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Two Families of Mixed Finite Elements for Second Order Elliptic Problems

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Summary. Two families of mixed finite elements, one based on triangles and the other on rectangles, are introduced as alternatives to the usual Raviart-Thomas-Nedelec spaces. Error estimates in $L^2(\Omega)$ and $H^{-s}(\Omega)$ are derived for these elements. A hybrid version of the mixed method is also considered, and some superconvergence phenomena are discussed.

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1. Introduction

The object of this paper is to introduce two families of space of mixed finite elements for second order elliptic problems. Over triangular decompositions of the domain these spaces will lie between corresponding Raviart-Thomas spaces [12, 14], will be of smaller dimension than the Raviart-Thomas space of the same index, and will provide asymptotic error estimates for the vector variable (i.e., the one for which mixed methods are designed to approximate well) of the same order as the corresponding Raviart-Thomas space. In addition, if the Fraeijs de Veubeke [8, 9] relaxation of the continuity of the normal component of the vector variable across interelement edges is introduced and an extension [1] of the Lagrange multiplier enforcing this continuity is made, then in all cases except for the lowest degree space in our family, the resulting superconvergent approximation of the scalar variable is asymptotically of the same order as that for the similarly modified version of the Raviart-Thomas method.

Our rectangular elements differ considerably from those of Raviart and Thomas, in that our vector elements are based on augmenting the space of vector polynomials of *total* degree k by exactly two additional vectors in place of augmenting the space of vector *tensor-products* of polynomials of degree k by $2k+2$ polynomials of higher degree. We also use a lower dimensional space for the scalar variable. The improved behavior of our elements over the Raviart-

Thomas elements listed above for triangular elements is valid for the rectangular elements, as well.

In the remainder of the Introduction the triangular elements will be described and a more precise summary of our results in this case will be presented. The rectangular case will be presented in Sect. 5.

Consider the Dirichlet problem (vectors will be denoted by bold face)

$$Lu = -\operatorname{div}(a\,\mathbf{grad}\,u) = f, \qquad \mathbf{x}\in\Omega, \tag{1.1a}$$

$$u = -g, \qquad \mathbf{x}\in\partial\Omega, \tag{1.1b}$$

where Ω is a bounded domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$ and $a=a(\mathbf{x})$ is a positive, smooth function on the closure of Ω. (In all that follows, L can be replaced by $Lu = -\operatorname{div}(a_0\,\mathbf{grad}\,u + \mathbf{a}_1 u) + \mathbf{a}_2\cdot\mathbf{grad}\,u + a_3 u$, provided that the corresponding Dirichlet problem is solvable; see [5, 6] for such an extension for the Raviart-Thomas method.) Let

$$c = c(\mathbf{x}) = a(\mathbf{x})^{-1}, \tag{1.2a}$$

$$\mathbf{q} = -a\,\mathbf{grad}\,u. \tag{1.2b}$$

Then, as usual for mixed methods, the operator L can be factored to give the first order system

$$c\mathbf{q} + \mathbf{grad}\,u = 0, \qquad \mathbf{x}\in\Omega, \tag{1.3a}$$

$$\operatorname{div}\mathbf{q} = f, \qquad \mathbf{x}\in\Omega, \tag{1.3b}$$

Let $\mathbf{V} = H(\operatorname{div};\Omega) = \{\mathbf{v}\in\mathbf{L}^2(\Omega) \mid \operatorname{div}\mathbf{v}\in L^2(\Omega)\}$ and $W = L^2(\Omega)$. The weak form of (1.3) appropriate for the mixed method is given by seeking $\{\mathbf{q}, u\}\in\mathbf{V}\times W$ such that

$$(c\mathbf{q}, \mathbf{v}) - (\operatorname{div}\mathbf{v}, u) = \langle g, \mathbf{v}\cdot\mathbf{n}\rangle, \qquad \mathbf{v}\in\mathbf{V}, \tag{1.4a}$$

$$(\operatorname{div}\mathbf{q}, w) = (f, w), \qquad w\in W, \tag{1.4b}$$

where $(\cdot,\cdot)$ indicates the inner product in $L^2(\Omega)$ or $\mathbf{L}^2(\Omega)$ and $\langle\cdot,\cdot\rangle$ that in $L^2(\partial\Omega)$ and $\mathbf{n}$ is the outer normal to $\partial\Omega$.

Let $\mathscr{T}_h = \{T\}$ be a triangulation of Ω such that

i) if $T\subset\Omega$, T has straight edges,
ii) if T is a boundary triangle, the boundary edge can be curved,
iii) all vertex angles exceed some $\theta_0 > 0$,
iv) if $T_1\cap T_2 \neq \emptyset$, then $T_1\cap T_2$ is either a vertex or a full edge of each,
v) $\operatorname{diam}(T) = h_T$, $\max_T h_T = h$.

We construct our triangular finite element spaces as follows. First, let k be a positive integer and let $P_k(T)$ denote the restriction of the set of all polynomials of total degree not greater than k to T. Then, set

$$\mathbf{V}(T) = \mathbf{V}^k(T) = \mathbf{P}_k(T), \tag{1.5a}$$

$$W(T) = W^{k-1}(T) = P_{k-1}(T), \tag{1.5b}$$

$$\mathscr{M}(T) = \mathscr{M}^k(T) = \mathbf{V}^k(T)\times W^{k-1}(T). \tag{1.5c}$$

Then, let

$$\mathbf{V}_h = V_h^k = \mathbf{V}(k, \mathcal{T}_h) = \{\mathbf{v} \in \mathbf{V} \mid \mathbf{v}|_T \in \mathcal{V}^k(T),\ T \in \mathcal{T}_h\} \tag{1.6a}$$

$$W_h = W_h^{k-1} = W(k-1, \mathcal{T}_h) = \{w \mid w|_T \in W^{k-1}(T),\ T \in \mathcal{T}_h\}, \tag{1.6b}$$

$$\mathcal{M}_h = \mathcal{M}_h^k = \mathcal{M}(k, \mathcal{T}_h) = \mathbf{V}_h^k \times W_h^{k-1}. \tag{1.6c}$$

Recall that, in (1.6a), $\mathbf{v} \in \mathbf{V}$ if and only if $\mathbf{v} \cdot \mathbf{n}_e$ is continuous across interior edges e. The space $\mathcal{M}_h^k$ over $\mathcal{T}_h$ will be our family of mixed finite element spaces over triangles, and we seek $\{q_h, u_h\} \in \mathcal{M}_h^k$ such that

$$(c\mathbf{q}_h, \mathbf{v}) - (\operatorname{div} \mathbf{v}, u_h) = \langle g, \mathbf{v} \cdot \mathbf{n} \rangle, \qquad \mathbf{v} \in \mathbf{V}_h^k, \tag{1.7a}$$

$$(\operatorname{div} \mathbf{q}_h, w) = (f, w), \qquad w \in W_h^{k-1}. \tag{1.7b}$$

Let us compare $\mathcal{M}_h^k$ with the corresponding Raviart-Thomas space $\mathcal{RT}_h^k$, which is constructed as follows. Let k be a nonnegative integer, and let [12, 14]

$$\mathbf{R}^k(T) = \mathbf{P}_k(T) \oplus \mathbf{x} P_k(T), \tag{1.8a}$$

$$S^k(T) = P_k(T); \tag{1.8b}$$

then form $\mathcal{RT}_h^k$ in the same manner as was used above for $\mathcal{M}_h^k$. Then,

$$\mathcal{RT}_h^{k-1} \subset \mathcal{M}_h^k \subset \mathcal{RT}_h^k, \qquad k \geqq 1. \tag{1.9}$$

Note that

$$\dim \mathcal{M}^k(T) = \tfrac{1}{2}(3k^2 + 7k + 4), \tag{1.10a}$$

$$\dim \mathbf{R}^k(T) \times S^k(T) = \dim \mathcal{M}^k(T) + 2k + 2; \tag{1.10b}$$

thus, the dimension of the space $\mathcal{RT}_h^k$ is significantly larger than that of $\mathcal{M}_h^k$, so that the solution of the linear algebraic system associated with $\mathcal{M}_h^k$ is simpler than that associated with $\mathcal{RT}_h^k$.

