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The Fixed Point Index of Fibre-Preserving Maps

Albrecht Dold (Heidelberg)

0. Introduction

For B a topological space, there is the category of *spaces over* B whose objects are maps $p: E \to B$ and whose morphisms *(maps over* B*)* are continuous maps $f: E_1 \to E_2$ such that $p_2 f = p_1$. We treat fixed point questions in this category. We make only mild assumptions on B (paracompact, locally compact), but more serious ones on $p: E \to B$ (namely ENR_B; cf. (1.1)); roughly speaking, we allow fibre bundles $p: E \to B$ with fibre a euclidean neighborhood retract, and more generally, $p|E'$: $E' \to B$, where E' is a fibrewise neighborhood retract of such a bundle $p: E \to B$.

We consider maps $f: V \to E$ over B, where V is an open subset of E $(pf = p|V)$, whose fixed point set, $\mathrm{Fix}(f) = \{v \in V \mid f(v) = v\}$, lies properly over B, i.e., $p|\mathrm{Fix}(f)$ is a proper map. Such an f is said to be *compactly fixed.* Two such maps are said to be equivalent, $f_0 \sim f_1$, if there is a compactly fixed map over $B \times [0, 1]$ whose parts over $B \times \{0\}$, $B \times \{1\}$ agree with f_0, f_1. Let FIX_B denote the set of equivalence classes $[f]$. Taking topological sums makes FIX_B into a commutative monoid, and our main result (4.3) asserts

$$\mathrm{FIX}_B \cong \pi^0_{\mathrm{stable}}(B \oplus pt),$$

the $0th$ *stable cohomotopy group* of $B \oplus a$ point.

This result has some interest also in the classical situation where B is a single point (hence E an ENR and $f: V \to E$ a map with compact fixed point set). Then $\pi^0_{\mathrm{stable}}(B \oplus pt) = \pi^0_{\mathrm{stable}}(\text{0-sphere}) = \mathbb{Z}$, and the passage to the equivalence class, $f \mapsto [f] \in \mathrm{FIX}_{pt} \cong \mathbb{Z}$, becomes the Lefschetz index. For general B again, we therefore call $[f] \in \mathrm{FIX}_B = \pi^0_{\mathrm{stable}}(B \oplus pt)$ the *index of* f, too. Going further still, if h is any multiplicative cohomology theory (with unit), we denote by $I(f) = I^h(f) \in h^0 B$ the image of $[f]$ under $\pi^*_{\mathrm{stable}}(B \oplus pt) \to hB$, and call this also the *index of* f (in h). We establish its main properties, mostly in analogy to the classical case $B = pt$.

The actual procedure in this paper is however in different order: After a preliminary section (§ 1) on the ENR_B-notion we give (in § 2) a direct definition of $I(f) \in h^0 B$ in more familiar terms, and we list its main properties (e.g., invariance under $\sim$). The next section (§ 3) essentially shows that I: $\mathrm{FIX}_B \to \pi^0_{\mathrm{stable}}(B \oplus pt)$ is surjective, and § 4 proves injectivity. The last section (§ 5) contains complementary remarks and problems.

1. Euclidean Neighborhood Retracts over B (ENR_B)

Let B denote a topological space. A continuous map p: $E \to B$ is called a *space over* B. If p: $E \to B$, p': $E' \to B$ are spaces over B then a continuous map f: $E \to E'$, such that $p'f = p$, is called a *map over* B. Spaces and maps over B form a category, $\mathfrak{Top}_B$, with an obvious homotopy structure (*homotopies over* B, or vertical homotopies). In particular, we have the notions of a subspace over B, a (neighborhood) retract over B, cofibration over B, and many other analogues from ordinary topology (compare [7, 8, 2], *et al.*). For every space Y the product $B \times Y$ is a space over B with respect to the first projection; this will always be understood in the following.

(1.1) *Definition.* A space over B, say p: $E \to B$, is called a *euclidean neighborhood retract*, abbreviated ENR_B, if there is a euclidean space $\mathbb{R}^n$, a continuous function τ: $B \times \mathbb{R}^n \to [0, 1]$, and maps $E \xrightarrow{i} \tau^{-1}(0, 1] \xrightarrow{r} E$ over B such that $ri = \mathrm{id}_E$. In other words, if (up to homeomorphism over B) E is a fibrewise retract of a numerically[1] defined open subset D of some $B \times \mathbb{R}^n$.

These ENR_B's are the objects on which we shall study fixed point theory in the following sections. In the present section we list some elementary properties of ENR_B's, mostly without proofs, or with indications only. In order to simplify the formulations we assume throughout the paper that B is *paracompact.*

(1.2) **Proposition.** *If* p: $E \to B$ *is* ENR_B *then* E *is paracompact.* – Indeed $B \times \mathbb{R}^n \times \mathbb{R}$ is paracompact because B is. The set $\tau^{-1}(0, 1]$ of (1.1) embeds as a closed set into $B \times \mathbb{R}^n \times \mathbb{R}$ via $(b, y) \mapsto (b, y, 1/\tau(b, y))$, and E is a closed set in $\tau^{-1}(0, 1]$ via i. □

(1.3) **Proposition.** $B \times \mathbb{R}^n$ *is* ENR_B. *Every numerically open*[2] *subset of an* ENR_B *is* ENR_B. *Every closed neighborhood retract over* B *of an* ENR_B *is* ENR_B. □

(1.4) **Proposition.** *If* $E \to B$ *is* ENR_B *and* $B' \to B$ *is a continuous map then* $E \times_B B' \to B'$ *is* $\mathrm{ENR}_{B'}$. *If* p: $E \to B$ *is* ENR_B *and* q: $F \to C$ *is* ENR_C *then* $p \times q$: $E \times F \to B \times C$ *is* $\mathrm{ENR}_{B \times C}$. *If* $E \to B$ *and* $E' \to B$ *are* ENR_B *then so is* $E \times_B E' \to B$. □

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[1] For many purposes, D could be an *arbitrary open subset* of $B \times \mathbb{R}^n$. If B is metric then every open subset of $B \times \mathbb{R}^n$ is of the form $\tau^{-1}(0, 1]$. – If B is a single point then $\mathrm{ENR}_B = \mathrm{ENR}$ (cf. [4], IV.8).

[2] i.e. of the form $\tau^{-1}(0, 1]$ for some τ, as in (1.1).

The definition of a *cofibration over B* uses the same words as the usual definition with all maps taken over B, and so does the characterisation of cofibrations (over B) as being "neighborhood deformation retracts"; compare [11, 2], *et al.* In particular, we get the following

(1.5) **Proposition.** *If* $p\colon E \to B$ *is* ENR_B *and* D *is a closed subset of* E, *with inclusion map* $i\colon D \to E$, *then* i *is a cofibration over* B *if and only if* $pi\colon D \to B$ *is* ENR_B. □

The property of being a cofibration is a (numerably-)local one (cf. [3]). The same result holds over B; combined with (1.5) it gives the following

(1.6) **Proposition.** *If* D *is a closed subset of* $B \times \mathbb{R}^n$ *and if every point of* D *has a neighborhood in* D *which is* ENR_B *then* D *itself is* ENR_B. □

(1.7) **Proposition.** *If* $\pi\colon D \to B$ *is a continuous map and* D *is a finite union of open subsets,* $D = \bigcup_{\nu=1}^{k} D_\nu$ *such that every* $\pi_\nu = \pi | D_\nu$ *is* ENR_B *then* π *itself is* ENR_B.

