Numer. Math. 46, 521–539 (1985)

Numerische Mathematik

On a Difference Scheme of Exponential Type for a Nonlinear Singular Perturbation Problem

Koichi Niijima

Department of Mathematics, Kyushu University, Fukuoka, Japan

Summary. A difference scheme of exponential type for solving a nonlinear singular perturbation problem is analysed. Although this scheme is not of monotone type, a L^1 convergence result is obtained. Relations between this scheme and Engquist-Osher scheme are also discussed.

Subject Classifications. AMS(MOS): 65J10; CR: G17.

1. Introduction

We consider a singular perturbation problem of the form

$$\varepsilon y'' - (f(y))' - b(x, y) = 0, \qquad 0 < x < 1, \tag{1.1a}$$

$$y(0) = A, \qquad y(1) = B, \tag{1.1b}$$

where ε is a small positive parameter. For solving this problem numerically, we derived in [3] the following difference scheme:

$$\begin{aligned} F_h(u)|_i \equiv \frac{\varepsilon}{h^2}\left[\int_{u_i}^{u_{i+1}} \xi\left(\frac{a(s)}{2}\rho\right) ds - \int_{u_{i-1}}^{u_i} \xi\left(\frac{a(s)}{2}\rho\right) ds\right] \\ -\frac{1}{2h}(f(u_{i+1}) - f(u_{i-1})) - b(x_i, u_i) = 0, \qquad i = 1, \dots, N-1, \end{aligned} \tag{1.2}$$

$$u_0 = A, \qquad u_N = B.$$

Here h denotes a mesh width $h = 1/N$ and $\xi(\mu)$ is a function defined by $\xi(\mu) = \mu \coth(\mu)$ in which $\mu = \frac{a(s)}{2}\rho$, where $a(s) = \frac{df(s)}{ds}$ and $\rho = \frac{h}{\varepsilon}$. But we could not analyse this scheme in [3]. In this paper, we shall give a L^1 convergence result for the scheme (1.2).

We put the following assumptions on $f(y)$ and $b(x, y)$:

H1. $f(y)$ belongs to $C^2(R)$ and $b(x, y)$ is in $C^1([0, 1] \times R)$, where $R = (-\infty, \infty)$,

H2. $b_y(x, y) \geqq \delta > 0$ on $[0, 1] \times R$.

Under these conditions, we show that (1.2) has a unique solution which converges to a correct solution of (1.1) in L^1 sense for all values of ε.

Recently, Osher [4] proposed a difference scheme for solving (1.1):

$$\begin{aligned} E_h(u)|_i \equiv \frac{\varepsilon}{h^2}(u_{i+1}-2u_i+u_{i-1}) - \frac{1}{h}\left(\int_{u_i}^{u_{i+1}} a_-(s)\,ds + \int_{u_{i-1}}^{u_i} a_+(s)\,ds\right) \\ -b(x_i, u_i)=0, \quad i=1, \ldots, N-1, \\ u_0=A, \quad u_N=B, \end{aligned} \tag{1.3}$$

where $a_-(s)=\min(a(s),0)$ and $a_+(s)=\max(a(s),0)$. This scheme is often called Engquist-Osher scheme (E-O scheme) which reproduces essential properties of the true solution, especially, interior shocks and boundary layers, very well. Our scheme is also sensitive to such phenomena.

Abrahamsson and Osher considered, in a more recent paper [1], a class of monotone difference schemes which include E-O scheme, and proved that the solutions of such schemes are of bounded variation uniformly in ε and h. This property plays an essential role in obtaining their L^1 convergence result. But since our scheme is not of monotone type in the sense of Abrahamsson and Osher, their method for getting a uniform bounded variation estimate is not applicable to our scheme. We devise in Sect. 4 a method to obtain this estimate. We remark there that our method can also be applied to E-O scheme. In Sect. 5, we give a L^1 convergence result. We then discuss how our scheme relates to E-O scheme. Section 6 is devoted to present some numerical results.

2. Solutions of the Continuous Problem

We summarise some known results concerning the continuous problem (1.1), which will be needed later.

Lemma 1. *The problem* (1.1) *has a unique solution* $y(x)$ *satisfying*

$$\max_{0\leqq x\leqq 1} |y(x)| \leqq \max\left(|A|, |B|, \frac{1}{\delta}\max_{0\leqq x\leqq 1}|b(x,0)|\right) \equiv C_0. \tag{2.1}$$

Lemma 2. *There exists a constant* C_1 *independent of* ε *such that for the solution* $y(x)$ *of* (1.1),

$$\int_0^1 |y'(x)|\,dx \leqq C_1. \tag{2.2}$$

These results have been obtained, for instance, by Lorenz [2].