The solution of the algebraic problem associated with (1.7) can be simplified by the introduction of a Lagrange multiplier to enforce the continuity of the normal component of $\mathbf{q}_h$ across interelement boundaries [1, 8, 9]. Let $\{e\}$ denote the collection of edges of all triangles $T \in \mathcal{T}_h$. Let

$$M_h = M_h^k = \{m \mid m|_e \in P_k(e) \text{ if } e \subset \Omega \text{ and } m|_e = 0 \text{ if } e \subset \partial\Omega\}, \tag{1.11a}$$

and let

$$\mathcal{V}_h = \mathcal{V}_h^k = \{\mathbf{v} \mid \mathbf{v}|_T \in \mathbf{V}^k(T),\ T \in \mathcal{T}_h\}. \tag{1.11b}$$

Note that, if $\mathbf{v} \in \mathcal{V}_h$, $\mathbf{v} \in \mathbf{V}_h$ if and only if

$$\sum_T \langle \mathbf{v} \cdot \mathbf{n}_T, m \rangle_{\partial T} = 0, \qquad m \in M_h^k, \tag{1.12}$$

where $\langle \cdot, \cdot \rangle_{\partial T}$ indicates the $L^2(\partial T)$ inner product; $(\cdot, \cdot)_T$ will indicate that in $L^2(T)$. Then, following Fraeijs de Veubeke [8, 9], we alter (1.7) to seek $\{\mathbf{q}_h, u_h, m_h\} \in \mathcal{V}_h^k \times W_h^{k-1} \times M_h^k$ such that

$$(c\mathbf{q}_h, \mathbf{v}) - \sum_T (\operatorname{div} \mathbf{v}, u_h)_T + \sum_T \langle \mathbf{v} \cdot \mathbf{n}_T, m_h \rangle_{\partial T} = \langle \mathbf{v} \cdot \mathbf{n}, g \rangle, \qquad \mathbf{v} \in \mathcal{V}_h^k, \tag{1.13a}$$

$$\sum_T (\operatorname{div} \mathbf{q}_h, w)_T = (f, w), \qquad w \in W_h^{k-1}, \tag{1.13b}$$

$$\sum_T \langle \mathbf{q}_h \cdot \mathbf{n}_T, p \rangle_{\partial T} = 0, \qquad p \in M_h^k. \tag{1.13c}$$

As a consequence of (1.12), the *function* $\mathbf{q}_h$ resulting from (1.13) coincides with that coming from (1.7), so that the pair $\{\mathbf{q}_h, u_h\}$ of (1.13) is the same as that of (1.7). Note that the *parameters* for the two $\mathbf{q}_h$'s are not the same, as the dimension of $\mathscr{V}_h^k$ is larger than that of $\mathbf{V}_h^k$.

The original object of the modification (1.13) was to permit the easy elimination of the $\mathbf{q}_h$-parameters (and, subsequently, the u_h-parameters) from the system of linear algebraic equations representing (1.13). The system remaining for the m_h-parameters is positive-definite [1], in place of the indefinite (saddle-point) system for (1.7). Arnold and Brezzi [1] noticed that the Lagrange multiplier corresponding to m_h for the Raviart-Thomas method contains new information about the scalar variable; a local post-processing of the pair $\{u_h, m_h\}$ leads to a function u_h^*, $u_h^*|_T \in P_{k+1}(T)$, such that

$$\|u-u_h^*\|_0 \leqq K\|u\|_{k+2+\delta_{k,0}} h^{k+2}, \tag{1.14}$$

while

$$\|u-u_h\|_0 \leqq K\|u\|_{k+1+\delta_{k,0}} h^{k+1}, \quad u_h \in \mathscr{R}\mathscr{T}_h^k. \tag{1.15}$$

We shall show that a similar post-processing can be applied to the solution of (1.13). For $k \geqq 2$, we shall see that (1.14) holds in our case, even though the direct approximation u_h to u is limited by the estimate

$$\|u-u_h\|_0 \leqq K\|u\|_k h^k, \quad u_h \in W_h^{k-1}, \quad k \geqq 1. \tag{1.16}$$

For $k=1$, $u_h^*|_T \in P_1(T)$ and

$$\|u=u_h^*\|_0 \leqq K\|u\|_2 h^2, \quad k=1. \tag{1.17}$$

A summary of our convergence results in $L^2(\Omega)$ and those for the corresponding Raviart-Thomas spaces is given in Table 1; it should be noted that a portion of the estimate given by (1.14) for the Raviart-Thomas spaces must be derived in a different way (as given below in the case of our spaces) than that employed in [1]. We shall derive error estimates in $H^{-s}(\Omega)$ as well; see Sect. 3.

Table 1. Asymptotic error estimates in $L^2(\Omega)$

	$\mathscr{M}_h^k$	$\mathscr{R}\mathscr{T}_h^k$
$\|\mathbf{q}-\mathbf{q}_h\|_0$	$K\|q\|_{k+1} h^{k+1}$	$K\|q\|_{k+1} h^{k+1}$
$\|u-u_h\|_0$	$K\|u\|_k h^k$	$K\|u\|_{k+1+\delta_{k,0}} h^{k+1}$
$\|u-u_h^*\|_0$	$K\|u\|_{k+1-\delta_{k,1}} h^{k+2-\delta_{k,1}}$	$K\|u\|_{k+2} h^{k+2}$

Throughout this paper we shall consider only the case in which the degree k is constant over the triangles $T \in \mathscr{T}_h$; however, it is possible to vary k over the triangles. Elsewhere [3], we shall develop transitional elements for both triangular and rectangular elements and discuss resulting procedures. Also elsewhere [4], we apply our elements in two different ways to problems of linear plane elasticity.

2. Projections and Approximations

The analysis of our mixed methods will be simplified [5, 6] by the existence of projections $\Pi_h = \Pi_h^k \colon H(\mathrm{div}, \Omega) \to \mathbf{V}_h^k$ and $P_h = P_h^{k-1} \colon L^2(\Omega) \to W_h^{k-1}$ such that the following diagram commutes:

$$\begin{array}{ccccc} H(\mathrm{div}, \Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & & \\ \downarrow{\scriptstyle \Pi_h^k} & & \downarrow{\scriptstyle P_h^{k-1}} & & \\ \mathbf{V}_h^k & \xrightarrow{\mathrm{div}} & W_h^{k-1} & \longrightarrow & 0 \end{array}$$

These projections will be constructed locally (i.e., on each $T \in \mathscr{T}_h$). Let P_h^{k-1} be the L^2-projection onto W_h^{k-1}, so that

$$(w - P_h^{k-1} w, z)_T = 0, \qquad z \in P_{k-1}(T),\ T \in \mathscr{T}_h. \tag{2.1}$$

Then, it is well known that

$$\|w - P_h^{k-1} w\|_{-s} \leqq Q\Big(\sum_T \|w\|_{j,T}^2 h_T^{2(s+j)}\Big)^{1/2} \tag{2.2}$$

for $0 \leqq s \leqq k$ and $0 \leqq j \leqq k$, where the index $-s$ indicates the norm in $H^{-s}(\Omega)$, which can be taken here to be the space dual to either $H^s(\Omega)$ or $H_0^s(\Omega)$.

The definition of $\Pi^k(T) = \Pi_h^k|_T$ will take two forms, one for straight-sided triangles and the other for a triangle with one curved side. First, let T have the three straight edges e_i, $i = 1, 2, 3$. Let $B_{k+1}(T) = \{p \in P_{k+1}(T) \mid p|_{\partial T} = 0\} = \lambda_1 \lambda_2 \lambda_3 P_{k-2}(T)$, where the λ_i's are the barycentric coordinates of T. Then, let

$$\langle (\mathbf{v} - \Pi^k(T)\mathbf{v}) \cdot \mathbf{n}_{e_i}, z \rangle_{e_i} = 0, \qquad z \in P_k(e_i),\ i = 1, 2, 3; \tag{2.3a}$$

$$(\mathbf{v} - \Pi^k(T)\mathbf{v}, \mathbf{grad}\, w)_T = 0, \qquad w \in P_{k-1}(T); \tag{2.3b}$$

$$(\mathbf{v} - \Pi^k(T)\mathbf{v}, \mathbf{curl}\, b)_T = 0, \qquad b \in B_{k+1}(T). \tag{2.3c}$$