This becomes a special case of (1.6) if we can embed D over B as a closed subset into some $B \times \mathbb{R}^n$. But every D_ν embeds (cf. (1.2)), and if we shrink the covering $\{D_\nu\}$ a little we can assume that the embedding of D_ν extends to a map $i_\nu\colon D_\nu \to B \times \mathbb{R}^{n_\nu}$ over B. Then

$$\{i_\nu\}_B\colon\ D \to B \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_k}$$

embeds D. □

(1.8) **Corollary.** *If* $\pi\colon D \to B$ *is a locally trivial fibration of finite type whose fibre is an (ordinary)* ENR *then* π *is* ENR_B.

Indeed, the assumption gives $D = \bigcup_{\nu=1}^{k} D_\nu$, where $D_\nu = B_\nu \times F$, B_ν is a numerically open set in B, and F is an ENR. □

Conversely, if $p\colon E \to B$ is ENR_B then every $p^{-1}b$ (where $b \in B$) must, of course, be an ENR. Also,

(1.9) **Proposition.** *If* $p\colon E \to B$ *is a proper map and is* ENR_B *then* p *is a Hurewicz-fibration.*

Proof. By Definition (1.1), we have $E \subset B \times \mathbb{R}^n$, and a vertical neighborhood retraction $r\colon D \to E$. For every $y \in E$ there are open sets $V_y \subset B$ and $W_y \subset \mathbb{R}^n$ such that $y \in (V_y \times W_y) \subset D$. Pick a point $b \in B$; as $p^{-1}b$ is compact, finitely many of the sets $V_y \times W_y$ ($y \in p^{-1}b$) cover $p^{-1}b$. If V is the (finite) intersection resp. W the union of the corresponding V_y resp. W_y then $p^{-1}b \subset (V \times W) \subset D$. Since a vertical retract of a fibration is itself a fibration, we see that $p | r(V \times W)\colon r(V \times W) \to V$ is a fibration. If we can show that $r(V \times W) \supset p^{-1}U$ for some neighborhood U of b, $U \subset V$, then we know that $p|p^{-1}U\colon p^{-1}U\to U$ is also a fibration, hence p is locally (with respect to B) a fibration, and therefore also globally. Now, the set $Y=E\cap(V\times W)$ is clearly open in E, and $(p^{-1}b)\subset Y\subset r(V\times W)$. Since p is proper it is closed; hence $p(E-Y)$ is closed in B, and $b\notin p(E-Y)$. Therefore $U=B-p(E-Y)$ is an open neighborhood of b, and

$$p^{-1}U=(E-p^{-1}p(E-Y))\subset Y\subset r(V\times W). \quad \square$$

A general ENR_B, $p\colon E\to B$, need not be a fibration (any numerically open subset E of $B\times\mathbb{R}^n$ is ENR_B). However, *locally* (with respect to E) *it always is a fibration*, in the following sense.

(1.10) **Proposition.** *If $p\colon E\to B$ is ENR_B then for every point $y\in E$ there are open neighborhoods Y of y in E and $U=p(Y)$ of $p(y)$ in B such that $p|Y\colon Y\to U$ is a Hurewicz-fibration.*

This is just about what the first part of the proof of (1.9) shows: Every $y\in E$ has a neighborhood in E which is a vertical retract of some $V_y\times W_y$, and vertical retracts of fibrations are fibrations. $\square$

One can now ask which local (w.r.t.E) fibrations $E\to B$ are ENR_B. I did not much pursue this question; the following is a partial answer.

(1.11) **Proposition.** *If $p\colon E\to B$ is locally (w.r.t.E) a fibration and if E and B are (ordinary)* ENR *then p is* ENR_B.

Sketch of Proof. Embed $j\colon E\to B\times E$, $j(y)=(p(y),y)$. Since E and $B\times E$ are ENR this is a cofibration and therefore admits a neighborhood retraction $r\colon W\to E$, $rj=\mathrm{id}$, where W is an open neighborhood of jE in $B\times E$. The two maps $\pi\colon W\subset B\times E\xrightarrow{\mathrm{proj}}B$ and $pr\colon W\to B$ may not coincide, but they agree on jE. Since B is ENR they are homotopic rel. jE in a neighborhood of jE, which we denote by the same letter W. If p is (globally) a Hurewicz-fibration then we can lift the homotopy (rel. jE), with initial position r. The final position of the lifted homotopy is a new retraction $r'\colon W\to E$ with $pr'=\pi$, i.e. a retraction over B. Therefore, p is ENR_B by (1.3).—In general, p is only locally a fibration, but then we can apply (1.6). $\square$

(1.12) **Corollary.** *Every submersion*[3] *$p\colon E\to B$ of C^1-manifolds with countable bases is* ENR_B—because it is locally a (trivial) fibration, by the implicit function theorem. $\square$

2. The (Fixed Point) Index-Homomorphism

(2.1) As before, we consider euclidean neighborhood retracts over B (ENR_B), say $p\colon E\to B$. We assume B to be *locally compact and paracompact*, or equivalently, a topological sum, $B=\bigoplus_{\lambda\in\Lambda}B_\lambda$, each summand

[3] $=C^1$-map whose rank equals $\dim(B)$ at every point of E.

of which is a countable union $B_\lambda = \bigcup_{j=1}^{\infty} B_\lambda^j$ of compact spaces B_λ^j. We consider continuous maps $f\colon V \to E$ such that V is an open subset of E, and $pf = p|V$. Such a map (over B) is said to be *compactly fixed* if $p|\mathrm{Fix}(f)\colon \mathrm{Fix}(f) \to B$ is a *proper* map, where $\mathrm{Fix}(f) = \{v \in V | f(v) = v\}$. I.e., f is compactly fixed if $(p^{-1} K) \cap (\mathrm{Fix}(f))$ is compact for every compact $K \subset B$. In close analogy to [4], VII.5 we shall now define an index-homomorphism $I_f\colon hB \to hB$, where h is any general cohomology theory. If h is a *multiplicative* cohomology theory and $1 \in h^0 B$ the unit element then $I(f) = I_f(1) \in h^0 B$ is also called the *index* of f; in this case, we'll find that $I_f(x) = I(f) \smile x$ for $x \in hB$.