3. Solutions of our Difference Scheme

In this section, we shall show the existence and uniqueness of solutions to the scheme (1.2). We also seek the range of the solution.

Theorem 1. *Let $y(x)$ be a solution of* (1.1). *For a positive constant r to be determined later, define an open ball V by*

$$V=\left\{v \mid \|v-y\|_1=h\sum_{i=0}^{N}|v_i-y(x_i)|<r,\ v=(v_0,\ldots,v_N)\right\}$$

and its closure by $\bar{V}$. Then (1.2) *has a solution in $\bar{V}$.*

Proof. As a tool of the proof, we use the contraction mapping theorem [5; p. 65] as employed in [3]. Choose a positive number k so that

$$k\left[\frac{1}{h}\max_{|z|\leqq C_0+r/h} a(z)\coth\left(\frac{a(z)}{2}\rho\right)+\max_{\substack{0\leqq x\leqq 1\\ |z|\leqq C_0+r/h}} b_y(x,z)\right]<1, \tag{3.1}$$

where C_0 denotes the constant defined in (2.1). With the above k, we define

$$G_h(u)|_i=\begin{cases}A, & i=0,\\ u_i+kF_h(u)|_i, & i=1,\ldots,N-1,\\ B, & i=N.\end{cases}$$

Introducing two vectors

$$u=(u_0,\ldots,u_N)$$

and

$$G_h(u)=(G_h(u)|_0,\ldots,G_h(u)|_N),$$

we consider an equation of the vector form

$$u=G_h(u), \tag{3.2}$$

which is equivalent to (1.2). In exactly the same way as the proof of Theorem 4 in [3], we can show that for any v and w in V,

$$\|G_h(v)-G_h(w)\|_1\leqq(1-k\delta)\|v-w\|_1.$$

Namely, the operator G_h is a contraction mapping in V with the contraction factor $1-k\delta$.

Next, we shall estimate the local truncation error $F_h(y)|_i$ to apply the contraction mapping theorem to (3.2). From now on, the symbol C is used as a generic constant independent of mesh points, ε and h. Remark in the first that $F_h(y)|_i$ can be decomposed as

$$\begin{aligned}F_h(y)|_i=L_h(y)|_i&+\frac{\varepsilon}{h^2}\int_{y_i}^{y_{i+1}}\left[\xi\left(\frac{a(s)}{2}\rho\right)-\xi\left(\frac{a(y_i)}{2}\rho\right)\right]ds\\&-\frac{\varepsilon}{h^2}\int_{y_{i-1}}^{y_i}\left[\xi\left(\frac{a(s)}{2}\rho\right)-\xi\left(\frac{a(y_i)}{2}\rho\right)\right]ds\\&-\frac{1}{2h}\int_{y_i}^{y_{i+1}}(y_{i+1}-s)\,a'(s)\,ds+\frac{1}{2h}\int_{y_{i-1}}^{y_i}(s-y_{i-1})\,a'(s)\,ds,\end{aligned} \tag{3.3}$$

where

$$L_h(y)|_i=\frac{\varepsilon}{h^2}\,\xi\left(\frac{a(y_i)}{2}\rho\right)(y_{i+1}-2y_i+y_{i-1})-a(y_i)\,\frac{y_{i+1}-y_{i-1}}{2h}-b(x_i,y_i).$$

Furthermore, the term $L_h(y)|_i$ may be written as

$$\begin{aligned} L_h(y)|_i &= \int_{x_{i-1}}^{x_{i+1}} K_i(t)(a(y(t))-a(y_i))\, y'(t)\, dt \\ &\quad + \int_{x_{i-1}}^{x_{i+1}} K_i(t)(b(t, y(t))-b(x_i, y_i))\, dt \\ &\equiv I_1+I_2 \end{aligned}$$

in which $K_i(x)$ is given by

$$K_i(x)=\begin{cases} \dfrac{\tau_i-1}{h(\tau_i+\tau_i^{-1})}\left[1-\exp\left(-a(y_i)\,\rho\,\dfrac{x-x_{i-1}}{h}\right)\right], & x_{i-1}\leqq x<x_i, \\ \dfrac{1-\tau_i^{-1}}{h(\tau_i+\tau_i^{-1})}\left[\exp\left(a(y_i)\,\rho\,\dfrac{x_{i+1}-x}{h}\right)-1\right], & x_i\leqq x<x_{i+1} \end{cases}$$

with $\tau_i=\exp(a(y_i)\,\rho)$. Such a technique has also been used in [3].

