Numer. Math. 49, 613–657 (1986)

Numerische Mathematik

The h, p and h-p Versions of the Finite Element Method in 1 Dimension

Part II. The Error Analysis of the h- and h-p Versions

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Summary. The paper is the second in the series of three devoted to the detailed analysis of the three basic versions of the finite element method in one dimensional setting. The first part [1] analyzed the p-version, the present one concentrates on the h and h-p versions.

Subject Classifications: AMS(MOS): 65N30; CR: G18.

1. Introduction

This paper is the second part in the series of three which address in detail the properties and performance of the h, p and h-p versions of the finite element method in one dimensional setting.

In general the h-version of FEM has the degree of elements fixed and the convergence is achieved by the refinement of the mesh. The p-version fixes the mesh and increases the degree of elements. The h-p version combines both approaches. In Part I [1] we have developed a basic tool for the error analysis and as an immediate application we studied the features of the p-version. In this part we will investigate the h and h-p version.

There is enormous literature devoted to the theory and practice of the h-version while the p and h-p versions are a very new developments [1]. The commercial code PROBE based on p and h-p versions was introduced in 1985. For the theoretical foundation of the h-p version in one dimensional setting we refer to [2, 3] where the approximation of the function x^α by piecewise polynomials of variable degrees and nodal points was studied.

In this paper we consider the same model problem as in Part I namely

$$-u''=f \tag{1.1}$$

$$u(0)=u(1)=0$$

* Supported by NSF Grant DMS-8315216

** Partially supported by UNR Contract N00014-85-K-0169

with the solution

$$u(x) = x^\alpha - x.$$

To ensure $u \in H^1$, we assume $\alpha > 1/2$. We assume that α is not an integer, so that the solution has a singularity of the type x^α at $x = 0$, which models the singularity caused by corners of the domain in two-dimensional problems.

Let $u_s(x)$ be the finite element solution of (1.1), and let

$$e(x) = u(x) - u_s(x)$$

be the error. We are interested in the relation between the magnitude of the error in energy norm, $\|e\|_E$, and the number N of degrees of freedom of the finite element space.

We denote the finite element space by

$$S \equiv S(\Sigma) \subset H^{01}(0,1)$$

which is determined by the mesh-degree combination $\Sigma = (\Delta, p)$, where

$$\Delta\colon \{0 = x_0^\Delta < x_1^\Delta < x_2^\Delta < \cdots < x_{m(\Delta)}^\Delta = 1\}$$
$$p\colon (p_1^\Delta, p_2^\Delta, \dots, p_{m(\Delta)}^\Delta)$$

in which p_i^Δ is the polynomial degree on the interval $I_i^\Delta = (x_{i-1}^\Delta, x_i^\Delta)$, $i = 1, 2, \dots, m(\Delta)$. Furthermore, we denote

$$h_i^\Delta \equiv |I_i^\Delta| = x_i^\Delta - x_{i-1}^\Delta$$
$$h(\Delta) = \max_{1 \leq i \leq m(\Delta)} h_i^\Delta.$$

The points x_i^Δ will be called the *nodal points*, the intervals I_i^Δ the elements. The restriction of $S(\Sigma)$ on I_i^Δ will also be called an *element* and p_i^Δ the *degree of the element* I_i^Δ. For the mesh Δ we will often write $x_i^\Delta \in \Delta$ or $I_i^\Delta \in \Delta$, etc., which will not lead to any misunderstanding. The number $m(\Delta)$ will be called the *cardinality* of the mesh Δ. Obviously, we have

$$N = \dim(S) = \sum_{i=1}^{m(\Delta)} p_i^\Delta - 1.$$

If there is no confusion we will drop the superscripts and write p_i, I_i, x_i, etc.; also we write m for $m(\Delta)$.

It is well known that in the case of our model problem $e(x_i) = 0$ and $\|e\|_E = \|e'\|_{L_2}$. Therefore our problem reduces to studying the approximation questions on every element separately. We denote the local error on a mesh interval $I = [a, b]$, associated with polynomial of degree p, by

$$\|e\|_{E(I)} \equiv E_p(I) \equiv E_p[a, b].$$

From Part I, we have the following theorem

Theorem 1.1. *Let $E_{p_i}(I_i)$ be the local error of the finite element solution of the model problem* (1.1),

$$I_i = [x_{i-1}, x_i],\ r_i = \frac{\sqrt{x_i} - \sqrt{x_{i-1}}}{\sqrt{x_i} + \sqrt{x_{i-1}}},\ h_i = x_i - x_{i-1},$$

then

$$E_{p_1}(I_1) \approx \frac{h_1^{\alpha-1/2}}{p_1^{2\alpha-1}}. \tag{1.2}$$

If $0 < r_i^2 < 1 - \frac{1}{p_i}$, $i \geqq 2$, *then*

$$E_{p_i}(I_i) \approx \frac{h_i^{\alpha-1/2}}{\sqrt{1-r_i^2}} \frac{r_i^{p_i+1-\alpha}}{p_i^\alpha} \left(\frac{1}{p_i^{\alpha-1/2}} + (1-r_i^2)^{\alpha-1/2}\right). \tag{1.3}$$

If $1 - \frac{1}{p_i} \leqq r_i^2 \leqq 1$, $i \geqq 2$ *then*

$$E_{p_i}(I_i) \approx h_i^{\alpha-1/2} \frac{r_i^{p_i+1-\alpha}}{p_i^{\alpha-1/2}} \left(\frac{1}{p_i^{\alpha-1/2}} + (1-r_i^2)^{\alpha-1/2}\right). \tag{1.4}$$

In the inequalities (1.2)–(1.4) the symbol $\approx$ means that the ratio of the left and the right hand side is bounded above and below by equivalency constants which merely depend on α.

Remark 1.1. If r_i is not close to 1, then (1.3) may be written as

$$E_{p_i}(I_i) \approx h_i^{\alpha-1/2} \left(\frac{1-r_i^2}{2r_i}\right)^{\alpha-1} \frac{r_i^{p_i}}{p_i^\alpha}. \tag{1.5}$$

Also we quote the following theorem from Part I (which is translated into the energy norm $\|e\|_E$):

Theorem 1.2. *Let x be given and $x > 0$, $\{I\}$ be a family of intervals containing x. Then*

$$\lim_{|I| \to 0} \frac{E_p(I)}{|I|^{p+1/2}} = C(\alpha, p) \frac{1}{x^{p+1-\alpha}} \tag{1.6}$$

where

$$C(\alpha, p) = \frac{\alpha \Gamma(\alpha) |\sin \pi\alpha|}{\sqrt{\pi}} \frac{\Gamma(p+1-\alpha)}{4^p \sqrt{2p+1}\, \Gamma(p+1/2)}$$

This limit is uniform with respect to $x \geqq \varepsilon$, $\varepsilon > 0$.
We will write

$$E_p(I) \cong C(\alpha, p) |I|^{p+1/2} \frac{1}{x^{p+1-\alpha}} \qquad (x \in I)$$

and denote $\cong$
by asymptotically equal.

2. The *h* Version of the Finite Element Method

2.1. *The Optimal Rate of Convergence of the h Version*

The h version assumes that the polynomial degree p of the elements is fixed; thus the number of degrees of freedom is

$$N = mp - 1$$

where m is the number of the elements. For simplicity, we let

$$N = mp. \tag{2.1}$$

The rate of convergence of the h extension is never better than N^{-p}. We have

Theorem 2.1. *Let $\alpha > 1/2$ be non-integer, then there is a constant $C = C(\alpha, p) > 0$ such that for any mesh $\Delta = \{0 = x_0 < x_1 < \cdots < x_m = 1\}$*

$$\|e\|_E \geqq CN^{-p}. \tag{2.2}$$

Proof. Let $h_i = x_i - x_{i-1}$. By Theorem 1.1 we have for $i \geqq 2$:

$$E_p[x_{i-1}, x_i] \geqq \begin{cases} C(\alpha)\,\dfrac{h_i^{\alpha-1/2}}{\sqrt{1-r_i^2}}\,\dfrac{r_i^{p+1-\alpha}}{p^{2\alpha-1/2}} & 0 < r_i^2 \leqq 1 - \dfrac{1}{p+1} \\ C(\alpha)\,h_i^{\alpha-1/2}\,\dfrac{r_i^{p+1-\alpha}}{p^{2\alpha-1}} & 1 - \dfrac{1}{p+1} \leqq r_i^2 \leqq 1 \end{cases} \tag{2.3}$$

where

$$r_i = \frac{\sqrt{x_i} - \sqrt{x_{i-1}}}{\sqrt{x_i} + \sqrt{x_{i-1}}}.$$

Since $r_i = \dfrac{h_i}{(\sqrt{x_i} + \sqrt{x_{i-1}})^2} \geqq \dfrac{h_i}{4}$, (2.3) gives

$$E_p[x_{i-1}, x_i] \geqq C(\alpha)\,\frac{h_i^{p+1/2}}{4^{p+1-\alpha} p^{2\alpha-1/2}} = C(\alpha, p)\, h_i^{p+1/2}. \tag{2.4}$$

Thus

$$\|e\|_E^2 \geqq \sum_{i=2}^{m} E_p[x_{i-1}, x_i]^2 \geqq C(\alpha, p)^2 \sum_{i=2}^{m} h_i^{2p+1}.$$

The right-hand side takes minimum if and only if $h_2 = h_3 = \cdots = h_m$. Since one must have $h_1 \to 0$ in order to obtain $\|e\|_E \to 0$, we can assume that $h_1 < 1/2$ and then

$$h_i \geqq \frac{1}{2m}, \qquad (i \geqq 2).$$

It follows that

$$\|e\|_E \geqq C(\alpha, p)\, m^{-p} = C(\alpha, p)\, N^{-p}$$

where $C(\alpha, p)$ is a generic constant only depending on α and p. □

In the following, we first discuss the performance of the general graded mesh, then the special mesh graded by the grading function $g(x) = x^\beta$ ($\beta > 0$) (which will be called radical mesh). The uniform mesh is a special case of the radical mesh with $\beta = 1$.

2.2. *The Graded Mesh*

$g(x)$ is called a mesh grading function if the nodal points of the mesh are such that

$$x_i = g\left(\frac{i}{m}\right) \qquad i = 0, 1, 2, \ldots, m.$$

We shall assume that the grading function satisfies the following conditions:

(G1) $g(0)=0$, $g(1)=1$.

(G2) g is continuous and strictly increasing.

We will confine ourselves to the special case

(G3) $g\in C^1(0,1)\cap C^0[0,1]$.

The continuity of g leads to

Lemma 2.1. *Let Δ be given by the grading function $g(x)$, satisfying (G1) ~ (G3), then*

$$\lim_{m\to\infty}\max_{1\leqq i\leqq m}|I_i|=0$$

where $m=m(\Delta)$ is the number of intervals in Δ. □

Let e_m be the error function of the finite element solution with a mesh of cardinality m graded by the grading function $g(x)$ and $\|e\|_{E(a,b)}$ denotes the error in energy norm on $[a,b]\subseteq I$.

Lemma 2.2. *Let $0<a<b<1$, then*

$$\lim_{m\to\infty} m^p\|e_m\|_{E(a,b)}=C(\alpha,p)\left\{\int_a^b[g'(t)]^{2p+1}[g(t)]^{-2(p+1-\alpha)}dt\right\}^{1/2}$$

where

$$C(\alpha,p)=\frac{\alpha\Gamma(\alpha)|\sin\pi\alpha|}{\sqrt{\pi}}\,\frac{\Gamma(p-\alpha+1)}{4^p\sqrt{2p+1}\,\Gamma(p+1/2)}. \tag{2.5}$$

Proof. Let $m\to\infty$, then by Theorem 1.2 for $x\geqq a$ one has

$$\begin{aligned}\|e_m\|_{E(a,b)}&\cong\left\{\sum_{[mg^{-1}(a)]\leqq i\leqq[mg^{-1}(b)]}E_p[x_{i-1},x_i]^2\right\}^{1/2}\\
&\cong C(\alpha,p)\left\{\sum_{[mg^{-1}(a)]\leqq i\leqq[mg^{-1}(b)]}\frac{h_i^{2p+1}}{x_i^{2(p+1-\alpha)}}\right\}^{1/2}\\
&\cong C(\alpha,p)\left\{\sum_{[mg^{-1}(a)]\leqq i\leqq[mg^{-1}(b)]}\frac{\left[g'\left(\frac{i}{m}\right)\frac{1}{m}\right]^{2p+1}}{\left[g\left(\frac{i}{m}\right)\right]^{2(p+1-\alpha)}}\right\}^{1/2}\\
&\cong\frac{1}{m^p}C(\alpha,p)\left\{\int_a^b[g'(t)]^{2p+1}[g(t)]^{-2(p+1-\alpha)}dt\right\}^{1/2}. \quad\square\end{aligned}$$

Lemma 2.3. *Suppose that*

$$\text{(G4)}\qquad \int_0^1[g'(t)]^{2p+1}[g(t)]^{-2(p+1-\alpha)}dt<\infty$$

and

$$\text{(G5)}\qquad g(t)=o\left(t^{\frac{1}{\alpha-1/2}}\right)\quad\text{as}\quad t\to 0$$

then

$$\lim_{m\to\infty} m^p \|e_m\|_E = C(\alpha,p)\left\{\int_0^1 [g'(t)]^{2p+1}[g(t)]^{-2(p+1-\alpha)}\,dt\right\}^{1/2}. \tag{2.6}$$

Proof. Under condition (G4), it is clear that

$$\lim_{m\to\infty} m^p \|e_m\|_{E(a,1)} = C(\alpha,p)\left\{\int_a^1 [g'(t)]^{2p+1}[g(t)]^{-2(p+1-\alpha)}\,dt\right\}^{1/2}$$

for any $a>0$. Thus it suffices to show that for any $\varepsilon>0$, there is $a>0$ such that

$$\overline{\lim_{m\to\infty}}\, m^p \|e_m\|_{E(0,a)} < \varepsilon .$$

In fact, by Theorem 1.1 we have $E_p[x_{i-1},x_i] \leqq C_1(\alpha,p)\, h_i^{\alpha-1/2}\, r_i^{p+1-\alpha}$, thus

$$\begin{aligned}
\|e_m\|^2_{E(0,a)} &\leqq \sum_{i=1}^{[mg^{-1}(a)]+1} E_p[x_{i-1},x_i]^2\\
&\leqq C_1(\alpha,p)\left\{h_1^{2\alpha-1} + \sum_{i=2}^{[mg^{-1}(a)]+1} h_i^{2\alpha-1}\left(\frac{h_i}{(\sqrt{x_{i-1}}+\sqrt{x_i})^2}\right)^{2(p+1-\alpha)}\right\}\\
&\leqq C_1(\alpha,p)\left\{h_1^{2\alpha-1} + \sum_{i=2}^{[mg^{-1}(a)]+1} h_i^{2\alpha-1}\, x_{i-1}^{-2(p+1-\alpha)}\right\}.
\end{aligned}$$

Thus by (G4), (G5) we obtain

$$\begin{aligned}
&\overline{\lim_{m\to\infty}}\,(m^p \|e_m\|_{E(0,a)})^2\\
&\quad\leqq C_1(\alpha,p)\left\{\lim_{m\to\infty}\left(m^{\frac{p}{\alpha-1/2}}\cdot g\left(\frac{1}{m}\right)\right)^{2\alpha-1} + \int_0^a [g'(t)]^{2p+1}[g(t)]^{-2(p+1-\alpha)}\,dt\right\}\\
&\quad= C_1(\alpha,p)\int_0^a [g'(t)]^{2p+1}[g(t)]^{-2(p+1-\alpha)}\,dt .
\end{aligned}$$

By (G4) we only need to choose a small enough and (2.6) follows. □

Lemma 2.4. *Let*

$$I[g] = \int_0^1 g(x)^{\sigma-n} g'(x)^n\,dx$$

with

$$\sigma = 2\alpha-1 \geqq 0,\ n = 2p+1,$$
$$g \in C^1(0,1)\cap C^0[0,1]$$

and

$$g(0)=0,\ g(1)=1,$$
$g(x)$ *is strictly increasing.*

Then the functional $I[g]$ has a unique minimizer

$$g(x)=x^{\beta}$$

with

$$\beta=\frac{n}{\sigma}=\frac{p+1/2}{\sigma-1/2}.$$

Proof. Let $G=g^{\sigma/n}$. Then

$$\begin{aligned}G'&=\frac{\sigma}{n}\,g^{\sigma/n-1}\,g' \qquad (\geqq 0)\\&=\frac{\sigma}{n}\,[g^{\sigma-n}(g')^{n}]^{1/n}.\end{aligned}$$

Thus

$$\begin{aligned}I[g]&=\int_0^1\left[\frac{n}{\sigma}\,G'(x)\right]^{n}dx\\&=\left(\frac{n}{\sigma}\right)^{n}\int_0^1[G'(x)]^{n}\,dx.\end{aligned}$$

Hence it suffices to consider the minimizer of the functional

$$J[G]=\int_0^1[G'(x)]^{n}\,dx \tag{2.7}$$

with the condition

$$\begin{gathered}G(0)=0,\ G(1)=1,\\G\in C^{1}(0,1)\cap C^{0}[0,1],\end{gathered} \tag{2.8}$$

and $G(x)$ is strictly increasing.