(2.2) As in [4] or [9] we first treat the case $E = B \times \mathbb{R}^n$, p = projection. If $K \subset B$ is compact then $(\mathrm{Fix}(f)) \cap (K \times \mathbb{R}^n)$ is compact and therefore contained in a set $K \times D_K$, where $D_K \subset \mathbb{R}^n$ is a closed ball around the origin. If B is a countable union of compact sets this easily yields a positive continuous function $\rho\colon B \to (0, +\infty) \subset \mathbb{R}$ such that

$$\mathrm{Fix}(f) \subset \{(b, y) \in B \times \mathbb{R}^n \,|\, \|y\| \le \rho(b)\} = E_\rho .$$

The same is true for general $B = \bigoplus B_\lambda$ because we can choose $\rho | B_\lambda$ for each λ independently. Clearly $(E - E_\rho)$ is a weak deformation retract of

$$(B \times \mathbb{R}^n, B \times (\mathbb{R}^n - 0)) = B \times (\mathbb{R}^n, \mathbb{R}^n - 0),$$

hence

$$h(E, E - E_\rho) \cong h(B \times (\mathbb{R}^n, \mathbb{R}^n - 0)).$$

We also remark that $\mathrm{Fix}(f)$ is closed *in* E (and not only in V) because this is locally so, namely in every $K \times \mathbb{R}^n$, where $K \subset B$ is compact. Therefore $(E, V, E - \mathrm{Fix}(f))$ is excisive, i.e. $h(E, E - \mathrm{Fix}(f)) \cong h(V, V - \mathrm{Fix}(f))$. Using these isomorphisms we now define the *index homomorphism* by the following composition

$$\begin{aligned} I_f\colon h_j B &\overset{\sigma^n}{\cong} h^{j+n}(B \times (\mathbb{R}^n, \mathbb{R}^n - 0)) \xrightarrow{(\iota - f)^*} h^{j+n}(V, V - \mathrm{Fix}(f)) \\ &\cong h^{j+n}(E, E - \mathrm{Fix}(f)) \to h^{j+n}(E, E - E_\rho) \\ &\cong h^{j+n}(B \times (\mathbb{R}^n, \mathbb{R}^n - 0)) \overset{\sigma^n}{\cong} h^j B, \end{aligned} \tag{2.3}$$

where σ^n is the n-fold suspension isomorphism, and $(\iota - f)\colon V \to B \times \mathbb{R}^n$, $(\iota - f)(b, y) = (b, y - \varphi(b, y))$, with φ the second component of f; $f(b, y) = (b, \varphi(b, y))$.

We now list some properties of I_f. Proofs are only given if they are sufficiently nontrivial, sufficiently different from corresponding proofs in [4], VII.5, and used in the sequel of this paper.

(2.4) **Localization in E.** *If W is an open subset of V such that* $\mathrm{Fix}(f) \subset W$ *then $f|W$ is compactly fixed, and $I_{f|W} = I_f$.* □

(2.5) **Localisation in *B*.** *The index homomorphism I_f factors as follows:*

$$hB \dashrightarrow h(B, B') \to hB, \quad \text{where } B' = B - p(\mathrm{Fix}(f)).$$

In particular, the composition of I_f with $hB \to hB'$ vanishes.

Sketch of Proof. Let $E' = p^{-1}B'$. Then we can modify the defining sequence of maps (2.3) from the 4th term on as follows:

$$h^{j+n}(E, E - \mathrm{Fix}(f)) \to h^{j+n}(E, (E - E_\rho) \cup E') \cong h^{j+n}((B, B') \times (\mathbb{R}^n, \mathbb{R}^n - 0))$$
$$\overset{\sigma^n}{\cong} h^j(B, B') \to h^j B. \quad \square$$

(2.6) **Units.** *If s: $B \to E$ is a section of p then the index homomorphism of sp: $E \to E$ agrees with the identity map of hB.* $\square$

(2.7) **Additivity.** *If $V = V_1 \cup V_2$, a union of open sets, such that $f_{12} = f | V_1 \cap V_2$ (and f) is compactly fixed then $f_1 = f | V_1, f_2 = f | V_2$ are compactly fixed, and $I_f + I_{f_{12}} = I_{f_1} + I_{f_2}$.* $\square$

The next property (2.8) of the index homomorphism concerns its behavior under a change of bases, i.e. a continuous map β: $B' \to B$, where B' is also locally compact and paracompact. We have a commutative diagram

$$\begin{array}{ccccc} V' \subset & E' & \xrightarrow{p'} & B' \\ \downarrow{\scriptstyle \beta_V} & \downarrow{\scriptstyle \beta_E} & & \downarrow{\scriptstyle \beta} \\ V \subset & E & \xrightarrow{p} & B \end{array}$$

where $E' = B' \times_B E = \{(b', y) \in B' \times E | \beta(b') = p(y)\}$, $V' = B' \times_B V = \beta_E^{-1}(V)$, and p', β_E, β_V are the projections.

(2.8) **Naturality in *B*.** *Let*

$$f' \colon V' \to E', \; f'(b', y) = (b', f(y)); \quad \text{i.e. } \beta_E f' = f \beta_V, \; p' f' = p' | V'.$$

Then f' is also compactly fixed, and

$$\beta^* \circ I_f = I_{f'} \circ \beta^*.$$

This follows by writing the defining sequence (2.3) for I_f under the defining sequence for $I_{f'}$, and inserting appropriate vertical arrows $\beta_?^*$ everywhere. $\square$

(2.9) **Homotopy Invariance.** *If g: $W \to F$ is a compactly fixed map over $B \times [0, 1]$, with projection $F \to B \times [0, 1]$, and if g_t: $W_t \to F_t$ is the part of g over $B \times \{t\} = B$, then $I_{g_t} = I_{g_0}$ for all $t \in [0, 1]$.*

Indeed, the part of g over $B \times \{t\} = B$, namely g_t, is obtained from g by the change of basis i_t: $B \to B \times [0, 1]$, $i_t(b) = (b, t)$, in the sense of (2.8).

Therefore, $i_t^* I_g = I_{g_t} i_t^*$. But i_t^* is isomorphic, and $i_t^* = i_0^*$, hence $I_{g_t} = i_0^* I_g (i_0^*)^{-1} = I_{g_0}$. □

(2.10) **Multiplicativity.** *If $f\colon V \to E$ and $f'\colon V' \to E$ are compactly fixed maps over B then so is $f \times_B f'\colon V \times_B V' \to E \times_B E'$, and*

$$I_{f \times_B f'} = I_f \circ I_{f'} = I_{f'} \circ I_f.$$

This follows from (2.14). We shall only need the following special case.

(2.11) **Stability.** *If $e\colon \mathbb{R} \to \mathbb{R}$ is constant (say $e(t)=0$) then $f \times e\colon V \times \mathbb{R} \to E \times \mathbb{R}$ (as a map over B with respect to $E \times \mathbb{R} \to E \xrightarrow{p} B$) is compactly fixed, and $I_{f \times e} = I_f$.*

This follows by writing the defining sequence (2.3) for I_f under the defining sequence for $I_{f \times e}$ and inserting vertical arrows σ everywhere except at the ends hB. Remark that $(\iota \times \mathrm{id}_{\mathbb{R}} - f \times e) = (\iota - f) \times \mathrm{id}_{\mathbb{R}}$. □

(2.12) **Commutativity.** *Assume $p\colon E \to B$, $p'\colon E' \to B$ are ENR_B, let $U \subset E$, $U' \subset E'$ open subsets, and $f\colon U \to E'$, $g\colon U' \to E$ continuous maps over B. If $gf\colon f^{-1} U' \to E$ is compactly fixed then so is $fg\colon g^{-1} U \to E'$, and $I_{fg} = I_{gf}$.*

Using 2.11, the proof is as for [4], VII, 5.9. □

(2.13) **Naturality in h.** *If $T\colon k \to h$ is a natural transformation of cohomology theories then $I_f^h T = T I_f^k$, where I_f^h, I_f^k denote the respective index homomorphisms.* □

(2.14) **The Index Element** (Proposition and Definition). *If h is a multiplicative cohomology theory then $I_f\colon hB \to hB$ is a homomorphism of hB-modules, i.e.*

$$I_f(x) = I_f(1) \smile x, \quad \textit{for all } x \in hB,$$

where $1 \in h^0 B$ is the unit element.