We start with the estimation of $L_h(y)|_i$. For latter use, we put

$$d_i=\int_{x_{i-1}}^{x_{i+1}} |y'(t)|\, dt.$$

The term I_1 is estimated as follows:

$$\begin{aligned} |I_1| &\leqq C\int_{x_{i-1}}^{x_{i+1}} K_i(t)\left|\int_{x_i}^{t} y'(s)\, ds\right| |y'(t)|\, dt \\ &\leqq C\int_{x_{i-1}}^{x_{i+1}} K_i(t)\int_{x_{i-1}}^{x_{i+1}} |y'(s)|\, ds\, |y'(t)|\, dt \\ &\leqq \frac{C}{h}\, d_i^2, \end{aligned}$$

where we used the estimate $\max_{|s|\leqq C_0} |a'(s)|\leqq C$ to get the first line, and the inequalities $0\leqq K_i(x)\leqq C/h$ in the last line.

The second term I_2 is bounded by $C(d_i+h)$ because of the equality $\int_{x_{i-1}}^{x_{i+1}} K_i(t)\, dt=1$ and assumption H1. Therefore, we have

$$|L_h(y)|_i|\leqq C\left(\frac{d_i^2}{h}+d_i+h\right). \tag{3.4}$$

Next, we shall estimate the remaining terms of (3.3). Noticing that $|(\mu\coth(\mu))'|\leqq 1$ holds for all μ, we have

$$\begin{aligned} \left|\xi\left(\frac{a(s)}{2}\,\rho\right)-\xi\left(\frac{a(y_i)}{2}\,\rho\right)\right| &\leqq \left|\int_{a(y_i)\,\rho/2}^{a(s)\,\rho/2} |(\mu\coth(\mu))'|\, d\mu\right| \\ &\leqq \frac{\rho}{2}\, |a(s)-a(y_i)| \\ &\leqq C\rho\, |s-y_i|. \end{aligned}$$

Hence we obtain

$$\left|\frac{\varepsilon}{h^2}\int_{y_i}^{y_{i+1}}\left[\xi\left(\frac{a(s)}{2}\rho\right)-\xi\left(\frac{a(y_i)}{2}\rho\right)\right]ds\right|\leqq\frac{C d_i^2}{h}$$

and

$$\left|\frac{\varepsilon}{h^2}\int_{y_{i-1}}^{y_i}\left[\xi\left(\frac{a(s)}{2}\rho\right)-\xi\left(\frac{a(y_i)}{2}\rho\right)\right]ds\right|\leqq\frac{C d_i^2}{h}.$$

It is easy to get

$$\left|\frac{1}{2h}\int_{y_i}^{y_{i+1}}(y_{i+1}-s)\,a'(s)\,ds\right|\leqq\frac{C d_i^2}{h}$$

and

$$\left|\frac{1}{2h}\int_{y_{i-1}}^{y_i}(s-y_{i-1})\,a'(s)\,ds\right|\leqq\frac{C d_i^2}{h}.$$

Combining these estimates with (3.4) yields

$$|F_h(y)|_i|\leqq C\left(\frac{d_i^2}{h}+d_i+h\right).$$

From this inequality, we can derive the estimation of $\|y-G_h(y)\|_1$. Indeed, we have

$$\begin{aligned}\|y-G_h(y)\|_1&=kh\sum_{i=1}^{N-1}|F_h(y)|_i|\\&\leqq Ck\left(\sum_{i=1}^{N-1}d_i^2+h\sum_{i=1}^{N-1}d_i+h\right)\\&\leqq Ck(4C_1^2+2C_1h+h),\end{aligned}$$

since it holds that $\sum_{i=1}^{N-1}d_i\leqq 2\int_0^1|y'(t)|\,dt\leqq 2C_1$ by Lemma 2 and that

$$\sum_{i=1}^{N-1}d_i^2\leqq\left(\sum_{i=1}^{N-1}d_i\right)^2\leqq 4C_1^2.$$

We thus get

$$\frac{1}{1-\kappa}\|y-G_h(y)\|_1\leqq\frac{C}{\delta}(4C_1^2+2C_1h+h),$$

where $\kappa=1-k\delta$. Since we can choose the radius r to be larger than the right hand side, the contraction mapping theorem is applicable to (3.2), and ensures that (3.2) has a solution in $\bar{V}$. This ends the proof of Theorem 1.