By the standard variational method, we have

$$\delta J[G]=\int_0^1 n\,[G'(x)]^{n-1}\,\delta G'\,dx.$$

This implies

$$G'(x)=\text{const.}$$

thus

$$G(x)=Cx+C_1$$

since $G(0)=0$, $G(I)=1$, we obtain

$$G(x)=x.$$

Hence

$$g(x)=x^{n/\sigma}.$$

It is easy to show that $G(x)=x$ is actually a minimizer of (2.7).
This shows that $g(x)=x^{n/\sigma}$ is the unique minimizer of $I[g]$. □

Remark 2.1. Lemma 2.4 gives immediately

$$\min I[g]=\int_0^1 (x^{n/\sigma})^{\sigma-n}\left(\frac{n}{\sigma}x^{(n/\sigma)-1}\right)^n dx$$
$$=\left(\frac{n}{\sigma}\right)^n=\left(\frac{p+1/2}{\alpha-1/2}\right)^{2p+1}. \quad \square \tag{2.9}$$

We now can state

Theorem 2.2. *Among all grading function $g(t)$ satisfying* (G1) ~ (G3)

$$g_{op}(x)=x^\beta \quad with \quad \beta=\frac{p+1/2}{\alpha-1/2} \tag{2.10}$$

is the optimal one. Precisely, with this grading function the limit

$$\lim_{m\to\infty} m^p \|e_m\|_E = C(\alpha,p)\left(\frac{p+1/2}{\alpha-1/2}\right)^{p+1/2} \tag{2.11}$$

attains the minimum.

Proof. Obviously, if

$$I[g]=\int_0^1 [g'(t)]^{2p+1}[g(t)]^{-2(p+1-\alpha)}\,dt=+\infty \quad \text{or} \quad m^p g\left(\frac{1}{m}\right)^{\alpha-1/2}\not\to 0,$$

then the rate of convergence for this mesh grading function $g(x)$ is worse. Thus one needs only to consider the grading function which satisfies (G4) and (G5). In this case, Lemma 2.3 gives

$$\lim_{m\to\infty} m^p\|e_m\|_E=A(\alpha,p)(I[g])^{1/2}.$$

Lemma 2.4 shows that the functional $I[g]$ has a unique minimizer $g_{op}(x)$ defined by (2.10), and the theorem follows. □

Remark 2.2. As $p\to\infty$, we have

$$C(\alpha,p)\left(\frac{p+1/2}{\alpha-1/2}\right)^{p+1/2}\cong\frac{\alpha\Gamma(\alpha)|\sin\pi\alpha|}{\sqrt{\pi}}\,\frac{\sqrt{2e}\,p^{p-\alpha+1/2}}{[4(\alpha-1/2)]^{p+1/2}} \tag{2.12}$$

and with $N=mp$, (2.11) leads to

$$\|e\|_E\approx C(\alpha)\,\frac{p^{2p-\alpha+1/2}}{[4(\alpha-1/2)N]^p}. \tag{2.13}$$

2.3. The Radical Mesh

Let us consider now special graded mesh given by the grading function

$$g(x)=x^\beta, \qquad (\beta>0). \tag{2.14}$$

We have the following theorem:

Theorem 2.3. *For the radical meshes given by the grading function* (2.14):

1) *if* $\beta > \frac{p}{\alpha - 1/2}$, *then*

$$\lim_{m \to \infty} m^p \|e_m\|_E = \frac{C(\alpha, p)\, \beta^{p+1/2}}{\sqrt{(2\alpha - 1)\beta - 2p}} \tag{2.15}$$

2) *if* $\beta = \frac{p}{\alpha - 1/2}$, *then*

$$\lim_{m \to \infty} \frac{m^p \|e_m\|_E}{\sqrt{\ln m}} = C(\alpha, p)\, \beta^{p+1/2}. \tag{2.16}$$

In 1) *and* 2) $C(\alpha, p)$ *is given by* (2.5).

3) *if* $\beta < \frac{p}{\alpha - 1/2}$, *then*

$$\lim_{m \to \infty} m^{\beta(\alpha - 1/2)} \|e_m\|_E = C_1(\alpha, \beta, p) \tag{2.17}$$

where $0 < C_1(\alpha, \beta, p) < \infty$ (*it has a more complicated expression than* $C(\alpha, p)$).

Proof. Let $g(x) = x^\beta$, then

$$\begin{aligned} G(x) &= [g'(x)]^{2p+1} [g(x)]^{-2(p+1-\alpha)} \\ &= \beta^{2p+1} x^{(2\alpha-1)\beta - (2p+1)}. \end{aligned}$$

$G \in L_1(0, 1)$ if and only if $\beta > \frac{p}{\alpha - 1/2}$, and in this case the conditions (G4), (G5) are satisfied. Thus for $\beta > \frac{p}{\alpha - 1/2}$ we have by Lemma 2.3:

$$\begin{aligned} \lim_{m \to \infty} m^p \|e_m\|_E &= C(\alpha, p) \left\{ \int_0^1 \beta^{2p+1} x^{(2p-1)\beta - (2p+1)} dx \right\}^{1/2} \\ &= \frac{C(\alpha, p)\, \beta^{p+1/2}}{\sqrt{(2\alpha - 1)\beta - 2p}}. \end{aligned} \tag{2.18}$$

In the case $\beta \leqq \frac{p}{\alpha - 1/2}$, we cannot use the previous lemma. Instead, we compute the estimates of the errors directly. Denoting by $a_n(i)$ the coefficients of the Legendre expansion of the solution on $[x_{i-1}, x_i]$, we have

$$E_p[x_{i-1}, x_i]^2 = \sum_{n=p}^{\infty} a_n(i)^2 \frac{2}{2n+1}.$$

If $i = 1$, then (cf. [1] Theorem 1)

$$\begin{aligned} |a_n(1)| &= \left(\frac{x_1}{2}\right)^{\alpha - 1/2} \frac{\alpha \Gamma(\alpha)^2 |\sin \pi\alpha|}{\pi} \frac{\Gamma(n - \alpha + 1)(2n+1)}{\Gamma(n + \alpha + 1)} \cdot 2^{\alpha - 1} \\ &= \left(\frac{1}{m}\right)^{\beta(\alpha - 1/2)} C_0(\alpha) \frac{\Gamma(n - \alpha + 1)}{\Gamma(n + \alpha + 1)} (n + 1/2) \end{aligned}$$

where

$$C_0(\alpha) = \frac{\sqrt{2}\,\alpha\Gamma(\alpha)^2\,|\sin\pi\alpha|}{\pi}.$$

Therefore

$$\begin{aligned} E_p[0, x_1] &= \left(\frac{1}{m}\right)^{\beta(\alpha-1/2)} C_0(\alpha) \left\{\sum_{n=p}^{\infty} \left(\frac{\Gamma(n-\alpha+1)}{\Gamma(n+\alpha-1)}\right)(n+1/2)\right\}^{1/2} \\ &\equiv A_{0,p}(\alpha)\left(\frac{1}{m}\right)^{\beta(\alpha-1/2)} \end{aligned}$$

where

$$0 < A_{0,p}(\alpha) \equiv C_0(\alpha)\left\{\sum_{n=p}^{\infty}\left(\frac{\Gamma(n-\alpha+1)}{\Gamma(n+\alpha+1)}\right)^2 (n+1/2)\right\}^{1/2} < \infty,$$

because

$$\left(\frac{\Gamma(n-\alpha+1)}{\Gamma(n+\alpha+1)}\right)^2 (n+1/2) \cong \frac{1}{n^{4\alpha-1}} \quad \text{and} \quad 4\alpha - 1 > 1.$$

If $i \geqq 2$, we have by Theorem 1.1

$$E_p[x_{i-1}, x_i] \leqq C(\alpha, p)\, h_i^{\alpha-1/2}\, r_i^{p+1-\alpha}$$

where

$$h_i = x_i - x_{i-1} = \left(\frac{1}{m}\right)^{\beta} [i^{\beta} - (i-1)^{\beta}]$$

and

$$r_i = \frac{i^{\beta/2} - (i-1)^{\beta/2}}{i^{\beta/2} + (i-1)^{\beta/2}}.$$

Hence

$$[m^{\beta(\alpha-1/2)}\|e_m\|_{E(x_1,1)}]^2 = \sum_{i=2}^{m} [E_p[x_{i-1}, x_i]\cdot m^{\beta(\alpha-1/2)}]^2 \equiv \sum_{i=2}^{m} b_i^2$$

where

$$\begin{aligned} b_i &= b(\alpha, \beta, p, i) \\ &\leqq C(\alpha, p)\,[i^{\beta} - (i-1)^{\beta}]^{\alpha-1/2}\, r_i^{p+1-\alpha} \\ &\leqq C(\alpha, \beta, p)\left(\frac{1}{i}\right)^{p+1/2-\beta(\alpha-1/2)}. \end{aligned}$$

Clearly, if $\beta < \frac{p}{\alpha - 1/2}$, then $p + 1/2 - \beta(\alpha - 1/2) > 1/2$ and $A_{1,p}(\alpha, \beta) \equiv \sum_{i=2}^{\infty} b_i^2 < \infty$. Thus

$$\lim_{m\to\infty} m^{\beta(\alpha-1/2)}\|e_m\|_E = \{A_{0,p}(\alpha)^2 + A_{1,p}(\alpha, \beta)^2\}^{1/2} < \infty.$$

We now consider the case $\beta = \frac{p}{\alpha - 1/2}$. The above result shows that

$$m^{\beta(\alpha-1/2)} \left\{ \sum_{i=2}^{m} E_{p+2}[x_{i-1}, x_i]^2 \right\}^{1/2} \to A_{1,p+1}(\alpha, \beta) < \infty$$

as $m \to \infty$. The terms $a_p(i)$ in the expression for $E_p[x_{i-1}, x_i]^2$ are (cf. [1], Theorem 1)

$$\begin{aligned} a_p(i)^2 \frac{2}{2p+1} &= \left[\left(\frac{h_i}{2} \right)^{\alpha - 1/2} \frac{\alpha \Gamma(\alpha) |\sin \pi \alpha|}{\sqrt{\pi} \cdot 2^{\alpha-1}} \frac{\Gamma(p-\alpha+1)}{\Gamma(p+1/2)} r_i^{p-\alpha+1} \Phi_{p,\alpha-1}(r_i^2) \right]^2 \frac{2}{2p+1} \\ &= \left[\left(\frac{1}{m} \right)^{\beta(\alpha-1/2)} C_2(\alpha, p) (i^\beta - (i-1)^\beta)^{\alpha-1/2} r_i^{p-\alpha+1} \Phi_{p,\alpha-1}(r_i^2) \right]^2 \end{aligned}$$

where

$$C_2(\alpha, p) = \frac{\alpha \Gamma(\alpha) |\sin \pi \alpha|}{4^{\alpha-1} \sqrt{\pi} \sqrt{2p+1}} \frac{\Gamma(p-\alpha+1)}{\Gamma(p+1/2)}.$$

Therefore

$$\sum_{i=2}^{\infty} a_p(i) \frac{2}{2p+1} = \left[\left(\frac{1}{m} \right)^{\beta(\alpha-1/2)} C_2(\alpha, p) \right]^2 \sum_{i=2}^{m} b_i$$

where

$$\begin{aligned} b_i &= (i^\beta - (i-1)^\beta)^{\alpha-1/2} r_i^{p-\alpha+1} \Phi_{p,\alpha-1}(r_i^2) \\ &\cong \beta^{\alpha-1/2} \left(\frac{\beta}{4} \right)^{p-\alpha+1} i^{-1/2} \quad \text{as } i \to \infty. \end{aligned}$$

By a calculus lemma

$$\lim_{m\to\infty} \frac{y_n}{z_n} = \lim_{m\to\infty} \frac{y_n - y_{n-1}}{z_n - z_{n-1}}$$

(provided $y_n \uparrow \infty$, $z_n \uparrow \infty$ and the right-hand side limit exists), we obtain

$$\lim_{m\to\infty} \frac{\sum_{i=1}^{m} b_i^2}{\sum_{i=2}^{m} \frac{1}{i}} = \lim_{m\to\infty} m b_m^2 = \left(\frac{\beta^{p+1/2}}{4^{p-\alpha+1}} \right)^2.$$

Hence we conclude

$$\begin{aligned} \lim_{m\to\infty} \frac{m^p \|e_m\|_E}{\sqrt{\ln m}} &= \lim_{m\to\infty} \left\{ \frac{A_{0,p}^2}{\ln m} + C_2(\alpha, p)^2 \frac{\sum_{i=1}^{m} b_i^2}{\ln m} + \frac{A_{1,p+1}^2}{\ln m} \right\}^{1/2} \\ &= C_2(\alpha, p) \frac{\beta^{p+1/2}}{4^{p-\alpha+1}} = C(\alpha, p) \beta^{p+1/2}. \quad \square \end{aligned}$$

Corollary 2.1. *For uniform meshes:*

1) *if* $p < \alpha - 1/2$, *then*

$$\lim_{m\to\infty} m^p \|e_m\|_E = \frac{C(\alpha,p)}{\sqrt{2\alpha-1-2p}} \tag{2.20}$$

2) *if* $p = \alpha - 1/2$, *then*

$$\lim_{m\to\infty} \frac{m^p \|e_m\|_E}{\sqrt{\ln m}} = C(\alpha,p) \tag{2.21}$$

3) *if* $p > \alpha - 1/2$, *then*

$$\lim_{m\to\infty} m^{\alpha-1/2} \|e_m\|_E = B(\alpha,p) \tag{2.22}$$

where $C(\alpha,p)$ *is given as before by* (2.5) *and* $0 < B(\alpha,p) < \infty$.