The element $I_f(1) \in h^0 B$ (which completely describes I_f) is called the *index of f*, and is denoted by $I(f)$, or $I^h(f)$ when appropriate.

Proof. In the sequence (2.3) which defines I_f all homomorphisms except σ^n are induced by maps over B, and are therefore hB-homomorphisms. The suspension isomorphism σ of a multiplicative cohomology theory has the form $\pm\sigma = - \times s =$ cross-product with $s = \sigma(1)$, and $\pm\sigma^n = - \times s \times s \ldots \times s = - \times s^n$; compare 3.1. But

$$x \times s^n = (x \times 1) \smile (1 \times s^n) = (p^* x) \smile (1 \times s^n),$$

hence $\sigma^n = - \times s^n$ is also an hB-homomorphism. □

So far we have developed the index theory only for the special ENR_B-spaces $B \times \mathbb{R}^n$. But using commutativity (2.12) we can extend it to the general case as in [4] VII, 5.10. I.e.,

(2.15) **Proposition and Definition.** *If $p: E \to B$ is any ENR_B, and $V \subset E$ is an open subset then every continuous map $f: V \to E$ over B admits a decomposition $f: V \xrightarrow{\alpha} U \xrightarrow{\beta} E$ over B, where U is open in some $B \times \mathbb{R}^n$. If f is compactly fixed then so is $g = \alpha\beta: \beta^{-1} V \to B \times \mathbb{R}^n$, hence $I_g: hB \to hB$ is defined. Moreover, I_g depends only on f, not on the decomposition $f = \beta\alpha$.*

By definition, $I_f = I_{\alpha\beta}$ is the *index-homomorphism of f* (resp. $I(f) = I(\alpha\beta) \in h^0 B$ the *index of f*, if h is multiplicative). *All the properties* (2.4)–(2.14), *formulated above for the special cases, continue to hold for the general case.* □

3. The Image of I in $h^0 B$

If h is a multiplicative cohomology theory, what is the image of $f \mapsto I(f)$; i.e., which elements in $h^0 B$ occur as indices of compactly fixed maps over B? We'll see that *precisely the stably spherical ones do.* We begin be recalling the term "stably spherical" and related notions.

(3.1) Let $s \in h^1(\mathbb{R}, \mathbb{R}-0) \cong \tilde{h}^1 S^1$ denote the image of $1 \in h^0(pt)$ under the isomorphisms $h^0(pt) \cong \tilde{h}^0(\mathbb{R}-0) \overset{\delta}{\cong} h^1(\mathbb{R}, \mathbb{R}-0)$, where $pt = a$ point. Then the general suspension isomorphism σ for h is given by multiplication with s, and its n-th iterate σ^n is given by multiplication with s^n, where $s^n = \sigma^n(1)$ is the image of $1 \in h^0(pt)$ under

$$h^0(pt) \overset{\sigma^n}{\cong} h^n(\mathbb{R}^n, \mathbb{R}^n - 0) \cong \tilde{h}^n(S^n).$$

This takes various forms, e.g.

$$\sigma^n = (- \times s^n)\colon\ h^0 B \cong h^n(B \times (\mathbb{R}^n, \mathbb{R}^n - 0)) = h^n(B \times \mathbb{R}^n, B \times (\mathbb{R}^n - 0)),$$

$$\sigma^n = (- \times s^n)\colon\ h^0 B \cong h^n(B \times S^n, B \times pt) = \tilde{h}^n(B \times S^n / B \times pt),$$

$$\sigma^n = (- \times s^n)\colon\ \tilde{h}^0 B \cong h^n(B \times S^n, B \vee S^n) = \tilde{h}^n(\Sigma^n B),$$

$$(\sigma^n, p^*)\colon\ h^0 B \oplus h^n B \cong h^n(B \times S^n),$$

where $p: B \times S^n \to B$ denotes projection.

(3.2) **Proposition and Definition.** *For $y \in h^0 B = \tilde{h}^0(B \oplus pt)$ the following properties are equivalent.*

(a) *There is a map $\psi_a: \Sigma^n(B \oplus pt) \to S^n$, for some n, such that $\psi_a^* s^n = \sigma^n(y)$.*

(b) *There is a map $\psi_b: (B \times S^n, B \times pt) \to (S^n, pt)$, for some n, such that $\psi_b^*(s^n) = \sigma^n(y)$.*

(c) *There is a map* $\psi_c\colon B\times S^n\to S^n$, *for some n, such that*

$$(\psi_c^*(s^n)-\sigma^n(y))\in im(p^*).$$

(d) *There is a map* $\psi_d\colon (B\times \mathbb{R}^n, B\times(\mathbb{R}^n-0))\to(\mathbb{R}^n,\mathbb{R}^n-0)$, *for some n, such that* $\psi_d^*(s^n)=\sigma^n(y)$.

If one, and therefore all of these properties hold then y is said to be *stably spherical*. – The proof is easy, and is left to the reader. □

(3.3) Another way of describing stably spherical classes is via *stable cohomotopy groups* $\pi^j_{\text{stable}}X=\varinjlim \pi(\Sigma^{k-j}X,S^k)$, where π on the right denotes homotopy classes, and $\Sigma=$ suspension. These constitute a reduced cohomology theory (cf. [6], p. 10), i.e. the groups $\gamma^j X=\pi^j_{\text{stable}}(X\oplus pt)$, and corresponding relative groups, form a cohomology theory γ, and $\tilde\gamma^j=\pi^j_{\text{stable}}$. The suspension isomorphism $\sigma\colon \tilde\gamma^j X\cong\tilde\gamma^{j+1}\Sigma X$ agrees with the obvious isomorphism $\pi^j_{\text{stable}}X\cong\pi^{j+1}_{\text{stable}}\Sigma X$. This cohomology theory is multiplicative, with unit element $1\in\gamma^0(pt)=\pi^0_{\text{stable}}(S^0)$ represented by the identity map of S^0. Similarly, $s^n=\sigma^n(1)\in\tilde\gamma^n S^n=\pi^n_{\text{stable}}S^n$ is represented by the identity map of S^n. The cohomology theory γ has the following

(3.4) **Universal Property.** *If h is any cohomology theory, and* $a\in h^0(pt)$, *then there is a unique transformation of cohomology theories* $\hat a\colon\gamma\to h$ *such that* $\hat a\colon\gamma(pt)\to h(pt)$ *takes* 1 *into* a.

In particular, if h is a multiplicative cohomology theory then there is a unique (multiplicative) transformation of cohomology theories $\varepsilon\colon\gamma\to h$ *such that* $\varepsilon(1)=1$. *In other words,* γ *is* the initial object *in the category of multiplicative cohomology theories.*

This is fairly obvious since γ is represented by spheres, with universal elements $s^n=\sigma^n(1)$. Explicitely, if $x\in\tilde\gamma^j X=\pi^j_{\text{stable}}X$ has the representative $\xi\colon\Sigma^{n-j}X\to S^n$ then $\hat a(x)=\sigma^{j-n}\xi^*\sigma^n(a)\in\tilde h^j X$. In particular, if h is multiplicative then $\varepsilon(x)=\sigma^{j-n}\xi^*(s^n)$. This together with (3.2)(a) shows

(3.5) **Proposition.** *If h is a multiplicative cohomology theory then* $y\in h^0 B=\tilde h^0(B\oplus pt)$ *is stably spherical if and only if y is in the image of* $\varepsilon\colon\gamma^0 B\to h^0 B$. □

The following lemma is used in the proofs of (3.7) and (4.8), but is also of independent interest.