This theorem guarantees the existence of solutions to (3.2), namely, (1.2). Next, we shall show that any solution of (1.2) lies in the ball

$$W=\{v=(v_0,\ldots,v_N)\mid\max_{i=0,\ldots,N}|v_i|\leqq C_0\},$$

where C_0 denotes the constant defined in (2.1).

Theorem 2. *Any solution of* (1.2) *lies in W. Moreover,* (1.2) *has only one solution in W.*

Proof. We rewrite (1.2) as follows:

$$F_h(u)|_i = \int_{u_i}^{u_{i+1}} \alpha(s)\,ds - \int_{u_{i-1}}^{u_i} \beta(s)\,ds - b(x_i, u_i) = 0, \qquad i=1, \dots, N-1,$$

$$u_0 = A, \qquad u_N = B, \tag{3.5}$$

where we put

$$\alpha(s) = \frac{\varepsilon}{h^2}\,\xi\left(\frac{a(s)}{2}\rho\right) - \frac{a(s)}{2h},$$

$$\beta(s) = \frac{\varepsilon}{h^2}\,\xi\left(\frac{a(s)}{2}\rho\right) + \frac{a(s)}{2h}.$$

We further apply the change of variables $s=(1-\theta)\,u_i+\theta u_{i+1}$ and $s=(1-\theta)\,u_{i-1}+\theta u_i$ to $\alpha(s)$ and $\beta(s)$, respectively, and put

$$p_{i+1} = \int_0^1 \alpha((1-\theta)\,u_i + \theta u_{i+1})\,d\theta,$$

$$q_i = \int_0^1 \beta((1-\theta)\,u_{i-1} + \theta u_i)\,d\theta.$$

Then (3.5) may be written as

$$F_h(u)|_i = p_{i+1}(u_{i+1}-u_i) - q_i(u_i - u_{i-1}) - b(x_i, u_i) = 0. \tag{3.6}$$

We apply Taylor's theorem to $b(x_i, u_i)$ and rewrite this equation as

$$-q_i\,u_{i-1} + (p_{i+1} + q_i + b_y(x_i, w_i))\,u_i - p_{i+1}\,u_{i+1} = -b(x_i, 0),$$

where w_i is a number between 0 and u_i. Since $\alpha(s)$ and $\beta(s)$ are positive for all s, it holds that $p_{i+1}>0$ and $q_i>0$. We also have $b_y(x_i, w_i) \geqq \delta$ by assumption H2. Therefore, the first assertion follows by the discrete maximum principle. The second assertion follows from the fact that the operator G_h is a contraction mapping also in W.

This theorem implies that the solutions of (1.1) and (1.2) exist in the same range.

4. Uniform Bounded Variation Estimate

Abrahamsson and Osher [1] considered three-point difference schemes of the form

$$\frac{\varepsilon}{h^2}(u_{i+1} - 2u_i + u_{i-1}) - \frac{1}{h}(g(u_{i+1}, u_i) - g(u_i, u_{i-1})) - b(x_i, u_i) = 0, \qquad i=1, \dots, N-1,$$

$$u_0 = A, \qquad u_N = B. \tag{4.1}$$

Here the flux function $g(u, v)$ satisfies the consistency condition

$$g(u, u)=f(u) \tag{4.2}$$

and the monotonicity condition

$$g_u(u, v)\leqq 0\leqq g_v(u, v). \tag{4.3}$$

They called this type of schemes monotone schemes and proved that the solution $u=(u_0, \ldots, u_N)$ to (4.1) fulfills

$$\sum_{i=1}^{N} |u_i - u_{i-1}| \leqq C_2, \tag{4.4}$$

where C_2 is a constant independent of ε and h. However, our scheme (1.2) is not a monotone scheme. To verify this, we rewrite (1.2) in the form

$$\frac{\varepsilon}{h^2}(u_{i+1}-2u_i+u_{i-1})-\frac{\varepsilon}{h^2}\left(\int_{u_i}^{u_{i+1}} \eta(s)\,ds-\int_{u_{i-1}}^{u_i} \zeta(s)\,ds\right)-b(x_i, u_i)=0 \tag{4.5}$$

in which

$$\eta(s)=1-\xi\left(\frac{a(s)}{2}\rho\right)+\frac{a(s)}{2}\rho=1-\frac{2\mu\exp(-\mu)}{\exp(\mu)-\exp(-\mu)},$$

$$\zeta(s)=1-\xi\left(\frac{a(s)}{2}\rho\right)-\frac{a(s)}{2}\rho=1-\frac{2\mu\exp(\mu)}{\exp(\mu)-\exp(-\mu)},$$