Next, let e_3 be curved. Then, define $\Pi^k(T)$ through the following degrees of freedom:

$$\langle (\mathbf{v} - \Pi^k(T)\mathbf{v}) \cdot \mathbf{n}_{e_i}, z \rangle_{e_i} = 0, \quad z \in P_k(e_i),\ i = 1, 2; \tag{2.4a}$$

$$(\mathrm{div}\,(\mathbf{v} - \Pi^k(T)\mathbf{v}), w)_T = 0, \quad w \in P_{k-1}(T); \tag{2.4b}$$

$$(\mathbf{v} - \Pi^k(T)\mathbf{v}, \mathbf{z})_T = 0, \quad \mathbf{z} \in \{\mathbf{y} \in P_k(T) \mid \mathrm{div}\, \mathbf{y} = 0 \text{ and } \mathbf{y} \cdot \mathbf{n} = 0 \text{ on } e_1 \cup e_2\}. \tag{2.4c}$$

Lemma 2.1. *The degrees of freedom* (2.3) *or* (2.4) *determine* $\Pi^k(T)$ *uniquely. Moreover, if* $\Pi_h^k|_T = \Pi^k(T)$ *for* $T \in \mathscr{T}_h$, *then* $\Pi_h^k \colon H(\mathrm{div}; \Omega) \to \mathbf{V}_h^k$ *and*

$$\mathrm{div}\, \Pi_h^k = P_h^{k-1}\, \mathrm{div} \tag{2.5}$$

on $H(\mathrm{div}; \Omega)$. *Finally, for* $1 \leqq r \leqq k+1$,

$$\|\mathbf{v} - \Pi_h^k \mathbf{v}\|_0 \leqq Q\Big(\sum_T \|v\|_{r,T}^2 h_T^{2r}\Big)^{1/2}. \tag{2.6}$$

Proof. Let $\mathbf{v}\in\mathbf{P}_k(T)$ have vanishing degrees of freedom given by (2.3). Then, $\mathbf{v}\cdot\mathbf{n}_T=0$ on ∂T. Since $\operatorname{div}\mathbf{v}\in P_{k-1}(T)$,

$$(\operatorname{div}\mathbf{v},\operatorname{div}\mathbf{v})=-(\mathbf{v},\mathbf{grad}\operatorname{div}\mathbf{v})_T+\langle\mathbf{v}\cdot\mathbf{n}_T,\operatorname{div}\mathbf{v}\rangle_{\partial T}=0,$$

and $\operatorname{div}\mathbf{v}=0$. Thus, $\mathbf{v}=\mathbf{curl}\,b$, where $b\in P_{k+1}(T)$ and $\frac{\partial b}{\partial t}=\mathbf{v}\cdot\mathbf{n}=0$, where $\mathbf{t}$ is the unit tangent vector along ∂T, so that b can be taken to lie in $B_{k+1}(T)$. Thus,

$$\|\mathbf{v}\|_{0,T}^2=(\mathbf{v},\mathbf{curl}\,b)_T=0,$$

and $\mathbf{v}=0$; consequently, (2.3) determines $\Pi^k(T)$ on straight-sided triangles.

Next, let T be a boundary triangle, and let $\mathbf{v}\in\mathbf{P}_k(T)$ have vanishing degrees of freedom for (2.4). Then, $\mathbf{v}\cdot\mathbf{n}=0$ on $e_1\cup e_2$ and $\operatorname{div}\mathbf{v}=0$ on T. Thus, (2.4c) implies that $\mathbf{v}=0$, as was to be shown.

Now, note that (2.3a) and (2.3b) imply that (2.4b) holds on straight-sided triangles; thus,

$$(\operatorname{div}(\mathbf{v}-\Pi_h^k\mathbf{v}),w)=0,\qquad w\in W_h^{k-1}. \tag{2.7}$$

Note also that, since $\operatorname{div}\mathbf{V}_h^k=W_h^{k-1}$,

$$(\operatorname{div}\mathbf{v},w-P_h^{k-1}w)=0,\qquad \mathbf{v}\in\mathbf{V}_h^k, \tag{2.8}$$

so that (2.5) holds. Finally, (2.6) follows from the Dupont-Scott [7] form of the Bramble-Hilbert lemma, since the vertex angles of the triangles in $\mathcal{T}_h$ are bounded below.

3. Error Analysis for the Mixed Method

Let us turn to the analysis of the error in the procedure of (1.7). Set

$$\mathbf{d}_h=\mathbf{q}-\mathbf{q}_h,\qquad \mathbf{e}_h=\Pi_h^k\mathbf{q}-\mathbf{q}_h,\qquad z_h=P_h^{k-1}u-u_h. \tag{3.1}$$

Then, subtracting (1.7) from (1.4) and applying (2.8) leads to the error equations

$$(c\mathbf{d}_h,\mathbf{v})-(\operatorname{div}\mathbf{v},z_h)=0,\qquad \mathbf{v}\in\mathbf{V}_h^k, \tag{3.2a}$$

$$(\operatorname{div}\mathbf{d}_h,w)=0,\qquad w\in W_h. \tag{3.2b}$$

Our error analysis will follow the development described by Douglas and Roberts [6], following Johnson and Thomée [17]; i.e., we shall base our argument on the duality lemmas to follow.

Lemma 3.1. *For* $s\geqq 0$,

$$\|z_h\|_{-s}\leqq K[\|\mathbf{d}_h\|_0 h^{\min(s+1,k+1)}+\|\operatorname{div}\mathbf{d}_h\|_0 h^{\min(s+2,k)}]. \tag{3.3}$$

Proof. Let $\psi\in H^s(\Omega)$, and let $\varphi\in H^{s+2}(\Omega)\cap H_0^1(\Omega)$ be such that $L^*\varphi=\psi$. Then, a calculation [5, 6] shows that

$$(z_h,\psi)=(c\,\mathbf{d}_h,a\,\mathbf{grad}\,\varphi-\Pi_h(a\,\mathbf{grad}\,\varphi))+(\operatorname{div}\mathbf{d}_h,\varphi-P_h\varphi),$$

so that (3.3) follows from (2.2), (2.6), and the assumed elliptic regularity for the Dirichlet problem for L^*.

Lemma 3.2. *For* $s \geqq 0$,

$$\|\operatorname{div} \mathbf{d}_h\|_{-s} \leqq K \|\operatorname{div} \mathbf{d}_h\|_0 h^{\min(s,k)}. \tag{3.4}$$

Proof. Let $\varphi \in H^s(\Omega)$. Then, by (3.2b),

$$(\operatorname{div} \mathbf{d}_h, \varphi) = (\operatorname{div} \mathbf{d}_h, \varphi - w), \qquad w \in W_h,$$

and (3.4) follows from (2.2).

Lemma 3.3. *For* $s \geqq 0$,

$$\|\mathbf{d}_h\|_{-s} \leqq K[\|\mathbf{d}_h\|_0 h^{\min(s,k+1)} + \|\operatorname{div} \mathbf{d}_h\|_0 h^{\min(s+1,k)}]. \tag{3.5}$$

Proof. (The argument below is simpler than that given in [6].) Let $\varphi \in \mathbf{H}^s(\Omega)$. Then, using (3.2b) and then (2.7),

$$\begin{aligned}(c\,\mathbf{d}_h, \varphi) &= (c\,\mathbf{d}_h, \Pi_h \varphi) + (c\,\mathbf{d}_h, \varphi - \Pi_h \varphi)\\ &= (\operatorname{div} \Pi_h \varphi, z_h) + (c\,\mathbf{d}_h, \varphi - \Pi_h \varphi)\\ &= (\operatorname{div} \varphi, z_h) + (c\,\mathbf{d}_h, \varphi - \Pi_h \varphi),\end{aligned}$$

so that

$$|(c\,\mathbf{d}_h, \varphi)| \leqq \|\varphi\|_s \{\|z_h\|_{-s+1} + \|\mathbf{d}_h\|_0 h^{\min(s,k+1)}\}.$$

Hence, (3.5) follows from (3.3).

