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The *h*, *p* and *h-p* Versions of the Finite Element Method in 1 Dimension

Part I. The Error Analysis of the *p*-Version

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Summary. This paper is the first one in the series of three which are addressing in detail the properties of the three basic versions of the finite element method in the one dimensional setting. The main emphasis is placed on the analysis when the (exact) solution has singularity of x^α-type. The first part analyzes the *p*-version, the second the *h*-version and general *h-p* version and the final third part addresses the problems of the adaptive *h-p* version.

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1. Introduction

In general, the *h-version* of the Finite Element Method is the standard version when the degree of elements is fixed and the convergence is achieved by the refinement of the mesh. The *p-version* fixes the mesh and the convergence is obtained by the increase of the degree of elements. The *h-p version* simultaneously refines the mesh and increases the degree of elements. In recent years, the *p* and *h-p* versions of the finite element method attracted large interest both in theory and computational practice. The commercial program PROBE [1] based on the *p*-version become available. This paper is the first in the series of three which will analyze in detail the properties of the *h*, *p* and *h-p* versions for solving the one dimensional problem, the solution of which has singularity of x^α-type. The first part analyzes the *p*-version, the second the *h*-version and general *h-p* versions and the final third one addresses the problem of the adaptive *h-p* version.

In this paper we will not discuss the two dimensional case; nevertheless the detailed results in one dimension are serving as guidelines to the two dimensional theory. For the analyses of the *p* and *h-p* finite element version for the

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two dimensional problem we refer to [2, 3]. For the computational implementation and engineering aspects of the p and h-p versions we refer to [4–9].

The p-version was studied theoretically first in [11]. The h-p version was addressed in [11], and for detailed analysis of the p-version in three dimension we refer to [12, 13]. For additional theoretical aspect of the p-version we refer also to [14, 15]. The p-version is in some sense related to the theory of spectral method. For these results we refer especially to [16–18].

In this paper we will consider the most simple model problem

$$-u''=f \tag{1.1}$$

$$u(0)=u(1)=0$$

with the solution

$$u_0(x)=(x-\xi)_+^\alpha-(1-\xi)^\alpha x-(-\xi)_+^\alpha(1-x) \tag{1.2}$$

where

$$(x-\xi)_+^\alpha=\begin{cases}(x-\xi)^\alpha & \text{if } x>\xi\\ 0 & \text{if } x\leqq\xi.\end{cases}$$

We will be interested in the accuracy of the finite element method measured in the energy norm $\|u\|_E^2=\int_0^1(u')^2\,dx$. We have to assume that $\alpha>\frac{1}{2}$ to get finite energy of the solution. The energy norm is in our case obviously equivalent to the H^1-norm of the standard Sobolev space H^1.

The x^α-type singularity of the solution is an analogue of the singularity of the solution of the two dimensional boundary value problems for elliptic partial differential equation occuring when the domain has corners. Solution of this type belongs to low order Sobolev spaces. This, in general, leads to low rate of convergence of the finite element method with quasiuniform meshes. On the other hand, taking into account the special structure of $u(x)$, one can achieve high (exponential) rate of convergence by proper design of the mesh and degrees of elements. Obviously, the results related to (1.1) can be generalized to the general case of two point boundary value problem.

For our model, the finite element method with C^0 elements gives exact solution in the nodal points and the analysis of the error of the finite element method reduces to the analysis of the best L_2-approximation of u' by piecewise polynomials.

Most of our results in this paper will provide both, the upper and lower bounds of the error of the finite element solution. In what it follows the following notation will be used. By $A\cong B$ we mean that A asymptotically equals B as some parameter tend to a certain limit. By $A\approx B$ we mean that A is equivalent to B, that is, there exists an *equivalency constant* $C>0$ (depending on some parameters to be indicated) such that

$$CB\leqq A\leqq C^{-1}B.$$

The paper is organized as follows: In Sects. 2–4 we study the properties of the Legendre expansion of the function $(x-\xi)_+^\alpha$. Section 5 addresses the Legendre expansion of more general functions. Section 6 discusses the performance of the p-version, and Sect. 7 deals with numerical computations.

2. The Legendre Expansion of $(x-\xi)_+^\alpha$

Let

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]$$

be the Legendre polynomials which form an orthogonal basis of $L_2[-1,1]$. For the properties of Legendre polynomials see [19–22].

We denote

$$u_\xi(x)=(x-\xi)_+^\alpha. \tag{2.1}$$

Suppose that its Legendre expansion is

$$u_\xi(x)\sim\sum_{n=0}^{\infty}a_n P_n(x). \tag{2.2}$$

The following theorem gives the expression for the coefficients of the expansion:

Theorem 1. *Let,* $\xi<1$, *then:*

1) *if* $\xi=-1$, $\alpha>-1$, *then*

$$a_0=\frac{1}{\alpha+1}2^\alpha, \tag{2.3a}$$

$$a_n=\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{(\alpha+1)(\alpha+2)\dots(\alpha+n+1)}(2n+1)\cdot 2^\alpha \tag{2.3b}$$

2) *if* $-1<\xi<1$, $\alpha>-1$, *then*

$$a_n=\frac{2n+1}{2}\frac{(1-\xi)^{\alpha+1}n!}{(\alpha+1)(\alpha+2)\dots(\alpha+n+1)}P_n^{(\alpha+1,-\alpha-1)}(\xi) \tag{2.4}$$

where $P_n^{(\alpha+1,-\alpha-1)}$ *is the Jacobi polynomial*

$$P_n^{(\mu,\nu)}(x)=\frac{(-1)^n}{2^n n!(1-x)^\mu(1+x)^\nu}\frac{d^n}{dx^n}[(1-x)^{n+\mu}(1+x)^{n+\nu}] \tag{2.5}$$

with $\mu=\alpha+1$, $\nu=-\alpha-1$,

3) *if* $\xi<-1$, *then*

$$a_n=\frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{(2n-1)!!}(2r)^{n-\alpha}\Phi_{n,\alpha}(r^2) \tag{2.6}$$

where

$$r=\frac{1}{-\xi+\sqrt{\xi^2-1}}\qquad(0<r<1) \tag{2.7}$$

and $\Phi_{n,\alpha}(x)$ is the Gauss hypergeometric function

$$\begin{aligned}\Phi_{n,\alpha}(x) &= F(n-\alpha, -\alpha-\tfrac{1}{2}; n+\tfrac{3}{2}; x)\\ &= 1+\sum_{k=1}^{\infty} \frac{(n-\alpha)(n-\alpha+1)\dots(n-\alpha+k-1)}{(n+\frac{3}{2})(n+\frac{5}{2})\dots(n+\frac{3}{2}+k-1)}\\ &\quad\cdot\frac{(-\alpha-\frac{1}{2})(-\alpha+\frac{1}{2})\dots(-\alpha-\frac{1}{2}+k-1)}{k!}x^k\\ &= 1+\sum_{k=1}^{\infty}\left[\sum_{j=1}^{k}\left(\frac{n-\alpha+j-1}{n+j+\frac{1}{2}}\right)\right](-1)^k\binom{\alpha+\frac{1}{2}}{k}x^k.\end{aligned} \tag{2.8}$$

Proof. We have

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} u_\xi(x) P_n(x)\,dx. \tag{2.9}$$

In the case $\xi = -1$

$$\begin{aligned}a_n &= \frac{2n+1}{2}\int_{-1}^{1}(1+t)^\alpha P_n(t)\,dt\\ &= \frac{2n+1}{2}\int_{-1}^{1}(1+t)^\alpha \frac{1}{2^n n!}[(t^2-1)^n]^{(n)}\,dt.\end{aligned}$$

Since $\alpha > -1$, for $k = 1, 2, \dots, n$ we have

$$\lim_{t\to\pm 1}(1+t)^{\alpha-k+1}[(t^2-1)^n]^{(n-k)} = 0,$$

integration by parts yields

$$\begin{aligned}a_n &= \frac{2n+1}{2^{n+1}n!}(-1)^n\alpha(\alpha-1)\dots(\alpha-n+1)\int_{-1}^{1}(1+t)^\alpha(t-1)^n\,dt\\ &= \frac{2n+1}{2^{n+1}}\frac{(-1)^{2n}\alpha(\alpha-1)\dots(\alpha-n+1)}{(\alpha+1)(\alpha+2)\dots(\alpha+n)}\int_{-1}^{1}(1+t)^{\alpha+n}\,dt\\ &= \frac{2n+1}{2^{n+1}}\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{(\alpha+1)(\alpha+2)\dots(\alpha+n+1)}2^{\alpha+n+1}\end{aligned}$$

and (2.3) follows.

For the case $-1 < \xi < 1$ we have

$$a_n = \frac{2n+1}{2}\int_{\xi}^{1}(t-\xi)^\alpha P_n(t)\,dt. \tag{2.10}$$

For $\operatorname{Re}\mu > 0$ and $0 < \gamma < 1$ one has

$$\int_{0}^{1}(1-x)^{\mu-1}P_n(1-\gamma x)\,dx = \frac{\Gamma(\mu)n!}{\Gamma(\mu+n+1)}P_n^{(\mu,-\mu)}(1-\gamma) \tag{2.11}$$

(see [22], p. 833). Setting $t = 1-(1-\xi)y$ we obtain from (2.10):

$$\begin{aligned}
a_n &= \frac{2n+1}{2}\int_1^0 [(1-\xi)(1-y)]^\alpha P_n(1-(1-\xi)y)[-(1-\xi)]\,dy \\
&= \frac{2n+1}{2}(1-\xi)^{\alpha+1}\int_0^1 (1-y)^{(\alpha+1)-1} P_n(1-(1-\xi)y)\,dy \\
&= \frac{2n+1}{2}(1-\xi)^{\alpha+1}\frac{\Gamma(\alpha+1)n!}{\Gamma(\alpha+n+2)} P_n^{(\alpha-1,-\alpha-1)}(\xi) \\
&= \frac{2n+1}{2}\,\frac{(1-\xi)^{\alpha+1}n!}{(\alpha+1)(\alpha+2)\dots(\alpha+n+1)} P_n^{(\alpha+1,-\alpha-1)}(\xi).
\end{aligned}$$

This proves (2.4).

Finally, for $\xi<-1$ integration by parts gives

$$\begin{aligned}
a_n &= \frac{2n+1}{2}\int_{-1}^1 (t-\xi)^\alpha P_n(t)\,dt \\
&= \frac{2n+1}{2}\int_{-1}^1 (t+|\xi|)^\alpha \frac{1}{2^n\cdot n!}[(t^2-1)^n]^{(n)}\,dt \\
&= (-1)^2\frac{2n+1}{2}\,\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{2^n\cdot n!}\int_{-1}^1 (t+|\xi|)^{\alpha-n}(t^2-1)^n\,dt.
\end{aligned}$$

Setting $t=\cos\theta$, we obtain

$$(-1)^n\int_{-1}^1 (t+|\xi|)^{\alpha-n}(t^2-1)^n\,dt=\int_0^\pi \frac{\sin^{2n+1}\theta}{(|\xi|+\cos\theta)^{n-\alpha}}\,\mathrm{d}\theta.$$

By [22], p. 384 we have

$$\int_0^\pi \frac{\sin^{2\mu-1}x}{(1+2a\cos x+a^2)^\nu}\,dx=B(\mu,\tfrac{1}{2})F(\nu,\nu-\mu+\tfrac{1}{2};\mu+\tfrac{1}{2};a^2)$$

with $\mathrm{Re}\,\mu>0$, $|a|<1$, and $F(\alpha,\beta;\nu;x)$ being the Gauss hypergeometric function. Let

$$r=\frac{1}{|\xi|+\sqrt{\xi^2-1}}\qquad (0<r<1)$$

thus

$$|\xi|=\frac{1+r^2}{2r}.$$

Setting $\mu=n+1$, $(\mu>0)$ and $\nu=n-\alpha$, we have

$$\begin{aligned}
\int_0^\pi \frac{\sin^{2n+1}\theta}{(|\xi|+\cos\theta)^{n-\alpha}}\,d\theta &= (2r)^{n-\alpha}\int_0^\pi \frac{\sin^{2(n+1)-1}\theta}{(1+2r\cos\theta+r^2)^{n-\alpha}}\,d\theta \\
&= (2r)^{n-\alpha}\frac{\Gamma(n+1)\sqrt{\pi}}{\Gamma(n+\frac{3}{2})}F(n-\alpha,\,-\alpha-\tfrac{1}{2};n+\tfrac{3}{2};r^2) \\
&= (2r)^{n-\alpha}\frac{2^{n+1}n!}{(2n+1)!!}\Phi_{n,\alpha}(r^2)
\end{aligned}$$

with $\Phi_{n,\alpha}$ defined in (2.8). (2.6) now follows easily. □

Remark 1. For $\xi=-1$, the expansion

$$(1+t)^\alpha=2^\alpha\left\{\frac{1}{\alpha+1}+\sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)\dots(\alpha-n+1)(2n+1)}{(\alpha+1)(\alpha+2)\dots(\alpha+n+1)}P_n(t)\right\} \tag{2.12}$$

is well known as Neumann-Stieljes series (see [20], pp. 240–244).

