{ij}\,a_j=0$ and $\lambda_{ijk}a_ja_k=-\beta'_i$. Thus if $Y=a_ie_i$, we have

$$Y^2=\lambda_{ijk}\,a_j\,a_k\,\hat{e}_i=-\beta'_i\,\hat{e}_i=-s^2.$$

So we set $2Y=X\in 2H$, and $X^2=-4s^2=-p_1(M)$, as required.

Sufficiency

By Theorem 10, M immerses smoothly in $\mathbb{R}^9$. Let ν be the normal bundle, A the total space of the corresponding disc bundle. Obtain M_0 as usual from M by removing the interior of a smoothly embedded disc D^6. Let A_0 be the sub-bundle with base M_0. We identify M with the zero cross-section in A. Our plan is to attach handles to A_0 along $A_0\cap\partial A$ to make it contractible, and hence a 9-disc. It will then follow that we can embed M_0 smoothly in D^9, meeting S^8 in ∂M_0. We can change this by a piecewise smooth isotopy near ∂M_0 to make it PL on ∂M_0. The embedding can then be completed by attaching a cone on ∂M_0 in the complementary disc to D^9 in S^9. As in the previous theorem, we can assume $H_3(M)=0$.

Note that any closed subset of ∂A not meeting the fibre S^2 over the central point of D^6 can be isotoped to lie in $A_0 \cap \partial A$: hence this is so for a subset of dimension $\leqq 5$.

We will first attach 3-cells to kill $H_2(A)$: we must select elements of $\pi_2(\partial A) \simeq H_2(\partial A)$. We choose a basis of the subgroup of $H_2(\partial A)$ annihilated by the cohomology class $s+Y$, where $X=2Y$ and s are as in the first part of the proof. Represent the chosen elements by disjointly embedded spheres, and perform framed surgery ([*11*], p. 520): note that by construction A is framed. Write C_0 for A_0 with the handles attached, and $C=C_0 \cup A$. Then, up to homotopy, C_0 is a wedge of 4-spheres, and C is obtained from it by attaching a 6-cell (by a trivial map: the functional Sq^2 vanishes since $Sq^2: H^4(M; \mathbb{Z}_2) \to H^6(M; \mathbb{Z}_2)$ does).

Now ∂C is simply-connected, and $H_2(\partial C)$ is free, $H_3(\partial C)$ vanishes. We assert that from these facts follows that an element of $H_4(\partial C)$ is spherical if and only if it is annihilated by the cup product of any two elements of $H^2(\partial C)$. It suffices to prove this for a CW complex K with only 2-cells and 4-cells. Then $\pi_4(K, K^2) \cong H_4(K)$, and we have an exact sequence

$$\pi_4 K \to \pi_4(K, K^2) \to \pi_3(K^2)$$

and can identify $\pi_3(K^2)$ with the subgroup of symmetric elements of $H_2(K) \otimes H_2(K)$, and the boundary map as the dual of the cup product $H^2(K) \otimes H^2(K) \to H^4(K)$. Our assertion follows.

Now $H^2(C)$ is generated by a class which we will call $s+Y$, and $(s+Y)^2 = 2sY$. Thus if $h_1, \ldots, h_{2n}$ is the base of $H_4(\partial C)$ dual to the base $\hat{e}_1, \ldots, \hat{e}_n, se_1, \ldots, se_n$ of $H^4(\partial C)$, $h_1, \ldots, h_n$ are spherical. They are dual to the cohomology classes $se_1, \ldots, se_n$, any two of which have product zero. Thus we can represent them by 4-spheres with intersection numbers zero; by an argument of [*11*] we can make all the 4-spheres disjointly embedded, with trivial normal bundles. These can be pushed into $C_0 \cap \partial C$. Attaching corresponding handles to C_0 makes it contractible; we verify easily that the boundary of the result is simply-connected. By a result of SMALE [*17*], we have a disc. This completes the proof.

The insufficiency of the condition of Theorem 12 (iii) is of interest. The obvious way of obtaining an embedding would be to embed M_0 by a general position argument, use that condition to obtain a nullhomotopy of ∂M_0 in the complement C of M_1, and then apply a general theorem. But the general theorem (Hudson, unpublished) gives an obstruction in $H_4(C) \cong \hat{H}$ to further progress. Our work suggests that the obstruction is in fact $\frac{1}{8}(p_1 + X^2)$.

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Department of Pure Mathematics
The University
Liverpool, 3

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