Now, let us bound $\|\mathbf{d}_h\|_0$ and $\|\operatorname{div} \mathbf{d}_h\|_0$. In (3.2a) take $v = \mathbf{e}_h$:

$$(c\,\mathbf{e}_h, \mathbf{e}_h) = (\operatorname{div} \mathbf{d}_h, z_h) - (c(\mathbf{q} - \Pi_h \mathbf{q}), \mathbf{e}_h) = -(c(\mathbf{q} - \Pi_h \mathbf{q}), \mathbf{e}_h),$$

so that

$$\|\mathbf{e}_h\|_0 \leqq K \|\mathbf{q} - \Pi_h \mathbf{q}\|_0$$

and

$$\|\mathbf{d}_h\|_0 \leqq \|\mathbf{e}_h\|_0 + \|\mathbf{q} - \pi_h \mathbf{q}\|_0 \leqq K \|\mathbf{q}\|_r h^r, \qquad 1 \leqq r \leqq k+1. \tag{3.6}$$

Next, note that

$$(\operatorname{div} \mathbf{e}_h, w) = (\operatorname{div} \mathbf{d}_h, w) = 0, \qquad w \in W_h,$$

so that $\operatorname{div} \mathbf{e}_h = \mathbf{0}$. Hence,

$$\|\operatorname{div} \mathbf{d}_h\|_0 = \|\operatorname{div}(\mathbf{q} - \Pi_h \mathbf{q})\|_0 \leqq K \|\operatorname{div} \mathbf{q}\|_r h^r, \qquad 0 \leqq r \leqq k. \tag{3.7}$$

Thus,

$$\begin{aligned}\|u - u_h\|_{-s} &\leqq \|z_h\|_{-s} + \|u - P_h^{k-1} u\|_{-s}\\ &\leqq K[\|\mathbf{q}\|_{r_1} h^{\min(r_1,k+1) + \min(s+1,k+1)}\\ &\quad + \|\operatorname{div} q\|_r h^{\min(r,k) + \min(s+2,k)}\\ &\quad + \|u\|_r h^{\min(r,k) + \min(s,k)}],\end{aligned}$$

where $r_1 \geqq 1$. This can be simplified by using the elliptic regularity for the Dirichlet problem for L^*. Then, it follows that ($|g|_j = \|g\|_{H^j(\partial\Omega)}$)

$$\|u - u_h\|_{-s} \leqq K(\|f\|_{r-2} + |g|_{r-1/2}) h^{r+s}, \qquad 2 \leqq r \leqq k+2,\ 0 \leqq s \leqq k-2. \tag{3.8}$$

Similarly,

$$\|\mathbf{q}-\mathbf{q}_h\|_{-s} \leqq K[\|\mathbf{q}\|_{r_1} h^{\min(r_1,k+1)+\min(s,k+1)} + \|\operatorname{div}\mathbf{q}\|_r h^{\min(r,k)+\min(s+1,k)}]$$
$$\leqq K(\|f\|_{r-1}+|g|_{r+1/2})h^{r+s}, \quad 1 \leqq r \leqq k+1, \ 0 \leqq s \leqq k-1. \tag{3.9}$$

Finally,

$$\|\operatorname{div}(\mathbf{q}-\mathbf{q}_h)\|_{-s} \leqq K\|\operatorname{div}\mathbf{q}\|_r h^{r+s} = K\|f\|_r h^{r+s}, \quad 0 \leqq r \leqq k, \ 0 \leqq s \leqq k. \tag{3.10}$$

Theorem 3.4. *Let* $\{\mathbf{q}_h, u_h\} \in \mathbf{V}_h^k \times W_h^{k-1}$ *be the solution of the mixed finite element method* (1.7). *Then, the estimates* (3.8), (3.9), *and* (3.10) *hold for the errors* $u-u_h$, $\mathbf{q}-\mathbf{q}_h$, *and* $\operatorname{div}(\mathbf{q}-\mathbf{q}_h)$, *respectively. In particular,*

$$\|u-u_h\|_{-k} + \|\mathbf{q}-\mathbf{q}_h\|_{-k+1} + \|\operatorname{div}(\mathbf{q}-\mathbf{q}_h)\|_{-k} \leqq K(\|f\|_k + |g|_{k+3/2})h^{2k} \tag{3.11}$$

and

$$\|u_h - P_h u\|_0 \leqq Q(\|f\|_k + |g|_{k+3/2})h^{\min(k+2,2k)}. \tag{3.12}$$

4. Analysis of the Hybrid Form of the Mixed Method

The object of this section is to associate with the Lagrange multiplier m_h, which was introduced to relax the continuity requirement on the normal component of the vector $\mathbf{v} \in \mathbf{V}_h^k$ (i.e., to produce a hydrid-mixed method), and the scalar variable u_h a new approximation u_h^* to u such that $u-u_h^*$ is more rapidly convergent to zero than $u-u_h$. The development here is closely analogous to that of Arnold and Brezzi [1]. In particular, we begin with an argument quite similar to that of Theorem 1.4 of [1]. Let

$$|m_h|_{0,h}^2 = \sum_{e \subset \Omega} \|m_h\|_{0,e}^2, \tag{4.1a}$$

$$|m_h|_{-1/2,h}^2 = \sum_{e \subset \Omega} |e| \, \|m_h\|_{0,e}^2, \tag{4.1b}$$

where $|e|$ denotes the length of the edge e. Also, let $Q_h = Q_h^k$ be the projection operator defined locally by $L^2(e)$-projection onto $P_k(e)$ for $e \subset \Omega$.

Lemma 4.1. *If* $\{\mathbf{q}_h, u_h, m_h\} \in \mathbf{V}_h^k \times W_h^{k-1} \times M_h^k$ *is the solution of* (1.13), *then*

$$\|m_h - Q_h^k u\|_{0,e} \leqq K\{h_T^{1/2}\|\mathbf{q}-\mathbf{q}_h\|_{0,T} + h_T^{-1/2}\|P_h^{k-1}u - u_h\|_{0,T}\} \tag{4.2}$$

and

$$|m_h - Q_h^k u|_{-1/2,h} \leqq K\{h\|\mathbf{q}-\mathbf{q}_h\|_0 + \|P_h^{k-1}u - u_h\|_0\}. \tag{4.3}$$

Moreover,

$$|m_h - Q_h^k u|_{-1/2,h} \leqq K(\|f\|_k + |g|_{k+3/2})h^{k+2-\max(2-k,0)} \tag{4.4}$$

for $k \geqq 1$.

Proof. The estimate (4.3) follows immediately from (4.2) and the assumed vertex angle condition for the triangles. The bound (4.4) follows from (4.3), (3.9), and (3.3). Thus, it suffices to prove (4.2).

Let $e \subset \Omega \cap T$ and define a vector $\mathbf{v} \in \mathbf{V}_h^k$ having support in T by requiring that

$$\mathbf{v} \cdot \mathbf{n}_e = m_h - Q_h^k u \quad \text{on } e, \tag{4.5a}$$

$$\mathbf{v} \cdot \mathbf{n}_T = 0 \quad \text{on } \partial T \setminus e, \tag{4.5b}$$

$$(\mathbf{v}, \mathbf{grad}\, w)_T = 0, \qquad w \in P_{k-1}(T), \tag{4.5c}$$

$$(\mathbf{v}, \mathbf{curl}\, w)_T = 0, \qquad w \in B_{k+1}(T); \tag{4.5d}$$

the existence and uniqueness of $\mathbf{v}$ is assured by Lemma 2.1. Moreover, a simple scaling argument shows that

$$h_T \|\mathbf{v}\|_{1,T} + \|\mathbf{v}\|_{0,T} \leqq K h_T^{1/2} \|m_h - Q_h^k u\|_{0,e}. \tag{4.6}$$

Now, use $\mathbf{v}$ as the test function in (1.13a):

$$(c\mathbf{q}_h, \mathbf{v})_T - (\operatorname{div} \mathbf{v}, u_h)_T + \langle m_h, m_h - Q_h u\rangle_e = 0.$$

Since

$$(c\mathbf{q}, \mathbf{v})_T - (\operatorname{div} \mathbf{v}, u)_T + \langle u, m_h - Q_h u\rangle_e = 0,$$

it follows that

$$\|m_h - Q_h u\|_{0,e}^2 = \langle m_h - u, m_h - Q_h u\rangle_e = (c(\mathbf{q} - \mathbf{q}_h), \mathbf{v})_T - (\operatorname{div} \mathbf{v}, z_h)_T,$$

and (4.2) follows from (4.6).

Let us recall some extension maps introduced by Arnold and Brezzi [1].

Lemma 4.2 [1]. *Let k be a nonnegative even integer. Then, there exists a map*

$$R^{k+1}(T): \prod_{i=1}^{3} L^2(e_i) \times L^2(T) \to P_{k+1}(T),$$

$u_h^* = R^{k+1}(T)\{m_h, u_h\}$, *given by*

$$\langle u_h^* - m_h, p\rangle_{e_i} = 0, \qquad p \in P_k(e_i),\ i = 1, 2, 3; \tag{4.7a}$$

$$(u_h^* - u_h, p)_T = 0, \qquad p \in P_{k-2}(T), \text{ for } k \geqq 2. \tag{4.7b}$$

Moreover,

$$\|u_h^*\|_{0,T} \leqq K\left[\|u_h\|_{0,T} + h_T^{1/2} \sum_{i=1}^{3} \|m_h\|_{0,e_i}\right]. \tag{4.8}$$

Arnold and Brezzi proved Lemma 4.2 only for straight-sided triangles. Since our boundary triangles can have a curved edge, we need to note that the same degrees of freedom continue to define $u_h^* \in P_{k+1}(T)$ uniquely, at least for small h, since as $h \to 0$ the curved edge deviates from the line segment joining its end point by $O(h^2)$. Hence, the matrix generated by (4.7) remains nonsingular for small h. An application of the argument of Dupont and Scott [7] shows that (4.8) continues to hold on the boundary triangles, again for small h. (We shall assume that h is this small in the argument below.)