Remark 2. Since

$$\alpha(\alpha+1)\dots(\alpha+n)=\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha)} \tag{2.13}$$

the above results can also be written in the following form:

1) if $\xi=-1$, $\alpha>-1$, then

$$\begin{aligned} a_n&=(-1)^n\frac{\Gamma(\alpha+1)\Gamma(n-\alpha)}{\Gamma(-\alpha)\Gamma(n+\alpha+2)}(2n+1)\cdot 2^\alpha \\ &=(-1)^{n-1}\frac{[\Gamma(1+\alpha)]^2\sin\pi\alpha}{\pi}\,\frac{\Gamma(n-\alpha)(2n+1)\cdot 2^\alpha}{\Gamma(\alpha+n+2)} \end{aligned} \tag{2.14}$$

2) if $-1<\xi<1$, $\alpha>-1$, then

$$a_n=\frac{\Gamma(1+\alpha)(1-\xi)^{\alpha+1}\Gamma(n+1)}{\Gamma(\alpha+n+2)}(n+\tfrac{1}{2})P_n^{(\alpha+1,-\alpha-1)}(\xi) \tag{2.15}$$

3) if $\xi>-1$, then

$$\begin{aligned} a_n&=(-1)^n\frac{\Gamma(\alpha+1)\Gamma(\frac{1}{2})}{\Gamma(-\alpha)2^n\Gamma(n+\frac{1}{2})}(2r)^{n-\alpha}\Phi_{n,\alpha}(r^2) \\ &=(-1)^{n-1}\frac{\Gamma(1+\alpha)\sin\pi\alpha}{2^\alpha\sqrt{\pi}}\,\frac{\Gamma(n-\alpha)}{\Gamma(n+\frac{1}{2})}r^{n-\alpha}\Phi_{n,\alpha}(r^2). \end{aligned} \tag{2.16}$$

Remark 3. In the special case $\xi=0$, we have $(\alpha>-1)$

$$\begin{aligned} a_{2n}&=(2n+\tfrac{1}{2})\frac{(-1)^n[1^2-(\alpha+1)^2][3^2-(\alpha+1)^2]\dots[(2n-1)^2-(\alpha+1)^2]}{(\alpha+1)(\alpha+2)\dots(\alpha+2n+1)} \\ &=(-1)^2(2n+\tfrac{1}{2})\frac{\Gamma(1+\alpha)}{\left(1+\frac{\alpha}{2}\right)\Gamma\left(-\frac{\alpha}{2}\right)}\,\frac{2^{2n}\Gamma\left(n+1+\frac{\alpha}{2}\right)\Gamma\left(n-\frac{\alpha}{2}\right)}{\Gamma(2n+\alpha+2)}, \end{aligned} \tag{2.17}$$

$$\begin{aligned} a_{2n+1}&=(2n+\tfrac{3}{2})\frac{(-1)^n[2^2-(\alpha+1)^2][4^2-(\alpha+1)^2]\dots[(2n)^2-(\alpha+1)^2]}{(\alpha+2)(\alpha+3)\dots(\alpha+2n+2)} \\ &=(-1)^2(2n+\tfrac{3}{2})\frac{\Gamma(1+\alpha)}{\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma\left(1-\frac{\alpha+1}{2}\right)}\,\frac{2^{2n+1}\Gamma\left(n+1+\frac{\alpha+1}{2}\right)\Gamma\left(n-\frac{\alpha-1}{2}\right)}{\Gamma(\alpha+2n+3)}. \end{aligned} \tag{2.18}$$

Proof. In fact, for the Jacobi polynomial $P_n^{(\alpha+1,-\alpha-1)}(t)$ the following recursion formula (see, for example [19], p. 71) holds:

$$\begin{aligned} &P_0^{(\alpha+1,-\alpha-1)}(t)=1\\ &P_1^{(\alpha+1,-\alpha-1)}(t)=1+\alpha+t\\ &P_{n+1}^{(\alpha+1,-\alpha-1)}(t)=\frac{2n+1}{n+1}\,t\,P_n^{(\alpha+1,-\alpha-1)}(t)-\frac{n^2-(\alpha+1)^2}{n(n+1)}P_{n-1}^{(\alpha+1,-\alpha-1)}(t).\end{aligned} \tag{2.19}$$

In particular,

$$\begin{aligned} &P_0^{(\alpha+1,-\alpha-1)}(0)=1\\ &P_1^{(\alpha+1,-\alpha-1)}(0)=1+\alpha\\ &P_{n+1}^{(\alpha+1,-\alpha-1)}(0)=-\frac{n^2-(\alpha+1)^2}{n(n+1)}P_{n-1}^{(\alpha+1,-\alpha-1)}(0).\end{aligned} \tag{2.20}$$

Equations (2.17) and (2.18) are obtained by discussing the cases for n even and n odd resp. □

Remark 4. Equations (2.17) and (2.18) can be combined into one formula:

$$a_n=(-1)^{\left[\frac{n-1}{2}\right]}\frac{\Gamma(1+\alpha)\phi_n(\alpha)}{\pi}(n+\tfrac{1}{2})\frac{2^n\Gamma\left(\frac{n+\alpha}{2}+1\right)\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma(n+\alpha+2)} \tag{2.21}$$

with

$$\phi_n(\alpha)=\begin{cases}\cos\dfrac{\pi\alpha}{2}, & \text{if } n \text{ is odd}\\[2ex] \sin\dfrac{\pi\alpha}{2}, & \text{if } n \text{ is even}\end{cases}$$

and $[x]$ being the largest integer which is not greater than x.

3. Estimates and Asymptotic Behaviour of the Coefficients of the Legendre Expansion

In this section we will obtain the asymptotic formulae and some estimates for the Legendre expansion. First, one can easily prove the following lemma by using Stirling's formula.

Lemma 1.

$$\frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)}=\frac{1}{n^{\beta-\alpha}}\left(1+O\left(\frac{1}{n}\right)\right) \qquad (n\to\infty) \tag{3.1}$$

where $O\left(\frac{1}{n}\right)$ *depends on* α, β.

We now prove the following theorem which is concerned with the asymptotic behaviour of the coefficients obtained in Sect. 2.

Theorem 2. *Let a_n be the coefficients of the Legendre expansion of $(x-\xi)_+^\alpha$ and $\alpha > -\frac{1}{2}$. Then*

1) *if* $\xi = -1$, *then*

$$a_n = (-1)^{n-1} \frac{C_0(\alpha)}{n^{2\alpha+1}} \left(1 + O\left(\frac{1}{n}\right)\right) \tag{3.2}$$

with

$$C_0(\alpha) = \frac{2^{\alpha+1}\Gamma(1+\alpha)^2 \sin \pi\alpha}{\pi},$$

2) *if* $-1 < \xi < 1$, *then*

$$a_n = \sqrt{\frac{2}{\pi}} \Gamma(\alpha+1) \left(\frac{\sin\theta}{n}\right)^{\alpha+\frac{1}{2}} \left\{\cos\left[(n+\tfrac{1}{2})\theta - (\alpha+\tfrac{3}{2})\frac{\pi}{2}\right] + O\left(\frac{1}{n}\right)\right\} \tag{3.3}$$

with $\theta = \arccos \xi$, *and* $O\left(\frac{1}{n}\right)$ *holding uniformly for* $|\xi| \leqq 1-\varepsilon$, $\varepsilon > 0$,

3) *if* $\xi < -1$, *then*

$$a_n = (-1)^{n-1} \frac{C_1(\alpha)}{n^{\alpha+\frac{1}{2}}} \left[(1-r^2)^{\alpha+\frac{1}{2}} + O\left(\frac{1}{n^\sigma}\right)\right] \cdot r^{n-\alpha} \tag{3.4}$$

where $C_1(\alpha) = \dfrac{\Gamma(1+\alpha)\sin\pi\alpha}{2^\alpha\sqrt{\pi}}$, $r = \dfrac{1}{|\xi| + \sqrt{\xi^2-1}}$ *and* $\sigma = \dfrac{\alpha+\frac{1}{2}}{\alpha+\frac{3}{2}} > 0$, $O\left(\dfrac{1}{n^\sigma}\right)$ *holds uniformly with respect to* $r \in (0, 1)$.

Proof. By Lemma 1 and (2.14), (3.2) follows immediately: for $\xi = -1$

$$\begin{aligned} a_n &= (-1)^{n-1} C_0(\alpha) \frac{\Gamma(n-\alpha)}{\Gamma(n+\alpha+2)} (n+\tfrac{1}{2}) \\ &= (-1)^{n-1} C_0(\alpha) \frac{1}{n^{2\alpha+1}} \left(1 + O\left(\frac{1}{n}\right)\right). \end{aligned}$$

For $-1 < \xi < 1$, we use the following asymptotic formula for Jacobi polynomial (see [19] p. 196).

$$P_n^{(\alpha,\beta)}(\cos\theta) = \frac{\cos(N\theta+\gamma)}{\sqrt{n\pi}\left(\sin\frac{\theta}{2}\right)^{\alpha+\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}} + O(n^{-\frac{3}{2}}) \tag{3.5}$$

where

$$N = n + (\alpha+\beta+1)/2,$$

$$\gamma = -(\alpha+\tfrac{1}{2})\frac{\pi}{2},$$

$$0 < \theta < \pi.$$

This formula holds for arbitrary real α and β and holds uniformly for $\theta \in [\varepsilon, \pi - \varepsilon]$, $\varepsilon > 0$. Thus, in particular, setting $\theta = \arccos\xi$, we obtain

$$P_n^{(\alpha+1,-\alpha-1)}(\xi)=\frac{\cos\left[(n+\frac{1}{2})\theta-(\alpha+\frac{3}{2})\frac{\pi}{2}\right]}{\sqrt{\pi n}\left(\sin\frac{\theta}{2}\right)^{\alpha+\frac{3}{2}}\left(\cos\frac{\theta}{2}\right)^{-\alpha-\frac{1}{2}}}+O(n^{-\frac{3}{2}})$$

$$=\frac{\sqrt{2}(\sin\theta)^{\alpha+\frac{1}{2}}\cos\left[(n+\frac{1}{2})\theta-(\alpha+\frac{3}{2})\frac{\pi}{2}\right]}{\sqrt{\pi n}(1-\cos\theta)^{\alpha+1}}+O(n^{-\frac{3}{2}}).$$

Since $(1-\xi)^{\alpha+1}=(1-\cos\theta)^{\alpha+1}$, by (2.15) we get

$$a_n=\frac{\Gamma(1+\alpha)(1-\xi)^{\alpha+1}\Gamma(n+1)}{\Gamma(n+\alpha+2)}(n+\tfrac{1}{2})P_n^{(\alpha+1,-\alpha-1)}(\xi)$$

$$=\frac{\Gamma(1+\alpha)(1-\cos\theta)^{\alpha+1}}{n^\alpha}\left(1+O\left(\frac{1}{n}\right)\right)$$

$$\cdot\left\{\sqrt{\frac{2}{\pi}}\frac{(\sin\theta)^{\alpha+\frac{1}{2}}}{\sqrt{\pi}}\frac{\cos\left[(n+\frac{1}{2})\theta-(\alpha+\frac{3}{2})\frac{\pi}{2}\right]}{(1-\cos\theta)^{\alpha+1}}+O\left(\frac{1}{n^{\frac{3}{2}}}\right)\right\}$$

$$=\sqrt{\frac{2}{\pi}}\Gamma(1+\alpha)\left(\frac{\sin\theta}{n}\right)^{\alpha+\frac{1}{2}}\left\{\cos\left[(n+\tfrac{1}{2})-(\alpha+\tfrac{3}{2})\frac{\pi}{2}\right]+O\left(\frac{1}{n}\right)\right\}.$$

This holds uniformly in $\theta\in[\varepsilon,\pi-\varepsilon]$.

For $\xi<-1$, (2.16) and Lemma 1 gives (uniformly in ξ)

$$a_n=(-1)^{n-1}C_1(\alpha)\frac{r^{n-\alpha}}{n^{\alpha+\frac{1}{2}}}\Phi_{n,\alpha}(r^2)\left(1+O\left(\frac{1}{n}\right)\right).$$

It remains to analyse $\Phi_{n,\alpha}(r^2)$. The series

$$\Phi_{n,\alpha}(r^2)=1+\sum_{k=1}^{\infty}\left(\prod_{j=1}^{\infty}\left(\frac{n-\alpha+j-1}{n+j+\frac{1}{2}}\right)\right)(-1)^k\binom{\alpha+\frac{1}{2}}{k}r^{2k}$$

converges uniformly in $r\in(0,1)$ since its general term is dominated by

$$\left|\binom{\alpha+\frac{1}{2}}{k}\right|=\left|\frac{\Gamma(k-\alpha-\frac{1}{2})}{\Gamma(-\alpha-\frac{1}{2})\Gamma(k+1)}\right|=O\left(\frac{1}{k^{\alpha+\frac{3}{2}}}\right)$$

as $k\to+\infty$ and $\alpha+\frac{1}{2}$ being non-integer. (If $\alpha+\frac{1}{2}$ is an integer, then $\Phi_{n,\alpha}(r^2)$ only contains finitely many terms), and because $\alpha+\frac{3}{2}>1$ for $\alpha>-\frac{1}{2}$. Thus

$$\lim_{n\to\infty}\Phi_{n,\alpha}(r^2)=1+\sum_{k=1}^{\infty}(-1)^k\binom{\alpha+\frac{1}{2}}{k}r^2=(1-r^2)^{\alpha+\frac{1}{2}}.$$