Proof. One only needs to notice $\beta = 1$ in this case. □

Remark 2.3. Although there is no simple expression for $C_1(\alpha,p,\beta)$ in (2.17), we can obtain various estimates. Because for $i \geqq 2$,

$$r_i = \frac{i^{\beta/2} - (i-1)^{\beta/2}}{i^{\beta/2} + (i-1)^{\beta/2}} \leqq \frac{2^{\beta/2}-1}{2^{\beta/2}+1}$$

are bounded away from 1 (β fixed), we have (see Remark 1.1)

$$E_p[x_{i-1}, x_i] \cong C_1(\alpha) \left(\frac{h_i}{2}\right)^{\alpha-1/2} \left(\frac{1-r_i^2}{2r_i}\right)^{\alpha-1} \frac{r_i^p}{p^\alpha}, \qquad (p \to \infty)$$

with $C_1(\alpha) = \dfrac{\alpha\Gamma(\alpha)\,|\sin \pi\alpha|}{\sqrt{\pi}}$. Because

$$h_i = x_i - x_{i-1} = \left(\frac{1}{m}\right)^\beta [i^\beta - (i-1)^\beta]$$

$$\frac{1-r_i^2}{2r_i} = \frac{2[i(i-1)]^{\beta/2}}{i^\beta - (i-1)^\beta},$$

we have for some constant $C(\alpha) > 0$,

$$m^{\beta(\alpha-1/2)} E_p[x_{i-1}, x_i] \leqq C(\alpha) \sqrt{i^\beta - (i-1)^\beta}\; [i(i-1)]^{\beta/2(\alpha-1)} \frac{r_i^p}{p^\alpha}$$

(this inequality holds for any m and p, i). Thus

$$\sum_{i=2}^{m} \{m^{\beta(\alpha-1/2)} E_p[x_{i-1}, x_i]\}^2 \leqq \frac{C(\alpha)^2}{p^{2\alpha}} \sum_{i=2}^{m} [i^\beta - (i-1)^\beta]\,[i(i-1)]^{\beta(\alpha-1)} r_i^{2p}.$$

Let $\beta \geqq 1$, then by using inequality

$$(1-x)^\beta \geqq 1 - \beta x$$

we have

$$[i^\beta-(i-1)^\beta]\,[i(i-1)]^{\beta(\alpha-1)}\,r_i^{2p}$$
$$=\frac{\left[1-\left(1-\frac{1}{i}\right)^\beta\right]^{2p+1}\left[1-\frac{1}{i}\right]^{2\beta(\alpha-1)}}{\left[1+\left(1-\frac{1}{i}\right)^\beta+2\left(1-\frac{1}{i}\right)^{\beta/2}\right]^{2p}}\,i^{\beta(2\alpha-1)}$$
$$\leqq\frac{\beta^{2p+1}}{4^{2p}}\,i^{\beta(2\alpha-1)-(2p+1)}.$$

If $1\leqq\beta\leqq\frac{p}{\alpha-1/2}$ then

$$\sum_{i=2}^{m} i^{\beta(2\alpha-1)-(2p+1)}\leqq\int_1^\infty x^{\beta(2\alpha-1)-(2p+1)}\,dx=\frac{1}{2p-\beta(2\alpha-1)},$$

thus

$$m^{\beta(\alpha-1/2)}\,\|e_m\|_E\leqq\left\{A_{0,p}^2+\frac{C(\alpha)^2}{p^{2\alpha}}\frac{\beta^{2p+1}}{4^{2p}}\frac{1}{2p-\beta(2\alpha-1)}\right\}^{1/2}$$

where $A_{0,p}$ is the same as in (2.19) and

$$A_{0,p}\leqq C(\alpha)\left\{\frac{1}{4\alpha-2}\frac{1}{p^{4\alpha-2}}\right\}^{1/2}=C(\alpha)\frac{1}{p^{2\alpha-1}}.$$

Therefore we obtain for $1\leqq\beta<\frac{p}{\alpha-1/2}$

$$m^{\beta(\alpha-1/2)}\,\|e_m\|_E\leqq C(\alpha)\max\left\{\frac{1}{p^{2\alpha-1}},\ \frac{1}{\sqrt{p-\beta(\alpha-1/2)}}\frac{\beta^{p+1/2}}{4^p p^\alpha}\right\}$$

especially when $\beta=1$, $p>\alpha-1/2$, we have

$$m^{\alpha-1/2}\,\|e_m\|_E\leqq C(\alpha)\max\left\{\frac{1}{p^{2\alpha-1}},\ \frac{1}{\sqrt{p-\alpha+1/2}}\frac{1}{4^p p^\alpha}\right\}.$$

3. The *h-p* Version

We shall discuss now the *h-p* version of the finite element method.

The *h-p* version increases the degree vector $\underline{p}$ simultaneously with the number m of the elements.

In this section we will deal only with some special combinations of mesh and degree vector. The discussion of optimal rate of convergence for general mesh-degree combinations is postponed to the next section.

3.1. The Geometric Mesh with Linear Degree Vector

Suppose that the mesh Δ is

$$\Delta = \{0 = x_0 < x_1 < x_2 < \cdots < x_m = 1\}$$

with

$$x_i = q^{m-i}, \qquad 0 < q < 1, \quad i = 1, 2, \ldots, m.$$

We will hereafter call the mesh a *geometric mesh* with ratio q. In this case

$$r_i = \frac{\sqrt{x_i} - \sqrt{x_{i-1}}}{\sqrt{x_i} + \sqrt{x_{i-1}}} = \frac{1 - \sqrt{q}}{1 + \sqrt{q}} \equiv r, \qquad \text{for all } i\text{'s}. \tag{3.1}$$

Let

$$\underline{p} = (p_1, p_2, \ldots, p_m)$$

be the corresponding degree vector to the mesh. Then we have by Remark 1.1

$$\|e\|_E^2 \approx \left[\frac{q^{(m-1)(\alpha-1/2)}}{p_1^{2\alpha-1}}\right]^2 + \left(\frac{1-r^2}{2r}\right)^{2(\alpha-1)} \sum_{i=2}^{m} \left[\frac{(q^{m-i} - q^{m-i+1})^{\alpha-1/2}}{p_i^{\alpha}} r^{p_i}\right]^2$$

$$= q^{2(m-1)(\alpha-1/2)} \left\{\frac{1}{p_1^{4\alpha-2}} + (1-q)\, q^{\alpha-1} \sum_{i=2}^{m} \frac{q^{(2\alpha-1)(1-i)} r^{2p_i}}{p_i^{2\alpha}}\right\}.$$

Denote

$$\mathscr{E}(m, \underline{p}) = \frac{1}{p_1^{4\alpha-2}} + (1-q)\, q^{\alpha-1} \sum_{i=2}^{m} \frac{q^{(2\alpha-1)(1-i)} r^{2p_i}}{p_i^{2\alpha}}, \tag{3.2}$$

$$\|e\|_E \approx \eta(m, \underline{p}) \equiv q^{(\alpha-1/2)(m-1)} \mathscr{E}(m, \underline{p})^{1/2}. \tag{3.3}$$

Clearly, for each $N \geqq 2$, there is $m \geqq 2$ and a degree vector $\underline{p}^{(m)} = \{p_i^{(m)}\}_{i=1}^m$ with $\sum_{i=1}^{m} p_i^{(m)} = N$ such that

$$\eta(m, \underline{p}^{(m)}) = \min\left\{\eta(k, \underline{p}) \,|\, 2 \leqq k \leqq N,\ \sum_{i=1}^{m} p_i = N,\ p_i \geqq 1\right\}.$$

Our first question is about the structure of $\underline{p}^{(m)}$ as $N \to \infty$. In order to simplify the problem we extend $\mathscr{E}(m, \underline{p})$ to the domain

$$D_m = \{\underline{p} \in \mathbf{R}^m \,|\, p_i > 0,\ i = 1, 2, \ldots, m\}.$$

Let $N \geqq m$, and

$$D'_{m,N} = \left\{\underline{p} \in D_m \,\middle|\, \sum_{i=1}^{m} p_i = N\right\}.$$

For each $N \geqq 2$, consider the minimization problem:

Find $(m_N, \underline{p}^{(m_N)})$ with $\underline{p}^{(m_N)} \in D'_{m,N}$ such that

$$\eta(m_N, \underline{p}^{(m_N)}) = \min\{\eta(k, \underline{p}) \,|\, 2 \leqq k \leqq N, \underline{p} \in D'_{k,N}\}.$$

Since for each m, $D'_{m,N}$ is a connected open set of an $(m-1)$-dimensional hyperplane of $\mathbf{R}^m$, and

$$\eta(m,\underline{p}) \geqq 0, \qquad \forall \underline{p} \in D'_{m,N}$$

$$\eta(m,p) \to \infty, \qquad \text{as } \underline{p} \to \underline{p}_0 \in \partial D'_{m,N}$$

it follows that for each $2 \leqq m \leqq N$, one can find a minimizer $\underline{p}^{(m,N)}$ of $\eta(m,\cdot)$ and $\underline{p}^{(m,N)}$ necessarily satisfies the following conditions obtained by Legrange multiplier method:

$$\frac{\partial \mathscr{E}(m,\underline{p}^{(m,N)})}{\partial p_1} = \frac{-(4\alpha-2)}{(p_1^{(m,N)})^{4\alpha-3}} + 2\lambda_N = 0 \tag{3.4}$$

$$\frac{\partial(m,\underline{p}^{(m,N)})}{\partial p_i} = 2Cq^{-(2\alpha-1)i}\,\frac{r^{2p_i^{(m,N)}}(p_i^{(m,N)}\ln r-\alpha)}{(p_i^{(m,N)})^{2\alpha+1}} + 2\lambda_N = 0 \ (i=2,\dots,m) \tag{3.5}$$

where $C=(1-q)\,q^{3\alpha-2}$. We find then m_N which minimizes the error.

We will call the sequence $\{\underline{p}^{(m_N)}\}_{N=2}^{\infty}$ the sequence of the optimal degree distributions. Clearly, an integer degree distribution $\underline{p}^{(N)}$ which satisfies $p_i^N \geqq 1$ and $|p_i^N - p_i^{(m_N)}| < 1$ will give a good rate of convergence which will be close to the extended case.

For the (extended) optimal degree distributions we have

Theorem 3.1. *As $N \to \infty$ the optimal degree distribution tends to be linear with a slope*

$$s_0 = (\alpha - 1/2)\,\frac{\ln q}{\ln r}.$$

This means, precisely, that for each fixed $i = 1,2,\dots,$

$$\lim_{N\to\infty}\,[p^{(m_N)}_{m_N-i} - p^{(m_N)}_{m_N-i-1}] = s_0\,.$$

Proof. First we notice that as $N \to \infty$ it is possible to obtain a rate of convergence $e^{-C\sqrt{N}}$ for some $C>0$. This will be proven in Theorem 3.2. From the expression of $\eta(m,\underline{p})$, it is easy to see that if m_N or $\max\limits_{1\leqq i\leqq m_N} p_i^{(m_N)}$ is bounded by some number, then we cannot achieve a rate of convergence better than $N^{-\sigma}$ for some $\sigma > 0$. Therefore, for the optimal degree distribution we must have

$$m_N \to \infty$$

$$\max_{1\leqq i\leqq m_N} p_i^{(m_N)} \to \infty$$

as $N \to \infty$. In fact, we can obtain an even stronger conclusion that for each $i = 1,2,3,\dots$ fixed:

$$p^{(m_N)}_{m_N-i} \to \infty \qquad (\text{as } N \to \infty),$$

which follows from

$$\eta(m,\underline{p}) > \sqrt{C}\,\frac{q^{(\alpha-1/2)(i-2)}\,r^{2p^{(m_N)}_{m_N-i}}}{(p^{(m_N)}_{m_N-1})^{2\alpha}}.$$

By (3.5) one has

$$Cq^{-(2\alpha-1)i}\frac{r^{2p_i^{(m_N)}}\left(p_i^{(m_N)}\ln\frac{1}{r}+\alpha\right)}{(p_i^{(m_N)})^{2\alpha+1}}=\lambda_N$$

for each $N\geqq 2$, $i=2,3,\ldots,m_N$.

Let $p_N(x)$, $0<x<\infty$ be the function implicitly defined by the equation

$$Cq^{-(2\alpha-1)i}\frac{r^{2p_N(x)}\left(p_N(x)\ln\frac{1}{r}+\alpha\right)}{(p_N(x))^{2\alpha+1}}=\lambda_N.$$

Consider the range of the function

$$g(y)=A\frac{r^{2y}\left(y\ln\frac{1}{r}+\alpha\right)}{y^{2\alpha+1}}$$

for any $A>0$. Since $\lim_{y\to 0^+}g(y)=+\infty$, $\lim_{y\to\infty}g(y)=0$, g: $(0,\infty)\to(0,\infty)$ is onto. Thus for any $\lambda_N>0$ (3.6) is solvable, and by implicit differentiation we find:

$$p_N'(x)=(\alpha-1/2)\frac{\ln q}{\ln r}\left[1+\frac{\alpha\left(p_N(x)\ln\frac{1}{r}+\alpha+1/2\right)}{p_N(x)\ln\frac{1}{r}\left(p_N(x)\ln\frac{1}{r}+\alpha\right)}\right]^{-1}. \tag{3.7}$$

This is well defined for all $p(x)>0$.

Observe that $p_N'(x)>0$, $p_N(i)=p_i^{(m_N)}$ for all $2\leqq i\leqq m_N$. We obtain that if $m_N-i-1\leqq x\leqq m_N-i$, then

$$p_{m_N-i-1}^{(m_N)}\leqq p_N(x)\leqq p_{m_N-i}^{(m_N)}. \tag{3.8}$$

By mean value theorem

$$p_{m_N}^{(m_N)}-p_{m_N}^{(m_N)}=p_N'(\xi_{N,i})$$

for some $m_N-i-1\leqq\xi_{N,i}\leqq m_N-i$. Since for any $i>0$ fixed by (3.8)

$$\lim_{N\to\infty}p_N(\xi_{N,i})=+\infty,$$

it follows from (3.7) that

$$\lim_{N\to\infty}(p_{m_N-i}^{(m_N)}-p_{m_N-i-1}^{(m_N)})=a_0=(\alpha-1/2)\frac{\ln q}{\ln r}.\quad\square$$

We now consider the case when a geometric mesh and linear degree vector are adopted. By this we mean

$$x_i=q^{m-i}$$

$$p_i=[1+s(i-1)]\qquad(i=1,2,\ldots,m).$$

(The value $s > 0$ will be called the slope.) In this case we can let

$$N = \frac{sm^2}{2}. \tag{3.9}$$

We have then

Theorem 3.2. *For the geometric mesh with ratio q combined with a linear degree vector of slope s, we have:*

1) *if* $s > s_0$, *then*

$$\|e\|_E \approx C(\alpha, q, s)\, q^{(\alpha - 1/2)\sqrt{2N/s}}; \tag{3.10}$$

2) *if* $s < s_0$, *then*

$$\|e\|_E \approx C(\alpha, q, s)\, r^{\sqrt{2Ns}}; \tag{3.11}$$

3) *if* $s = s_0$, *then*

$$\|e\|_E \approx C(\alpha, q)\, e^{-\sqrt{(\alpha - 1/2)N}\sqrt{2\ln q \ln r}}, \tag{3.12}$$

where $r = \dfrac{1 - \sqrt{q}}{1 + \sqrt{q}}$ *and* $s_0 = (\alpha - 1/2)\dfrac{\ln q}{\ln r}$ *is the optimal slope in the sense that the exponential rate attends maximum (with same q).*

Furthermore, the optimal geometric mesh and linear degree vector combination is given by

$$\begin{cases} q_{op} = (\sqrt{2} - 1)^2 \\ s_{op} = 2^\alpha - 1\,. \end{cases} \tag{3.13}$$

In this case

$$\|e\| \approx C(\alpha)\, [(\sqrt{2} - 1)^2]^{\sqrt{(\alpha - 1/2)N}}. \tag{3.14}$$

In (3.10) ~ (3.14), *the equivalence constants depend on* (α, q, s), (α, q) *and* α *respectively.*