where $\mu=\frac{a(s)}{2}\rho$. If we put

$$g(u, v)=\frac{\varepsilon}{h}\left(\int_0^u \eta(s)\,ds-\int_0^v \zeta(s)\,ds\right)+f(0),$$

then (4.5) is reduced to the form (4.1). This flux function fulfills certainly the consistency relation (4.2) because of $g(u, u)=\int_0^u a(s)\,ds+f(0)=f(u)$. But the monotonicity condition (4.3) is not satisfied, since $g_u(u, v)=\frac{\varepsilon}{h}\eta(u)$ and $g_v(u, v)=-\frac{\varepsilon}{h}\zeta(v)$ may have both signs. Thus another methods are required to establish the uniform bounded variation estimate (4.4) for solutions to our scheme. The following theorem gives such one method.

Theorem 3. *The solution* $u=(u_0, \ldots, u_N)$ *of* (1.2) *satisfies the estimate* (4.4).

Proof. It is convenient to use the form (3.6) instead of (1.2). From (3.6), we get for $i=2, \ldots, N-1$,

$$\begin{aligned}0&=F_h(u)|_i-F_h(u)|_{i-1}\\&=p_{i+1}(u_{i+1}-u_i)+q_{i-1}(u_{i-1}-u_{i-2})\\&\quad-(p_i+q_i)(u_i-u_{i-1})-[b(x_i, u_i)-b(x_{i-1}, u_{i-1})].\end{aligned}$$

Using further the Taylor's expansion

$$b(x_i, u_i) - b(x_{i-1}, u_{i-1}) = b_x(\bar{x}_i, u_i)\, h + b_y(x_{i-1}, \bar{u}_i)(u_i - u_{i-1}),$$

we have

$$\begin{aligned}&(b_y(x_{i-1}, \bar{u}_i) + p_i + q_i)(u_i - u_{i-1})\\ &= q_{i-1}(u_{i-1} - u_{i-2}) + p_{i+1}(u_{i+1} - u_i) - b_x(\bar{x}_i, u_i)\, h.\end{aligned} \tag{4.6}$$

We now define $\operatorname{sgn}(z)$ by

$$\operatorname{sgn}(z) = \begin{cases} z/|z| & \text{for } z \neq 0, \\ 0 & \text{for } z = 0. \end{cases}$$

On multiplying (4.6) by $\operatorname{sgn}(u_i - u_{i-1})$ and summing the resulting equality over $i = 2, \ldots, N-1$, we obtain after some changes of subscripts,

$$\begin{aligned}&\sum_{i=2}^{N-1} (\delta + p_i + q_i)|u_i - u_{i-1}|\\ &\quad \leqq \sum_{i=3}^{N-1} p_i |u_i - u_{i-1}| + \sum_{i=2}^{N-2} q_i |u_i - u_{i-1}|\\ &\quad + p_N(u_N - u_{N-1}) \operatorname{sgn}(u_{N-1} - u_{N-2})\\ &\quad + q_1(u_1 - u_0) \operatorname{sgn}(u_2 - u_1) + C_3.\end{aligned}$$

Here C_3 is a constant independent of ε and h such that $|b_x(\bar{x}_i, u_i)| \leqq C_3$. Since some terms in the above inequality are cancelled out from both sides, we get

$$\begin{aligned}&\delta \sum_{i=2}^{N-1} |u_i - u_{i-1}| + p_2 |u_2 - u_1| + q_{N-1} |u_{N-1} - u_{N-2}|\\ &\quad \leqq C_3 + q_1(u_1 - u_0) \operatorname{sgn}(u_2 - u_1) + p_N(u_N - u_{N-1}) \operatorname{sgn}(u_{N-1} - u_{N-2}).\end{aligned} \tag{4.7}$$

We need again (3.6) for $i = 1$ and $i = N-1$. Multiplying (3.6) for $i = 1$ by $\operatorname{sgn}(u_2 - u_1)$, we have

$$p_2 |u_2 - u_1| = q_1(u_1 - u_0) \operatorname{sgn}(u_2 - u_1) + b(x_1, u_1) \operatorname{sgn}(u_2 - u_1). \tag{4.8}$$

Next, we multiply (3.6) for $i = N-1$ by $\operatorname{sgn}(u_{N-1} - u_{N-2})$ to get

$$\begin{aligned}q_{N-1} |u_{N-1} - u_{N-2}| &= p_N(u_N - u_{N-1}) \operatorname{sgn}(u_{N-1} - u_{N-2})\\ &\quad - b(x_{N-1}, u_{N-1}) \operatorname{sgn}(u_{N-1} - u_{N-2}).\end{aligned} \tag{4.9}$$

Combining (4.7), (4.8) and (4.9) gives

$$\delta \sum_{i=2}^{N-1} |u_i - u_{i-1}| \leqq C_3 + 2\, C_4,$$

where $C_4 = \max\limits_{\substack{0 \leqq x \leqq 1 \\ |z| \leqq C_0}} |b(x, z)|$. The desired result follows from this inequality and two estimates $|u_1 - u_0| \leqq C_0 + |A|$ and $|u_N - u_{N-1}| \leqq C_0 + |B|$.