We shall estimate now the difference

$$R_n=|\Phi_{n,\alpha}(r^2)-(1-r^2)^{\alpha+\frac{1}{2}}|.$$

Since

$$\prod_{j=1}^{k}\left(\frac{n-\alpha+j-1}{n+j+\frac{1}{2}}\right)=\frac{\Gamma(n-\frac{3}{2})\Gamma(n-\alpha+k)}{\Gamma(n-\alpha)\Gamma(n+k+\frac{3}{2})}=\left(\frac{n}{n+k}\right)^{\alpha+\frac{3}{2}}\left(1+O\left(\frac{1}{n}\right)\right)$$

holds uniformly in k, and since this product decreases as k increases,

$$R_n=\left|\sum_{k=1}^{\infty}\left[\prod_{j=1}^{k}\left(\frac{n-\alpha+j-1}{n+j+\frac{1}{2}}\right)-1\right](-1)^k\binom{\alpha+\frac{1}{2}}{k}r^{2k}\right|\leqq\sum_1+\sum_2$$

where

$$\begin{aligned}\sum_1&=\sum_{k=1}^{N}\left|\left[\prod_{j=1}^{k}\frac{n-\alpha+j-1}{n+j+\frac{1}{2}}-1\right]\binom{\alpha+\frac{1}{2}}{k}r^{2k}\right|\\&\leqq\left[1-\prod_{j=1}^{N}\left(\frac{n-\alpha+j-1}{n+j+\frac{1}{2}}\right)\right]\sum_{k=1}^{\infty}\left|\binom{\alpha+\frac{1}{2}}{k}\right|\\&\leqq C_1\left(1-\left(\frac{n}{n+N}\right)^{\alpha+\frac{3}{2}}\right),\end{aligned}$$

$$\begin{aligned}\sum_2&=\sum_{k=N+1}^{\infty}\left|\left[\prod_{j=1}^{k}\left(\frac{n-\alpha+j-1}{n+j+\frac{1}{2}}\right)-1\right]\binom{\alpha+\frac{1}{2}}{k}r^{2k}\right|\\&\leqq\sum_{k=N+1}^{\infty}\left|\binom{\alpha+\frac{1}{2}}{k}\right|\leqq C_2\frac{1}{N^{\alpha+\frac{1}{2}}},\end{aligned}$$

and C_1, C_2 are independent of n and N. Choose $N=[n^{\frac{1}{\alpha+\frac{3}{2}}}]$, then as $n\to\infty$ we obtain

$$1-\left(\frac{n}{n+N}\right)^{\alpha+\frac{3}{2}}=1-\left(1+\frac{1}{n^{\frac{\alpha+\frac{1}{2}}{\alpha+\frac{3}{2}}}}\right)^{-(\alpha+\frac{3}{2})}=O\left(\frac{1}{n^{\sigma}}\right)$$

with $\sigma=\dfrac{\alpha+\frac{1}{2}}{\alpha+\frac{3}{2}}>0$. We see that $\sum_1$ and $\sum_2$ have then the same order $O\left(\frac{1}{n^{\sigma}}\right)$, thus

$$R_n=O\left(\frac{1}{n^{\sigma}}\right)$$

which uniformly holds for $r\in(0,1)$. Therefore we can write

$$\Phi_{n,\alpha}(r^2)=(1-r^2)^{\alpha+\frac{1}{2}}+O\left(\frac{1}{n^{\sigma}}\right)$$

and Eq. (3.4) follows. □

Remark 5. In the case $\xi=0$, using (2.21) and Lemma 1 one can easily obtain

$$a_n=(-1)^{\left[\frac{n-1}{2}\right]}\sqrt{\frac{2}{\pi}}\,\Gamma(1+\alpha)\,\phi_n(\alpha)\frac{1}{n^{\alpha-\frac{1}{2}}}(1+O(\tfrac{1}{2}))\tag{3.6}$$

where

$$\phi_n(\alpha)=\begin{cases}\cos\dfrac{\pi\alpha}{2}&\text{if } n \text{ is odd,}\\[2ex]\sin\dfrac{\pi\alpha}{2}&\text{if } n \text{ is even.}\end{cases}$$

We shall now prove the inequalities of a_n which are important in our further error analysis.

Theorem 3. *If* $\xi \leqq -1$ *and* $n > \alpha$ $(\alpha > -\frac{1}{2})$,

$$|a_n| \approx \frac{r^{n-\alpha}}{n^{\alpha+\frac{1}{2}}} \left(\frac{1}{n^{\alpha+\frac{1}{2}}} + (1-r^2)^{\alpha+\frac{1}{2}}\right) \tag{3.7}$$

with the equivalence constant depending only on α *and* $r = \dfrac{1}{|\xi| + \sqrt{\xi^2 - 1}}$. *(Here* $r = 1$ *is admissible.)*

Proof. Since

$$a_n = (-1)^{n-1} C(\alpha) \frac{r^{n-\alpha}}{n^{\alpha+\frac{1}{2}}} \Phi_{n,\alpha}(r^2) \left(1 + O\left(\frac{1}{n}\right)\right)$$

holds uniformly in r as $n \to \infty$, it suffices to estimate $\Phi_{n,\alpha}(r^2)$.

Using the integral representation (see [23], p. 259):

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}\, dt$$

for $\operatorname{Re} c > \operatorname{Re} a > 0$, $|\arg(1-z)| < \pi$, we have for $n > \alpha$

$$\begin{aligned} \Phi_{n,\alpha}(r^2) &= F(n-\alpha, -\alpha-\tfrac{1}{2}; n+\tfrac{3}{2}; r^2) \\ &= \frac{\Gamma(n+\frac{3}{2})}{\Gamma(\alpha+\frac{3}{2})\Gamma(n-\alpha)} \int_0^1 t^{n-\alpha-1}(1-t)^{\alpha+\frac{1}{2}}(1-r^2 t)^{\alpha+\frac{1}{2}}\, dt. \end{aligned} \tag{3.8}$$

Writing $(1-r^2 t)^{\alpha+\frac{1}{2}} = [(1-t) + (1-r^2)t]^{\alpha+\frac{1}{2}}$ and noting that for $0 < \alpha + \frac{1}{2} \leqq 1$, the function $x^{\alpha+\frac{1}{2}}$ is concave. We can use Jensen's inequality ([24], p. 28) for a, $b \geqq 0$, $r > s > 0$

$$(a^r + b^r)^{1/r} \leqq (a^s + b^s)^{1/s}, \tag{3.9}$$

to obtain for $0 < \alpha + \frac{1}{2} \leqq 1$, $r = 1$, $s = \alpha + \frac{1}{2}$:

$$\begin{aligned} &2^{\alpha-\frac{1}{2}}[(1-t)^{\alpha+\frac{1}{2}} + (1-r^2)^{\alpha+\frac{1}{2}} t^{\alpha+\frac{1}{2}}] \\ &= 2^{\alpha+\frac{1}{2}} \left(\frac{(1-t)^{\alpha+\frac{1}{2}} + [(1-r^2)t]^{\alpha+\frac{1}{2}}}{2}\right) \\ &\leqq 2^{\alpha+\frac{1}{2}} \left(\frac{(1-t) + (1-r^2)t}{2}\right)^{\alpha+\frac{1}{2}} = [(1-t) + (1-r^2)t]^{\alpha+\frac{1}{2}} \\ &\leqq (1-t)^{\alpha+\frac{1}{2}} + (1-r^2)^{\alpha+\frac{1}{2}} t^{\alpha+\frac{1}{2}}. \end{aligned} \tag{3.10}$$

For $\alpha + \frac{1}{2} \geqq 1$ the function $x^{\alpha+\frac{1}{2}}$ is convex, using once more the Jensen's inequality with $r = \alpha + \frac{1}{2}$, $s = 1$, we get

$$\begin{aligned} (1-t)^{\alpha+\frac{1}{2}} + (1-r^2)^{\alpha+\frac{1}{2}} t^{\alpha+\frac{1}{2}} &\leqq [(1-t) + (1-r^2)t]^{\alpha+\frac{1}{2}} \\ &= 2^{\alpha+\frac{1}{2}} \left(\frac{(1-t) + (1-r^2)t}{2}\right)^{\alpha+\frac{1}{2}} \\ &\leqq 2^{\alpha+\frac{1}{2}} \frac{(1-t)^{\alpha+\frac{1}{2}} + (1-r^2)^{\alpha+\frac{1}{2}} t^{\alpha+\frac{1}{2}}}{2} \\ &= 2^{\alpha-\frac{1}{2}}[(1-t)^{\alpha+\frac{1}{2}} + (1-r^2)^{\alpha+\frac{1}{2}} t^{\alpha+\frac{1}{2}}]. \end{aligned} \tag{3.11}$$

Hence we get by using (3.8) and Lemma 1

$$\frac{\Gamma(n+\frac{3}{2})}{\Gamma(\alpha+\frac{3}{2})\Gamma(n-\alpha)}\int_0^1 t^{n-\alpha-1}(1-t)^{\alpha+\frac{1}{2}}\{(1-t)^{\alpha+\frac{1}{2}}+(1-r^2)^{\alpha+\frac{1}{2}}t^{\alpha+\frac{1}{2}}\}\,dt$$
$$=\frac{\Gamma(n+\frac{3}{2})}{\Gamma(\alpha+\frac{3}{2})\Gamma(n-\alpha)}\left[\frac{\Gamma(n-\alpha)\Gamma(2\alpha+2)}{\Gamma(n+\alpha+2)}+(1-r^2)^{\alpha+\frac{1}{2}}\frac{\Gamma(n+\frac{1}{2})\Gamma(\alpha+\frac{3}{2})}{\Gamma(n+\alpha+2)}\right]$$
$$=\frac{2^{2\alpha+1}\Gamma(1+\alpha)}{\sqrt{\pi}}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+\alpha+2)}+(1-r^2)^{\alpha+\frac{1}{2}}\frac{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}{\Gamma(n-\alpha)\Gamma(n+\alpha+2)}$$
$$=\left(D(\alpha)\frac{1}{n^{\alpha+\frac{1}{2}}}+(1-r^2)^{\alpha+\frac{1}{2}}\right)\left(1+O\left(\frac{1}{n}\right)\right)$$

where

$$D(\alpha)=\frac{2^{2\alpha+1}\Gamma(1+\alpha)}{\sqrt{\pi}} \tag{3.12}$$

and the term $O\left(\frac{1}{n}\right)$ is uniform with respect to r, thus we have (using the notation $\approx$)

$$\Phi_{n,\alpha}(r^2)\approx\frac{1}{n^{\alpha+\frac{1}{2}}}+(1-r^2)^{\alpha+\frac{1}{2}} \tag{3.13}$$

and inequality (3.7) follows easily. ((3.7) is valid for $r=1$ by the Eq. (3.2).) □

Remark 6. The estimate (3.13) is true under the condition $n>\alpha$. We now consider the case when $n\leqq\alpha$. In the proof of Theorem 1 we have

$$\int_0^\pi \frac{\sin^{2(n+1)-1}\theta}{(1+2r\cos\theta+r^2)^{n-\alpha}}\,d\theta=\frac{2^{n+1}n!}{(2n+1)!!}\Phi_{n,\alpha}(r^2).$$

For $n\leqq\alpha$, it follows that

$$\Phi_{n,\alpha}(r^2)=\frac{(2n+1)!!}{2(2n)!!}\int_0^\pi\frac{\sin^{2(n+1)-1}\theta}{(1+2r\cos\theta+r^2)^{n-\alpha}}\,d\theta$$
$$\geqq\frac{(2n+1)!!}{2(2n)!!}\int_0^{\pi/2}\sin^{2n+1}\theta\,d\theta=\tfrac{1}{2}.$$

On the other hand, apparently we have

$$\Phi_{n,\alpha}(r^2)\leqq\frac{(2n+1)!!}{2(2n)!!}\int_0^\pi 4^{\alpha-n}\sin^{2(n+1)-1}\theta\,d\theta\leqq 4^\alpha.$$

Therefore we have

$$\tfrac{1}{2}\leqq\Phi_{n,\alpha}(r^2)\leqq 4^\alpha \quad \text{if } n\leqq\alpha \tag{3.14}$$

and

$$|a_n|\approx\frac{r^{n-\alpha}}{n^{\alpha+\frac{1}{2}}} \quad \text{if } n\leqq\alpha. \tag{3.15}$$

Theorem 4. *If* $-1 \leqq \xi \leqq 1$, *then there is a constant* $C>0$ *independent of* n *and* ξ, *such that*

$$|a_n| \leqq \begin{cases} C\theta^{2+2}n & 0 \leqq \theta \leqq \lambda n^{-1} \\ C\left(\dfrac{\delta}{n}\right)^{\alpha+\frac{1}{2}} & \lambda n^{-1} \leqq \theta \leqq \pi - \lambda n^{-1} \\ C n^{-(2\alpha+1)} & \pi - \lambda n^{-1} \leqq \theta \leqq \pi \end{cases} \tag{3.16}$$

with $\theta = \arccos \xi$, $\lambda > 0$ *fixed* (C *may depend on* λ), *and* $\delta = \min(\theta, \pi - \theta)$.

Proof. For any real α, β and $c>0$ fixed,

$$P_n^{(\alpha,\beta)}(\cos\theta) = \begin{cases} \theta^{-\alpha-\frac{1}{2}} O(n^{-\frac{1}{2}}), & cn^{-1} \leqq \theta \leqq \pi/2, \\ O(n^\alpha), & 0 \leqq \theta \leqq cn^{-1} \end{cases} \tag{3.17}$$

(see [19], p. 169). In particular,

$$P_n^{(\alpha+1,-\alpha-1)}(\cos\theta) = \begin{cases} \theta^{-\alpha-\frac{3}{2}} O(n^{-\frac{1}{2}}), & cn^{-1} \leqq \theta \leqq \pi/2 \\ O(n^{\alpha+1}), & 0 \leqq \theta \leqq cn^{-1}. \end{cases} \tag{3.18}$$

For $\pi/2 \leqq \theta \leqq \pi$, we use $\pi - \theta$ instead of u. Because $\cos\theta = -\cos(\pi-\theta)$ and

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x) \tag{3.19}$$

(see [19], p. 59), we have

$$\begin{aligned} P_n^{(\alpha+1,-\alpha-1)}(\cos\theta) &= (-1)^n P_n^{(-\alpha-1,\alpha+1)}(\cos(\pi-\theta)) \\ &= \begin{cases} (\pi-\theta)^{\alpha+\frac{1}{2}} O(n^{-\frac{1}{2}}), & \pi - cn^{-1} \geqq \theta \geqq \pi/2, \\ O(n^{-\alpha-1}), & \pi \geqq \theta \geqq \pi - cn^{-1}. \end{cases} \end{aligned} \tag{3.20}$$

Noticing that for $0 \leqq \theta \leqq \pi$, $1-\xi = 1-\cos\theta = O(\theta^2)$, (2.15) gives

$$\begin{aligned} a_n &= \frac{\Gamma(1+\alpha)(1-\xi)^{\alpha+1}\,\Gamma(n+1)}{\Gamma(\alpha+n+2)} (n+\tfrac{1}{2}) P_n^{(\alpha+1,-\alpha-1)}(\xi) \\ &= O(\theta^{2\alpha+2} n^{-\alpha}) P_n^{(\alpha+1,-\alpha-1)}(\xi) \\ &= \begin{cases} \theta^{\alpha+\frac{1}{2}} O(n^{-\alpha-\frac{1}{2}}), & cn^{-1} \leqq \theta \leqq \pi/2 \\ \theta^{2\alpha+2} O(n), & 0 \leqq \theta \leqq cn^{-1} \\ (\pi-\theta)^{\alpha+\frac{1}{2}} O(n^{-\alpha-\frac{1}{2}}), & \pi - cn^{-1} \geqq \theta \geqq \pi/2 \\ O(n^{-(2\alpha+1)}), & \pi \geqq \theta \geqq \pi - cn^{-1} \end{cases} \end{aligned}$$

and (3.16) follows. □

4. Error Analysis of the Best L_2-Approximation of $(x-a)_+^\alpha$ on $[a,b]$

We know that the best L_2-approximation is given by the partial sum of Legendre expansion. Let $E_p([a,b],d)$ denote the error of the best L_2-

approximation on $[a,b]$ of the function $(x-d)_+^\alpha$, $(d<b)$ by polynomials of degree p. If $[a,b]=[-1,1]$, then we will write simply $E_p(\xi)\equiv E_p([-1,1],\xi)$.