(3.6) **Lemma.** *Let* $f\colon V\to E$ *be a compactly fixed map over B, as in* (2.1). *Then there exists a neighborhood* $W\subset V$ *of* Fix(f) *and a compactly fixed map* $f'\colon E\times\mathbb{R}\to E\times\mathbb{R}$ *over B such that* Fix$(f')=$Fix$(f)\times\{0\}$, *and* $f'(w,t)=(f(w),0)$ *for* $w\in W$, $t\in\mathbb{R}$. *In particular,* f' *is a* globally defined *compactly fixed map with the same index as* f (cf. (2.11) and (2.4)).

Proof. Assume first $E=B\times\mathbb{R}^n$, with projection maps $p\colon E\to B$, $q\colon E\to\mathbb{R}^n$. Put $\varphi=qf\colon V\to\mathbb{R}^n$. Then Fix$(f)=\{y\in E|\varphi(y)=q(y)\}$, and we have to construct W and $\varphi'\colon E\times\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}$ such that

$$\{\varphi'(y,t)=(q(y),t)\}\Leftrightarrow\{\varphi(y)=q(y)\ \text{and}\ t=0\},$$

and $\varphi'(w,t)=(\varphi(w),0)$ for $w\in W$. Choose a continuous function $\tau\colon E\to[0,1]$ such that $\tau^{-1}(0)$ is a neighborhood of $E-V$ and $W=\tau^{-1}(1)$ is a neighborhood of $\mathrm{Fix}(f)$. Define $\varphi'\colon E\times\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}$ as follows.

$$\varphi'(y,t)=\begin{cases}[q(y)-\tau(y)\,(q(y)-\varphi(y)),\ t+1-\tau(y)\,(t+1)] & \text{if } y\in V,\\ [q(y),\ t+1] & \text{if } y\in\tau^{-1}(0).\end{cases}$$

For $y\in W=\tau^{-1}(1)$ we have $\varphi'(y,t)=[\varphi(y),0]$, as required. If $\varphi'(y,t)=[q(y),t]$ then comparing second components shows $\tau(y)\neq 0$, hence $y\in V$; therefore comparing first components gives $\tau(y)\,(q(y)-\varphi(y))=0$, hence $q(y)=\varphi(y)$, hence $y\in\mathrm{Fix}(f)$ and $\tau(y)=1$; comparing second components again gives $t=0$. Altogether, $\varphi'(y,t)=[q(y),t]\Rightarrow q(y)=\varphi(y)$ and $t=0$. The converse is clear.

This takes care of the special case $E=B\times\mathbb{R}^n$, which is all we need for the proofs of (3.7) and (4.8). The general case is reduced to the special case by retraction arguments as in (2.15); we omit the details. □

(3.7) **Theorem.** *Let h a multiplicative cohomology theory, and B a locally compact paracompact space. The elements of $h^0 B$ which occur as indices of compactly fixed maps over B are precisely the stable spherical ones; i.e. the image of I coincides with the image of* $\varepsilon\colon \pi^0_{\text{stable}}(B\oplus pt)\to h^0 B$.

Proof. Let $x=I(f)\in h^0 B$; we have to show that x is stably spherical[4]. By the very definition (2.15) of $I(f)$ we can assume that f is of the form $f\colon V\to B\times\mathbb{R}^n$, where V is open in $E=B\times\mathbb{R}^n$. By Lemma (3.6) we can assume that f is globally defined, i.e. $f\colon B\times\mathbb{R}^n\to B\times\mathbb{R}^n$. Consider then the following diagram

$$(3.8)\qquad \begin{array}{ccccccc}
h^0 B\cong h^n(B\times(\mathbb{R}^n,\mathbb{R}^n-0)) & \cong & h^n(E,E-E_\rho) & \xleftarrow{(\iota-f)^*} & h^n(B\times(\mathbb{R}^n,\mathbb{R}^n-0)) & \cong & h^0 B\\
\| & & \nwarrow{\scriptstyle\psi^*} & & \nearrow{\scriptstyle q^*} & & \uparrow\\
h^0 B\cong h^n(B\times(\mathbb{R}^n,\mathbb{R}^n-0)) & & \xleftarrow[\psi^*]{} & h^n(\mathbb{R}^n,\mathbb{R}^n-0) & & \cong & h^0(p,t),
\end{array}$$

where E_ρ and $(\iota-f)$ are as in (2.2) and (2.3), q denotes the second projection and $\psi\colon(B\times\mathbb{R}^n,B\times(\mathbb{R}^n-0))\to(\mathbb{R}^n,\mathbb{R}^n-0)$ is defined as follows.

$$\psi(b,z)=\begin{cases} z-qf(b,z)=(q(\iota-f))\,(b,z) & \text{for } \|z\|\geqq\rho(b),\\ \dfrac{\|z\|}{\rho(b)}\,(q(\iota-f))\left(b,\rho(b)\dfrac{z}{\|z\|}\right) & \text{for } \|z\|\leqq\rho(b).\end{cases}$$

---

[4] As the referee points out, this is immediate from (2.13) and (3.5). I've retained the proof because it is instructive and the diagram (3.8) is also needed for the converse.

In words, ψ agrees with $q(\iota-f)$ on the exterior of the tube E_ρ, and radially extends $q(\iota-f)|\dot{E}_\rho$ from the boundary $\dot{E}_\rho$ to the interior of E_ρ. The two maps $q(\iota-f)$, ψ: $(E, E-E_\rho)\to(\mathbb{R}^n, \mathbb{R}^n-0)$ are homotopic rel. $(E-E_\rho)$, by linear deformation. Therefore, $(\iota-f)^* q^*=\psi^*$; furthermore, $\psi(B\times(\mathbb{R}^n-0))\subset(\mathbb{R}^n-0)$. This explains the diagram (3.8), and shows that it is commutative. Following $1\in h^0$ along the upper row gives $I(f)=x$, by Definition (2.3), (2.14) of $I(f)$. Following 1 along the lower row then shows that x is spherical, by (3.2) (d).

Assume now $x\in h^0 B$ is spherical. Then $\sigma^n(x)=\psi^*(s^n)$ for some n and some map ψ: $B\times(\mathbb{R}^n, \mathbb{R}^n-0)\to(\mathbb{R}^n, \mathbb{R}^n-0)$. Define f: $B\times\mathbb{R}^n\to B\times\mathbb{R}^n$ by $f(b,z)=(b, z-\psi(b,z))$. Then $\mathrm{Fix}(f)=\{(b,z)|\psi(b,z)=0\}=\psi^{-1}(0)$, and this is contained in $B\times\{0\}$, hence f is compactly fixed. The diagram (3.8) above now shows $x=I(f)$ (using $\sigma^n(x)=\psi^*(s^n)$, and the definition of $I(f)$). □

4. Homotopy Invariance as the Fundamental Property of I; the Monoid FIX_B

If h is the cohomology theory defined by stable cohomotopy groups then, trivially, every class $y\in h^0 B$ is stably spherical (cf. (3.5)) hence $f\mapsto I(f)$ is surjective by (3.7). We now show that it is also *injective up to homotopy*, i.e. if $I(f_0)=I(f_1)$ then f_0, f_1 are equivalent in the sense of (2.9). In order to emphasize the cobordism character of this result we introduce the "monoid of fixed point situations" FIX_B, as follows.