Proof. We have by (3.3)

$$\begin{aligned} \|e\|_E &\approx q^{(m-1)(\alpha - 1/2)} \left\{1 + (1 - q)\, q^{\alpha - 1} \sum_{i=2}^{m} \frac{q^{(2\alpha - 1)(1 - i)}\, r^{2(1 + s(i-1))}}{(1 + s(i-1))^{2\alpha}}\right\}^{1/2} \\ &= q^{(m-1)(\alpha - 1/2)} \left\{1 + (1 - q)\, q^{\alpha - 1} \sum_{i=2}^{m} \frac{e^{(i-1)(2s\ln r - (2\alpha - 1)\ln q)}}{(1 + s(i-1))^{2\alpha}}\right\}^{1/2}. \end{aligned} \tag{3.15}$$

If $2s\ln r - (2\alpha - 1)\ln q < 0$, i.e.,

$$s > \frac{(\alpha - 1/2)\ln q}{\ln r} = s_0$$

the sum in the bracket converges as $m \to \infty$, thus

$$\begin{aligned} \|e\|_E &\approx C(\alpha, q, s)\, q^{(m-1)(\alpha - 1/2)} \\ &= Cq^{(\alpha - 1/2)\sqrt{2N/s}}\,. \end{aligned}$$

If $s < s_0$, i.e., $2s\,ln\,r - (2\alpha - 1)\,ln\,q > 0$, the quantity in the bracket is of order

$$e^{(m-1)(2s\,ln\,r-(2\alpha-1)\,ln\,q)} = r^{2s(m-1)}q^{-(m-1)(2\alpha-1)}$$

(as $m \to \infty$), thus

$$\begin{aligned}\|e\|_E &\approx C(\alpha, q, s)\,r^{s(m-1)}\\ &= C r^{\sqrt{2sN}}.\end{aligned}$$

If $s = s_0$, (3.15) gives

$$\begin{aligned}\|e\| &\approx C(\alpha, q)\,q^{(\alpha-1/2)\sqrt{\frac{2N\,ln\,r}{(\alpha-1/2)\,ln\,q}}}\\ &= Ce^{-\sqrt{(\alpha-1/2)N}\,\sqrt{2\,ln\,q\,ln\,r}}.\end{aligned}$$

We now show that $s = s_0$ gives a better rate of convergence and hence it is the optimal slope. Indeed, we have: if $s > s_0$, then

$$q^{(\alpha-1/2)\sqrt{2N/s}} = e^{(\alpha-1/2)\,ln\,q\,\sqrt{2N/s}} > e^{(\alpha-1/2)\,ln\,q\,\sqrt{2N/s_0}} = e^{-\sqrt{(\alpha-1/2)N}\,\sqrt{2\,ln\,q\,ln\,r}};$$

if $s < s_0$, then

$$r^{\sqrt{2sN}} = e^{ln\,r\,\sqrt{2sN}} > e^{ln\,r\,\sqrt{2s_0N}} = e^{-\sqrt{(\alpha-1/2)N}\,\sqrt{2\,ln\,ln\,r}}.$$

Thus in either case, the rate of convergence is not better than that when $s = s_0$.

Now suppose with each q we associate its optimal slope s_0. Then it can be seen readily that the optimal rate of convergence (among those with geometric mesh and linear degree combination) is achieved if the quantity

$$\psi(q) = ln\,q\,ln\,r = ln\,q\,ln\,\frac{1-\sqrt{q}}{1+\sqrt{q}}$$

reaches its maximum. By the lemma of K. Scherer and R. DeVore [2, 3] stated below, the function $\psi(q)$ has a unique maximum at $q_0 = (\sqrt{2} - 1)^2$, and

$$\psi(q_0) = 2\,[ln(\sqrt{2} - 1)]^2.$$

Therefore we conclude that the optimal-geometric-mesh-linear-degree combination is given by

$$q = q_0 = (\sqrt{2} - 1)^2,$$

and

$$s = s_0 = \frac{(\alpha - 1/2)\,ln\,q_0}{ln\,\frac{1-\sqrt{q_0}}{1+\sqrt{q_0}}} = 2\alpha - 1$$

thus

$$\begin{aligned}\|e\|_E &\approx C(\alpha)\,e^{-\sqrt{(\alpha-1/2)N}\,\sqrt{4(ln(\sqrt{2}-1))^2}}\\ &= C(\alpha)\,[(\sqrt{2} - 1)^2]^{\sqrt{(\alpha-1/2)N}}.\end{aligned}$$

The important lemma we quoted from [3] is

Lemma 3.1. (K. Scherer and R. DeVore). *For* $x \in (0, \infty)$ *the function* $F(x) = \left(\frac{1-\delta^x}{1+\delta^x}\right)^x$, $(0<\delta<1)$ *has a unique minimum at* $x = x_0 = \frac{ln(\sqrt{2}-1)}{ln\,\delta}$, *for which* $F(x_0) = (\sqrt{2}-1)^{\frac{ln(\sqrt{2}-1)}{ln\,\delta}}$. *In particular, if* $\delta = \sqrt{2}-1$, *then* $F(x) \geqq \sqrt{2}-1$; *the equality holds if and only if* $x = 1$.

Corollary 3.1. *The function*

$$\psi(x) = ln\,x\,ln\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right) \qquad 0<x<1$$

has a unique maximum at

$$x = x_0 = (\sqrt{2}-1)^2$$

for which

$$\psi(x_0) = 2\,[ln(\sqrt{2}-1)]^2 .$$

Proof. Let $x = \delta^{2y}$ with $\delta = \sqrt{2}-1$, we obtain

$$\psi(x) = \psi(\delta^{2y}) = ln\,\delta^2\,ln\left(\frac{1-\delta^y}{1+\delta^y}\right)^y .$$

Since $ln\,\delta^2 < 0$, $\left(\frac{1-\delta^y}{1+\delta^y}\right)$ has a unique minimum at $y=1$ implies $\psi(\delta^{2y})$ has a unique maximum at $y=1$, thus $\psi(x)$ has a unique maximum at $x = x_0 = \delta^2$. □

Remark 3.1. For the optimal combination of geometric mesh and linear degree vector, the estimate (3.14) can be written as

$$\|e\|_E \approx e^{-1.7626\sqrt{(\alpha-1/2)N}} . \tag{3.16}$$

We will see in Section 4 that this exponential rate of convergence is the best possible one. Thus we may say that this mesh-degree h-p version is the *near optimal* one.

An important case is the bisected geometric mesh, i.e., $q = 1/2$. If we choose the optimal slope for this mesh, i.e., we choose

$$s_0 = \frac{ln\,1/2}{ln\,\frac{1-1/\sqrt{2}}{1+1/\sqrt{2}}}(\alpha-1/2) = \frac{ln\,2}{2\,ln(1+\sqrt{2})}(\alpha-1/2)$$
$$= 0.3932\,(\alpha-1/2),$$

then we have

$$\|e\|_E \approx e^{-1.5632\sqrt{(\alpha-1/2)N}} . \tag{3.17}$$

If for $q=1/2$, we choose $s=1$. Then

$$\|w\|_E \approx \begin{cases} e^{-0.9803(\alpha-1/2)\sqrt{N}}, & \text{if } \alpha < 3.0432 \\ e^{-2.4929\sqrt{N}}, & \text{if } \alpha > 3.0432. \end{cases} \tag{3.18}$$

(The above estimates have equivalence constants depending only on α.)

3.2. The Geometric Mesh with Uniformly Distributed Degree Vector

We now consider the case that the polynomial degree is p on every element. This is important in the higher dimensional case for it makes it easier to construct conforming basis functions and to deal with data management process. We will show that this mesh-degree combination can also give an exponential rate of convergence and has a similar feature as in the case of the linear degree vector.

Theorem 3.3. *For the geometric mesh with ratio q combined with uniformaly distributed degree p, the relation between the optimal choice of p and the number of elements m in the mesh is asymptotically linear, i.e.,*

$$p \cong s_0 m \qquad (\text{as } m \to \infty)$$

with s_0 being the same as in Theorem 3.2.

Proof. We have by (3.3)

$$\|e\|_E \approx \left\{ \frac{q^{(2\alpha-1)(m-1)}}{p^{4\alpha-2}} + \frac{(1-q)q^{\alpha-1}}{p^{2\alpha}} r^{2p} \sum_{i=2}^{m} q^{2(\alpha-1/2)(m-i)} \right\}^{1/2}$$

$$= \left\{ \frac{q^{(2\alpha-1)(m-1)}}{p^{4\alpha-2}} + \frac{(1-q)q^{\alpha-1}\left(1-q^{(2\alpha-1)(m-1)}\right)}{1-q^{2\alpha-1}} \frac{r^{2p}}{p^{2\alpha}} \right\}^{1/2}.$$

Clearly, to obtain $\|e\|_E \to 0$ as $N \to \infty$ it is necessary that $p \to +\infty$, and an exponential rate of convergence is achieved if and only if both m and p tend to $+\infty$. The optimal rate of convergence for a given q is obtained by minimizing the function

$$f(m,p) = \frac{q^{(2\alpha-1)m}}{p^{2\alpha-2}} + Cr^{2p}$$

under the constraint $mp=N$.

By Lagrange multiplier method the necessary condition is

$$\frac{\partial f}{\partial m} = (2\alpha-1)\ln q \frac{q^{(2\alpha-1)m}}{p^{2\alpha-2}} - \lambda p = 0$$

and

$$\frac{\partial f}{\partial p} = -\frac{2\alpha-2}{p} \frac{q^{(2\alpha-1)m}}{p^{2\alpha-2}} + Cr^{2p} 2\ln r - \lambda m = 0.$$

Therefore

$$\frac{(2\alpha-1)\,ln\,q}{p^{2\alpha-1}}\,q^{(2\alpha-1)m}=\lambda=-\frac{2\alpha-2}{m}\,\frac{q^{(2\alpha-1)m}}{p^{2\alpha-1}}+\frac{2C\,ln\,r}{m}\,r^{2p},$$

$$\left[(2\alpha-1)nq+\frac{2\alpha-2}{m}\right]\frac{q^{(2\alpha-1)m}}{p^{2\alpha-1}}=\frac{2C\,ln\,r}{m}\,r^{2p},$$

or

$$(2\alpha-1)\,m\,ln\,q+ln\left[\frac{(2\alpha-1)\,m\,ln\,q+2\alpha-2}{2C\,ln\,r}\right]=2p\,ln\,r+ln\,p^{2\alpha}.$$

Hence

$$m\left[(2\alpha-1)\,ln\,q+0\left(\frac{ln\,m}{m}\right)\right]=p\left[2\,ln\,r+0\left(\frac{ln\,p}{p}\right)\right].$$

As $m,p\to\infty$, we have

$$p\cong(\alpha-1/2)\,\frac{ln\,q}{ln\,r}\,m\equiv s_0\,m.\quad\square$$

Theorem 3.4. *For the geometric mesh with ratio q and the uniformly distributed degree p related with the number of elements m by $p=sm$, we have*

1) *if $s>s_0$, then*

$$\|e\|_E\approx C(\alpha,q)\,\frac{q^{(\alpha-1/2)\sqrt{N/s}}}{\sqrt{sN}^{2\alpha-1}};\tag{3.20}$$

2) *if $s<s_0$, then*

$$\|e\|_E\approx C(\alpha,q)\,\frac{r^{\sqrt{sN}}}{\sqrt{sN}^{\alpha}};\tag{3.21}$$

3) *if $s=s_0$, then one gets optimal rate of convergence for a given q*

$$\|e\|\approx C(\alpha,q)\,\frac{e^{-\sqrt{(\alpha-1/2)N}\,\sqrt{ln\,r\,ln\,q}}}{\sqrt{N^{\sigma}}}\tag{3.22}$$

where

$$s_0=\frac{(\alpha-1/2)\,ln\,q}{ln\,r},\qquad r=\frac{1-\sqrt{q}}{1+\sqrt{q}},$$

and

$$\sigma=\min(2\alpha-1,\alpha).$$

The optimal combination is also given by

$$q=q_{op}=(\sqrt{2}-1)^2.$$

$$s=s_{op}=2\alpha-1.$$

Similar to the linear degree vector case, for the optimal combination we have

$$\|e\|_E \approx C(\alpha)\,\frac{[(\sqrt{2}-1)^2]^{\sqrt{(\alpha-1/2)N/2}}}{\sqrt{N^\sigma}}. \tag{3.23}$$

In the above, the equivalence constants depend on (α, q), α *respectively.*

Proof. Let $p = sm$, then (3.19) becomes

$$\|e\|_E \approx \left\{\frac{q^{(2\alpha-1)(m-1)}}{p^{4\alpha-2}} + C(\alpha,q)\,\frac{r^{2p}}{p^{2\alpha}}\right\}^{1/2}$$

$$\approx C(\alpha,q)\left\{\frac{q^{(2\alpha-1)\sqrt{N/s}}}{(sN)^{2\alpha-1}} + \frac{r^{2\sqrt{sN}}}{(sN)^\alpha}\right\}^{1/2}.$$

If $s > s_0$, then

$$q^{(2\alpha-1)\sqrt{N/s}} < q^{(2\alpha-1)\sqrt{N/s_0}} = e^{-2\sqrt{(\alpha-1/2)N\ln r \ln q}}$$
$$= r^{2\sqrt{s_0 N}} > r^{2\sqrt{sN}}$$

thus

$$\|e\|_E \approx C(\alpha,q)\,\frac{q^{(\alpha-1/2)\sqrt{N/s}}}{\sqrt{sN}^{2\alpha-1}}.$$

If $s < s_0$, then

$$q^{(2\alpha-1)\sqrt{N/s}} < q^{(2\alpha-1)\sqrt{N/s_0}} = r^{2\sqrt{s_0 N}} < r^{2\sqrt{sN}}$$

thus

$$\|e\|_E \approx C(\alpha,q)\,\frac{r^{\sqrt{sN}}}{\sqrt{sN}^\alpha}.$$

Clearly the optimal factor s is $s = s_0 = (\alpha - 1/2)\,\dfrac{\ln q}{\ln r}$ for which

$$\|e\|_E \approx C(\alpha,q)\,\frac{e^{-\sqrt{(\alpha-1/2)N}\sqrt{\ln r \ln q}}}{\sqrt{sN^\sigma}}$$

with $\sigma = \min(2\alpha-1, \alpha)$.

By the corollary of Lemma 3.1 we see the optimal q is also $q_0 = (\sqrt{2}-1)^2$ as in the linear degree vector case. □

Remark 3.2. We have:

For $q = q_0 = (\sqrt{2}-1)^2$, $s = s_0 = 2\alpha - 1$.

$$\|e\|_E \approx \frac{1}{\sqrt{N^\sigma}}\,e^{-1.2464\sqrt{(\alpha-1/2)N}}, \tag{3.24}$$

$$\sigma = \min(2\alpha-1, \alpha)$$

For $q=1/2$, $s=s_0=0.3932(\alpha-1/2)$,

$$\|e\|_E \approx \frac{e^{-1.1054\sqrt{(\alpha-1/2)N}}}{\sqrt{N^\sigma}} \tag{3.25}$$

$$\sigma=\min(2\alpha-1,\alpha).$$

For $q=1/2$, $s=1$.

$$\|e\|_E \approx \begin{cases} \dfrac{1}{\sqrt{N^{2\alpha-1}}}e^{-0.6931(\alpha-1/2)\sqrt{N}}, & \text{if } \alpha<3.0432 \\ \dfrac{1}{\sqrt{N^\alpha}}e^{-1.7627\sqrt{N}}, & \text{if } \alpha>3.0432. \end{cases} \tag{3.26}$$

One can see that the exponent is exactly $1/\sqrt{2}$ times the exponent in the linear degree vector case. (The above estimates have equivalence constants depending on α only.)