First, we consider the interval $[-1,1]$ and the function is

$$u_\xi(x)=(x-\xi)_+^\alpha.$$

Let

$$(x-\xi)_+^\alpha \sim \sum_{n=0}^{\infty} a_n P_n(x), \qquad \text{on } [-1,1], \tag{4.1}$$

then

$$E_p(\xi)=\left\| \sum_{n=p+1}^{\infty} a_n P_n(x) \right\|_{L_2(-1,1)} = \left\{ \sum_{n=p+1}^{\infty} a_n^2 \frac{2}{2n+1} \right\}^{\frac{1}{2}}. \tag{4.2}$$

In the general case of the interval $[a,b]$ and the function $u_d(x)=(x-d)_+^\alpha$ the following relation can be easily obtained:

Lemma 2.

$$E_p([a,b],d)=h^{\alpha+\frac{1}{2}}E_p(\xi)$$

where

$$\xi=\frac{d-c}{h}, \qquad c=\frac{a+b}{2}, \qquad h=\frac{b-a}{2}.$$

Now we are going to obtain asymptotic formulae for the error $E_p(\xi)$ and $E_p([a,b],d)$.

Theorem 5. *If* $\xi=-1$, *then*

$$E_p(-1)=C_0(\alpha)\frac{2^{\alpha+\frac{1}{2}}}{(p+1)^{2\alpha+1}}\left(1+O\left(\frac{1}{p}\right)\right) \qquad (p\to\infty) \tag{4.3}$$

where

$$C_0(\alpha)=\frac{\Gamma(1+\alpha)^2\,|\sin\pi\alpha|}{\sqrt{2\alpha+1}\,\pi}.$$

Consequently,

$$E_p([a,b],a)=C_0(\alpha)\frac{(b-a)^{\alpha+\frac{1}{2}}}{(p+1)^{2\alpha+1}}\left(1+O\left(\frac{1}{p}\right)\right) \tag{4.4}$$

or

$$E_p([a,b],a)\approx\frac{(b-a)^{\alpha+\frac{1}{2}}}{(p+1)^{2\alpha+1}} \tag{4.5}$$

with the equivalence constant depending on α *but not* p.

Proof.

$$E_p(-1)=\left\{\sum_{n=p+1}^{\infty} a_n^2\frac{2}{2n+1}\right\}^{\frac{1}{2}} = \frac{2^{\alpha+1}\Gamma(1+\alpha)^2\,|\sin\pi\alpha|}{\pi}\left\{\sum_{n=p+1}^{\infty}\frac{1+O(1/n)}{n^{4\alpha+3}}\right\}^{\frac{1}{2}}$$

$$=\frac{2^{\alpha+1}\Gamma(1+\alpha)^2|\sin\pi\alpha|}{\pi}\frac{1}{\sqrt{4\alpha+2}}\frac{1}{(p+1)^{2\alpha+1}}\left(1+O\left(\frac{1}{p}\right)\right)$$

$$=\frac{\Gamma(1+\alpha)^2|\sin\pi\alpha|}{\pi\sqrt{2\alpha+1}}\frac{2^{\alpha+\frac{1}{2}}}{(p+1)^{2\alpha+1}}\left(1+O\left(\frac{1}{p}\right)\right).$$

Noting that $h=\frac{b-a}{2}$, (4.4) follows readily by using Lemma 2. □

Theorem 6. *If* $\xi<-1$, *then*

$$E_p(\xi)=C_1(\alpha)\left(\frac{1-r^2}{2r}\right)^{\alpha}\frac{r^{p+1}}{(p+1)^{\alpha+1}}\left(1+O\left(\frac{1}{p^{\sigma}}\right)\right) \tag{4.6}$$

with

$$\sigma>0,\qquad r=\frac{1}{|\xi|+\sqrt{\xi^2-1}},\qquad C_1(\alpha)=\frac{\Gamma(1+\alpha)|\sin\pi\alpha|}{\sqrt{\pi}},$$

and the term $O\left(\frac{1}{p^{\sigma}}\right)$ *is uniform with respect to* $\xi<-1-\varepsilon$ $(\varepsilon>0)$. *Consequently,*

$$E_p([a,b],d)=C_1(\alpha)\left(\frac{b-a}{2}\right)^{\alpha+\frac{1}{2}}\left(\frac{1-r^2}{2r}\right)^{\alpha}\frac{r^{p+1}}{(p+1)^{\alpha+1}}\left(1+O\left(\frac{1}{p^{\sigma}}\right)\right) \tag{4.7}$$

with $r=\frac{\sqrt{b-d}-\sqrt{a-d}}{\sqrt{b-d}+\sqrt{a-d}}=\frac{b-a}{b+a-2d+2\sqrt{(b-d)(a-d)}}$ *for* $d<a$.

First we need to prove an auxiliary lemma

Lemma 3. *If* $0<s<1$, $\sigma>0$, *then*

$$\sum_{n=N}^{\infty}\frac{s^n}{n^{\sigma}}=\frac{s^N}{N^{\sigma}}\frac{1}{1-s}\left(1+O\left(\frac{\ln N}{N}\right)\right). \tag{4.8}$$

The term $O\left(\frac{\ln N}{N}\right)$ *is uniform in* $s\in[0,1-\varepsilon]$, $\varepsilon>0$.

Proof. Observe that

$$0<\frac{\frac{s^N}{N^{\sigma}}\frac{1}{1-s}-\sum_{n=N}^{\infty}\frac{s^n}{n^{\sigma}}}{\frac{s^N}{N^{\sigma}}\frac{1}{1-s}}=\frac{\sum_{n=N+1}^{n}\left(\frac{1}{N^{\sigma}}-\frac{1}{n^{\sigma}}\right)s^n}{\frac{s^N}{N^{\sigma}}\frac{1}{1-s}}$$

$$=(1-s)\sum_{k=1}^{\infty}\left(1-\left(\frac{N}{N+k}\right)^{\sigma}\right]s^k$$

$$\leqq(1-s)\left[1-\left(\frac{N}{N+m}\right)^{\sigma}\right]\sum_{k=1}^{m}s^k+(1-s)\sum_{k=m+1}^{\infty}s^K$$

$$=\left[1-\left(\frac{N}{N+m}\right)^{\sigma}\right](s-s^{m+1})+s^{m+1}$$

$$=s\left[1-(1-s^m)\left(\frac{N}{N+m}\right)^{\sigma}\right]\equiv R_{N,m}. \tag{4.9}$$

Choose $m=\dfrac{\ln N}{\ln\frac{1}{s}}$, then for $N\to\infty$ we obtain

$$R_{N,m}\leqq 1-\left(1+\frac{m}{N}\right)^{-\sigma}+s^m=\frac{\sigma\ln N}{N\ln\frac{1}{s}}+\frac{1}{N}+O\left(\left(\frac{\ln N}{N}\right)^2\right)$$
$$=O\left(\frac{\ln N}{N}\right)$$

and this term $O\left(\frac{\ln N}{N}\right)$ holds uniformly with respect to $0<s\leqq 1-\varepsilon$, $\varepsilon>0$. □

Proof of Theorem 6. Lemma 1 and Lemma 2 give

$$E_p(\xi)=\left\{\sum_{n=p+1}^{\infty}a_n^2\frac{2}{2n+1}\right\}^{\frac{1}{2}}$$
$$=\frac{\Gamma(1+\alpha)|\sin\pi\alpha|}{\sqrt{\pi}}\frac{(1-r^2)^{\alpha+\frac{1}{2}}}{(2r)^\alpha}\left\{\sum_{n=p+1}^{\infty}\frac{r^{2n}}{n^{2\alpha+1}}\frac{2}{2n+1}\right\}^{\frac{1}{2}}\left(1+O\left(\frac{1}{p^\sigma}\right)\right)$$
$$=C_1(\alpha)\frac{(1-r^2)^{\alpha+\frac{1}{2}}}{(2r)^\alpha}\left\{\frac{1}{1-r^2}\frac{r^{2(p+1)}}{(p+1)^{2\alpha+2}}\right\}^{\frac{1}{2}}\left(1+O\left(\frac{\ln p}{p}\right)\right)\left(1+O\left(\frac{1}{p^\sigma}\right)\right)$$
$$=C_1(\alpha)\left(\frac{1-r^2}{2r}\right)^\alpha\frac{r^{p+1}}{(p+1)^{\alpha+1}}\left(1+O\left(\frac{1}{p^\sigma}\right)\right)$$

where $\sigma=\dfrac{\alpha+\frac{1}{2}}{\alpha+\frac{3}{2}}\in(0,1)$ and the term $O\left(\frac{1}{p^\sigma}\right)$ is uniform with respect to $0<r\leqq 1-\varepsilon$, $\varepsilon>0$. Considering the general case we obtain

$$\xi=\frac{d-c}{h}=\frac{2d-a-b}{b-a},$$
$$r=\frac{1}{|\xi|+\sqrt{\xi^2-1}}=\frac{b-a}{b+a-2d+2\sqrt{(b-d)(a-d)}}=\frac{\sqrt{b-d}-\sqrt{a-d}}{\sqrt{b-d}+\sqrt{a-d}}$$

$(a<d)$, and (4.7) easily follows. □

Remark 7. If $d<a$, the singular point is outside of the interval $[a,b]$, then the error $E_p([a,b],d)$ reduces exponentially by the above theorem, and the rate is characterized by the ratio r. This value depends only on the ratio of the length of the interval and the distance between the position of the singularity and the interval, but is independent of α. In Sect. 5 we will see that this property also holds for more general functions. In fact, let $|I|=b-a$, $\delta=a-d=\text{dist.}(d,[a,b])$, then

$$\xi=\frac{d-c}{h}=-\left(1+\frac{2(a-d)}{b-a}\right)=-(1+2\lambda)$$

where $\lambda=\dfrac{\delta}{|I|}$. Therefore,

$$r=\frac{1}{|\xi|+\sqrt{\xi^2-1}}=\frac{1}{1+2\lambda+2\sqrt{\lambda(1+\lambda)}}. \tag{4.10}$$

Thus r is geometric invariant of the error.

We are now going to obtain the estimate which is uniform with respect to $\xi \leqq -1$. We need first the following lemma:

Lemma 4. *Let* $0<s\leqq 1$, $\sigma>0$, $N\geqq 2$.

1) *If* $0<s\leqq 1-\frac{1}{N}$, *then*

$$\sum_{n=N}^{\infty}\frac{s^n}{n^\sigma}\approx\frac{s^N}{N^\sigma}\frac{1}{1-s}, \tag{4.11}$$

2) *if* $1-\frac{1}{N}\leqq s\leqq 1$, *then*

$$\sum_{n=N}^{\infty}\frac{s^N}{n^{\sigma+1}}\approx\frac{s^N}{N^\sigma}. \tag{4.12}$$

These inequalities hold uniformly in s and N.

Proof. As in the proof of Lemma 3, we have

$$0<\frac{\frac{s^N}{N^\sigma}\frac{1}{1-s}-\sum_{n=N}^{\infty}\frac{s^n}{n^\sigma}}{\frac{s^N}{N^\sigma}\frac{1}{1-s}}\leqq s\left[1-(1-s^m)\left(\frac{N}{N+m}\right)^\sigma\right]\equiv R_{N,m}.$$

If $0<s\leqq 1\leqq -\frac{1}{N}$, then for any $m\geqq 1$

$$R_{N,m}\leqq 1-\left(\frac{N}{N+m}\right)^\sigma\left[1-\left(1-\frac{1}{N}\right)^m\right].$$

Choose $m=N$ and notice that $\left(1-\frac{1}{N}\right)^N\uparrow\frac{1}{e}$, we get

$$0<\frac{\frac{s^N}{N^\sigma}\frac{1}{1-s}-\sum_{n=N}^{\infty}\frac{s^n}{n^\sigma}}{\frac{s^N}{N^\sigma}\frac{1}{1-s}}\leqq 1-\frac{1}{2^\sigma}\left(1-\frac{1}{e}\right),$$

thus

$$\frac{s^N}{N^\sigma}\frac{1}{1-s}\geqq\sum_{n=N}^{\infty}\frac{s^n}{n^\sigma}\geqq\frac{1}{2^\sigma}\left(1-\frac{1}{e}\right)\frac{s^N}{N^\sigma}\frac{1}{1-s}$$

and (4.11) follows.