Our method for proving this theorem can be applied to E-O scheme. It is only remarked that E-O scheme (1.3) is reduced to the form (3.5), if we define $\alpha(s)$ and $\beta(s)$ by

$$\alpha(s)=\frac{\varepsilon}{h^2}-\frac{a_-(s)}{h},$$

$$\beta(s)=\frac{\varepsilon}{h^2}+\frac{a_+(s)}{h}$$

both of which are positive for all s.

5. L^1 Convergence Result

By virtue of Theorems 2 and 3, we can derive the same L^1 convergence result as in [4] and [1]. Let $U_h^\varepsilon(x)$ be a step function defined by

$$U_h^\varepsilon(x)=u_i, \qquad x_i \leqq x < x_{i+1}, \qquad i=0, \ldots, N-1,$$

where $u=(u_0, \ldots, u_N)$ is a solution of (1.2). A key point in our analysis is to show that the family $\{U_h^\varepsilon\}$ is precompact in $L^1(0, 1)$, as was so in [4] and [1]. Although this result was proved by Sanders [6], we shall offer an alternate proof of it.

Lemma 3. *The family* $\{U_h^\varepsilon\}$ *is precompact in* $L^1(0, 1)$.

Proof. For any $\gamma>0$, we can choose sufficiently large positive integers n and M such that

$$\frac{C_0}{M}+\frac{C_2}{n} \leqq \gamma \tag{5.1}$$

in which C_0 and C_2 are constants defined in (2.1) and (4.4), respectively. We divide the interval $[0, 1]$ into n equidistant subintervals, and define grid points by

$$0=t_0<t_1<\ldots<t_n=1.$$

Define a set S of step functions $T(x)$ having the following form

$$T(x)=\frac{m}{M} C_0, \qquad t_j \leqq x < t_{j+1}, \qquad j=0, \ldots, n-1,$$

where m is an integer with $|m| \leqq M$. This set is finite. We show that any element U_h^ε in the family $\{U_h^\varepsilon\}$ is approximated by some function in S. For the function $U_h^\varepsilon(x)$, we introduce an auxiliary step function $\hat{U}_h^\varepsilon(x)$:

$$\hat{U}_h^\varepsilon(x)=U_h^\varepsilon(t_j), \qquad t_j \leqq x < t_{j+1}, \qquad j=0, \ldots, n-1.$$

We first show that $\hat{U}_h^\varepsilon$ is approximated by some function in S. Since $\max_{0 \leqq x \leqq 1} |U_h^\varepsilon(x)| \leqq C_0$, there exists an integer m_j such that

$$\frac{m_j}{M} C_0 \leqq U_h^\varepsilon(t_j) < \frac{m_j+1}{M} C_0.$$

Therefore, a step function

$$T_a(x) = \frac{m_j}{M} C_0, \quad t_j \leqq x < t_{j+1}, \quad j = 0, \ldots, n-1$$

gives an approximant of $\hat{U}_h^\varepsilon(x)$, because it holds that

$$\int_0^1 |\hat{U}_h^\varepsilon(x) - T_a(x)| \, dx = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left(U_h^\varepsilon(t_j) - \frac{m_j}{M} C_0 \right) dx < \frac{C_0}{M}. \tag{5.2}$$

Next, we estimate

$$J \equiv \int_0^1 |U_h^\varepsilon(x) - \hat{U}_h^\varepsilon(x)| \, dx.$$

The term J may be written as

$$J = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} |U_h^\varepsilon(x) - U_h^\varepsilon(t_j)| \, dx.$$