Arnold and Brezzi did not find a simple expression for a corresponding map for odd k; however, for $k=1$ and $k=3$ they found the following maps, which also satisfy (4.8). For $k=1$, let $R^2(\mathscr{T})$ be given by letting $u_h^*|_T \in P_2(T)$ be determined by the relations

$$\langle u_h^* - m_h, 1\rangle_{e_i} = 0, \qquad i = 1, 2, 3; \tag{4.9a}$$

$$(u_h^* - u_h, p)_T = 0, \qquad p \in P_1(T); \tag{4.9b}$$

this definition of $R^2(\mathscr{T})$ so given for completeness only, as it is not used below to extend m_h and u_h when $k=1$. For $k=3$, let $u_h^*|_T \in P_4(T)$ be given by

$$\langle u_h^* - m_h, p\rangle_{e_i} = 0, \quad p \in P_2(e_i),\ i=1,2,3; \tag{4.10a}$$

$$(u_h^* - u_h, p)_T = 0, \quad p \in P_2(T). \tag{4.10b}$$

For $k \in \{0,1,2,3,4,6,8,\ldots\}$, let R_h^{k+1} denote the extension of the corresponding $R^{k+1}(T)$ to all $T \in \mathscr{T}_h$, and let

$$u_h^* = \begin{cases} R_h^1\{m_h, u_h\}, & k=1, \\ R_h^{k+1}\{m_h u_h\}, & k=2,3,4,6,8,\ldots; \end{cases} \tag{4.11}$$

interpret m_h as $-g$ on edges $e_i \subset \partial\Omega$. Also, set

$$u_h^\# = \begin{cases} R_h^1\{u,u\}, & k=1, \\ R_h^{k+1}\{u,u\}, & k=2,3,4,6,\ldots. \end{cases} \tag{4.12}$$

It is easy [1] to see that

$$\|u - u_h^\#\|_0 \leqq \begin{cases} K\|u\|_2 h^2, & k=1, \\ K\|u\|_{k+2} h^{k+2}, & k=2,3,4,6,\ldots. \end{cases} \tag{4.13}$$

Set

$$z_h^* = u_h^* - u_h^\#. \tag{4.14}$$

The cases $k=1$, $k=3$, and k even must be considered separately. Let us start with $k=1$, so that

$$\langle z_h^*, 1\rangle_{e_i} = \langle m_h - Q_h^0 u, 1\rangle_{e_i} = \langle m_h - Q_h^1 u, 1\rangle_{e_i}, \quad i=1,2,3, \tag{4.15}$$

as $\langle (Q_h^1 - Q_h^0)u, 1\rangle_{e_i} = 0$, for $e_i \subset \Omega$. If $e_i \subset \partial\Omega$, $\langle z_h^*, 1\rangle_{e_i} = 0$. Thus, by (4.8),

$$\|z_h^*\|_{0,T} \leqq K h_T^{1/2} \sum_{i=1}^{3} \|Q_h^1 u - m_h\|_{0,e_i},$$

and, by Lemma 4.1 and (4.4),

$$\|z_h^*\|_0 \leqq K(\|f\|_1 + |g|_{5/2})h^2, \quad k=1. \tag{4.16}$$

Thus,

$$\|u - u_h^*\|_0 \leqq K(\|f\|_1 + |g|_{5/2})h^2, \quad k=1. \tag{4.17}$$

Next, let $k=3$. Then,

$$\langle z_h^*, p\rangle_{e_i} = \langle m_h - Q_2^2 u, p\rangle_{e_i} = \langle m_h - Q_h^3 u, p\rangle_{e_i}, \quad p \in P_2(e_i),\ i=1,2,3,$$

and

$$(z_h^*, p)_T = (u_h - P_h^2 u, p)_T, \quad p \in P_2(T).$$

Consequently,

$$\|z_h^*\|_{0,T} \leqq K[\|u_h - P_h^2 u\|_{0,T} + h_T^{1/2}\|m_h - Q_h^3 u\|_{0,\partial T}]. \tag{4.18}$$

By (3.3), (3.6), (3.7), and (4.4),

$$\|z_h^*\|_0 \leqq K(\|f\|_3 + |g|_{9/2})h^5, \quad k=3. \tag{4.19}$$

By (4.13) and (4.19),

$$\|u-u_h^*\|_0 \leqq K(\|f\|_3+|g|_{9/2})h^5, \qquad k=3. \tag{4.20}$$

Finally, let us consider the case of positive, even k. Then,

$$\langle z_h^*, p\rangle_{e_i} = \langle m_h - Q_h^k u, p\rangle_{e_i}, \qquad p\in P_k(e_i),\ i=1,2,3;$$
$$(z_h^*, p)_T = (u_h - P_h^{k-2}u, p)_T = (u_h - P_h^{k-1}u, p)_T, \qquad p\in P_{k-2}(T),$$

so that

$$\|z_h^*\|_{0,T} \leqq K[\|u_h - P_h^{k-1}u\|_{0,T} + h_T^{1/2}\|m_h - Q_h^k u\|_{0,\partial T}]$$

and

$$\|z_h^*\|_0 \leqq K(\|f\|_k + |g|_{k+3/2})h^{k+2},$$

as in (4.19) above. Thus,

$$\|u-u_h^*\|_0 \leqq K(\|f\|_k+|g|_{k+3/2})h^{k+2}, \qquad k=2,4,\ldots. \tag{4.21}$$

The results above can be summarized in the following theorem.

Theorem 4.3. *Let the function* u_h^* *be defined by* (4.11). *Then,*

$$\|u-u_h^*\|_0 \leqq \begin{cases} K(\|f\|_1+|g|_{5/2})h^2, & k=1,\\ K(\|f\|_k+|g|_{k+3/2})h^{k+2}, & k=2,3,4,6,\ldots. \end{cases} \tag{4.22}$$

We have left to show that a function u_h^*, $u_h^*|_T\in P_{k+1}(T)$, can be associated with our approximate solution $\{\mathbf{q}_h, u_h, m_h\}$ such that the inequality (4.22) holds for odd $k\geqq 5$. Presumably, ad hoc choices of degrees of freedom can be found for each such k, in the somewhat unlikely event that someone wants to use polynomials of these degrees. However, for all $k\geqq 2$, it is possible to offer as an alternative procedure a local version of the Nitsche [13] procedure.

Now, let $T\in\mathscr{T}_h$ and define $u_h^*, u_h^*|_T\in P_{k+1}(T)$, triangle-by-triangle as the solution of the equations

$$\begin{aligned} A(u_h^*, p) = A_{\sigma,T}(u_h^*, p) &= (a\,\mathbf{grad}\,u_h^*, \mathbf{grad}\,p)_T - \left\langle a\frac{\partial u_h^*}{\partial n}, p\right\rangle_{\partial T} \\ &\quad - \left\langle u_h^*, a\frac{\partial p}{\partial n}\right\rangle_{\partial T} + \sigma h_T^{-1}\langle u_h^*, p\rangle_{\partial T} \\ &= (f,p)_T - \left\langle m_h, a\frac{\partial p}{\partial n}\right\rangle_{\partial T} + \sigma h_T^{-1}\langle m_h, p\rangle_{\partial T} \end{aligned} \tag{4.23}$$

for $p\in P_{k+1}(T)$, where σ is a constant depending on $k+1$, the coefficient a, and the minimum angle constraint for $T\in\mathscr{T}_h$. We shall outline the proof of the following theorem, since its proof is just a variant of the usual proof of the convergence of the Nitsche method on all of Ω.