In order to show (4.12), observe that

$$\begin{aligned}\sum_{n=N}^{\infty}\frac{s^N}{n^{\sigma+1}}&\leqq s^N\sum_{n=N}^{\infty}\frac{1}{n^{\sigma+1}}\leqq s^N\int_{N-1}^{\infty}\frac{dt}{t^{\sigma+1}}\\&=\frac{s^N}{\sigma}\frac{1}{(N-1)^\sigma}=\frac{s^N}{\sigma}\left(\frac{N}{N-1}\right)^\sigma\frac{1}{N^\sigma}\\&\leqq\frac{2^\sigma}{\sigma}\frac{s^N}{N^\sigma}\qquad\text{for } N\geqq 2.\end{aligned}$$

If $1 \geqq s \geqq 1 - \frac{1}{N}$, then

$$\begin{aligned}\sum_{n=N}^{\infty} \frac{s^n}{n^{\sigma+1}} &\geqq \int_N^{\infty} \frac{s^x}{x^{\sigma+1}} dx \\ &= \int_0^{\infty} \frac{s^{y+N}}{(y+N)^{\sigma+1}} dy \qquad (y = x - N) \\ &\geqq \frac{s^N}{N^\sigma} \int_0^{\infty} \frac{(1-1/N)^y}{(1+y/N)^\sigma} \frac{dy}{N+y} \\ &= \frac{s^N}{N^\sigma} \int_0^{\infty} \frac{(1-1/N)^{Nt}}{(1+t)^{\sigma+1}} dt \qquad (t = y/N)\end{aligned}$$

since $(1-1/N)^N \uparrow \frac{1}{e}$ as $N \to \infty$, $\left(1 - \frac{1}{N}\right)^N \geqq \frac{1}{4}$ for $N \geqq 2$. Thus

$$\sum_{n=N}^{\infty} \frac{s^n}{n^{\sigma+1}} \geqq \frac{s^N}{N^\sigma} \int_0^{\infty} \frac{4^{-t}}{(1+t)^{\sigma+1}} dt.$$

This proves (4.12). □

Theorem 7. *Let* $\xi < -1$, $p+1 > \alpha$, *then*

1) *if* $0 < r^2 \leqq 1 - \frac{1}{p+1}$, *then*

$$E_p(\xi) \approx \frac{1}{\sqrt{1-r^2}} \frac{r^{p+1-\alpha}}{(p+1)^{\alpha+1}} \left[\frac{1}{(p+1)^{\alpha+\frac{1}{2}}} + (1-r^2)^{\alpha+\frac{1}{2}}\right] \tag{4.13}$$

2) *if* $1 - \frac{1}{p+1} \leqq r^2 \leqq 1$, *then*

$$E_p(\xi) \approx \frac{r^{p+1-\alpha}}{(p+1)^{\alpha+\frac{1}{2}}} \left[\frac{1}{(p+1)^{\alpha+\frac{1}{2}}} + (1-r^2)^{\alpha+\frac{1}{2}}\right]. \tag{4.14}$$

Inequalities (4.13) *and* (4.14) *hold uniformly in r and p, where*

$$r = \frac{1}{|\xi| + \sqrt{\xi^2 - 1}}.$$

In the case of general interval $[a, b]$ the inequalities have to be modified by multiplication with a factor $(b-a)^{\alpha+\frac{1}{2}}$.

Proof. The results are the direct consequence of Theorem 3, Lemma 4, and the simple inequality

$$\frac{1}{\sqrt{2}}(\sqrt{x} + \sqrt{y}) \leqq \sqrt{x+y} \leqq \sqrt{x} + \sqrt{y}. \quad \square$$

Remark 8. By Remark 6 these estimates are also valid for $p+1 \leqq \alpha$.

Now we shall explore the error behavior for the case $-1<\xi<1$. As we see in Theorem 2 the coefficients a_n behave in a more complicated manner, hence we cannot obtain a simple asymptotic formula as in the other cases. However, we have the following estimates:

Theorem 8. *For $-1<\xi<1$, there exist a constant $C>0$, which depends only on α such that*

$$E_p(\xi) \leqq \begin{cases} C\left(\dfrac{\delta}{p+1}\right)^{\alpha+\frac{1}{2}}, & 0\leqq\theta\leqq\pi-\dfrac{1}{p+1}, \\ C\left(\dfrac{1}{p+1}\right)^{2\alpha+1}, & \pi-\dfrac{1}{p+1}\leqq\theta\leqq\pi, \end{cases} \tag{4.15}$$

where $\delta=\min(\theta,\pi-\theta)$, $\theta=\arccos\xi$.

For the error in the general interval $[a,b]$, $a<d<b$, the right-hand side has to be multiplied by $(b-a)^{\alpha+\frac{1}{2}}$, and

$$\theta=\arccos\frac{2d-a-b}{b-a}.$$

Proof. By Theorem 4 we can write

$$|a_n| \leqq \begin{cases} C\dfrac{1}{n^{2\alpha+1}} & \pi-\dfrac{2}{n}\leqq\theta\leqq\pi \\ C\left(\dfrac{\delta}{n}\right)^{\alpha+\frac{1}{2}} & 0\leqq\theta\leqq\pi-\dfrac{2}{n} \end{cases}$$

with $C=C(\alpha)$, $\delta=\min(\pi-\theta,\theta)$.

Therefore, if $\pi-\dfrac{1}{p+1}\leqq\theta\leqq\pi$, then

$$\begin{aligned} E_p(\xi) &= \left\{\sum_{n=p+1}^{\infty} a_n^2\frac{2}{2n+1}\right\}^{\frac{1}{2}} \\ &\leqq C\left\{\sum_{n=p+1}^{2p}\frac{1}{n^{4\alpha+3}}+\sum_{n=2p+1}^{\infty}\frac{\left(\frac{1}{p}\right)^{2\alpha+1}}{n^{2\alpha+2}}\right\}^{\frac{1}{2}} \\ &\leqq C\left(\frac{1}{p+1}\right)^{2\alpha+1}. \end{aligned}$$

The case when $0\leqq\theta\leqq\pi-\dfrac{1}{p+1}$ follows similarly. □

For the other side of the inequality we have the following result.

Theorem 9. *Let $\varepsilon\leqq\theta\leqq\pi-\varepsilon$, $\varepsilon>0$, then there is $C=(\alpha,\varepsilon)>0$ such that*

$$E_p(\xi)\geqq C\left(\frac{\delta}{p+1}\right)^{\alpha+\frac{1}{2}} \quad (\delta=\min(\theta,\pi-\theta)) \tag{4.16}$$

the estimate is uniform with respect to θ ($\delta=\arccos\xi$).

Proof. By Theorem 1

$$E_p(\xi)=\sqrt{\frac{2}{\pi}}\Gamma(1+\alpha)(\sin\theta)^{\alpha+\frac{1}{2}}\left\{\sum_{n=p+1}^{\infty}\frac{\cos^2\left[(n+\frac{1}{2})\theta-(\alpha+\frac{3}{2})\frac{\pi}{2}\right]}{n^{2\alpha+2}}+O\left(\frac{1}{n^{2\alpha+3}}\right)\right\}^{\frac{1}{2}}$$

$$=\sqrt{\frac{2}{\pi}}\Gamma(1+\alpha)(\sin\theta)^{\alpha+\frac{1}{2}}\left\{\sum_{n=p+1}^{\infty}\frac{\cos^2\left[(n+\frac{1}{2})\theta-(\alpha+\frac{3}{2})\frac{\pi}{2}\right]}{n^{2\alpha+2}}\right\}^{\frac{1}{2}}+O\left(\frac{1}{p^{\alpha+1}}\right). \tag{4.17}$$

The term $O\left(\frac{1}{p^{\alpha+1}}\right)$ is uniform with respect to $\theta\in[\varepsilon,\pi-\varepsilon]$.

It can be readily seen that (4.16) is proved if we show that

$$\max_{m=n,n+1}\left\{\cos^2\left[(m+\tfrac{1}{2})\theta-(\alpha+\tfrac{3}{2})\frac{\pi}{2}\right]\right\}\geqq\sin^2\frac{\varepsilon}{2}. \tag{4.18}$$

We prove this by contradiction. Let $\gamma=\frac{\varepsilon}{\pi}$, thus $\sin\frac{\varepsilon}{2}=\cos\left(\frac{\pi}{2}-\gamma\frac{\pi}{2}\right)$. If (4.18) is not true, then for $m=n$ and $m=n+1$ we have

$$\cos^2\left[(m+\tfrac{1}{2})\theta-(\alpha+\tfrac{3}{2})\frac{\pi}{2}\right]<\cos^2\left(\frac{\pi}{2}-\gamma\frac{\pi}{2}\right).$$

Thus there are integers $k(n)$ and $k(n+1)$ such that

$$-\gamma<(2n+1)\frac{\theta}{\pi}-(\alpha+\tfrac{1}{2}+2k(n))<\gamma,$$

$$-\gamma<(2n+3)\frac{\theta}{\pi}-(\alpha+\tfrac{1}{2}+2k(n+1))<\gamma.$$

Therefore

$$-\gamma<\frac{\theta}{\pi}-(k(n+1)-k(n))<\gamma.$$

Because $\gamma\leqq\frac{\theta}{\pi}$ and $\gamma\leqq 1-\frac{\theta}{\pi}$, we get

$$k(n+1)-k(n)>\frac{\theta}{\pi}-\gamma\geqq\frac{\theta}{\pi}-\frac{\theta}{\pi}=0,$$

$$k(n+1)-k(n)<\frac{\theta}{\pi}+\gamma\leqq\frac{\theta}{\pi}+1-\frac{\theta}{\pi}=1.$$

Since $k(n+1)-k(n)$ is an integer, we have the desired contradiction. □

Remark 9. It is easy to obtain the estimate for the case $\xi=0$:

$$E_0(0)\cong\frac{C(\alpha)}{(p+1)^{\alpha+\frac{1}{2}}} \tag{4.19}$$

where

$$C(\alpha)=\frac{\Gamma(1+\alpha)}{\sqrt{(2\alpha+1)\pi}}.$$

In fact by (3.6) we have

$$\begin{aligned}
E_p(0) &\cong \sqrt{\frac{2}{\pi}}\,\Gamma(1+\alpha)\left\{\sum_{n=p+1}^{\infty}\phi_n(\alpha)^2\frac{1}{n^{2\alpha+2}}\right\}^{\frac{1}{2}}\\
&=\sqrt{\frac{2}{\pi}}\,\Gamma(1+\alpha)\left\{\sum_{k=\left[\frac{p+1}{2}\right]}^{\infty}\frac{\left(\sin\frac{\pi\alpha}{2}\right)^2}{(2k)^{2\alpha+2}}+\sum_{k=\left[\frac{p+1}{2}\right]}^{\infty}\frac{\left(\cos\frac{\pi\alpha}{2}\right)^2}{(2k+1)^{2\alpha+2}}\right\}^{\frac{1}{2}}\\
&\cong\sqrt{\frac{2}{\pi}}\,\Gamma(1+\alpha)\left\{\frac{1}{2^{2\alpha+2}}\sum_{k=\left[\frac{p+1}{2}\right]}^{\infty}\frac{1}{k^{2\alpha+2}}\right\}^{\frac{1}{2}}\\
&\cong\frac{\Gamma(1+\alpha)}{\sqrt{(2\alpha+1)\pi}}\,\frac{1}{(p+1)^{\alpha+\frac{1}{2}}}. \quad \square
\end{aligned}$$

At last we consider the problem of the asymptotic behavior of $E_p([a,b],\xi)$ as the size $|I|$ of the interval approaches zero. For simplicity let $\xi=0$. We have the following theorem:

Theorem 10. *Let $x>0$ and let $\{I\}$ be a family of intervals containing x. Then*

$$\lim_{|I|\to 0}\frac{E_p(I,0)}{|I|^{p+\frac{1}{2}}}=C(\alpha,p)\frac{1}{x^{p+1-\alpha}} \tag{4.20}$$

where

$$C(\alpha,p)=\frac{\Gamma(1+\alpha)|\sin\pi\alpha|}{\sqrt{\pi}}\,\frac{\Gamma(p+1-\alpha)}{4^{p+1}\sqrt{2p+3}\,\Gamma(p+\frac{3}{2})}.$$

This limit is uniform with respect to $x\geqq\varepsilon$, $\varepsilon>0$.

Proof. Let $x\geqq\varepsilon$, $\varepsilon>0$. We may assume that the intervals $I=[a,b]$ in which x lies are away from zero:

$$0<a\leqq x\leqq b.$$

First consider the ratio r in the coefficient a_n of (2.16). In this case we have

$$r=r(a,b)=\frac{|I|}{a+b+2\sqrt{ab}}.$$

Noting $|a+b+2\sqrt{ab}-4x|\leqq 3|I|$, we have

$$\left|\frac{4x\,r(a,b)}{|I|}-1\right|=\frac{|a+b+2\sqrt{ab}-4x|}{a+b+2\sqrt{ab}}\leqq\frac{3|I|}{4x-3|I|},$$

thus

$$\left|\frac{r(a,b)}{|I|}-\frac{1}{4x}\right|\leqq\frac{3|I|}{4x(4x-3|I|)}.$$

Choose $|I|<\varepsilon$, then the above estimate becomes

$$\left|\frac{r(a,b)}{|I|}-\frac{1}{4x}\right|\leqq\frac{3|I|}{4\varepsilon(4\varepsilon-3\varepsilon)}=\frac{3}{4\varepsilon^2}|I|$$

hence $\lim\limits_{|I|\to 0}\frac{r(a,b)}{|I|}=\frac{1}{4x}$ uniformly hold with respect to $x\geqq\varepsilon$, $\varepsilon>0$.