3.3. *The Uniform Mesh*

Now we consider uniform mesh with a degree vector selected arbitrarily. In this case $x_i=\frac{i}{m}$. Since $\sum_{i=1}^{m} p_i=N$ and $p_i\geqq 1$, we have $m\leqq N$, $p_i\leqq N$. A very rough estimate shows that

$$\|e\|_E \geqq E_p[x_0,x_1] \approx \frac{x_1^{\alpha-1/2}}{p_1^{2\alpha-1}}=\frac{1}{m^{\alpha-1/2}p_1^{2\alpha-1}}\geqq\frac{1}{N^{3(\alpha-1/2)}}. \tag{3.27}$$

Therefore for uniform mesh the rate of convergence is never better than an algebraic one regardless of the degree vector.

3.4. *The Optimal Rate of Convergence for Uniformly Distributed Degree Vector*

We know from Section 2.2 that for a fixed degree p as the numb r of elements $m\to\infty$ the optimal graded mesh is a radical mesh with the grading function x^β, $\beta=\frac{p+1/2}{\alpha-1/2}$. Therefore the h-p version with uniformly distributed degree the geometric mesh is not optimal. We can expect that using the optimal radical mesh would give a better rate of convergence. We will study therefore the envelope of the error curves of the optimal radical meshes.

Theorem 3.5. *There is an h-p version with uniformly distributed degree which has the error estimate*:

$$\|e\|_E \approx \frac{1}{\sqrt{N^{\alpha-1/2}}}e^{-4/e\sqrt{(\alpha-1/2)N}}=\frac{1}{\sqrt{N^{\alpha-1/2}}}e^{-1.4715\sqrt{(\alpha-1/2)N}}. \tag{3.28}$$

As $N \to \infty$, the meshes tend to be geometric with a ratio $q \cong e^{-4/e^2} = 0.5820$, and the relation between degree p and the number of elements m tends to be linear; $p \cong 4/e^2(\alpha - 1/2)m = 0.5413(\alpha - 1/2)m$.

Proof. By Remark 2.4 the optimal radical mesh has a rate of convergence

$$\|e\|_E \approx \frac{p^{2p-\alpha+1/2}}{(4(\alpha-1/2)N)^p} \qquad (N = mp).$$

Let $x = 4(\alpha - 1/2)N$, $\sigma = \alpha - 1/2$, we now seek the envelope of the family of curves

$$f(x,p) = \frac{p^{2p-\sigma}}{x^p} \tag{3.29}$$

where p is considered to be the parameter. To this end we need to solve the simultaneous equations

$$\begin{cases} y = f(x,p) \\ \dfrac{\partial f}{\partial p}(x,p) = 0 \end{cases}$$

where

$$\frac{\partial f}{\partial p} = \frac{p^{2p-\sigma}}{x^p}\left(2\ln p + 2 - \frac{\sigma}{p} - \ln x\right).$$

Thus

$$\frac{\sigma}{p} = \ln\frac{(ep)^2}{x}.$$

Since $x = 4(\alpha - 1/2)N \to \infty$ implies $p \to \infty$, we see that

$$\frac{(ep)^2}{x} \to 1,$$

thus

$$p \cong \frac{\sqrt{x}}{e} = \frac{\sqrt{4(\alpha-1/2)N}}{e} \qquad (N \to \infty)$$

and

$$\|e\|_E \approx \frac{(\sqrt{x}/e)^{2\sqrt{x}/e-\sigma}}{x^{\sqrt{x}/e}} \approx \frac{1}{\sqrt{N^{\alpha-1/2}}}\, e^{-4/e\sqrt{(\alpha-1/2)N}}$$

(with equivalence constant depending only on α).

Since $N = mp$, we get

$$p \cong \frac{\sqrt{x}}{e} = \frac{4}{e^2}(\alpha - 1/2)m \qquad (m \to \infty),$$

Table 1. $\|e\|_E \approx \frac{1}{\sqrt{N^\sigma}} e^{-\kappa\sqrt{(A-1/2)N}}$

Method	q	s	κ	σ
G-mesh L-degree	q	$(\alpha-1/2)\frac{\ln q}{\ln r}$	$\sqrt{2\ln q \ln r}$	0
	$\frac{1}{2}$	$0.3932\,(\alpha-1/2)$	1.5632	0
	$(\sqrt{2}-1)^2$	$2\alpha-1$	1.7627	0
G-mesh U-degree	q	$(\alpha-1/2)\frac{\ln q}{\ln r}$	$\sqrt{\ln q \ln r}$	$\min(\alpha, 2\alpha-1)$
	$\frac{1}{2}$	$0.3932\,(\alpha-1/2)$	1.1054	$\min(\alpha, 2\alpha-1)$
	$(\sqrt{2}-1)^2$	$2\alpha-1$	1.2464	$\min(\alpha, 2\alpha-1)$
R-mesh U-degree	e^{-4/e^2}	$4/e^2(\alpha-1/2)$	1.4715	$\alpha-1/2$
q and s are asymptotic values				

and for $i=1,2,\ldots$ fixed

$$x_{m-i}=\left(\frac{m-i}{m}\right)^\beta=\left(1-\frac{i}{m}\right)^{\frac{4/e_2(\alpha-1/2)m+1/2}{\alpha-1/2}} \rightarrow (e^{-4/e_2})^i \qquad (m\rightarrow\infty).$$

Thus the mesh tends to be geometric with a ratio $q=e^{-4/e_2}$. □

Remark 3.3. We can obtain a more precise asymptotic relation between p and m:

$$p=\frac{4}{e^2}(\alpha-(\alpha-1/2)m+(\alpha-1/2)+\frac{e^2}{8}(\alpha-7/2)\frac{1}{m}+\cdots. \qquad (3.30)$$

Remark 3.4. One sees that the rate of convergence given by (3.28) is better than using geometric mesh. The mesh is a radical one, which has a very strong refinement in the neighborhood of the singularity:

$$x_i=\left(\frac{i}{m}\right)^\beta=\left(\frac{i}{m}\right)^{4m/e^2}.$$

In summary, for our model problem the *h-p* version can achieve an exponential rate of convergence. The geometric mesh with a ratio $q=(\sqrt{2}-1)^2$ combined with a linear degree vector with a slope $s=2\alpha-1$ is near optimal. The same ratio $q=(\sqrt{2}-1)^2$ is also best for the geometric mesh with uniform degree vector. This optimal ratio $(\sqrt{2}-1)^2$ is independent of the strength of the singularity. The following table gives a summary of the various *h-p* extensions. We express the rate as

$$\|e\|_E \approx \frac{1}{\sqrt{N^\sigma}} e^{-\kappa\sqrt{(\alpha-1/2)N}}$$

and

R-mesh	– the optimal radical mesh,
G-mesh	– the geometric mesh,
L-degree	– the linear degree vector,
U-degree	– the uniform degree vector,
q	– the ratio for a geometric mesh,
s	– the slope for a linear degree vector or the factor in the relation $p = sm$ for the uniform degree vector.

4. The Optimal Rate of Convergence for Arbitrary Mesh-degree Combination

We now answer the equation what is the possible optimal rate of convergence in all possible mesh-degree combinations.

Let $S = (\Delta, \underline{p})$ be an arbitrary mesh-degree combination and

$$\mathscr{E}(\Delta, \underline{p}) = \left\{ \sum_{i=1}^{m(\Delta)} E_{p_i}[x_{i-1}, x_i]^2 \right\}^{1/2}$$

be the error in energy norm of the finite element solution of the model problem (1.1).

Define for $N \geqq k$

$$E_{N,k} = \inf \left\{ \mathscr{E}(\Delta, p) \,\middle|\, \sum_{i=1}^{p} p_i = N,\ m(\Delta) = k \right\}, \tag{4.1}$$

if $N < k$, let $E_{N,k} = \infty$ (N be an integer). In what will follow we make the following convention:

$$\infty \cdot 0 = \infty .$$

Clearly, $E_{N,k}$ is the minimal error among the combinations having k elements with the number of degrees of freedom being N.

Furthermore, we define

$$E_N = \inf_{k \geqq 1} \{E_{N,k}\}, \tag{4.2}$$

and E_N is then the smallest error of the errors related to the mesh-degree combinations having the same degree of freedom N.

For simplicity we allow the *degenerated case*, i.e., $x_{i-1} = x_i$, for some $1 \leqq i \leqq m(\Delta)$. Clearly, this will not change $E_{N,k}$ and E_N.

In Section 3 we already found an upper bound for E_N:

$$E_N \leqq C(\alpha) q_0^{\sqrt{(\alpha - 1/2)N}} \tag{4.3}$$

where $q_0 = (\sqrt{2} - 1)^2$. To obtain the lower bound of E_N is more difficult. In [3], K. Scherer solved the problem for L_∞ approximation. Out of his result we can easily obtain a lower bound for E_N:

$$E_N \geqq C(\alpha) N^{-(2\alpha+1)} q_0^{\sqrt{(\alpha - 1/2)N}} .$$

Therefore the upper bound (4.3) is optimal in the sense of the exponential decay factor.

Using ideas of [3], we will give a direct proof of an improved lower bound of E_N:

$$E_N \geqq C(\alpha) \frac{1}{\sqrt{N^{\alpha-1/2}}} q_0^{\sqrt{(\alpha-1/2)N}}. \tag{4.4}$$

The idea of the proof is to study the operator $\mathscr{T}$ which transmits the sequence $\{E_{N,k}\}_{N=1}^{\infty}$ to $\{E_{N,k+1}\}_{N=1}^{\infty}$. From the monotonicity of this operator one can construct a lower bound for E_N.

For our model problem $E_p[a,b]$ ($[a,b] \in \Delta$) is the error of best L_2-approximation of $x^{\alpha-1}$ on $[a,b]$ with polynomial degree $p-1$. From this observation one can easily obtain

Lemma 4.1. *For $\lambda > 0$*

$$E_p[\lambda a, \lambda b] = \lambda^{\alpha-1/2} E_p[a,b]. \quad \square \tag{4.5}$$

We now observe that $\mathscr{E}(\Delta, \underline{p})$ is a continuous function on a compact set

$$\{(x_1, x_2, \ldots, x_{k-1}; p_1 p_2, \ldots, p_k) | 0 \leqq x \leqq 1,\ 1 \leqq p_i \leqq N\}$$

with the constraint

$$0 = x_0 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_k = 1$$

$$\sum_{i=1}^{k} p_i = N, \qquad 1 \leqq k \leqq N.$$

We have

Lemma 4.2. *For any k, N given ($k \leqq N$), there are mesh-degree combinations $\Sigma = (\Delta, \underline{p})$ for which $E_{N,k}$, E_N are actually attained.* $\square$

Lemma 4.3. *Let $E_{n,k+1}$ ($N \geqq k+1$) is achieved with $(\Delta, \underline{p})$ where*

$$\Delta = \{0 = x_0 \leqq x_1 \leqq \cdots \leqq x_k \leqq x_{k+1} = 1\}$$

$$\underline{p} = (p_1, p_2, \ldots, p_k, p_{k+1})$$

$$m(\Delta) = k+1, \ \sum_{i=1}^{k+1} p_i = N,$$

then

$$E_{N,k+1}^2 = x_k^{2(\alpha-1/2)} E_{N-p_{k+1},k}^2 + E_{p_{k+1}}[x_k, 1]^2. \tag{4.6}$$

If E_N is achieved with $(\Delta, \underline{p})$ given above, then

$$E_N^2 = x_k^{2(\alpha-1/2)} E_{N-p_{k+1}}^2 + E_{p_{k+1}}[x_k, 1]^2. \tag{4.7}$$

Proof. It is easy to see from Lemma 4.1 and Lemma 4.2 that for $\lambda > 0$

$$E_{N,k}^{[\lambda a, \lambda b]} = \lambda^{\alpha-1/2} E_{N,k}^{[a,b]}$$

$$E_N^{[\lambda a, \lambda b]} = \lambda^{\alpha-1/2} E_N^{[a,b]}$$

where $E_{N,k}^{[a,b]}$ and $E_N^{[a,b]}$ are defined similarly as $E_{N,k}$ and E_N but on $[a,b] \subseteq [0,1]$ instead on $[0,1]$.

Suppose that $E_{N,k+1}$ is achieved with $(\Delta, \underline{p})$, then

$$\begin{aligned}
E_{N,k+1} &= \sum_{i=1}^{k} E_{p_i}[x_{i-1}, x_i]^2 + E_{p_{k+1}}[x_k, 1]^2 \\
&\geqq (E_{N-p_{k+1},k}^{[0,x_k]})^2 + E_{p_{k+1}}[x_k, 1]^2 \\
&= x_k^{2(\alpha-1/2)} E_{N-p_{k+1},k} + E_{p_{k+1}}[x_k, 1]^2 \\
&= \sum_{i=1}^{k} E_{p'}[x'_{i-1}, x'_i] + E_{p_{k+1}}[x_k, 1]^2 \\
&\geqq E_{N,k+1}^2
\end{aligned}$$

where $\{0 = x'_0 \leqq x'_1 \leqq \cdots \leqq x'_k = x_k\}$ and $\{p'_1, \ldots, p'_k\}$ with $\sum_{i=1}^{k} p'_i = N - p_{k+1}$ are the mesh and degree vector for which $E_{N-p_{k+1},k}^{[0,x_k]}$ is achieved.

Similarly, if E_N is achieved with $(\Delta, \underline{p})$, then

$$\begin{aligned}
E_N^2 &= \sum_{i=1}^{k} E_{p_i}[x_{i-1}, x_i]^2 + E_{p_{k+1}}[x_k, 1]^2 \\
&\geqq (E_{N-p_{k+1}}^{[0,x_k]})^2 + E_{p_{k+1}}[X_k, 1]^2 \\
&= x_k^{2(\alpha-1/2)} E_{N-p_{k+1}}^2 + E_{p_{k+1}}[x_k, 1]^2 \\
&= \sum_{i=1}^{k'} E_{p'_i}[x'_{i-1}, x'_i]^2 E_{p_{k=1}}[x_k, 1]^2 \\
&\geqq E_N^2
\end{aligned}$$

where $\{0 = x'_0 \geqq x'_1 \leqq \cdots \leqq x'_{k'} = x_k\}$ and $\{p'_1, \ldots, p'_{k'}\}$ are the mesh and degree vector for which $E_{N-p_{k+1}}^{[0,x_k]}$ is achieved. It may happen that $k' \neq k$, but still

$$\sum_{i=1}^{k'} p'_i = N - p_{k+1}. \quad \square$$

From this lemma we get immediately

Lemma 4.4. *For* $N \geqq 2$,

$$E_{N,k+1}^2 = \inf_{\substack{0 \leqq a \leqq 1 \\ 1 \leqq v \leqq N-1}} \{a^{2\alpha-1} E_{N-v,k}^2 + E_v[a,1]^2\} \tag{4.8}$$

$$E_N^2 = \inf_{\substack{0 \leqq a \leqq 1 \\ 1 \leqq v \leqq N-1}} \{a^{2\alpha-1} E_{n-v}^2 + E_v[a,1]^2\} \tag{4.9}$$

and there are pairs (a, v) *for which these infima are actually achieved.* $\square$

We now define the operator $\mathcal{T}$ as follows:

Definition 4.1. Suppose $\{e_N\}_{N=1}^{\infty} = \boldsymbol{e}$ is a non-negative, non-increasing sequence (which may also take the "value" ∞). Let $\mu = \alpha - 1/2 (> 0)$ be given. Define

$$(\mathcal{T}\boldsymbol{e})_1 = \infty \tag{4.10a}$$

$$(\mathscr{T}e)_N = \inf_{\substack{0 \leqq a \leqq 1 \\ 1 \leqq v \leqq N-1}} \{(a^\mu e_{N-v})^2 + E_v[a, 1]^2\}^{1/2}, \qquad N \geqq 2, \tag{4.10b}$$

with the convention $\infty \cdot 0 = \infty$, where $(e)_N = e_N$ is the N-th component of e.