(4.1) *Definition.* If B is a locally compact paracompact space let $\mathfrak{F}_B$ denote the set of (fibrewise homeomorphism classes of) compactly fixed maps f: $V\to E$ over B, in the sense of (2.1). Two elements of $\mathfrak{F}_B$, say f_0: $V_0\to E_0$ and f_1: $V_1\to E_1$, are called *equivalent*, in symbols $f_0\sim f_1$, if a compactly fixed g: $W\to F$ over $B\times[0,1]$ exists, as in (2.9), such that $g_0=f_0$, $g_1=f_1$. This is an equivalence relation in $\mathfrak{F}_B$ (transitivity by glueing); the equivalence class of f is denoted by $[f]$, and the set of equivalence classes by $\mathrm{FIX}_B=\mathfrak{F}_B/\sim\,=\{[f]\}$.

If f_1: $V_1\to E_1$, $f_2\to E_2$ are in $\mathfrak{F}_B$ then so is the topological sum $f_1\oplus f_2$: $V_1\oplus V_2\to E_1\oplus E_2$. Addition $\oplus$ is compatible with $\sim$, and therefore induces an addition $+$ in FIX_B, namely $[f_1]+[f_2]=[f_1\oplus f_2]$. This turns FIX_B into a commutative associative monoid with neutral element $[\emptyset]$. If h is a multiplicative cohomology theory then the index (2.15) defines a map I: $\mathfrak{F}_B\to h^0 B$. This map is compatible with $\sim$ by (2.9), and is additive by (2.7). Therefore *I defines a homomorphism* (denoted by the same letter)

$$(4.2)\qquad I\colon \mathrm{FIX}_B\to h^0 B,\qquad I[f]=I(f),$$

and the main result of §§3–4 is as follows.

(4.3) **Theorem.** *If h is the cohomology theory defined by stable cohomotopy groups* (cf. (3.3)) *then* (4.2) *is isomorphic, i.e.,*

$$I: \mathrm{FIX}_B \cong \pi^0_{\mathrm{stable}}(B \oplus pt).$$

As remarked above, we know already (3.7) that I is surjective. We now prepare the proof of injectivity by some preliminary considerations.

(4.4) **Lemma.** *If $f_0: V_0 \to E_0, f_1: V_1 \to E_1$ are compactly fixed maps over B such that E_0 is a numerically open*[2] *part of E_1, $\mathrm{Fix}(f_1) \subset V_0 \subset V_1$, and $f_0(v) = f_1(v)$ for $v \in V_0$, then $f_0 \sim f_1$.*

Proof. $F = E_0 \times [0,1] \cup E_1 \times (0,1]$ is an $\mathrm{ENR}_{B \times [0,1]}$,

$$W = V_0 \times [0,1] \cup V_1 \times (0,1]$$

is an open subset of F, and $g = (f_0 \times \mathrm{id}, f_1 \times \mathrm{id}): W \to F$ is a compactly fixed map over $B \times [0,1]$ whose parts over $B \times \{0\}$ resp. $B \times \{1\}$ are f_0 resp. f_1. □

(4.5) Assume now $E \overset{i}{\subset} D \xrightarrow{q} B$ are ENR_B, and E is a fibrewise retract of D, i.e. there exists $r: D \to E$ such that $ri = \mathrm{id}$, $qir = q$. Then i has the (fibrewise) homotopy extension property over B (compare [2], Chapter I). Therefore, the mapping cylinder

$$Z = \{(y,t) \in D \times [0,1] \mid y \in E \text{ or } t = 0\}$$

is a retract over B of $D \times [0,1]$. In particular, $Z \to B$, $(y,t) \mapsto q(y)$, is an ENR_B (cf. (1.3)).—In the following, we'll sometimes identify D with $\{(y,t) \in Z \mid t = 0\}$, and E with $\{(y,t) \in Z \mid t = 1\}$.

(4.6) **Lemma.** *With the notation of* (4.5), *if $g: W \to Z$ is a compactly fixed map such that $g(W) \subset D = \{(y,t) \in Z \mid t = 0\}$ resp. $g(W) \subset E = \{y,t) \in Z \mid t = 1\}$ then $g \sim (g_0 = g \mid W \cap D: W \cap D \to D)$ resp. $g \sim (g_1 = g \mid W \cap E: W \cap E \to E)$.*

Proof. If $g(W) \subset D$ consider the space

$$F = \{(y,t,\tau) \in D \times [0,1] \times [0,1] \mid (y,t) \in Z \text{ and } t \leqq \tau\}.$$

The map

$$\rho: Z \times [0,1] \to F, \quad \rho(y,t,\tau) = (y, \mathrm{Min}(t,\tau), \tau),$$

is a retraction which is fibrewise with respect to $(y,t,\tau) \mapsto (q(y), \tau)$. *With* respect to this projection F is therefore $\mathrm{ENR}_{B \times [0,1]}$. Furthermore, $(W \times [0,1]) \cap F$ is an open subset of F, and

$$\gamma: (W \times [0,1]) \cap F \to F, \quad \gamma(y,t,\tau) = (g(y,t), \tau)$$

is a compactly fixed map whose parts over $\tau = 0$ and $\tau = 1$ agree with g_0 and g, hence $g_0 \sim g$.

If $g(W) \subset E$ we replace F by

$$\{(y, t, \tau) \in D \times [0, 1] \times [0, 1] \mid (y, t) \in Z \text{ and } t \geqq \tau\},$$

and proceed as before. □

(4.7) **Lemma.** *As in (4.5), let $i: E \subset D$ be ENR_B and $r: D \to E$ a retraction over B; $r\,i = \mathrm{id}$, $q\,i\,r = q$. If $f: V \to E$ is a compactly fixed map over B then so is $f' = i f r: r^{-1} V \to D$, and $f' \sim f$.*

Proof. Clearly $\mathrm{Fix}(f') = \mathrm{Fix}(f)$, so f' is compactly fixed. Consider the mapping cylinder $Z \subset D \times [0, 1]$ over B, as in (4.5), and the retraction $s: Z \to D$, $s(y, t) = y$. Define

$$(s^{-1} r^{-1} V) \times [0, 1] \to Z \times [0, 1], \quad \text{by } (y, t, \tau) \mapsto (f'(y), \tau, \tau).$$

This is clearly a compactly fixed map over $B \times [0, 1]$. Its parts over $\tau = 0$ $\tau = 1$ are equivalent with f' resp. f, by Lemma (4.6). Hence $f' \sim f$. □

(4.8) **Proposition.** *Every element ξ of FIX_B has a representative of the form $f: B \times \mathbb{R}^n \to B \times \mathbb{R}^n$ such that*

(i) $\|y - \varphi(b, y)\| = \|y\|$,

(ii) $\varphi(b, \lambda y) = \lambda \varphi(b, y)$

for all $b \in B$, $y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^+ = [0, +\infty)$, where φ is the second component of f, i.e. $f(b, y) = (b, \varphi(b, y))$. – Note in particular that (i) implies $\mathrm{Fix}(f) = B \times \{0\}$.