The subinterval $(t_j, t_{j+1}]$ includes at most finite mesh points x_j. We denote them by $x_{j,1}, \ldots, x_{j,l_j}$. We thus have

$$\begin{aligned} J_j &\equiv \int_{t_j}^{t_{j+1}} |U_h^\varepsilon(x) - U_h^\varepsilon(t_j)| \, dx = \int_{x_{j,1}}^{t_{j+1}} |U_h^\varepsilon(x) - u_{j-1, l_{j-1}}| \, dx \\ &= \sum_{i=1}^{l_j - 1} \left(\int_{x_{j,i}}^{x_{j,i+1}} + \int_{x_{j,l_j}}^{t_{j+1}} \right) |U_h^\varepsilon(x) - u_{j-1, l_{j-1}}| \, dx \\ &= h \sum_{i=1}^{l_j - 1} |u_{j,i} - u_{j-1, l_{j-1}}| + (t_{j+1} - x_{j,l_j}) |u_{j,l_j} - u_{j-1, l_{j-1}}| \\ &\leqq \frac{1}{n} \left\{ |u_{j,1} - u_{j-1, l_{j-1}}| + \sum_{i=1}^{l_j - 1} |u_{j,i+1} - u_{j,i}| \right\}, \end{aligned}$$

where in the first line we used the fact that $U_h^\varepsilon(x) = U_h^\varepsilon(t_j) = u_{j-1, l_{j-1}}$ hold for $t_j < x < x_{j,1}$. Hence we get

$$J \leqq \frac{1}{n} \sum_{i=1}^{N} |u_i - u_{i-1}| \leqq \frac{C_2}{n}. \tag{5.3}$$

Combining (5.1), (5.2) and (5.3) yields

$$\int_0^1 |U_h^\varepsilon(x) - T_a(x)| \, dx < \gamma$$

which implies that $\{U_h^\varepsilon\}$ is precompact in $L^1(0, 1)$.

From this lemma, we obtain the following results.

Theorem 4. *The family $\{U_h^\varepsilon\}$ with $h \to 0$ has a converging subsequence in $L^1(0, 1)$ to $U_0^{\bar{\varepsilon}}$. If $\bar{\varepsilon} > 0$, then $U_0^{\bar{\varepsilon}}$ is a solution of (1.1) for $\varepsilon = \bar{\varepsilon}$. If $\bar{\varepsilon} = 0$, then U_0^0 is a weak solution of (1.1a) for $\varepsilon = 0$.*

Proof. By Lemma 3, we can pick up a subsequence $\{U_{h_\nu}^{\varepsilon_\nu}\}_{\nu=1,2,\ldots}$ from the family $\{U_h^\varepsilon\}$ with $h \to 0$ such that

$$U_{h_\nu}^{\varepsilon_\nu} \to U_0^{\bar{\varepsilon}} \quad \text{in } L^1(0,1) \tag{5.4}$$

as $\varepsilon_\nu \to \bar{\varepsilon}$ and $h_\nu \to 0$. For simplicity, we put $\varepsilon = \varepsilon_\nu$ and $h = h_\nu$.

When $\bar{\varepsilon} > 0$, we shall prove that

$$\int_0^1 |U_0^{\bar{\varepsilon}}(x) - y^{\bar{\varepsilon}}(x)|\, dx = 0, \tag{5.5}$$

where $y^{\bar{\varepsilon}}(x)$ denotes a solution of (1.1) for $\varepsilon = \bar{\varepsilon}$. Define a step function $Y^\varepsilon(x)$ by

$$Y^\varepsilon(x) = y^\varepsilon(x_i), \qquad x_i \leqq x < x_{i+1}, \qquad i = 0, \ldots, N-1$$

for the solution $y^\varepsilon(x)$ of (1.1). Then it follows from Lemma 2 that

$$\begin{aligned}\int_0^1 |Y^\varepsilon(x) - y^\varepsilon(x)|\, dx &\leqq h \int_0^1 |y^{\varepsilon\prime}(x)|\, dx \\ &\leqq C_1 h.\end{aligned} \tag{5.6}$$

We recall here the proof of Theorem 1 in which we have obtained the estimate

$$\|y^\varepsilon - G_h(y^\varepsilon)\|_1 \leqq Ck \left(\sum_{i=1}^{N-1} d_i^2 + h \sum_{i=1}^{N-1} d_i + h \right). \tag{5.7}$$

Since we have $d_i = \int_{x_{i-1}}^{x_{i+1}} |y^{\varepsilon\prime}(x)|\, dx \leqq 2h \max_{0 \leqq x \leqq 1} |y^{\varepsilon\prime}(x)|$, (5.7) yields

$$\frac{1}{1-\kappa} \|y^\varepsilon - G_h(y^\varepsilon)\|_1 \leqq C(\varepsilon) h$$

with $\kappa = 1 - k\delta$, where $C(\varepsilon)$ is a constant such that $\lim_{\varepsilon \to 0} C(\varepsilon) = \infty$. But since $\lim_{\varepsilon \to \bar{\varepsilon}} C(\varepsilon) = C(\bar{\varepsilon}) < \infty$ because of $\bar{\varepsilon} > 0$, the contraction mapping theorem gives

$$h \sum_{i=1}^{N-1} |u_i - y^\varepsilon(x_i)| \leqq C(\varepsilon) h$$

which implies

$$\int_0^1 |U_h^\varepsilon(x) - Y^\varepsilon(x)|\, dx \leqq C(\varepsilon) h. \tag{5.8}$$