By (2.16)

$$a_n=(-1)^{n-1}\frac{\Gamma(1+\alpha)\sin\pi\alpha}{\sqrt{\pi}\cdot 2^\alpha}\frac{\Gamma(n-\alpha)}{\Gamma(n+\frac{1}{2})}r^{n-\alpha}\Phi_{n,\alpha}(r^2).$$

Recall that $\Phi_{n,\alpha}(r^2)$ has a majorant series $\sum\limits_{k=1}^{\infty}\frac{C}{k^{\alpha+\frac{3}{2}}}<\infty$ (see proof of Theorem 2), thus $\lim\limits_{r\to 0}\Phi_{n,\alpha}(r^2)=1$ holds uniformly in n. Therefore we have

$$\begin{aligned}\sum_{n=p+2}^{\infty}a_n^2\frac{2}{2n+1}&\leqq C(\alpha)\sum_{n=p+2}^{\infty}\left[\frac{\Gamma(n-\alpha)}{\Gamma(n+\frac{1}{2})}\right]^2\frac{2}{2n+1}r^{2(n-\alpha)}\\&\leqq C(\alpha)\frac{1}{1-r^2}r^{2(p+2-\alpha)},\end{aligned}$$

it follows that if $x\geqq\varepsilon$ and $|I|$ is small, then

$$\begin{aligned}E_{p+1}(I,0)^2&=\left(\frac{|I|}{2}\right)^{2\alpha+1}\left\{\sum_{n=p+2}^{\infty}a_n^2\frac{2}{2n+1}\right\}\\&\leqq C(\alpha)\frac{|I|^{2\alpha+1}r^{2(p+2-\alpha)}}{1-r^2}\\&\leqq C(\alpha,\varepsilon)|I|^{2p+5}.\end{aligned}$$

Hence

$$\begin{aligned}\left(\frac{E_p(I,0)}{|I|^{p+\frac{3}{2}}}\right)^2&=\frac{1}{|I|^{2p+3}}\left(\frac{|I|}{2}\right)^{2\alpha+1}\cdot\left(a_{p+1}^2\frac{2}{2p+1}+E_{p+1}(I,0)^2\right)\\&=\left(\frac{\Gamma(1+\alpha)\sin\pi\alpha}{\sqrt{\pi}\,2^\alpha}\frac{\Gamma(p+1-\alpha)}{\Gamma(p+\frac{3}{2})}\right)^2\frac{1}{2^{2\alpha}}\frac{1}{2p+3}\left(\frac{r}{|I|}\right)^{2(p+1-\alpha)}+O(|I|^2)\\&\to\left(\frac{\Gamma(1+\alpha)\sin\pi\alpha}{\sqrt{\pi}}\frac{\Gamma(p+1-\alpha)}{4^{p+1}\sqrt{2p+3}\,\Gamma(p+\frac{3}{2})}\frac{1}{x^{p+1-\alpha}}\right)^2.\quad\square\end{aligned}$$

Remark 10. For the constant $C(\alpha,p)$ in (4.20), we have

$$C(\alpha,p)\cong\frac{\Gamma(1+\alpha)|\sin\pi\alpha|}{\sqrt{2\pi}}\frac{1}{4^{p+1}p^{\alpha+1}}\tag{4.21}$$

as $p\to\infty$.

5. The Best L_2-Approximation of Analytic Functions Which has an x^α-Type Singularity

We now extend our results to a more general case: the class of analytic function which has an x^α-type singularity. Precisely, we will discuss the function which is analytic as a function of complex variable except at one point $x_0 \in \mathbb{R} \setminus [a,b]$, at which it satisfies the following growth condition for $|z-x_0| \leqq K$, $K>0$:

$$|u(z)-u_0| > K|z-x_0|^\alpha \quad (\alpha > -\tfrac{1}{2}, \text{ noninteger}), \tag{5.1}$$

where

$$u_0 = \begin{cases} u(x_0) & \text{if } \alpha < 0 \\ 0 & \text{if } \alpha < 0. \end{cases}$$

The case that the singular point x_0 is real and outside of the interval $[a,b]$ is most interesting. We can obtain an estimate which has very similar character as that for the function $(x-\xi)_+^\alpha$. We have the following results:

Theorem 11. *Let $u \in L^2(a,b)$ satisfy the above condition* (5.1), *and let*

$$I=[a,b], \qquad d=\text{dist.}(x_0, I),$$

$$h=\tfrac{1}{2}|I|=\frac{b-a}{2}.$$

Then the error of best L_2-approximation by polynomials of degree p can be estimated as follows: if $\varepsilon \leqq r \leqq 1-\varepsilon$ $(\varepsilon>0)$, then

$$E_p[a,b] \leqq K\,C(\alpha)\,h^{\alpha+\frac{1}{2}} \left(\frac{1-r^2}{2r}\right)^\alpha r^{p+1} \tag{5.2}$$

where

$$r \equiv r\left(\frac{d}{h}\right) = \frac{1}{1+\frac{d}{h}+\sqrt{\frac{d}{h}\left(2+\frac{d}{h}\right)}}.$$

Proof. First of all, we make the linear transformation as before:

$$x=c+ht, \qquad t \in [-1,1]$$

$$c=\frac{a+b}{2}, \qquad h=\frac{b-a}{2}.$$

It maps $[-1,1]$ onto $[a,b]$. Therefore for the function

$$w(t)=u(c+ht)$$

defined on $[-1,1]$, the singular point will be

$$\xi_0=\frac{x_0-c}{h}$$

and

$$\delta \equiv \operatorname{dist}(\xi_0, [-1,1]) = \frac{1}{h}\operatorname{dist}(x_0, [a,b]) = \frac{d}{h},$$

and the growth condition (5.1) becomes

$$|w(\zeta) - u_0| \leqq K\,|(c+h\zeta)-(c+h\xi_0)|^\alpha = Kh^\alpha |\zeta - \xi_0|^\alpha. \tag{5.3}$$

As usual, we expand $w(t)$ into the Legendre series

$$w(t) \sim \sum_{n=0}^{\infty} a_n P_n(t) \qquad (t \in [-1,1])$$

then

$$E_p[-1,1] = \left\{ \sum_{n=p+1}^{\infty} a_n^2 \frac{2}{2n+1} \right\}^{\frac{1}{2}}. \tag{5.4}$$

Without loss of generality, we may assume $x_0 < a$, thus $\xi_0 < -1$ and $\xi_0 = -1 - \delta$.

Since $w(t)$ is analytic except at $\xi_0 \notin [-1,1]$, we have

$$\begin{aligned} a_n &= \frac{2n+1}{2} \int_{-1}^{1} w(t) P_n(t)\, dt \\ &= \frac{2n+1}{2} \frac{1}{2^n \cdot n!} \int_{-1}^{1} w(t) [(t^2-1)^n]^{(n)}\, dt \\ &= \frac{2n+1}{2} \frac{(-1)^n}{2^n \cdot n!} \int_{-1}^{1} w^{(n)}(t)(t^2-1)^n\, dt \\ &= \frac{2n+1}{2} \frac{1}{2^n \cdot n!} \int_{-1}^{1} \left\{ \frac{n!}{2\pi i} \oint_\gamma \frac{w(\zeta)}{(\zeta - t)^{n+1}}\, d\zeta \right\} (1-t^2)^n\, dt \end{aligned}$$

where γ is any contour to which t is interior and ξ_0 is exterior, and it is positively directed.

Now we choose γ to be the circle centered at t with a radius $R = R(t) < |t - \xi_0|$, then we have the following estimates;

$$\begin{aligned} \left| \frac{1}{2\pi i} \oint_\gamma \frac{w(\zeta)}{(\zeta - t)^{n+1}}\, d\zeta \right| &= \left| \frac{1}{2\pi i} \oint_\gamma \frac{w(\zeta) - u_0}{(\zeta - t)^{n+1}}\, d\zeta \right| \\ &\leqq \frac{1}{2\pi} \oint_\gamma \frac{|w(\zeta) - u_0|}{R^{n+1}}\, ds \\ &\leqq \frac{Kh}{R^n} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |\zeta - \xi_0|^\alpha\, d\theta \right\} \end{aligned}$$

where $\zeta = t + \mathrm{Re}^{i\theta}$.

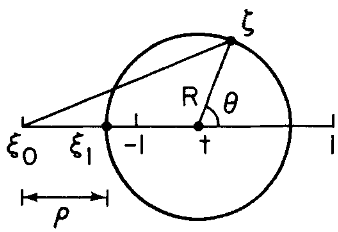

Fig. 1.

Let ξ_1 be the intersection of the circle to the segment $[\xi_0, t]$, and let $\rho=\xi_1-\xi_0$ be the distance of ξ_0 from the circle, then

$$M(\rho)=\frac{1}{2\pi}\int_0^{2\pi}|\zeta-\xi_0|^\alpha\, d\theta$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\rho^2+\left(2R\cos\frac{\theta}{2}\right)^2+4\rho R\cos^2\frac{\theta}{2}\right]^{\alpha/2} d\theta.$$

If $-\frac{1}{2}<\alpha<0$, the integral is dominated by

$$\left|2R\cos\frac{\theta}{2}\right|^\alpha,$$

and if $\alpha>0$, it is continuous in ρ. If we set $R_0=t-\xi_0=R+\rho$, we can let $\rho\to 0^+$ so that $R\to R_0^-$, and obtain

$$M=\lim_{\rho\to 0^+} M(\rho)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|2R\cos\frac{\theta}{2}\right|^\alpha d\theta=C(\alpha)R_0^\alpha$$

with

$$C(\alpha)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|2\cos\frac{\theta}{2}\right|^\alpha d\theta.$$

Because

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|2\cos\frac{\theta}{2}\right|^\alpha d\theta=\frac{2^{\alpha+1}}{\pi}\int_0^{\pi/2}(\sin\phi)^\alpha\, d\phi,$$

using the fact that for $0\leqq\phi\leqq\frac{\pi}{2}$,

$$\frac{2}{\pi}\phi\leqq\sin\phi\leqq\phi,$$

we obtain

$$\frac{2^\alpha}{\alpha+1}=\frac{2^{\alpha+1}}{\pi}\frac{2^\alpha}{\pi}\cdot\frac{1}{\alpha+1}\frac{\pi^{\alpha+1}}{2}$$
$$\leqq C(\alpha)\leqq\frac{2^{\alpha+1}}{\pi}\frac{1}{\alpha+1}\frac{\pi^{\alpha+1}}{2}=\frac{\pi^\alpha}{\alpha+1},$$

this holds for all $\alpha > -1$. Therefore we have

$$|a_n| \leqq \frac{2n+1}{2} \frac{1}{2^n} \int_{-1}^{1} \frac{K h^\alpha C(\alpha)}{R^{n-\alpha}} (1-t^2)^n \, dt$$
$$= K C(\alpha) h^\alpha \frac{2n+1}{2^{n+1}} \int_{-1}^{1} \frac{(1-t^2)^n}{(t+1+\delta)^{n-\alpha}} dt. \tag{5.5}$$

In Sect. 2 we obtained

$$\int_{-1}^{1} \frac{(1-t^2)^n}{(t+1+\delta)^{n-\alpha}} dt = (2r)^{n-\alpha} \frac{2^{n+1} n!}{(2n+1)!!} \Phi_{n,\alpha}(r^2)$$

with $\Phi_{n,\alpha}$ defined by (2.8), and

$$r = \frac{1}{1+\delta+\sqrt{(1+\delta)^2-1}} = \frac{1}{1+\delta+\sqrt{\delta(2+\delta)}},$$

thus (5.5) gives

$$|a_n| \leqq K C(\alpha) h^\alpha (2r)^{n-\alpha} \frac{n!}{(2n-1)!!} \Phi_{n,\alpha}(r^2)$$
$$= K C(\alpha) h^\alpha \left(\frac{(2n)!!}{(2n-1)!!}\right) r^{n-\alpha} \Phi_{n,\alpha}(r^2).$$

Now all estimates of $\Phi_{n,\alpha}$ obtained in Sect. 3 can be applied here, for example, we have

$$\Phi_{n,\alpha}(r^2) = (1-r^2)^{\alpha+\frac{1}{2}} + O\left(\frac{1}{n^\sigma}\right)$$

with $0<r<1$, $\sigma>0$, uniformly hold in r. Since we also have the asymptotic equality

$$\frac{(2n)!!}{(2n-1)!!} \cong \sqrt{\frac{\pi n}{2}}$$

it follows that

$$|a_n| \leqq K C(\alpha) h^\alpha \sqrt{n}\, r^{n-\alpha} (1-r^2)^{\alpha+\frac{1}{2}}.$$

At last, we obtain for $\varepsilon \leqq r \leqq 1-\varepsilon$ ($\varepsilon > 0$):

$$E_p[-1,1]^2 = \sum_{n=p+1}^{\infty} a_n^2 \frac{2}{2n+1}$$
$$\leqq \sum_{n=p+1}^{\infty} [K C(\alpha)]^2 h^{2\alpha} (1-r^2)^{2\alpha+1} r^{2(n-\alpha)}$$
$$= \left[K C(\alpha) h^\alpha \left(\frac{1-r^2}{2r}\right)^\alpha r^{p+1}\right]^2$$

and

$$E_p[a,b] = \sqrt{2h}\, E_p[-1,1]$$
$$\leqq K C(\alpha) h^{\alpha+\frac{1}{2}} \left(\frac{1-r^2}{2r}\right)^\alpha r^{p+1}. \quad \square$$

Remark 11. As in the case discussed in previous sections, if x_0 is an endpoint, then

$$E_p[a,b] \leqq C \frac{h^{\alpha+\frac{1}{2}}}{(p+1)^{2\alpha+1}}$$

and if x_0 is interior to $[a,b]$, then

$$E_p[a,b] \leqq C \left(\frac{h}{p+1}\right)^{\alpha+\frac{1}{2}}.$$

Finally, the case when x_0 is not real also gives an exponential rate of convergence:

$$E_p[a,b] \leqq C h^{\alpha+\frac{1}{2}} r^p.$$

Here r is determined as follows: let ρ_0 be the sum of semi-axes of the ellipse D_{ρ_0} which has the foci at ± 1 and passes through the point $\xi_0 = \frac{x_0 - c}{h}$, with $c = \frac{a+b}{2}$, $h = \frac{b-a}{2}$, then

$$\rho_0 = \tfrac{1}{2}\{|\xi_0 - 1| + |\xi_0 + 1| + \sqrt{(|\xi_0 - 1| + |\xi_0 + 1|)^2 - 4}\}$$

and $\frac{1}{\rho_0} < r < 1$. This result follows easily from Theorem 9.1.1, of [19], p. 245.