Clearly, the operator $\mathscr{T}$ has the following properties:

Lemma 4.5. *$\mathscr{T}$ is well defined for any non-negative, non-increasing sequence $e = \{e_N\}_{N=1}^\infty$, and for any $N \geqq 2$, there exists a pair (a, v) for which $(\mathscr{T}e)_N = \{(a^\mu e_{N-v})^2 + E_v[a, 1]^2\}$.* □

Lemma 4.6. *$\mathscr{T}$ maps non-negative, non-increasing sequence $\{e_N\}_{N=1}^\infty$ to a non-negative, non-increasing sequence $\mathscr{T}\{e_N\}_{N=1}^\infty$, and it is monotone,* i.e., *if $e_1 \leqq e_2$, then $\mathscr{T}e_1 \leqq \mathscr{T}e_2$. (The inequality is used in the componentwise sense,* i.e., *$e_1 \leqq e_2$ if and only if $e_N^{(1)} \leqq e_N^{(2)}$ for all N, where $e_i = \{e_N^{(i)}\}_{N=1}^\infty$, $i = 1, 2$).*

Proof. Let $m_N = (\mathscr{T}e)_N$, and suppose that there is a pair (a_1, v_1) such that

$$m_N = \{(a_1^\mu e_{N-v_1})^2 + E_{v_1}[a_1, 1]^2\}^{1/2}$$

and suppose that for (a_2, v_2)

$$m_{N+1} = \{(a_2^\mu e_{N-v_2+1})^2 + E_{v_2}[a_2, 1]^2\}^{1/2}.$$

By definition we have

$$\begin{aligned}(\mathscr{T}e)_{N+1} = m_{N+1} &\leqq \{(a_1^\mu e_{N-v_1+1})^2 + E_{v_1}[a_1, 1]^2\}^{1/2} \\ &\leqq \{(a_1^\mu e_{N-v_1)^2 + E_{v_1}}[a_1, 1]^2\}^{1/2} \\ &= m_N = (\mathscr{T}e)_N.\end{aligned}$$

Suppose now $e_1 \leqq e_2$, and for some pairs (a_i, v_i), $i = 1, 2$,

$$m_N^{(i)} = \{(a_i e_{N-v_i}^{(i)})^2 + E_{v_i}[a_i, 1]^2\}^{1/2}.$$

We then have

$$\begin{aligned}(\mathscr{T}e_1)_N = m_N^{(1)} &= \{(a_1^\mu e_{N-v_1}^{(1)})^2 + E_{v_1}[a_1, 1]^2\}^{1/2} \\ &\leqq \{(a_2^\mu e_{N-v_2}^{(1)})^2 + E_{v_2}[a_2, 1]^2\}^{1/2} \\ &\leqq \{(a_2^\mu e_{N-v_2}^{(2)} + e_{v_2}[a_2, 1]^2\}^{1/2} \\ &= m_N^{(2)} = (\mathscr{T}e_2)_N.\end{aligned}$$

Remark 4.1. By Lemma 4.6 the power of the operator $\mathscr{T}$ is also well defined:

$$\mathscr{T}^0 e = e$$

$$\mathscr{T}^n e = \mathscr{T}(\mathscr{T}^{n-1} e).$$

Lemma 4.7. *Let E_n, $E_{v,k}$ be defined as before. Then*

$$E_N = \inf_{n \geqq 0} \{\mathscr{T}^n \{E_{v,1}\}_{v=1}^\infty)_N\}, \qquad N \geqq 1. \tag{4.11}$$

Proof. By Lemma 4.4, it is clear that

$$(\mathscr{T}\{E_{v,k}\}_{v=1}^\infty)_N = E_{N,k+1},$$
$$(k = 1, 2, \ldots;\ N \geqq 2).$$

Thus

$$E_N = \inf_{n \geqq 1} \{E_{N,k}\} = \inf_{n \geqq 0} \{(\mathcal{T}^n \{E_{\nu,1}\}_{\nu=1}^{\infty})_N\}, \qquad (n \geqq 1).$$

The relation holds trivially for $N=1$. □

From the lemma, we easily obtain the following important corollary which allows us to find a good lower bound for E_N:

Lemma 4.8. *If* $e = \{e_N\}_{N=1}^{\infty}$ *is such*

1) $(\mathcal{T} e)_N \geqq e_N$
2) $e_N \leqq E_{N,1} = E_N[0,1]$

then $e_N \leqq E_N$ *for all* N.

Proof. By Lemma 4.6, 1) implies

$$e_N \leqq (\mathcal{T}^n e)_N, \qquad \text{for} \quad n \geqq 0,\ N \geqq 1$$

and by 2) we obtain

$$e_N \leqq (\mathcal{T}^n \{E_{\nu,1}\}_{\nu=1}^{\infty})_N, \qquad (\text{all } n \geqq 0).$$

Therefore $e_N \leqq \inf_{n \geqq 0} \{(\mathcal{T}^n \{E_{\nu,1}\}_{\nu=1}^{\infty})_N\} = E_N$. □

We now are about to find a fine lower bound for E_N. Recall Theorem 1.1 that for $[a,b] \in \Delta$, and $r = \dfrac{\sqrt{b} - \sqrt{a}}{\sqrt{b} + \sqrt{a}}$

1) if $0 < r^2 \leqq 1 - \dfrac{1}{p+1}$, then

$$E_p[a,b] \geqq C(\alpha) \frac{(b-a)^{\alpha - 1/2}}{\sqrt{1-r^2}} \frac{r^{p+1-\alpha}}{p^{\alpha}} \left(\frac{1}{p^{\alpha - 1/2}} + (1-r^2)^{\alpha - 1/2}\right)$$

2) if $1 - \dfrac{1}{p+1} < r^2 \leqq 1$, then

$$E_p[a,b] \geqq C(\alpha)(b-a)^{\alpha - 1/2} \frac{r^{p+1-\alpha}}{p^{\alpha - 1/2}} \left(\frac{1}{p^{\alpha - 1/2}} + (1-r^2)^{\alpha - 1/2}\right).$$

Let $\lambda = \dfrac{a}{b}$, then

$$(b-a)^{\mu} = b^{\mu}(1-\lambda)^{\mu} = b^{\mu}(1+\sqrt{\lambda})^{2\mu} \left(\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}\right)^{\mu} \geqq b^{\mu} r^{\mu}.$$

These inequalities may be written together as

$$E_p[a,b] \geqq C(\alpha) b^{\mu} \frac{r^{p+1/2}}{p^{\tau}} \qquad (\mu = \alpha - 1/2) \tag{4.12}$$

with $\tau = \max\{2\alpha - 1, \alpha\}$, and if we know that $\varepsilon \leqq r \leqq 1 - \varepsilon$ for some $\varepsilon > 0$, we can obtain $\tau = \alpha$ (in this case constant depends also on ε).

We are interested in the special case of (4.12) when $b=1$:

$$E_\nu[a,1] \geqq C_0(\alpha)\,\frac{r^{\nu-1/2}}{\nu^\tau} \qquad \tau=\max(\alpha, 2\alpha-1), \tag{4.13}$$

if $\varepsilon \leqq r \leqq 1-\varepsilon$, then

$$E_\nu[a,1] \geqq C_0(\alpha,\varepsilon)\,\frac{r^{\nu-1/2}}{\nu^\alpha}, \tag{4.14}$$

where $r=r(a)=\dfrac{1-\sqrt{a}}{1+\sqrt{a}}$.

We now state our main result:

Theorem 4.1. *E_N has a lower bound of the form*

$$e_N = C(\alpha)\,\frac{q_0^{\sqrt{\mu N}}}{\sqrt{N^\mu}} \tag{4.15}$$

with $\mu=\alpha-1/2\,(>0)$, $q_0=(\sqrt{2}-1)^2$.

Proof. It suffices to show that there is a constant $C=C(\alpha)$, such that $\{e_N\}_1^\infty$ satisfies the conditions of Lemma 4.8.

Note

$$E_{N,1}=E_N[0,1] \approx \frac{1}{N^{2\alpha-1}} = \frac{1}{N^{2\mu}}$$

since

$$\frac{e_N}{E_{N,1}} \approx N^{3\mu/2}\, q_0^{\sqrt{\mu N}} \to 0 \qquad \text{as } N\to\infty,$$

we can find a constant $C_1(\alpha)>0$ such that $e_N \geqq E_{N,1}$ for $C \leqq C_1(\alpha)$, $N \geqq 1$.

If we can show $\mathcal{T}\{e_N\}_{N=1}^\infty \geqq \{e_N\}_{N=1}^\infty$, then Lemma 4.8 claims that e_N is a lower bound of E_N. Therefore it now suffices to show that for some $0<C<C_1(\alpha)$, (independent of N) one has:

$$\left(\frac{C}{\sqrt{N^\mu}}\, q_0^{\sqrt{\mu N}}\right)^2 \leqq \left(\frac{Ca^\mu}{\sqrt{N-\nu^\mu}}\, q_0^{\sqrt{\mu(N-\nu)}}\right)^2 + E_\nu[a,1]^2. \tag{4.16}$$

This must hold for all $0\leqq a\leqq 1$, $1\leqq \nu \leqq N-1$. (4.16) is equivalent to

$$a^{2\mu}\sqrt{\frac{N}{n-\nu}}^{\,2\mu}\, q_0^{2(\sqrt{\mu(N-\nu)}-\sqrt{\mu N})} + C^{-2}\sqrt{N}^{2\mu} q_0^{-2\sqrt{\mu N}}\, E_\nu[a,1]^2 \geqq 1. \tag{4.17}$$

For simplicity we denote:

$$x=\sqrt{a}, \quad \sigma=4\mu, \quad \delta=\sqrt{q_0}=\sqrt{2}-1, \quad \gamma=C^{-2},$$

then $0\leqq x\leqq 1$, $\sigma>0$, $0<\delta<1$, $\gamma \geqq C_1(\alpha)^{-2}$, we can write (4.17) as

$$\begin{aligned}\phi_N(x,\nu) &\equiv x^\sigma\left(\frac{N}{N-\nu}\right)^\mu \delta^{2(\sqrt{\sigma(N-\nu)}-\sqrt{\sigma N})} + \gamma N^\mu \delta^{-2\sqrt{\sigma N}}\, E_\nu[x^2,1]^2 \\ &= x^\sigma\left(\frac{N}{N-\nu}\right)^\mu \delta^{\frac{-2\sqrt{\sigma\nu}}{\sqrt{N-\nu}+\sqrt{N}}} + \gamma N^\mu \delta^{-2\sqrt{\sigma N}}\, E_\nu[x^2,1]^2 \geqq 1.\end{aligned} \tag{4.18}$$

We must show (4.18) holds for $0\leqq x\leqq 1$, $1\leqq \nu\leqq N-1$, $N\geqq 1$.

First, we claim that there exist $\varepsilon=\varepsilon(\alpha)>0$, $\gamma_1=\gamma_1(\alpha,\varepsilon)>0$, such that

$$\text{if } \frac{\sqrt{N}}{v}>\varepsilon^{-1} \quad \text{or} \quad \frac{\sqrt{N}}{v}<\varepsilon \quad \text{and} \quad \gamma \geqq \gamma_1,$$

then

$$\phi_N(x,v)\geqq 1.$$

There are two cases:

Case 1. $\frac{\sqrt{N}}{v}<\varepsilon.$

If $\left(\frac{N}{N-v}\right)^{\mu} x^{\sigma}\,\delta^{\frac{-2\sqrt{\sigma}\,v}{\sqrt{N-v}+\sqrt{N}}}>1$, then $\phi_N(x,N)\geqq 1$, otherwise we have
$1\geqq\left(\frac{N}{N-v}\right)^{\mu} x^{\sigma}\,\delta^{\frac{-2\sqrt{\sigma}\,v}{\sqrt{N-v}+\sqrt{N}}}>x^{\sigma}\,\delta^{-\frac{\sqrt{\sigma}\,v}{\sqrt{N}}}$

thus

$$x<\delta^{\frac{v}{\sqrt{\sigma N}}}<\delta^{\frac{1}{\sqrt{\sigma}\,\varepsilon}}, \tag{4.19}$$

and if $\varepsilon\to 0$, then $x\to 0$.

Note that if x is small enough, then

$$\left(\frac{1-x}{1+x}\right)^{1/x}>\frac{1}{9}.$$

Thus there is $\varepsilon_1>0$, such that for $0<\varepsilon\leqq\varepsilon_1$ and $\frac{\sqrt{N}}{v}<\varepsilon$ we obtain by using (4.18), (4.13) and (4.19):

$$\begin{aligned}
\phi_N(x,v) &\geqq \gamma N^{\mu}\delta^{-2\sqrt{\sigma N}}E_v[x^2,1]^2\\
&\geqq \gamma C_0(\alpha)N^{\mu-2\tau}\delta^{-2\sqrt{\sigma N}}\left(\frac{1-x}{1+x}\right)^{2v+1}\\
&\geqq \gamma C_0(\alpha)N^{\mu-2\tau}\delta^{-2\sqrt{\sigma N}}\left(\frac{1}{9}\right)^{(2v+1)x}\\
&\geqq \gamma C_0(\alpha)N^{\mu-2\tau}\delta^{-2\sqrt{\sigma N}}\left(\frac{1}{9}\right)^{(2v+1)\delta^{\frac{v}{\sqrt{\sigma N}}}}\\
&= \gamma C_0(\alpha)N^{\mu-2\tau}\delta^{-2\sqrt{\sigma N}+(2v+1)\delta^{\frac{v}{\sqrt{\sigma N}}}\cdot\frac{\ln 1/9}{\ln\delta}}\\
&= \gamma C_0(\alpha)N^{\mu-2\tau}\delta^{-2\sqrt{\sigma N}\left(1-\kappa\left(\frac{v}{\sqrt{\sigma N}}+\frac{1}{2\sqrt{\sigma N}}\right)\delta^{\frac{v}{\sqrt{\sigma N}}}\right)}
\end{aligned}$$

with $\kappa=\frac{\ln 1/9}{\ln\delta}>0.$

Because $\frac{\nu}{\sqrt{\sigma N}} \geqq \frac{1}{\sqrt{\sigma \varepsilon}} \to +\infty$ as $\varepsilon \to 0$, we have

$$\left(\frac{\nu}{\sqrt{\sigma N}} + \frac{1}{2\sqrt{\sigma N}}\right) \delta^{\frac{\nu}{\sqrt{\sigma N}}} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Thus there is $\varepsilon_2 = \varepsilon_2(\alpha) > 0$, such that for $0 < \varepsilon \leqq \varepsilon_2$ and

$$\frac{\sqrt{N}}{\nu} < \varepsilon,\ \kappa\left(\frac{\nu}{\sqrt{N}} + \frac{1}{2\sqrt{\sigma N}}\right) \delta^{\frac{1}{\sqrt{\sigma N}}} < \frac{1}{2}.$$

Therefore

$$\begin{aligned} \phi_N(x, \nu) &\geqq \gamma C_0(\alpha) N^{\mu - 2\tau} \delta^{-2\sqrt{\sigma N}(1 - 1/2)} \\ &= \gamma C_0(\alpha) N^{\mu - 2\tau} \delta^{-\sqrt{\sigma N}}. \end{aligned}$$

However, $N^{\mu - 2\tau} \delta^{-\sqrt{\sigma N}} \to +\infty$ as $N \to +\infty$; hence there is $\gamma_1 = \gamma_1(\alpha) > 0$, such that for $\gamma \geqq \gamma_1$

$$\gamma C_0(\alpha) N^{\mu - 2\tau} \delta^{-\sqrt{\sigma N}} \geqq 1.$$

Hence for $0 < \varepsilon \leqq \min(\varepsilon_1, \varepsilon_2)$, $\gamma \geqq \gamma_1$, and $\frac{\sqrt{N}}{\nu} < \varepsilon$, we have $\phi_N(x, \nu) \geqq 1$.

Case 2. $\frac{\sqrt{N}}{\nu} > \varepsilon^{-1}$.