Theorem 4.4. *Let* $k\geqq 2$ *and* u_h^* *be defined triangle-by-triangle by* (4.23). *Then,*

$$\|u-u_h^*\|_0 \leqq K(\|f\|_k+|g|_{k+3/2})h^{k+2}. \tag{4.24}$$

Proof. Recall that the Nitsche form A, while not coercive over $H^1(T)$, is coercive over $P_{k+1}(T)$ for sufficiently large σ. In particular,

$$A(p,p) \geqq \rho(\|\mathbf{grad}\, p\|_{0,T}^2 + h_T^{-1}\|p\|_{0,\partial T}^2), \quad p \in P_{k+1}(T), \tag{4.25}$$

for a fixed choice of such σ. The error equation for (4.23) is

$$\begin{aligned} A(u-u_h^*, p) &= \left\langle m_h - u, a\frac{\partial p}{\partial n}\right\rangle_{\partial T} + \sigma h_T^{-1}\langle u - m_h, p\rangle_{\partial T} \\ &= \left\langle m_h - Q_{h,a}^k u, a\frac{\partial p}{\partial n}\right\rangle_{\partial T} + \sigma h_T^{-1}\langle u - m_h, p\rangle_{\partial T}, \quad p \in P_{k+1}(T), \end{aligned} \tag{4.26}$$

where $Q_{h,a}^k$ is the L^2-projection on each edge into $P_k(e)$ with respect to the weight function a. Take $\hat{u} \in P_{k+1}(T)$, shift u to p^*, and choose $p = \hat{u} - u_h^*$:

$$\begin{aligned} A(\hat{u}-u_h^*, \hat{u}-u_h^*) = A(\hat{u}-u, \hat{u}-u_h^*) &+ \left\langle m_h - Q_{h,a}^k u, a\frac{\partial}{\partial n}(\hat{u}-u_h^*)\right\rangle_{\partial T} \\ &+ \sigma h_T^{-1}\langle u - m_h, \hat{u}-u_h^*\rangle_{\partial T}. \end{aligned} \tag{4.27}$$

By the same inverse property as was used to show (4.25), plus properly choosing $\hat{u}$ to approximate u on T and applying Lemma 4.1, we see that

$$A(\hat{u}-u_h^*, \hat{u}-u_h^*) \leqq K[\|u\|_{k+2,T}^2 h^{2k+2} + \|\mathbf{q}-\mathbf{q}_h\|_{0,T}^2 + h_T^{-2}\|P_h^{k+1}u - u_h\|_{0,T}^2], \tag{4.28}$$

from which it follows that

$$\sum_T \{\|\mathbf{grad}(u-u_h^*)\|_{0,T}^2 + h_T^{-1}\|u-u_h^*\|_{0,\partial T}^2\} \leqq K\{\|f\|_k^2 + |g|_{k+3/2}^2\} h^{2k+2}, \quad k \geqq 2. \tag{4.29}$$

It is now appropriate to use a version of the standard duality argument, on each triangle separately, to develop an L^2-estimate for $u - u_h^*$. Let

$$-\operatorname{div}(a\, \mathbf{grad}\, \varphi) = \hat{u} - u_h^*, \quad x \in T, \tag{4.30a}$$

$$\varphi = 0, \quad x \in \partial T, \tag{4.30b}$$

so that $\|\varphi\|_{2,T} \leqq K\|\hat{u}-u_h^*\|_{0,T}$ and $\|\varphi\|_{1,T} \leqq K\|\varphi\|_{2,T} h$, as the piecewise-linear interpolant of φ on T vanishes. Hence,

$$\begin{aligned} \|\hat{u}-u_h^*\|_{0,T}^2 &= A(\hat{u}-u_h^*, \varphi) \\ &\leqq K A(\hat{u}-u_h^*, \hat{u}-u_h^*)^{1/2}\|\hat{u}-u_h^*\|_{0,T} h; \end{aligned}$$

consequently, the desired bound (4.24) follows easily.

The choice of a Galerkin-like procedure such as (4.23) is quite natural to find u_h^* since we are looking for an approximation to the scalar variable u and have the right-hand side of the differential equation and an accurate approximation to u on the edges of the triangles. Note that Theorems 4.3 and 4.4 complete the justification of Table 1.

We should like to remark that, while the usual reason for choosing a mixed method is to obtain the vector variable $\mathbf{q}$ accurately, the solution of the mixed method clearly contains information about the scalar variable to an order in h that would permit evaluating $-a\,\mathbf{grad}\,u_h^*$ to find a second approximation of $\mathbf{q}$ of the same order of accuracy as that given by $\mathbf{q}_h$.

Let us consider the algebraic equations generated by (1.13). The system takes the form (where the triple $\{\mathbf{q}_h, u_h, m_h\}$ is represented by the parameters $\{\varphi_h, \mathscr{u}_h, \mathscr{m}_h\}$)

$$\mathscr{A}\varphi_h+\mathscr{B}\mathscr{u}_h+\mathscr{C}\mathscr{m}_h=g, \tag{4.31 a}$$

$$\mathscr{B}^* q_h=f, \tag{4.31 b}$$

$$\mathscr{C}^* q_h=0. \tag{4.31 c}$$

The matrix $\mathscr{A}$ is block diagonal, with positive definite diagonal blocks of size $\dim \mathbf{V}^k(T)$. The first step in the solution of (4.31) consists of eliminating φ_h, leading then to an equation of the form

$$\mathscr{B}^*\mathscr{A}^{-1}\mathscr{B}\mathscr{u}_h+\mathscr{B}^*\mathscr{A}^{-1}\mathscr{C}\mathscr{m}_h=f'. \tag{4.32}$$

The matrix $\mathscr{B}^*\mathscr{A}^{-1}\mathscr{B}$ is again block diagonal, with blocks of the size $\dim W_h^{k-1}$, and $\mathscr{u}_h$ can be eliminated to produce a system of the form

$$\mathscr{D}\mathscr{m}_h=f''. \tag{4.33}$$

The matrix $\mathscr{D}$ is symmetric positive definite; its graph connects the parameters for $\mathscr{m}_h$ on an edge e to those for $\mathscr{m}_h$ on the other four sides of the two triangles containing e. A variety of iterative or direct methods can be employed to find the solution of (4.33), from which $\mathbf{q}_h, u_h$, and u_h^* (if needed) can be evaluated. The advantage of the procedure outlined above (over and above the possibility of finding u_h^*) instead of treating the linear equation arising from the standard mixed method (1.7) is that $\mathscr{D}$ is positive definite in place of the indefinite system for (1.7).

If the Raviart-Thomas-Nedelec method is modified [1] to correspond to (4.31), the resulting matrix $\mathscr{D}$ has exactly the same structure as the one above; however, both of the elimination steps for φ_h and $\mathscr{u}_h$ involve blocks of noticeably larger size than those above.

5. Rectangular Elements

Our rectangular elements are based on polynomials of some fixed *total* degree, rather than of the same degree in each variable; consequently, the local dimension of our space is much smaller than that of the corresponding Raviart-Thomas space. Let $\mathscr{T}_h=\{R\}$, where the rectangles R are such that $\operatorname{diam}(R)=h_R\leqq h$ and the ratio of the side lengths of R is bounded by a constant independent of R and h. Let $k\geqq 1$.

We again base the scalar space on P_{k-1}:

$$W^{k-1}(R)=P_{k-1}(R). \tag{5.1}$$

We would like to use vector polynomials of degree k for our vector space; however, this choice fails the stability condition [2] necessary for optimal order estimates. Thus, we augment these polynomials by adding a space of polynomials of degree $k+1$ of dimension two. Let

$$\mathbf{V}^k(R)=\mathbf{P}_k(R)\oplus Span(\mathbf{curl}\, x^{k+1}y, \mathbf{curl}\, x y^{k+1}). \tag{5.2}$$

Again, define $\mathcal{M}_h^k=\mathbf{V}_h^k\times W_h^{k-1}$ through the analogue of (1.6) and then seek $\{\mathbf{q}_h, u_h\}\in\mathcal{M}_h^k$ satisfying (1.7). The analysis of this resulting mixed method is faciliated, as before, by the existence of projections $\Pi_h^k: H(\mathrm{div}, \Omega)\to \mathbf{V}_h^k$ and $P_h^{k-1}: L^2(\Omega)\to W_h^{k-1}$. As before, let P_h^{k-1} denote L^2-projection:

$$(w-P_h^{k-1}w, z)_R=0, \qquad z\in P_{k-1}(R),\ R\in\mathcal{T}_h. \tag{5.3}$$

Next, let $\Pi_h^k|_R$ be defined by the following degrees of freedom when R has no curved edge:

$$\langle(\mathbf{q}-\Pi_h^k\mathbf{q})\cdot\mathbf{n}_{e_i}, p\rangle_{e_i}=0, \qquad p\in P_k(e_i),\ i=1,2,3,4; \tag{5.4a}$$

$$(\mathbf{q}-\Pi_h^k\mathbf{q}, \mathbf{v})_R=0, \qquad \mathbf{v}\in\mathbf{P}_{k-2}(R). \tag{5.4b}$$