6. Error Analysis of p-Version of FEM for the Model Problem

Since the error of the finite element solution u of the model problem is exactly the error of the best L_2-approximation of u' by piecewise polynomials (see Sect. 1), the results of previous sections give the error analysis for the p version of FEM with only one element. One only needs to notice that $u'(x) = \alpha(x - \xi)_+^{\alpha-1} - \text{const.}$ so that the error estimates will be obtained by replacing α in the previous results by $\alpha - 1$, and p by $p-1$, and taking into account the change of length of the interval. Thus the following results follow easily from Theorems 5–9.

Theorem 12. *Let $E_p(\xi)$ be the error of the finite element solution of the model problem* (1.1) *when using only one element $I = [0,1]$ itself. (ξ is the position of the singularity.) Then for $\alpha > \frac{1}{2}$, one has*

1) *if $\xi = 0$, then*

$$E_p(0) = C_0(\alpha) \frac{1}{p^{2\alpha-1}} \left(1 + O\left(\frac{1}{p}\right)\right) \qquad (p \to \infty)$$

with $C_0(\alpha) = \frac{\alpha \Gamma(\alpha)^2 |\sin \pi \alpha|}{\pi \sqrt{2\alpha - 1}}$,

2) *if $\xi < 0$, then*

$$E_p(\xi) = C_1(\alpha) \left(\frac{1-r^2}{2r}\right)^{\alpha-1} \frac{r^p}{p^\alpha} \left(1 + O\left(\frac{1}{p^\sigma}\right)\right)$$

where $\sigma>0$, $r=\frac{\sqrt{1-\xi}-\sqrt{-\xi}}{\sqrt{1-\xi}+\sqrt{-\xi}}$, $C_1(\alpha)=\frac{\alpha\Gamma(\alpha)|\sin\pi\alpha|}{\sqrt{\pi}\,2^{\alpha-\frac{1}{2}}}$ *and the term* $O\left(\frac{1}{p^\sigma}\right)$ *is uniform with respect to* $\xi<-\varepsilon$, $(\varepsilon>0)$.

And we have the estimation:

if $0<r^2\leqq 1-\frac{1}{p}$, *then*

$$E_p(\xi)\approx\frac{1}{\sqrt{1-r^2}}\,\frac{r^{p+1-\alpha}}{p^\alpha}\left[\frac{1}{p^{\alpha-\frac{1}{2}}}+(1-r^2)^{\alpha-\frac{1}{2}}\right]$$

if $1-\frac{1}{p}\leqq r^2<1$, *then*

$$E_p(\xi)\approx\frac{r^{p+1-\alpha}}{p^{\alpha-\frac{1}{2}}}\left[\frac{1}{p^{\alpha-\frac{1}{2}}}+(1-r^2)^{\alpha-\frac{1}{2}}\right]$$

where equivalency constants depend only on α,

3) *if* $0<\xi<1$, *then there exists a constant* $C>0$ *depending only on* α *such that*

$$E_p(\xi)\leqq\begin{cases}C\left(\dfrac{\delta}{p}\right)^{\alpha-\frac{1}{2}} & 0<\theta\leqq\pi-\dfrac{1}{p}\\[2mm] C\left(\dfrac{1}{p}\right)^{2\alpha-1} & \pi-\dfrac{1}{p}\leqq\theta<\pi\end{cases}$$

where $\delta=\min(\theta,\pi-\theta)$, $\theta=\arccos(2\xi-1)$.

On the other hand, if $\varepsilon\leqq\theta\leqq\pi-\varepsilon$, $\varepsilon>0$, *then there is a constant* $C=C(\alpha,\varepsilon)>0$ *such that*

$$E_p(\xi)\geqq C\left(\frac{\delta}{p}\right)^{\alpha-\frac{1}{2}}. \quad \square$$

Consider now the p version with more than one intervals and with the singularity of the solution located in any of the nodal points of the mesh. As $p\to\infty$ the rate of convergence will be the same as in the case $\xi=0$, i.e., $\frac{1}{p^{2\alpha-1}}$. If ξ is in the interior of any mesh interval, then the rate of convergence for $p\to\infty$ will be the same as when $0<\xi<1$ and it is half in the exponent of the case above, i.e. $1/p^{\alpha-\frac{1}{2}}$. However, these rates of convergence appear only if p is large. For small p, the relation of the error in dependence on the number of total degrees of freedom of the finite element space has in general two phases. If the mesh is properly designed (in general, there is sufficiently strong refinment around the singularity), then the rate of error reduction will be exponential in the beginning phase. When p is large enough, this rate becomes algebraic. If the refinement of the mesh is not strong enough (for example, the uniform mesh), then the exponential part of the error reduction cannot even appear. If ξ lies outside of the interval then (also for one element mesh) only the exponential phase appears. Typically, when ξ lies inside the interval the graph of the error in dependence on N in double logarithmic scale is S-shaped (actually reflected S). The first part is the exponential phase and the second the algebraic one (see Fig. 3a, b). In practice we would like to achieve desired accuracy in the exponential phase. Using Sect. 5 the results can be extended to the general case of a function with a singularity of type x^{α}.

7. Numerical Results

In the previous section we have shown various estimates characterizing the error behavior of the p-version of FEM. Here we will numerically analyze the accuracy of the estimates, the range of asymptotic validity, etc. The error in this chapter is measured in the energy norm. As said in Sect. 6, the estimates are obtained from the estimates of the error in L_2-norm by replacing α by $\alpha-1$ and p by $p-1$.

In the tables, p is the polynomial degree of the approximation, and the error of the finite element solution (when only one element is used) is denoted by $E_p \equiv E_p(\alpha, \xi)$. Let $E_p^A = E_p^A(\alpha, \xi)$ be the error given by certain asymptotic formulae, $E_p^B = E_p^B(\alpha, \xi)$ be given by some estimates. We will compute the ratios

$$R_p^A = \frac{E_p}{E_p^A}, \quad R_p^B = \frac{E_p}{E_p^B}.$$

These ratios will be called *numerical constants* which reflect the quality of the estimates, the range of the asymptotic validity, etc.

In the case $\xi = 0$, the asymptotic formula is (cf. Theorem 12)

$$E_p^A = \frac{C_0(\alpha)}{p^{2\alpha-1}} \tag{7.1}$$

where

$$C_0(\alpha) = \frac{\alpha \Gamma(\alpha)^2 |\sin \pi \alpha|}{\pi \sqrt{2\alpha - 1}}$$

and we have

$$\lim_{p \to \infty} R_p^A = 1.$$

The numerical results are shown by Table 1 for $\alpha = 0.7$ and $\alpha = 3.5$. We can see that for a small α (the singularity is strong) formula (7.1) gives very good results even for small p. Thus in this case the asymptotic range is quite large. For α large (the singularity is weak or the function is "smoother"), the asymptotic range shift to large p, but for small p the accuracy of the asymptotic formula is still quite good. It is also seen that there is a big gain of the error reduction when p increases from 1 to 2, 3, especially for large α.

In the case $\xi < 0$, the error reduces exponentially. By Theorem 12 we have the following asymptotic formula:

$$E_p^A = C_1(\alpha) \left(\frac{1 - r^2}{2r} \right)^{\alpha - 1} \frac{r^p}{p^{\alpha}}, \tag{7.2}$$

where

$$C_1(\alpha) = \frac{\alpha \Gamma(\alpha) |\sin \pi \alpha|}{\sqrt{\pi}\, 2^{\alpha - \frac{1}{2}}}$$

Table 1. $\xi=0$

p	$\alpha=0.7$		$\alpha=3.5$	
	E_p	R_p^A	E_p	R_p^A
1	4.743*E*-1	0.9877	1.021	0.2032
2	3.627*E*-1	0.9967	3.402*E*-1	4.335
3	3.090*E*-1	0.9985	3.093*E*-2	4.488
4	2.756*E*-1	0.9992	2.379*E*-3	1.940
5	2.522*E*-1	0.9995	4.760*E*-4	1.480
6	2.344*E*-1	0.9996	1.400*E*-4	1.300
7	2.204*E*-1	0.9997	5.154*E*-5	1.208
8	2.090*E*-1	0.9998	2.210*E*-5	1.153
9	1.994*E*-1	0.9998	1.057*E*-5	1.118
10	1.912*E*-1	0.9999	5.495*E*-6	1.094
11	1.840*E*-1	0.9999	3.053*E*-6	1.077
12	1.777*E*-1	0.9999	1.790*E*-6	1.064
13	1.722*E*-1	1.000	8.999*E*-7	1.054
14	1.671*E*-1	1.000	6.978*E*-7	1.046
15	1.626*E*-1	1.000	4.585*E*-7	1.040

and

$$r=\frac{\sqrt{1-\xi}-\sqrt{-\xi}}{\sqrt{1-\xi}+\sqrt{-\xi}}$$

here

$$\lim_{p\to\infty} R_p^A=1.$$

In this case we also have the estimates

$$E_p^B=\begin{cases}\dfrac{C_2(\alpha)}{\sqrt{1-r^2}}\dfrac{r^{p+1-\alpha}}{p^\alpha}\left[\dfrac{D(\alpha)}{p^{\alpha-\frac{1}{2}}}+(1-r^2)^{\alpha-\frac{1}{2}}\right] & \text{if } 0<r^2\leqq 1-\dfrac{1}{p}\\ C_2(\alpha)\dfrac{r^{p+1-\alpha}}{p^{\alpha-\frac{1}{2}}}\left[\dfrac{D(\alpha)}{p^{\alpha-\frac{1}{2}}}+(1-r^2)^{\alpha-\frac{1}{2}}\right] & \text{if } 1-\dfrac{1}{p}\leqq r^2\leqq 1\end{cases} \tag{7.3}$$

with

$$C_2(\alpha)=C_1(\alpha)/2^{\alpha-1},$$

$$D(\alpha)=\frac{2^{2\alpha-1}\Gamma(\alpha)}{\sqrt{\pi}}.$$

Here, $D(\alpha)$ is the number obtained in proving Theorem 3 (see (3.12)). $C_2(\alpha)$ is chosen so that as $p\to\infty$, the first formula asymptotically agrees with (2), therefore we also have $\lim_{p\to\infty} R_p^B=1$. It is shown in Table 2(a)–2(c) that the asymptotic behavior is not too simple (it will be described later). Note the two parts of the formula E_p^B coincide at $r^2=1-\frac{1}{p}$. For $r=1$ (i.e. $\xi=0$), the formulae (7.2) and (7.3) differ. In fact, if we write (7.2) by $E_p^A(0)$, then

$$\frac{E_p^B(0)}{E_p^A(0)}=\sqrt{4\alpha-2}.$$

Table 2a. $\xi = -0.0005$ ($r = 0.9563$)

p	$\alpha = 0.7$			$\alpha = 3.5$		
	E_p	R_p^A	R_p^B	E_p	R_p^A	R_p^B
1	4.192*E*-1	0.4780	0.6334	1.022	3078.0	0.0549
2	2.945*E*-1	0.5705	0.5806	3.404*E*-1	12130.0	1.224
3	2.323*E*-1	0.6251	0.5440	3.091*E*-2	4761.0	1.324
4	1.928*E*-1	0.6636	0.5162	2.373*E*-3	1047.0	0.5972
5	1.647*E*-1	0.6929	0.4939	4.738*E*-4	476.9	0.4753
6	1.433*E*-1	0.7165	0.4752	1.390*E*-4	276.9	0.4352
7	1.268*E*-1	0.7359	0.4593	5.106*E*-5	182.4	0.4212
8	1.126*E*-1	0.7524	0.4453	2.180*E*-5	130.0	0.4187
9	1.010*E*-1	0.7666	0.4329	1.038*E*-5	97.80	0.4224
10	9.116*E*-2	0.7791	0.4218	5.373*E*-6	76.56	0.4297

Table 2b. $\xi = -0.05$ ($r = 0.6417$)

p	$\alpha = 0.7$			$\alpha = 3.5$		
	E_p	R_p^A	R_p^B	E_p	R_p^A	R_p^B
1	2.190*E*-1	0.7478	0.4700	1.153	15.40	0.0340
2	9.671*E*-2	0.8362	0.4321	3.580*E*-1	84.33	1.129
3	5.063*E*-2	0.8765	0.4706	2.912*E*-2	44.18	1.934
4	2.640*E*-2	0.9003	0.4962	1.937*E*-3	12.53	1.227
5	1.475*E*-2	0.9162	0.5150	3.254*E*-4	7.165	1.253
6	8.434*E*-3	0.9276	0.5298	7.844*E*-5	5.097	1.366
7	4.904*E*-3	0.9362	0.5417	2.322*E*-5	4.031	1.482
8	2.886*E*-3	0.9429	0.5518	7.853*E*-6	3.390	1.576
9	1.716*E*-3	0.9484	0.5604	2.920*E*-6	2.966	1.640
10	1.027*E*-3	0.9528	0.5678	1.165*E*-6	2.666	1.677