Proof. If E is an ENR_B then, by definition E is a retract over B of a numerically open[2] subset D of some $B \times \mathbb{R}^m$, say $E \xrightarrow{i} D \xrightarrow{r} E$. Every compactly fixed map $g: V \to E$ is equivalent, by (4.7), to $g' = i g r: r^{-1} V \to D$, and this (by (4.4)) is equivalent to the composition $r^{-1} V \xrightarrow{g'} D \subset B \times \mathbb{R}^m$, which we still denote by g'. By (3.6), there is an open neighborhood $W \subset r^{-1} V$ of $\mathrm{Fix}(g')$ and a compactly fixed $g'': B \times \mathbb{R}^{m+1} \to B \times \mathbb{R}^{m+1}$, such that $\mathrm{Fix}(g'') = \mathrm{Fix}(g') \times \{0\}$ and $g''(w, t) = (g'(w), 0)$ for $w \in W$. The former implies (by (4.4)) that $g'' \sim g'' | W \times \mathbb{R}$, the latter implies (4.7) that $g'' | W \times \mathbb{R} \sim g' | W$, and this (by (4.4) again) is $\sim$ to g'.

So far we've shown that every $\xi \in \mathrm{FIX}_B$ has a representative g'' of the form $B \times \mathbb{R}^n \to B \times \mathbb{R}^n$; it remains to achieve (i) and (ii). Recall (2.2) that there is a continuous positive function. $\rho: B \to (0, +\infty)$ such that $\mathrm{Fix}(g'') \subset E_\rho$; in other words, all fixed points in $\{b\} \times \mathbb{R}^n$ have norm smaller than $\rho(b)$. Under the homeomorphism $(b, y) \mapsto (b, 2\rho(b)\, y)$ the map g'' transforms into a new map, denoted by $g: B \times \mathbb{R}^n \to B \times \mathbb{R}^n$ again, all of whose fixed points have norm smaller than 1; in fact, $\mathrm{Fix}(g) \subset E_{\frac{1}{2}}$.

Now we proceed somewhat as after (3.8). We let $\psi: B \times \mathbb{R}^n \to \mathbb{R}^n$ the second component of g, thus $g(b, y) = (b, \psi(b, y))$. Then

$$\mathrm{Fix}(g) = \{(b, y) \mid \psi(b, y) = y\};$$

and $\psi(b, y)=y \Rightarrow \|y\|<1$. We define a map G over $B \times [0, 1]$ as follows.

$$G: B \times [0, 1] \times \mathbb{R}^n \to B \times [0, 1] \times \mathbb{R}^n, \qquad G(b, t, y)=(b, t, \Psi(b, t, y)),$$

where $\Psi: B \times [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ is given by $\Psi(b, t, y)=$

$$\begin{cases} (1-t)[y-e_t(y)+\psi(b, e_t(y))]+t[y-\|y\|\, e_1(e_t(y)-\psi(b, e_t(y)))] & \text{for } \|y\| \geqq 1; \\ (1-t)\psi(b, y)+t[y-\|y\|\, e_1(e_t(y)-\psi(b, e_t(y)))] & \text{for } \|y\| \leqq 1, \end{cases}$$

and $e_t(x)$ stands for $(1-t)x+t\dfrac{x}{\|x\|}$; note that $y=0$ implies $\Psi(b, t, y)=(1-t)\psi(b, y)$, and $\|y\|=1 \Rightarrow e_t(y)=y$. We have

$$\operatorname{Fix}(G)=\{(b, t, y) \mid \Psi(b, t, y)=y\}.$$

If $(b, t, y) \in \operatorname{Fix}(G)$ and $\|y\| \geqq 1$ then the definition of Ψ shows

$$(1-t)[\psi(b, e_t(y))-e_t(y)]+t\,\|y\|\, e_1[\psi(b, e_t(y))-e_t(y)]=0;$$

since $[\psi(b, e_t(y))-e_t(y)] \neq 0$ for $\|y\| \geqq 1$, this means

$$(1-t)\,\|\psi(b, e_t(y))-e_t(y)\|+t\,\|y\|=0,$$

which is impossible. Thus $\operatorname{Fix}(G) \subset \{(b, t, y) \mid \|y\| \leqq 1\}$, hence G is compactly fixed. The part of G over $t=0$ agrees with g $(G_0=g)$, the part of G over $t=1$, say $G_1=f$, is given by $f(b, y)=(b, \varphi(b, y))$ with

$$\varphi(b, y)=y-\|y\|\, e_1[e_1(y)-\psi(b, e_1(y))].$$

This clearly satisfies (i) and (ii), and $f \sim g$ represents ξ. □

Proof of Theorem 4.3. It suffices to show injectivity of I. Let $\xi, \eta \in \mathrm{FIX}_B$ such that $I\xi=I\eta$. Choose representatives $f: B \times \mathbb{R}^m \to B \times \mathbb{R}^m$, $g: B \times \mathbb{R}^n \to B \times \mathbb{R}^n$ of ξ, η as in (4.8); in particular, $I(f)=I(g)$. But the index $I(f) \in \pi^0_{\text{stable}}(B \oplus pt)$ of a map $f: B \times \mathbb{R}^m \to B \times \mathbb{R}^m$ as in (4.8) (i.e. with properties (i), (ii)) is precisely the stable homotopy class defined by $\varphi_{\divideontimes}: B \times \mathbb{R}^m \to \mathbb{R}^m$, or rather by $\varphi^S_{\divideontimes}=\varphi_{\divideontimes} \mid B \times S^{m-1}: B \times S^{m-1} \to S^{m-1}$, where φ is the second component of f $(f(b, y)=(b, \varphi(b, y)))$, and $\varphi_{\divideontimes}(b, y)=y-\varphi(b, y)$; this is clear from the definition of the suspension isomorphism in π^0_{stable} and the definition of I. Therefore the maps $\varphi^S_{\divideontimes}: B \times S^{m-1} \to S^{m-1}$, $\psi^S_{\divideontimes}: B \times S^{n-1} \to S^{n-1}$, corresponding to f, g, must represent the same element of $\pi^0_{\text{stable}}(B \oplus pt)$, i.e. for some μ, $\nu>0$ such that $m+\nu=n+\mu$ the maps

$$(\varphi^S_{\divideontimes}) *_B \mathrm{id}: B \times S^{m-1+\nu} \to S^{m-1+\nu},$$

$$(\psi^S_{\divideontimes}) *_B \mathrm{id}: B \times S^{n-1+\mu} \to S^{n-1+\mu}$$

are homotopic, where $*_B$ denotes the (fibrewise) join with $S^{\nu-1}$ resp. $S^{\mu-1}$. By radially extending these maps from spheres to all of euclidean space we can also say that

$$\varphi_{*} \times \mathrm{id}\colon\ B \times \mathbb{R}^{m+\nu} \to \mathbb{R}^{m+\nu} = \mathbb{R}^m \times \mathbb{R}^\nu$$

is homotopic to

$$\psi_{*} \times \mathrm{id}\colon\ B \times \mathbb{R}^{n+\mu} \to \mathbb{R}^{n+\mu} = \mathbb{R}^n \times \mathbb{R}^\mu$$

by a homotopy

$$\theta\colon\ B \times [0,1] \times \mathbb{R}^k \to \mathbb{R}^k, \qquad (k = m+\nu = n+\mu)$$

such that $\|\theta(b,t,y)\| = \|y\|$ (and $\theta(b,0,y) = (\varphi_{*} \times \mathrm{id})(b,y)$, $\theta(b,1,y) = (\psi_{*} \times \mathrm{id})(b,y)$); in particular, $\theta(b,t,y) = 0 \Leftrightarrow y = 0$. Define

$$D\colon\ B \times [0,1] \times \mathbb{R}^k \to B \times [0,1] \times \mathbb{R}^k, \qquad D(b,t,y) = \big(b,t,y - \theta(b,t,y)\big).$$