As before, if $x^\sigma D^{\frac{-2\sqrt{\sigma}\,\nu}{\sqrt{N-1}+\sqrt{N}}} \geqq 1$, then $\phi_N(x, \nu) \geqq 1$, otherwise we have

$$x < \delta^{\frac{\nu}{\sqrt{\sigma N}}}$$

Therefore we can assume that

$$1 - x > 1 - \delta^{\frac{\nu}{\sqrt{\sigma N}}}$$

and obtain

$$\begin{aligned} \phi_N(x, \nu) &\geqq \gamma N^\mu \delta^{-2\sqrt{\sigma N}} E_\nu[x^2, 1]^2 \\ &\geqq \gamma C_0(\alpha) N^{\mu - 2\tau} \delta^{-2\sqrt{\sigma N}} \left(\frac{1 - x}{1 + x}\right)^{2\nu + 1} \\ &\geqq \gamma C_0(\alpha) N^{\mu - 2\tau} \delta^{-2\sqrt{\sigma N}} \left(\frac{1 - \delta^{\frac{\nu}{\sqrt{\sigma N}}}}{2}\right)^{2\nu + 1} \\ &= \gamma C_0(\alpha) N^{\mu - 2\tau} \delta^{-2\sqrt{\sigma N} + (2\nu + 1) \ln\left(\frac{1 - \delta^{\frac{\nu}{\sqrt{\sigma N}}}}{2}\right) / \ln \delta} \\ &= \gamma C_0(\alpha) N^{\mu - 2\tau} \delta^{-2\sqrt{\sigma N}\left(1 - \frac{\left(\frac{\nu}{\sqrt{\sigma N}} + \frac{1}{2\sqrt{\sigma N}}\right) \ln\left(\frac{1 - \delta^{\frac{\nu}{\sqrt{\sigma N}}}}{2}\right)}{\ln \delta}\right)}. \end{aligned}$$

In this case, one has $\frac{v}{\sqrt{N}} < \varepsilon \to 0$ as $\varepsilon \to 0$, and because

$$\lim_{\varepsilon \to 0} \varepsilon \, ln \left(\frac{1-\delta^\varepsilon}{2} \right) = 0,$$

there is $\varepsilon_3 = \varepsilon_3(\alpha) > 0$, such that if $0 < \varepsilon \leqq \varepsilon_3$ and $\frac{\sqrt{N}}{v} > \varepsilon^{-1}$, then

$$\frac{\left(\frac{v}{\sqrt{\sigma N}} + \frac{1}{2\sqrt{\sigma N}} \right) ln \left(\frac{1-\delta^{\frac{v}{\sqrt{\sigma N}}}}{2} \right)}{ln\,\delta} \leqq \frac{1}{2}.$$

Hence $\phi_N(x,v) \geqq \gamma C_0(\alpha)^{\mu-2\tau} \delta^{-\sqrt{\sigma N}}$. Again, we see that if $\gamma \geqq \gamma_1$ with $0 < \varepsilon < \min(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ we get $\phi_N(x,v) \geqq 1$. This proves our claim.

It remains to analyze the case $\varepsilon \leqq \frac{\sqrt{N}}{v} \leqq \varepsilon^{-1}$ with $\varepsilon > 0$ chosen above. We claim that in this case there is $\gamma_2 = \gamma_2(\alpha) > 0$ such that if $\gamma \geqq \gamma_2$ then $\phi_N(x,v) \geqq 1$.

As before, if $x^\sigma \delta^{\frac{-2\sqrt{\sigma v}}{\sqrt{N-v}+\sqrt{N}}} \geqq 1$, then $\phi_N(x,v) \geqq 1$. Otherwise we have

$$x \leqq \delta^{\frac{v}{\sqrt{\sigma N}}} \leqq \delta^{\frac{\varepsilon}{\sqrt{\sigma}}} \equiv \eta_1 < 1. \tag{4.20}$$

On the other hand, if we have

$$\gamma N^\mu \delta^{-2\sqrt{\sigma N}} E_N[x^2, 1]^2 \geqq 1,$$

then we also get $\phi_N(x,v) \geqq 1$. Therefore we can only consider the case

$$1 > \gamma N^\mu \delta^{-2\sqrt{\sigma N}} E_N[x^2,1]^2 \geqq \gamma C_0(\alpha) N^{\mu-2\tau} \delta^{-2\sqrt{\sigma N}} \left(\frac{1-x}{1+x} \right)^{2v+1}.$$

As was proved, for $\gamma \geqq \gamma_1$ we have

$$\gamma C_0(\alpha) N^{\mu-2\tau} \delta^{-\sqrt{\sigma N}} \geqq 1.$$

Thus

$$\delta^{\sqrt{\sigma N}} \geqq \left(\frac{1-x}{1+x} \right)^{2v+1} \geqq \delta^{(2v+1)\frac{ln\,\delta}{ln\,x}}.$$

(The last inequality is obtained by using Lemma 3.1). Hence

$$\sqrt{\sigma N} \leqq (2v+1) \frac{ln\,\delta}{ln\,x} \leqq 3v \frac{ln\,\delta}{ln\,x}$$

and thus

$$x \geqq \delta^{\frac{3}{\sqrt{\sigma N}}\varepsilon} = \eta_2 > 0. \tag{4.21}$$

Therefore we can restrict ourselves to consider only the case $0<\eta_2 \leqq x \leqq \eta_1 < 1$. This allows us to use estimate (4.14) in which $\tau=\alpha$. Consequently, we obtain:

$$\phi_{N,\nu}(x,\nu) \geqq x^{\sigma}\,\delta^{-\frac{\sqrt{\sigma}\,\nu}{\sqrt{N}}} + \gamma C_0(\alpha,\varepsilon)\left(\frac{\sqrt{N}}{\nu}\right)^{2\alpha} \frac{1}{\sqrt{N}}\left(\frac{1-x}{1+x}\right)^{2\nu+1} \delta^{-2\sqrt{\sigma N}}$$

$$> x^{\sigma}\,\delta^{\frac{\sqrt{\sigma}\,\nu}{\sqrt{N}}} + \gamma C_0(\alpha,\varepsilon)\,\varepsilon^{2\alpha}\left(\frac{1-\eta_1}{1+\eta_2}\right)\frac{1}{\sqrt{N}}\,\delta^{-2\sqrt{\sigma N}+\frac{2\nu\ln\delta}{\ln x}}. \tag{4.22}$$

Denote $C(\alpha,\varepsilon)=C_0(\alpha,\varepsilon)\,\varepsilon^{2\alpha}\left(\frac{1-\eta_1}{1+\eta_2}\right)$. Suppose that $x^{\sigma}\,\delta^{-\frac{\sqrt{\sigma N}}{\sqrt{N}}}<1$ (otherwise $\phi_N(x,\nu)\geqq 1$), we find

$$y \equiv \frac{\nu}{\sqrt{\sigma N}}\ln\delta - \ln x > 0.$$

Using the inequality for $x>0$:

$$e^{-x} > 1-2x,$$

$$e^{x} > 1+x,$$

(4.22) gives

$$\phi_N(x,\nu) \geqq e^{\sigma\ln x - \frac{\sqrt{\sigma}\,\nu}{\sqrt{N}}\ln\delta} + \gamma C(\alpha,\varepsilon)\,\frac{1}{\sqrt{N}}\,e^{\left(-2\sqrt{\sigma N}-\frac{2\nu\ln\delta}{\ln q}\right)\ln^{\delta}}$$

$$= e^{-\sigma y} + \gamma C(\alpha,\varepsilon)\,\frac{1}{\sqrt{N}}\,t^{\frac{2\sqrt{\sigma N}\ln\delta}{\ln x}y}$$

$$\geqq 1-2\sigma y + \gamma C(\alpha,\varepsilon)\,\frac{1}{\sqrt{N}}\left(1+\frac{2\sqrt{\sigma N}\ln\delta}{\ln x}\,y\right)$$

$$\geqq 1+\left(\gamma C(\alpha,\varepsilon)\,\frac{2\sqrt{\sigma}\ln\delta}{\ln\eta_2} - 2\sigma\right)y.$$

Since $\frac{\ln\delta}{\ln\eta_2}>0$, it follows that there is $\gamma_2>0$, if $\gamma\geqq\gamma_2$, then

$$\gamma C(\alpha,\varepsilon)\,\frac{2\sqrt{\sigma}\ln\delta}{\ln\eta_2} - 2\sigma > 0.$$

Choose $\gamma \geqq \max(\gamma_1,\gamma_2)=\gamma(\alpha)$, we will conclude

$$\phi_N(x,\nu)\geqq 1$$

for all $0\leqq x\leqq 1$, $1\leqq \nu \leqq N-1$, $N\geqq 1$. This completes our proof. □

Remark 4.2. In K. Scherer's theory (see [3]) the operator $\mathscr{T}$ is defined differently. It is based on the fact that the local errors are equilibrated in the best L_∞-

approximation. From this fact it was shown that the optimal mesh-degree combination tends to be a geometric mesh with ratio $q_0 = (\sqrt{2}-1)^2$ associated with linear degree vector with slope $s_0 = 2\alpha - 1$. More precisely, as $N \to \infty$ one has

$$x_{m-i} \to q_0^i$$
$$p_{m-i} - p_{m-i-1} \to 2\alpha - 1$$

for $i = 0, 1, 2, \ldots$.

5. The Numerical Performance of Various Versions of the FEM

The problem as before we will consider

$$\begin{aligned} -u'' &= f \\ u(0) &= u(1) = 0 \end{aligned} \tag{5.1}$$

with solution

$$u(x) = x^\alpha - x.$$

The relative error in the energy norm is denoted by $\|e\|$. We are mainly concerned with the relation between $\|e\|$ and N the number of degrees of freedom. (In the graphs, N will always be the abscissa and $\|e\|$ the ordinate, but with different scales.)

5.1. *The h-Version* ($\xi = 0$)

(1) The h-version with uniform mesh

Let

$$E_m = E_m(\alpha, p) = \frac{1}{m^\mu p^{2\alpha - 1}}, \quad (p \neq \alpha - 1/2), \quad \alpha > -1/2 \tag{5.2}$$

($\mu = \min(p, \alpha - 1/2)$, p the polynomial degree and m the number of elements) be the estimate of the error as in Part I. We denote

$$C_m = \frac{\|e\|}{E_m}$$

the "numerical constant" of the estimate. By Corollary 2.1, C_m converges to a limit $C(\alpha, p)$ as $m \to \infty$, and $C(\alpha, p)$ is bounded above and below by constants (>0) which only depend on α. It possibly has another limit when $p \to \infty$.

Table 2 shows the results for $\alpha = 0.7$ and $p = 1, 4$. Fig. 1 shows the graph in the $\log N - \log \|e\|$ scale for $\alpha = 2.9$. Because $N \approx mp$ the error E_m can also be written in the form

$$E_m = \frac{1}{N^\mu p^{2\alpha - \mu - 1}}.$$

Since $2\alpha - 1 - \mu \geqq \alpha - 1/2 > 0$, an increase of degree of element reduces the error. This is also clearly shown in Fig. 1.

(2) The h version with radical mesh

Table 2. ($\alpha = 0.7$)

m	$p=1$ ($\mu=0.2$)		$p=4$ ($\mu=0.2$)	
	$\|e\|$	C_m	$\|e\|$	C_m
2	0.3742	0.4298	0.2168	0.4336
4	0.3262	0.4304	0.1887	0.4335
8	0.2840	0.4305	0.1643	0.4336
16	0.2473	0.4306	0.1430	0.4335
32	0.2153	0.4306	0.1245	0.4335
64	0.1874	0.4305	0.1084	0.4336
128	0.1632	0.4307	0.09435	0.4335
256	0.1420	0.4305	0.08214	0.4335
512	0.1237	0.4307	0.07151	0.4336

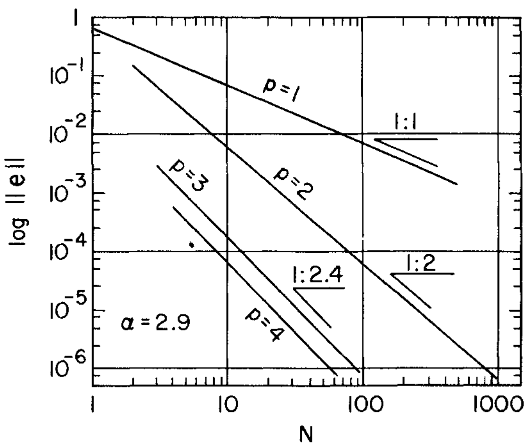

Fig. 1

Let the mesh grading function (see Section 3.1) be

$$g(x) = x^{\beta}$$

then

$$x_i = g\left(\frac{i}{m}\right) \qquad i = 0, 1, 2, \ldots, m.$$

Let

$$E_m = E_m(\alpha, p) = \begin{cases} \dfrac{1}{m^{\mu}}, & \text{if } p \neq \beta(\alpha - 1/2) \\ \dfrac{\sqrt{\ln m}}{m^{p}}, & \text{if } p = \beta(\alpha - 1/2) \end{cases} \tag{5.3}$$

where $\mu = \min(p, \beta(\alpha - 1/2))$.

Table 3. ($\alpha = 1.1$, $p = 2$)

m	$\beta = 2$ ($\mu = 1.2$)		$\beta_c = 3.333$ ($\mu = 2$)	
	$\|e\|$	C_m	$\|e\|$	C_m
2	0.1776 *E*-1	0.04080	0.1658 *E*-1	0.07966
4	0.7859 *E*-2	0.04148	0.5570 *E*-2	0.07569
8	0.3440 *E*-2	0.04171	0.1691 *E*-2	0.07505
16	0.1500 *E*-2	0.04179	0.4869 *E*-3	0.07486
32	0.6533 *E*-3	0.04181	0.1360 *E*-3	0.07481
64	0.2844 *E*-3	0.04182	0.3721 *E*-4	0.07474
128	0.1238 *E*-3	0.04182	0.1004 *E*-4	0.07468
256	0.5390 *E*-4	0.04183	0.2684 *E*-5	0.07470
512	0.2346 *E*-4	0.04183	0.7115 *E*-6	0.07468

m	$\beta_{op} = 4.167$ ($\mu = 2$)		$\beta = 5$ ($\mu = 2$)	
	$\|e\|$	C_m	$\|e\|$	C_m
2	0.1998 *E*-1	0.07792	0.2410 *E*-1	0.0964
4	0.6480 *E*-2	0.1037	0.7865 *E*-2	0.1258
8	0.1822 *E*-2	0.1166	0.2162 *E*-2	0.1384
16	0.4818 *E*-3	0.1233	0.5585 *E*-3	0.1430
32	0.1238 *E*-3	0.1268	0.1411 *E*-3	0.1445
64	0.3137 *E*-4	0.1285	0.3541 *E*-4	0.1450
128	0.7894 *E*-5	0.1293	0.8861 *E*-5	0.1452
256	0.1980 *E*-5	0.1298	0.2216 *E*-5	0.1452
512	0.4959 *E*-6	0.1300	0.5540 *E*-6	0.1452

According to Theorem 2.3, the numerical constant

$$C_m = \frac{\|e\|}{E_m}$$

converges to a limit $C(\alpha, \beta, p)$ as $m \to \infty$.

The same theorem shows that the optimal rate of convergence m^{-p} occurs for $\beta > \frac{p}{\alpha - 1/2}$. We will say

$$\beta_c = \frac{p}{\alpha - 1/2}$$

is the *critical* β. The *optimal* β is given by

$$\beta_{op} = \frac{p + 1/2}{\alpha - 1/2}$$

which gives the minimum of the limit $C(\alpha, \beta, p)$.

Table 3 and Fig. 2 show the accuracy for different β's when $\alpha = 1.1$ and $p = 2$.

Figure 3 shows the accuracy obtained by different p and optimal mesh.