If R has one curved edge, assume that this edge is labelled e_4. Then, we can modify (5.4) in a fashion similar to (2.4). Let Π_h^k be determined on such a boundary element by the relations

$$\langle(\mathbf{q}-\Pi_h^k\mathbf{q})\cdot\mathbf{n}_{e_i}, p\rangle_{e_i}=0, \qquad p\in P_k(e_i),\ i=1,2,3; \tag{5.5a}$$

$$(\mathrm{div}(\mathbf{q}-\Pi_h^k\mathbf{q}), w)_R=0, \qquad w\in P_{k-1}(R); \tag{5.5b}$$

$$(\mathbf{q}-\Pi_h^k\mathbf{q}, \mathbf{v})_R=0, \qquad \mathbf{v}\in\{\mathbf{y}\in\mathbf{V}^k(R): \mathrm{div}\,\mathbf{y}=\mathbf{0} \text{ and } \mathbf{y}\cdot\mathbf{n}_R=\mathbf{0} \text{ on } e_1\cup e_2\cup e_3\}. \tag{5.5c}$$

Lemma 5.1. *The degrees of freedom of* (5.4) *or* (5.5) *determine* Π_h^k. *Moreover,* $\mathrm{div}\,\Pi_h^k=P_h^{k-1}\,\mathrm{div}$ *as a map from* $H(\mathrm{div};\Omega)$ *to* W_h^{k-1}, *so that*

$$(\mathrm{div}(\mathbf{q}-\Pi_h^k\mathbf{q}), w)=0, \qquad w\in W_h^{k-1}, \tag{5.6a}$$

$$(\mathrm{div}\,\mathbf{v}, w-P_h^{k-1}w)=0, \qquad \mathbf{v}\in\mathbf{V}_h^k. \tag{5.6b}$$

Also,

$$\|\mathbf{q}-\Pi_h^k\mathbf{q}\|_0\leqq K\|\mathbf{q}\|_r h^r, \qquad 1\leqq r\leqq k+1; \tag{5.7a}$$

$$\|w-P_h^{k-1}w\|_{-s}\leqq K\|w\|_r h^{r+s}, \qquad 0\leqq s\leqq k,\ 0\leqq r\leqq k. \tag{5.7b}$$

Proof. To establish unisolvence for (5.4), it suffices to treat the case $R=[0,1]^2$. Let $\mathbf{q}\in\mathbf{V}^k(R)$ have vanishing degrees of freedom. Since $\mathbf{q}=(q_1, q_2)$ with

$$q_1(x,y)=a_1y^k+a_2xy^k+q_1'(x,y),$$
$$q_2(x,y)=b_1x^k+b_2x^ky+q_2'(x,y),$$

where the degrees of q_1' and q_2' in y and x, respectively, are less than k, it follows from (5.4a) that $a_1=a_2=b_1=b_2=0$, so that q contains no terms coming from the **curl** of $x^{k+1}y$ or xy^{k+1}. It then follows from (5.4a) that $\mathbf{q}\cdot\mathbf{n}=0$ on ∂R. Thus,

$$q_1=x(1-x)q_1''(x,y), \qquad q_1''\in P_{k-2}(R),$$

and (5.4b) implies that q_1 vanishes. Similarly, $q_2=0$, and the projection is well-defined.

Next, note that

$$(\operatorname{div}(\mathbf{q}-\Pi_h^k\mathbf{q}), w)_R = -(\mathbf{q}-\Pi_h^k\mathbf{q}, \mathbf{grad}\, w)_R + (\langle \mathbf{q}-\Pi_h^k\mathbf{q})\cdot\mathbf{n}, \mathbf{w}\rangle_{\partial R}$$
$$=0, \qquad w\in P_{k-1}(R),$$

and that $\operatorname{div}\mathbf{V}^k(R)=P_{k-1}(R)$.

Now, let R be a boundary rectangle. It suffices to take R having vertices at $(0,0)$, $(1,0)$, $(0,1)$, and $(\hat{x},1)$, so that e_4 connects $(1,0)$ and $(\hat{x},1)$. As above, if $\mathbf{q}\in\mathbf{V}^k(R)$ has vanishing degrees of freedom (5.5), (5.5a) applied to the upper and lower edges of R implies that $b_1=b_2=0$ and $q_2=0$ on these edges. Also, q_1 vanishes on the edge $x=0$, so that $\mathbf{q}\cdot\mathbf{n}_R=0$ on $e_1\cup e_2\cup e_3$. Moreover, (5.5b) forces $\operatorname{div}\mathbf{q}$ to vanish, so that (5.5c) implies that $\mathbf{q}=0$; i.e., Π_h^k is uniquely determined on a boundary rectangle.

Since (5.6a) holds rectangle-by-rectangle, the remainder of the lemma follows easily.

A glance through the development of Sect. 3 shows that the arguments there used only the properties (5.6) and (5.7) and not the specific form of the spaces $\mathbf{V}_h^k$ and W_h^{k-1}. Hence, the error estimates of Theorem 3.4 are valid when the space based on triangular elements is replaced by one based on rectangular elements or, in fact, one based on mixing rectangular and triangular elements, since they have been designed to fit across straight-edges parallel to one of the axes.

It is again possible and profitable to introduce a hybrid version of the mixed procedure for rectangular elements. Again let $\mathscr{V}_h^k$ denote those vectors lying in $\mathbf{V}^k(R)$ for all $R\in\mathscr{T}_h$, and let M_h^k be the set of functions reducing to polynomials of degree k on each interior edge and to zero (or $-g$, when being used later to extend m_h and u_h to u_h^*) on boundary edges. Let $\{\mathbf{q}_h, u_h, m_h\}$ again denote the solution of (1.13), now for rectangular elements, and note that Lemma 4.1 is valid also for the rectangular case.

The Lagrange multiplier m_h can again be exploited to produce a superconvergent approximation u_h^* to u. The local Nitsche procedure (4.23) can be applied to the rectangular decomposition, and the error estimate (4.24) can again be derived for $k\geqq 2$; in fact, the proof of Theorem 4.4 covers this case. It is also possible to give an analogue of the Lemma 4.2 of Arnold and Brezzi in the rectangular case, this time for all $k\geqq 1$.

Lemma 5.2. *Let*

$$Q^{k+1}(R)=P_{k+1}(R)\oplus\operatorname{Span}\{x^{k+1}y, xy^{k+1}, q^{k+1}\}, \tag{5.8}$$

where

$$q^{k+1}(x,y)=\begin{cases} x^{k+2}y-xy^{k+2}, & k \text{ odd}, k\geqq 1,\\ x^{k+2}-y^{k+2}, & k \text{ even}, k\geqq 2.\end{cases} \tag{5.9}$$

Let $u_h^*\in Q^{k+1}(R)$ satisfy

$$\langle u_h^*-m_h, p\rangle_{e_i}=0, \qquad p\in P_k(e_i),\ i=1,2,3,4; \tag{5.10a}$$

$$(u_h^*-u_h, w)_R=0, \qquad w\in P_{k-3}(R)+\operatorname{Span}(l_{k-1}(x)l_{k-1}(y)), \tag{5.10b}$$

where l_{k-1} *is the Laguerre polynomial* (*or, more precisely, the ultraspherical Gegenbauer polynomial* $C_{k-1}^{(3/2)}$, [10]) *of degree* $k-1$; i.e., *on the interval* $[-1,1]$, *these polynomials are orthogonal with respect to the weight function* $1-t^2$:

$$\int_{-1}^{1} l_i(t)l_j(t)(1-t^2)dt=\delta_{ij}, \tag{5.11a}$$

$$l_0(t)=2^{-1/2}. \tag{5.11b}$$

Then, u_h^* *is uniquely determined by the degrees of freedom* (5.10) *and*

$$\|u_h^*\|_{0,R} \leqq K\{\|u_h\|_{0,R}+h_R^{1/2}\|m_h\|_{0,\partial R}\}. \tag{5.12}$$

Proof. The number of degrees of freedom is $\frac{1}{2}(k^2+5k+12)=\dim P_{k+1}+3=\dim Q^{k+1}(R)$; thus, the lemma will follow if we can demonstrate unisolvence. For this purpose we can take R to be the square $[-1,1]^2$ and renormalize l_{k-1} so that $l_{k-1}(1)=1$.