Table 2c. $\xi = -1$ ($r = 0.1716$)

p	$\alpha = 0.7$			$\alpha = 3.5$		
	E_p	R_p^A	R_p^B	E_p	R_p^A	R_p^B
1	3.729*E*-2	0.8224	0.4233	4.716	2.491	0.0191
2	4.302*E*-3	0.8982	0.4865	5.981*E*-1	20.83	1.197
3	5.744*E*-4	0.9284	0.5215	1.696*E*-2	14.23	2.428
4	8.196*E*-5	0.9447	0.5440	3.619*E*-4	4.842	1.587
5	1.216*E*-5	0.9550	0.5602	1.855*E*-5	3.160	1.542
6	1.851*E*-6	0.9620	0.5728	1.323*E*-6	2.486	1.546
7	2.865*E*-7	0.9671	0.5831	1.130*E*-7	2.127	1.539
8	4.495*E*-8	0.9710	0.5916	1.085*E*-8	1.896	1.509
9	7.124*E*-9	0.9738	0.5987	1.135*E*-9	1.746	1.480
10	1.124*E*-9	0.9639	0.5974	1.264*E*-10	1.638	1.448

Table 3a. $\alpha=0.7$

p	$\xi=0.05$		$\xi=0.005$		$\xi=0.0005$	
	E_p	R_p^B	E_p	R_p^B	E_p	R_p^B
1	5.191*E*-1	0.8518	4.791*E*-1	0.7862	4.748*E*-1	0.7792
2	4.804*E*-1	1.040	3.779*E*-1	0.8184	3.643*E*-1	0.7888
3	4.802*E*-1	1.159	3.373*E*-1	0.8589	3.120*E*-1	0.7946
4	4.726*E*-1	1.208	3.182*E*-1	0.9091	2.804*E*-1	0.8011
5	4.517*E*-1	1.208	3.091*E*-1	0.9658	2.589*E*-1	0.8089
6	4.239*E*-1	1.175	3.053*E*-1	1.026	2.435*E*-1	0.8182
7	3.987*E*-1	1.114	3.040*E*-1	1.087	2.319*E*-1	0.8288
8	3.835*E*-1	1.126	3.038*E*-1	1.118	2.230*E*-1	0.8407
9	3.786*E*-1	1.138	3.038*E*-1	1.145	2.161*E*-1	0.8539
10	3.784*E*-1	1.162	3.034*E*-1	1.168	2.106*E*-1	0.8681

p	$\xi=0.25$		$\xi=0.5$		$\xi=0.75$	
	E_p	R_p^B	E_p	R_p^B	E_p	R_p^B
1	6.508*E*-1	1.099	7.410*E*-1	1.216	7.483*E*-1	1.264
2	6.289*E*-1	1.220	5.973*E*-1	1.126	5.877*E*-1	1.140
3	5.399*E*-1	1.136	5.773*E*-1	1.180	5.372*E*-1	1.130
4	5.130*E*-1	1.143	5.271*E*-1	1.141	5.341*E*-1	1.190
5	5.082*E*-1	1.184	5.174*E*-1	1.171	4.929*E*-1	1.149
6	4.742*E*-1	1.146	4.884*E*-1	1.147	4.735*E*-1	1.144
7	4.613*E*-1	1.150	4.821*E*-1	1.168	4.721*E*-1	1.177
8	4.588*E*-1	1.175	4.622*E*-1	1.149	4.499*E*-1	1.152
9	4.386*E*-1	1.150	4.571*E*-1	1.166	4.384*E*-1	1.149
10	4.305*E*-1	1.152	4.426*E*-1	1.151	4.375*E*-1	1.171

The ratio R_p^B in this case is bounded above and below by constants which depends on α.

It can be seen in Table 2 that the asymptotic range of (7.2) depends on α and ξ. When α increases or ξ gets close to the approximation interval $[0,1]$, the asymptotics are shifter toward large p. If α is large and ξ is close to -1, E_p^A is large, while E_p^B gives good estimation in the sense that for large range of both α and ξ the ratio R_p^B is quite stable.

For the case $0<\xi<1$, we use the estimate:

$$E_p^B=\begin{cases}\left(\dfrac{\sqrt{1-\xi^2}}{p}\right)^{\alpha-\frac{1}{2}} & 0<\theta\leqq\pi-\dfrac{1}{p}\\ \dfrac{1}{p^{2\alpha-1}} & \pi-\dfrac{1}{p}\leqq\theta\leqq\pi\end{cases}\tag{7.4}$$

where $\theta=\arccos\xi$. In this case we cannot have an accurate asymptotic formula as for $\xi\leqq0$. The numerical constant $R_p^B=E_p/E_p^B$ no longer shows a monotonic behaviour. Tables 3(a, b) are made for $\alpha=0.7$ and 3.5 and ξ ranging from 0.0005 to 0.75. The numerical results show that the ratio R_p^B is stable (it is bounded above and below by constants which depend on α). For small α, the range of R_p^B is quite small (this means that E_p^B gives very good estimation). When α is large, this estimate is not so good but we still have the right rate of

Table 3b. $\alpha=3.5$

p	$\xi=0.05$		$\xi=0.005$		$\xi=0.0005$	
	E_p	R_p^B	E_p	R_p^B	E_p	R_p^B
1	8.960E-1	2.048	1.008	2.304	1.019	2.330
2	3.211E-1	46.97	3.384E-1	49.50	3.400E-1	49.74
3	3.334E-2	24.84	3.114E-2	51.88	3.095E-2	51.57
4	3.159E-3	5.580	2.436E-3	22.81	2.424E-3	22.33
5	8.295E-4	2.862	4.983E-4	17.80	4.778E-4	17.07
6	3.238E-4	1.930	1.508E-4	16.08	1.410E-4	15.03
7	1.501E-4	1.421	5.749E-5	15.46	5.211E-5	14.01
8	7.319E-5	1.034	2.566E-5	10.70	2.241E-5	13.43
9	3.567E-5	0.7170	1.287E-5	7.640	1.076E-5	13.07
10	1.984E-5	0.5474	7.070E-6	5.758	5.622E-6	12.85

p	$\xi=0.25$		$\xi=0.5$		$\xi=0.75$	
	E_p	R_p^B	E_p	R_p^B	E_p	R_p^B
1	4.795E-1	1.687	1.552E-1	0.3548	2.091E-2	0.0736
2	2.279E-1	6.414	9.953E-2	1.820	1.711E-2	0.4817
3	4.760E-2	4.525	4.289E-2	2.647	1.202E-2	1.142
4	5.351E-3	1.205	8.816E-3	1.290	6.987E-3	1.574
5	1.593E-3	0.7009	3.630E-3	1.037	3.097E-3	1.362
6	1.513E-3	1.150	1.897E-3	0.9364	9.555E-4	0.7263
7	8.019E-4	0.9679	1.128E-3	0.8843	6.864E-4	0.8283
8	3.633E-4	0.6546	7.293E-4	0.8534	5.736E-4	1.033
9	3.599E-4	0.9233	5.001E-4	0.8334	2.932E-4	0.7521
10	2.551E-4	0.8975	3.901E-4	0.8196	2.127E-4	0.7485

convergence. An interesting fact is that the error E_p^B for fixed p is nearly symmetric in ξ (about $\xi=\frac{1}{2}$), yet the function $(x-\xi)_+^\alpha$ is *not* symmetric. We also see that the error (for fixed p) has the maximum at $\xi=\frac{1}{2}$ (the middle of the interval). It tends to be smaller when ξ moves to the endpoint of the interval.

The graph of the error function $E_p(\alpha, \xi)$ for $\alpha=1.5$ is shown in Fig. 2. It is very clear that if the singularity is located outside of the interval of approximation, then the rate of convergence is remarkably increased. This fact is important in designing a right mesh for the FEM. It will be shown in the second part of the paper that a strong refinement around the singularity will greatly reduce the error for the same number of degrees of freedom.

So far we discussed the p-version with one element only ($m=1$). Let us now address briefly the case when the number of elements $m \geqq 2$. We assume that the solution is $u(x)=x^\alpha - x$ (i.e., the case $\xi=0$). Figures 3(a) and 3(b) show the cases when $\alpha=0.7$ and $\alpha=1.1$, respectively. In the figures, $m=2\sim 10$, and the meshes are made by geometric progression with a ratio $q=0.15$. The dotted curve is the error for the nearly optimal mesh-degree combination (cf. Part II). In Fig. 4 we compare the cases in p-version for uniform mesh and geometric mesh ($\alpha=0.7$).

It is clearly seen that the p-version has often two phases. The first phase (when p is small) has approximately an exponential rate. The second phase

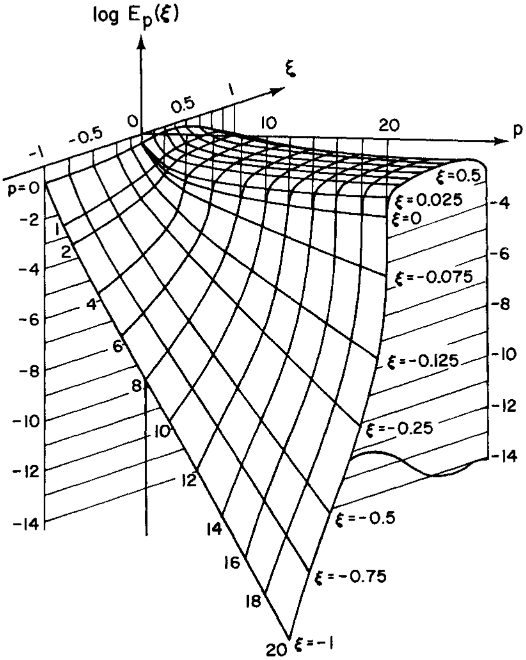

Fig. 2. p-version with one element ($\alpha = 1.5$)

(when p is large) tends to be algebraic. This figure will be called an S-curve (actually, it is a reflected S). For the geometric meshes, when having the same number of elements, if q is small, the range of exponential phase is enlarged. However, it does not mean that the smaller q is the better the result, since the curve is also shifted up. Roughly speaking, when α is small or the required accuracy is high, it is better to use smaller q. When α is large or required accuracy is low, large q is preferable. But the difference for $q = 0.1 \sim 0.3$ is not too large if the required accuracy is not too high (say $1\% \sim 5\%$).

Secondly, for a small (solution is highly unsmooth) one needs more intervals than in the case of α large (solution smooth). For example, when $\alpha = 0.7$, using $m = 2$ or $m = 5$ one even cannot obtain the accuracy of 5% in practice since the polynomial degree cannot go too high (say, $p = 10$). Figures 3(a, b) show all these curves lie above the curve for the optimal h-p extension (see also Part II). Likely there is an envelope for these p-version curves, and this envelope lies above the optimal h-p extension curve.

It is very clear that the mesh-design is crucial to achieve a good rate of convergence. If the singularity is present, then the uniform mesh with few elements is not acceptable. Figure 4 shows that the uniform mesh performs badly. There is no "exponential" phase at all and for $\alpha = 0.7$ it cannot get even 5% accuracy for hundreds of times many degrees of freedom comparing with the case when a geometric mesh ($q = 0.15$, $m = 20$) was used.

In practice (see [5, 7, 9]) one would like to achieve the required accuracy at the end of the "exponential" physe. It is more advantageous to overrefine mesh than to underrefine it. Since the p-version is using hierarchical elements [4], as long as a mesh is given the increase of N is not too expensive, while

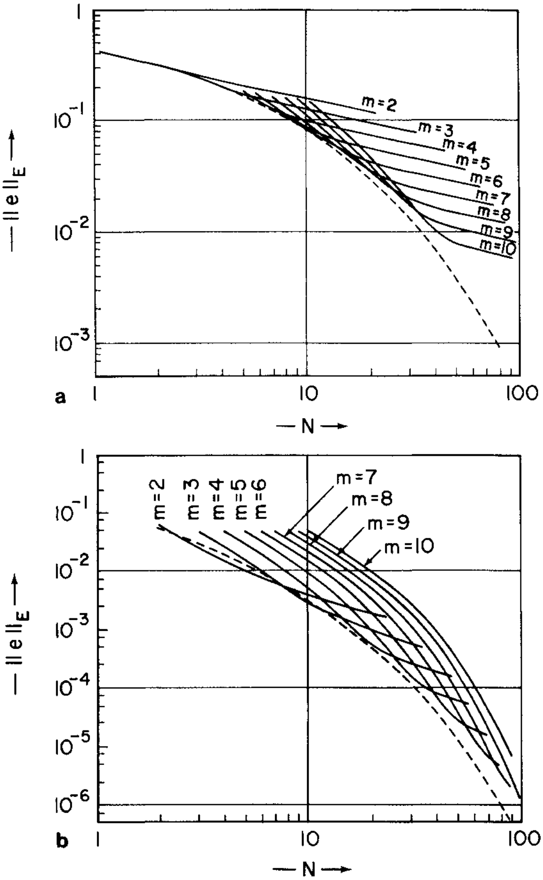

Fig. 3a. ($\alpha=0.7$) **b.** ($\alpha=1.1$)

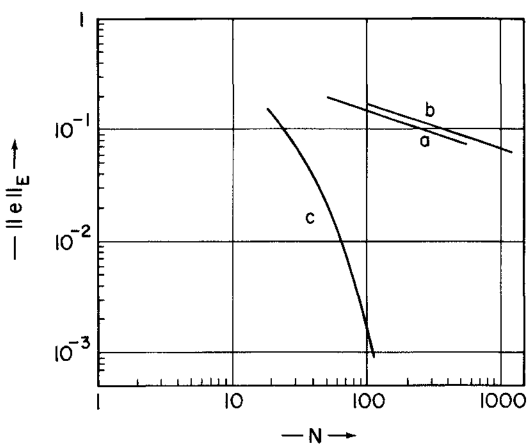

Fig. 4. ($\alpha=0.7$); *a*: uniform mesh ($m=100$); *b*: uniform mesh ($m=50$); *c*: geometric mesh ($m=20$, $q=0.15$)

for unsmooth solutions a strongly refined mesh is very important to get the desired accuracy.

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