Table 4 shows the mesh parameter of the optimal radical meshes ($m = 256$). Table 5 shows the ratio $\|e\|/\|e\|_{op}$ for different β's where $\|e\|_{op}$ is the error for $\beta = \beta_{op}$.

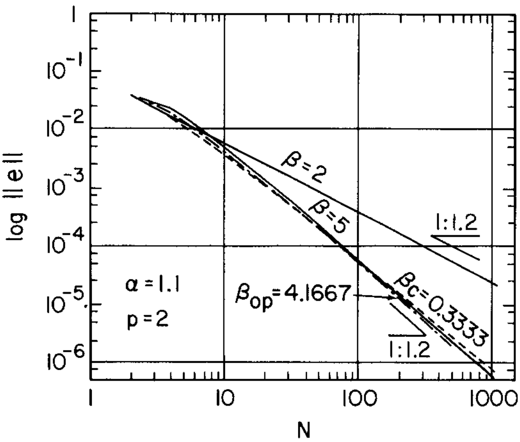

Fig. 2

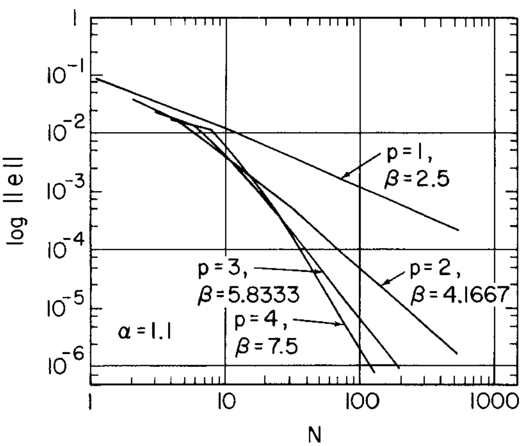

Fig. 3

We see that only for low accuracy it is good to use underrefined mesh. For an accuracy about 1 % the use of higher elements with overrefined mesh is advantageous. However, for highly unsmooth solutions (α is small) the refinement for both critical and optimal β is so strong that there are difficulties with the implementation (see Table 4).

5.2. h-p Version

(1) Geometric mesh with linear distributed degree vector (*G-L*)

In this case q denotes the ratio of geometric mesh and s the slope of degree vector. For each q the optimal slope s is given by

$$s_0 = (\alpha - 1/2)\,\frac{ln\,q}{ln\,r} \qquad \left(r = \frac{1-\sqrt{q}}{1+\sqrt{q}}\right).$$

Table 4. Mesh parameters

α	p	β_c	β_{op}	$\beta=\beta_c$		$\beta=\beta_{op}$	
				$\lvert I\rvert_{min}$	$\lvert I\rvert_{max}$	$\lvert I\rvert_{min}$	$\lvert I\rvert_{max}$
0.7	1	5	7.5	9.09 *E*-13	1.94 *E*-2	8.67 *E*-19	2.89 *E*-2
	2	10	12.5	8.27 *E*-25	3.84 *E*-2	7.89 *E*-31	4.77 *E*-2
	3	15	17.5	7.52 *E*-37	5.70 *E*-2	7.17 *E*-43	6.62 *E*-2
	4	20	22.5	6.84 *E*-49	7.53 *E*-2	6.53 *E*-55	8.43 *E*-2
	5	25	27.5	6.22 *E*-61	9.32 *E*-2	5.93 *E*-67	1.02 *E*-1
1.1	1	1.667	2.5	9.69 *E*-5	6.50 *E*-3	9.54 *E*-7	9.74 *E*-3
	2	3.333	4.167	9.39 *E*-9	1.30 *E*-2	9.24 *E*-11	1.62 *E*-2
	3	5	5.833	9.09 *E*-13	1.94 *E*-2	9.95 *E*-15	2.26 *E*-2
	4	6.667	7.5	8.81 *E*-17	2.58 *E*-2	8.67 *E*-19	2.89 *E*-2
	5	8.333	9.167	8.45 *E*-21	3.21 *E*-2	8.40 *E*-23	3.52 *E*-2

Table 5. $\|e\|/\|e\|_{op}$ $(p=2)$

α	m	β						
		1	4	7	$10^{(c)}$	$12.5^{(op)}$	15	$\|e\|_{op}$
0.7	2	0.94	0.67	0.71	0.97	1.00	1.03	0.6383
	4	1.45	0.69	0.61	0.75	1.00	1.13	0.3609
	8	3.41	1.07	0.68	0.74	1.00	1.17	0.1336
	16	10.82	2.09	0.91	0.80	1.00	1.16	0.0394

α	m	β						
		1	2	$3.333^{(c)}$	$4.167^{(op)}$	5	7	$\|e\|_{op}$
1.1	8	6.15	1.89	0.93	1.00	1.19	1.87	0.1996 *E*-1
	16	15.34	3.11	1.01	1.00	1.16	1.82	0.5278 *E*-2
	32	39.38	5.27	1.10	1.00	1.14	1.79	0.1356 *E*-2
	64	102.6	9.07	1.19	1.00	1.12	1.77	0.3436 *E*-3
	128	268.9	16.25	1.27	1.00	1.12	1.76	0.8647 *E*-4
	256	705.3	27.15	1.35	1.00	1.12	1.75	0.2169 *E*-4

The rate of convergence is then

$$E_N = E_N(\alpha, q, s) = \begin{cases} q^{(\alpha-1/2)\sqrt{2N/s}} & \text{if } s > s_0 \\ r^{\sqrt{2Ns}} & \text{if } s < s_0 \\ e^{-\kappa\sqrt{(\alpha-1/2)N}} & \text{if } s = s_0 \end{cases} \tag{5.4}$$

where $\kappa = \sqrt{2 ln q ln r}$ and N stands for the number of degrees of freedom of the finite element spaces. The optimal rate of convergence is achieved when $q=(\sqrt{2}-1)^2 \approx 1.7627$. (Cf. Theorem 3.2).

Table 6. (*G-L*, $\alpha = 0.7$)

$q = 0.5$, $\kappa = 1.5632$, $s = 0.07864$			$q = 0.3$, $\kappa = 1.7211$, $s = 0.1957$		
N	$\lVert e \rVert$	C_N	N	$\lVert e \rVert$	C_N
2	0.3742	2.024	2	0.3427	1.018
5	0.2509	1.198	9	0.1120	1.127
10	0.1357	1.238	15	0.5717 *E*-1	1.127
15	0.8525 *E*-1	1.278	26	0.2299 *E*-1	1.164
20	0.6695 *E*-1	1.526	36	0.1154 *E*-1	1.169
			44	0.7184 *E*-2	1.185

$q_{op} = 0.1715$, $\kappa = 1.7627$, $s = 0.4$			$q = 0.1$, $\kappa = 1.7366$, $s = 0.7032$		
N	$\lVert e \rVert$	C_N	N	$\lVert e \rVert$	C_N
2	0.3187	0.9718	2	0.3063	0.9187
11	0.7518 *E*-1	1.027	10	0.7443 *E*-1	0.8677
25	0.1962 *E*-1	1.010	18	0.3149 *E*-1	0.8496
46	0.4929 *E*-2	1.035	35	0.8374 *E*-2	0.8287
66	0.1736 *E*-2	1.049	57	0.2244 *E*-2	0.7898
81	0.8671 *E*-3	1.046	75	0.9246 *E*-2	0.7709

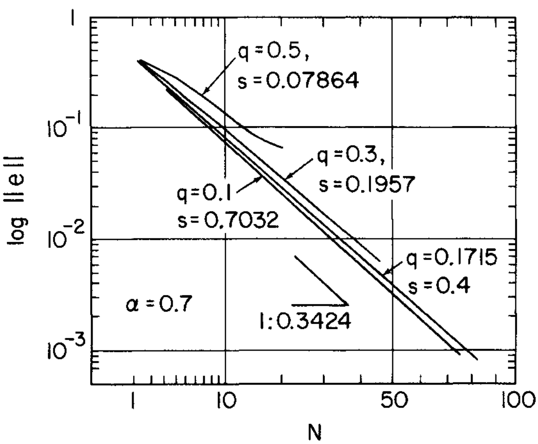

Fig. 4

The numerical constant

$$C_N = \frac{\lVert e \rVert}{E_N}$$

is bounded above and below by constants which depends only on α, q and s.

Table 6 shows the performance of the *h-p* version for various q and corresponding optimals s for $\alpha = 0.7$. Figure 4 drawn in $\sqrt{N} - \log \lVert e \rVert$ scale shows

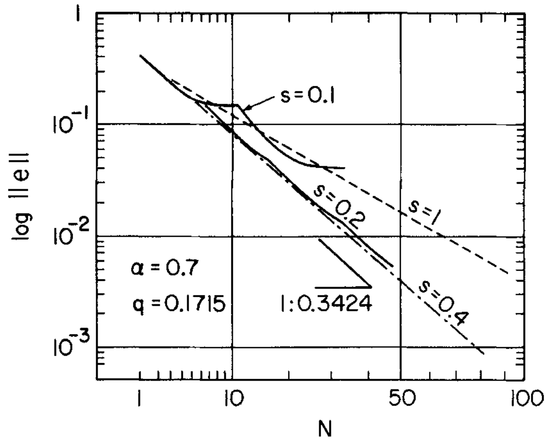

Fig. 5

Table 7. (G-U, $\alpha=0.7$)

$q=0.5$ $\kappa'=1.1054$ $s=0.07864$			$q=0.3$ $\kappa'=1.2170$ $s=0.1957$		
N	$\|e\|$	C_N	N	$\|e\|$	C_N
2	0.3742	0.9596	2	0.3427	0.9431
5	0.2509	1.331	12	0.1002	1.575
10	0.1357	1.450	33	0.2554 *E*-1	1.980
26	0.6246 *E*-1	2.430	45	0.1063 *E*-1	1.552
32	0.4153 *E*-1	2.289	64	0.6815 *E*-2	2.273
40	0.2451 *E*-1	2.032	80	0.2798 *E*-2	1.687

$q_{op}=0.1715$ $\kappa'=1.2465$ $s=0.4$			$q=0.1$ $\kappa'=1.2280$ $s=0.7032$		
N	$\|e\|$	C_N	N	$\|e\|$	C_N
2	0.3187	0.8935	2	0.2130	0.5932
15	0.6935 *E*-1	1.550	20	0.4110 *E*-1	1.367
32	0.2160 *E*-1	1.707	35	0.1530 *E*-1	1.368
55	0.6905 *E*-2	1.753	63	0.5131 *E*-2	1.710
78	0.3140 *E*-2	1.983	88	0.1964 *E*-2	1.626
112	0.1031 *E*-2	1.961	108	0.1166 *E*-2	1.807

the error behaviour of this case. Figure 5 shows the accuracy for various s, and optimal q and $\alpha=0.7$. In the used scale the error behaves linearly. The slope shown in the figure characterizes the theoretically best rate of convergence.

(2) Geometric mesh with uniformly distributed degree vector (G-U)

In this case the degree p is a multiple of the number m of elements:

$$p=sm. \tag{5.5}$$

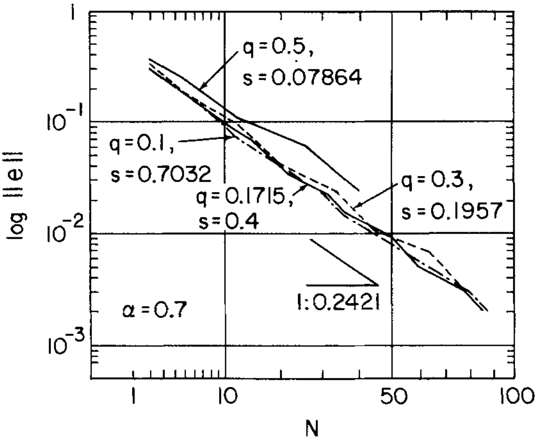

Fig. 6

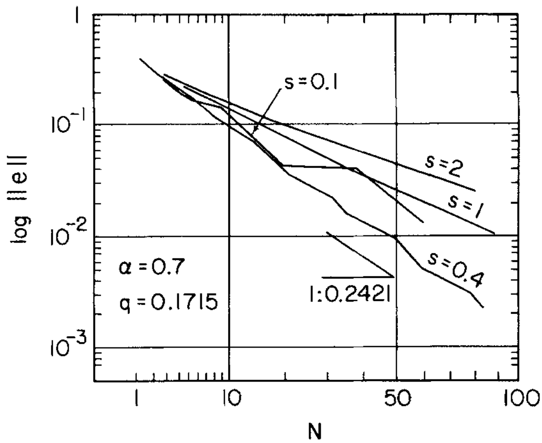

Fig. 7

The rate of convergence is given by

$$E_N = E_N(\alpha, q, s) = \begin{cases} \dfrac{q^{(\alpha-1/2)\sqrt{N/s}}}{\sqrt{s}N^{2\alpha-1}} & s > s_0 \\[2ex] \dfrac{r^{\sqrt{Ns}}}{\sqrt{s}N^{\alpha}} & s < s_0 \\[2ex] \dfrac{1}{\sqrt{s}N^{\sigma}}\, e^{\kappa'\sqrt{(\alpha-1/2)N}} & s = s_0 \end{cases}$$

where $\sigma = \min(\alpha, 2\alpha-1)$, $\kappa' = \sqrt{ln\, q\, ln\, r}$, s_0 is the optimal factor in (5.5) which is the same as the optimal slope in the $(G\text{-}L)$ case:

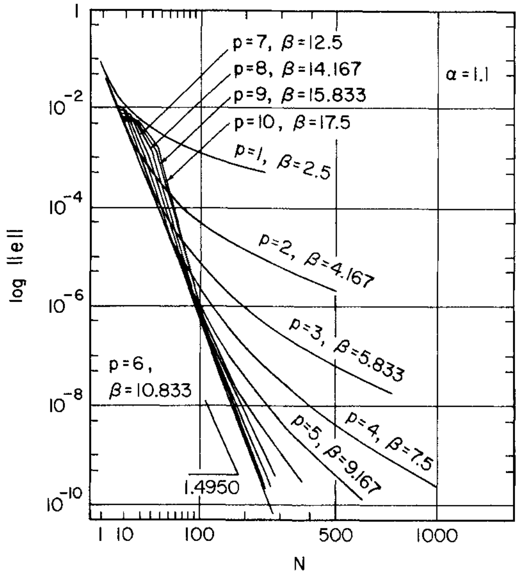

Fig. 8

$$s_0 = (\alpha - 1/2)\,\frac{ln\, q}{ln\, r}, \qquad r = \frac{1-\sqrt{q}}{1+\sqrt{q}}.$$

(Cf. Theorem 3.4.)

Note the exponents in (5.6) are the same as in (5.4) but multiplied by $1/\sqrt{2}$. The optimal q and s of the geometric mesh are

$$q = (\sqrt{2}-1)^2 \quad \text{and} \quad s = 2\alpha - 1.$$

Table 7 shows the accuracy and the values of the numerical constant for $\alpha = 0.7$, various q and corresponding optimal s.

Figure 6 shows in the $\sqrt{N} - \log\|e\|$ scale the behaviour of the method. Figure 7 shows the accuracy for various s when the q is optimal, the error behaviour linearly and the slope shown in the figure characterizes the maximally possible rate of convergence.

In Section 3.4 it was shown that the curves for radical meshes (p fixed and $m \to \infty$) have an envelope p. If we let p be a parameter which tends to ∞, this envelope has slope

$$\frac{4}{e}\sqrt{a - 1/2}/ln\,10 \;(\approx 1.4950 \text{ for } \alpha = 1.1)$$

in the $\sqrt{N} - \log\|e\|$ scale. Figure 8 shows the envelope of the error curves for optimal radical meshes when p increases. The slope shown in the figure is the optimal rate of the h-p version with uniform p.

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Received May 30, 1985/April 24, 1986