Consider first the case of even k, and let $z\in Q^{k+1}(R)$,

$$z(x,y)=p(x,y)+c_1x^{k+1}y+c_2xy^{k+1}+c_3(x^{k+2}-y^{k+2}),$$

with $p\in P_{k+1}(R)$, have vanishing degrees of freedom:

$$\langle z,w\rangle_{e_i}=0, \quad w\in P_k(e_i), \quad i=1,2,3,4; \tag{5.13a}$$

$$(z,w)_R=0, \quad w\in P_{k-3}(R)+\operatorname{Span}(l_k-1(x)l_{k-1}(y)). \tag{5.13b}$$

Let L_k denote the Legendre polynomial of degree k on $[-1,1]$, again normalized so that $L_k(1)=1$. Then, (5.13a) implies that

$$z|_{e_i}=a_iL_{k+1}(t)+b_iL_{k+2}(t), \quad i=1,2,3,4. \tag{5.14}$$

In fact, $b_i=0$. To see this, label the top of R to be e_1 and let the remaining edges be numbered in the clockwise direction. Assume that $b_1=1$. Then, $c_3=1$, so that $b_1=b_3=1$ and $b_2=b_4=-1$. Then, continuity of z at the vertices implies, as L_{k+1} is odd and L_{k+2} even, that

$$\begin{aligned} a_1+1&=a_2-1,\\ -a_2-1&=a_3+1,\\ -a_3+1&=-a_4-1,\\ a_4-1&=-a_1+1, \end{aligned}$$

from which it follows that $a_1=a_1-8$; hence, $b_i=0$. So,

$$z|_{e_i}=a_iL_{k+1}(t), \quad a_1=a_2=-a_3=-a_4=a.$$

Set

$$\mathscr{L}_{k+1}(x,y)=xL_{k+1}(y)+yL_{k+1}(x)-xy, \tag{5.15}$$

so that $\mathscr{L}_{k+1}\in Q^{k+1}(R)$ and

$$z=a\mathscr{L}_{k+1}+(1-x^2)(1-y^2)w, \tag{5.16}$$

where, a priori, $w \in P_{k-2}(R)$. Let $b(x, y)=(1-x^2)(1-y^2)$ and write w in the form

$$w(x, y)=q+\sum_{r+s=k-2} c_{rs} x^r y^s, \quad q \in P_{k-3}(R).$$

Since $b(x, y) x^r y^s$ contains the factor $x^{r+2} y^{s+2}$, which does not belong to Q^{k+1}, it follows that $w \in P_{k-3}$ and

$$w(x, y)=\sum_{r+s \leqq k-3} c_{rs} l_r(x) l_s(y).$$

Now, apply (5.13b) to (5.16). Since

$$(x L_{k+1}(y)+y L_{k+1}(x), p)_R=0$$

for $p \in P_{k-3}(R) \oplus \operatorname{Span}(l_{k-1}(x) l_{k-1}(y))$,

$$(-a x y+b w, p)_R=0, \quad p \in P_{k-3}(R) \oplus \operatorname{Span}(l_{k-1}(x) l_{k-1}(y)). \tag{5.17}$$

The relations of (5.17) can be represented in matricial form as

$$\left[\begin{array}{c|cccc} A_{11} & & & 0 & \\ \hline & D_{11} & & & \\ & & \ddots & & 0 \\ B_j & & & \ddots & \\ & 0 & & & D_{nn} \end{array}\right]\left[\begin{array}{c} -a \\ \hline \\ c_{rs} \\ \\ \end{array}\right]=0, \tag{5.18}$$

where

$$A_{11}=(x y, l_{k-1}(x) l_{k-1}(y))_R=\left(\int_{-1}^{1} l_{k-1}(t) t\, dt\right)^2 \tag{5.19}$$

and $D_{ii}>0$. Consequently, unisolvence follows if $A_{11} \neq 0$. Since $t=c l_1(t)$, it suffices to show that, for odd $j \geqq 3$,

$$I_j=\int_{-1}^{1} l_1(t) l_j(t) dt=\int_{-1}^{1} l_1(t) l_j(t) t^2 dt \neq 0. \tag{5.20}$$

Assume not; i.e., assume $I_j=0$ for some odd $j \geqq 3$. Recall the recursion formula

$$l_{r+2}(t)=(t^2-c_r) l_r(t)-d_r l_{r-2}(t), \quad l_{-1}(t)=0.$$

Then, with $r=1$,

$$\int_{-1}^{1} l_3(t) l_j(t) dt=\int_{-1}^{1}(t^2-c_1) l_1(t) l_j(t) dt=0,$$

and, if $j \neq 3$,

$$\int_{-1}^{1} l_3(t) l_j(t) t^2 dt=0.$$

A recursive application of this argument shows that

$$\int_{-1}^{1} l_j(t)^2\,dt=0,$$

a contradiction; hence, $I_j \neq 0, j$ odd.

This completes the proof of Lemma 5.2 for even k.

Now, let k be odd. Let

$$z(x,y)=p(x,y)+c_1 x^{k+1}y+c_2 x y^{k+1}+c_3(x^{k+2}y-xy^{k+2}),$$

$p\in P_{k-3}(R)$, be an element of $Q^{k+1}(R)$ having vanishing degrees of freedom. Then, (5.10a) implies that

$$z|_{e_i}=a_i L_{k+1}(t)+b_i L_{k+2}(t), \qquad i=1,2,3,4;$$

again continuity of z at the vertices implies that $b_i=0$ and $a_1=a_2=a_3=a_4=a$. Now let

$$\mathscr{L}_{k+1}(x,y)=L_{k+1}(x)+L_{k+1}(y)-1.$$

Then, $\mathscr{L}_{k+1}\in Q^{k+1}$ and $\mathscr{L}_{k+1}|_{e_i}=L_{k+1}$. Thus,

$$z=a\mathscr{L}_{k+1}+b(x,y)w, \qquad w\in P_{k-1}.$$

A similar argument to the one above shows that $w\in P_{k-3}$, which leads us to the following analogue of (5.17):

$$(-a+bw,p)_R=0, \qquad p\in P_{k-3}(R)\oplus \operatorname{Span}(l_{k-1}(x)l_{k-1}(y)). \tag{5.21}$$

For this case, unisolvence follows if

$$\begin{aligned}A_{11}=(1,l_{k-1}(x)l_{k-1}(y))_R&=\left(\int_{-1}^{1} l_{k-1}(t)\,dt\right)^2\\&=\text{const.}\left(\int_{-1}^{1} l_0(t)l_{k-1}(t)\,dt\right)^2\neq 0.\end{aligned}$$

An argument of the same type as above shows that, if

$$\int_{-1}^{1} l_j(t)\,dt=0$$

for some even j, then $l_j\equiv 0$. Thus, the proof of Lemma 5.2 is completed in all cases.

For $k\geqq 2$, let $u_h^*|_R\in Q^{k+1}(R)$ be determined by (5.10), where $\{\mathbf{q}_h, u_h, m_h\}$ is the solution of (1.13). For $k=1$, let $u_h^*|_R\in Q^1(R)$. Then, it is clear that

$$\|u-u_h^*\|_0\leqq\begin{cases}K(\|f\|_0+|g|_{3/2})h^2, & k=1,\\ K(\|f\|_k+|g|_{k+3/2})h^{k+2}, & k\geqq 2.\end{cases} \tag{5.22}$$

Let is turn to the computational aspects of the method (1.13) associated with rectangular elements; i.e., consider the equations (4.31) for the rectangular case. Note that

$$\dim \mathscr{V}^k(R)=(k+1)(k+2)+2=k^2+3k+4,$$

whereas

$$\dim \mathscr{R}\mathscr{T}^k(R)=2(k+1)(k+2)=2k^2+6k+4,$$

which is essentially twice as great. Thus, the reduction from (4.31) to (4.32) is significantly less expensive for our elements than for the Raviart-Thomas ones. Also,

$$\dim W^{k-1}(R)=\tfrac{1}{2}(k^2-k),$$

versus $(k+1)^2$ for the Raviart-Thomas scalar part. Again, the elimination of the u_h-parameters is cheaper for our elements. The matrix $\mathscr{D}$ of (4.33) again has the same structure for both our and the Raviart-Thomas rectangular elements. The form of $\mathscr{D}$ lends itself to the development of efficient iterative methods for the solution of (4.33); this issue will be discussed elsewhere.

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