Then D is a compactly fixed map over $B \times [0,1]$ (in fact, $\mathrm{Fix}(D) = B \times [0,1] \times \{0\}$) whose parts over $B \times \{0\}$ resp. $B \times \{1\}$ agree with

$$f \times \{0\}\colon\ (B \times \mathbb{R}^m) \times \mathbb{R}^\nu \to (B \times \mathbb{R}^m) \times \mathbb{R}^\nu$$

resp.

$$g \times \{0\}\colon\ (B \times \mathbb{R}^n) \times \mathbb{R}^\mu \to (B \times \mathbb{R}^n) \times \mathbb{R}^\mu,$$

where $\{0\}$ denotes the constant map zero. Therefore $f \times \{0\} \sim g \times \{0\}$. But $f \times \{0\} \sim f$ and $g \times \{0\} \sim g$ by (4.7), hence $\xi = \eta$. □

5. Remarks and Problems

(5.1) **Axiomatic Characterisation of the Index.** This is just what (4.3) provides: *If J is a function which assigns to every compactly fixed $f\colon V \to E$ over B an element $J(f)$, invariant under homotopy* (2.9), *then J is determined by its values on a set of representatives for the elements of* $\pi^0_{\mathrm{stable}}(B \oplus pt)$; more accurately, by its values on maps

$$f\colon\ B \times \mathbb{R}^n \to B \times \mathbb{R}^n, \qquad f(b,z) = \big(b, z - \psi(b,z)\big),$$

as at the very end of Section 3 (or as in (4.8)), where ψ ranges over a set of representatives for $\pi^0_{\mathrm{stable}}(B \oplus pt)$.

Any such J is also invariant under localisation (2.4), is commutative in the sense of (2.12), etc. In other words, *homotopy-invariance alone (plus suitable normalisation on the special f above) suffices to characterise the index; all the other properties are consequences.* If one is willing to assume additivity (2.7) as an axiom then one needs normalization only on a set of special f representing *generators* for $\pi^0_{\mathrm{stable}}(B \oplus pt)$.

Even in the classical case where $B = pt$ and $\pi^0_{\mathrm{stable}}(B \oplus pt) = \mathbb{Z}$ this result is of interest: It says that the classical Lefschetz fixed point index for ENRs is characterised by homotopy-invariance alone together with normalization. For normalization one can take the power maps $z \mapsto z^n$ of the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, putting $J(z \mapsto z^n) = 1 - n$. Or if one assumes additivity then it suffices to normalise on the identity map of a point. Note however that "homotopy-invariance" has to be understood in the sense of (2.9) (or of [1]), and not just in the more usual sense of topology (compare also [5]).

(5.2) The problem of *calculating the index* or of proving a *Lefschetz-Hopf type theorem* is very intriguing. The Lefschetz-Hopf-theorem should express $I(f)$ in terms of $f^*: hE \to hE$ provided $f: E \to E$ is a *globally defined* (compactly fixed) map over B and $p: E \to B$ is *proper* (or at least $\mathrm{im}(f) \to B$ is proper). I don't know whether there is such a theorem in general, i.e. whether $I(f)$ is determined by f^* (if f is globally defined and p is proper). Certainly one would have to use the module structure of hE over hB and the fact that f^* is a homomorphism of hB-modules. One is led to try the Lefschetz-trace $\Lambda f^* \in h^0 B$ of the hB-endomorphism f^*. I convinced myself that it does coincide with $I(f)$ for some simple examples of ENR_Bs, such as (5.3) below. But in general Λf^* may not even be defined (if hE is a complicated hB-module); when it is defined I still don't know whether it always agrees with $I(f)$.

If $h = H$ is ordinary cohomology then, of course, Λ does give the right answer but the result is not new because then

$$I(f) \in H^0(B; \mathbb{Z}) = \mathrm{Hom}(H_0 B, \mathbb{Z})$$

is the classical index of $p^{-1} b \to p^{-1} b$, viewed as an integral valued function of $b \in B$. In this context Knill's Theorem 1 in [9] should be mentioned; it deals with ordinary (co-)homology only but it provides a computable invariant (under ordinary homotopy, not $\sim$) which is richer than $I(f) \in H^0(B; \mathbb{Z})$.

(5.3) *Example.* Let $B = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, and consider the map

$$f: B \times S^1 \to B \times S^1, \quad f(b, z) = (b, b \cdot z).$$

This is a globally defined compactly fixed map over B, and $p: B \times S^1 \to B$, $p(b, z) = b$, is proper. I have verified geometrically (and not without pains) that $[f] \in \mathrm{FIX}_B$ is the non-zero element of $\pi^0_{\text{stable}} S^1 = \mathbb{Z}_2$. On the other hand, for any cohomology theory h the hB-module $h(B \times S^1)$ has the basis $\{1 \otimes 1, 1 \otimes s\}$, and

$$f^*(1 \otimes 1) = 1 \otimes 1, \quad f^*(1 \otimes s) = (1 + \bar{v} \cdot s) \cdot (1 \otimes s) + s \cdot (1 \oplus 1),$$

where $\bar{v} \in h^{-1}(\text{point})$ corresponds to $v^*(s^2) \in \tilde{h}^2 S^3$ under 3-fold suspension, and $v: S^3 \to S^2$ is the Hopf-map. (In order to understand the term $\bar{v} \cdot s$ one suspends f and calculates $(\Sigma f)^*$, remembering the Hopf-construction.)

It follows that the trace of f^* equals $\Lambda(f)=(\bar{v}\cdot s)\in\tilde{h}^0 S^1$; in particular, if h is stable cohomotopy then $\Lambda(f)=[v]\in\pi^0_{\text{stable}} S^1$.

(5.4) The example above raises other questions, too. Suppose, for instance, G is a Lie group operating on a compact manifold M. Then

$$G\times M\to G\times M, \qquad (g,z)\mapsto(g,g\cdot z),$$

is a compactly fixed map over G; it represents (by (4.3)) an element of $\pi^0_{\text{stable}}(G\oplus pt)$, resp. of $\pi^0_{\text{stable}} G$ if $\chi M=0$. Are these interesting elements, in particular if $M=G$ and the operation is by left translation?

(5.5) As stable cohomotopy is isomorphic, via the Pontrjagin-Thom construction, to *stably framed cobordism* Ω^*_{fr}, we also get from Theorem (4.3) an isomorphism of FIX_B with $\Omega^0_{fr}B$. It should be easy to describe this isomorphism directly (without cohomotopy) by geometric constructions in the spirit of Quillen's geometric description of complex cobordism (cf. [10], Section 1). In fact, the very definition (4.2) of FIX_B and some of the arguments in § 4 are clearly cobordism style. The question arises whether fixed point theory or cobordism might profit from this connection.

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A. Dold
Mathematisches Institut der Universität
D-6900 Heidelberg 1
Im Neuenheimer Feld 288
Federal Republic of Germany

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