Combining (5.6) with (5.8) leads to

$$\int_0^1 |U_h^\varepsilon(x) - y^\varepsilon(x)|\, dx \leqq (C_1 + C(\varepsilon)) h. \tag{5.9}$$

In the last, we notice that

$$\int_0^1 |y^\varepsilon(x) - y^{\bar{\varepsilon}}(x)|\, dx \to 0 \qquad \text{as } \varepsilon \to \bar{\varepsilon} \tag{5.10}$$

which follows from the continuity of $y^\varepsilon(x)$ with respect to the parameter ε.

Thus the inequality

$$\int_0^1 |U_0^{\bar{\varepsilon}}(x)-y^{\bar{\varepsilon}}(x)|\,dx \leqq \int_0^1 |U_0^{\bar{\varepsilon}}(x)-U_h^{\varepsilon}(x)|\,dx+\int_0^1 |U_h^{\varepsilon}(x)-y^{\varepsilon}(x)|\,dx$$
$$+\int_0^1 |y^{\varepsilon}(x)-y^{\bar{\varepsilon}}(x)|\,dx$$

together with (5.4), (5.9) and (5.10) implies (5.5).

When $\bar{\varepsilon}=0$, it is sufficient to prove that U_0^0 satisfies the following equation

$$\int_0^1 [\phi'(x) f(U_0^0(x))-\phi(x)\, b(x, U_0^0(x))]\,dx=0 \tag{5.11}$$

for any $\phi\in C_0^\infty(0,1)$. To do this, we shall find relations between our difference operator F_h and E-O difference operator E_h. Observing $a_+(s)=(|a(s)|+a(s))/2$ and $-a_-(s)=(|a(s)|-a(s))/2$, the term $E_h(u)|_i$ takes the form

$$E_h(u)|_i=\frac{\varepsilon}{h^2}(u_{i+1}-2u_i+u_{i-1})$$
$$+\frac{1}{2h}\left[\int_{u_i}^{u_{i+1}}(|a(s)|-a(s))\,ds-\int_{u_{i-1}}^{u_i}(|a(s)|+a(s))\,ds\right]-b(x_i, u_i).$$

Using this expression, we can get a relation between F_h and E_h as follows:

$$F_h(u)|_i=E_h(u)|_i-R_h(u)|_i. \tag{5.12}$$

Here $R_h(u)|_i$ denotes

$$R_h(u)|_i=\frac{\varepsilon}{h^2}\left(\int_{u_i}^{u_{i+1}}\lambda(s)\,ds-\int_{u_{i-1}}^{u_i}\lambda(s)\,ds\right)$$

in which

$$\lambda(s)=1-\frac{2|\mu(s)|\exp(-|\mu(s)|)}{\exp(|\mu(s)|)-\exp(-|\mu(s)|)}$$

with $\mu(s)=\frac{a(s)}{2}\rho$.

We now extend $U_h^\varepsilon(x)$ so that it takes the value u_0 for $-h\leqq x<0$, and the value u_N for $1\leqq x\leqq 1+h$, and denote the extended function again by $U_h^\varepsilon(x)$. Multiplying $F_h(U_h^\varepsilon)|_i=0$ by $h\phi(x_i)$ and summing over $i=0,\ldots,N$, we get from (5.12),

$$h\sum_{i=0}^N E_h(U_h^\varepsilon)|_i\,\phi(x_i)-h\sum_{i=0}^N R_h(U_h^\varepsilon)|_i\,\phi(x_i)=0.$$

In the same way as in [4], we see that $\lim\limits_{\substack{h\to 0\\ \varepsilon\to 0}}\left[h\sum_{i=0}^N E_h(U_h^\varepsilon)|_i\,\phi(x_i)\right]$ equals to the left hand side of (5.11). So it suffices to prove that

$$\lim_{\substack{h\to 0\\ \varepsilon\to 0}}\left[h\sum_{i=0}^N R_h(U_h^\varepsilon)|_i\,\phi(x_i)\right]=0. \tag{5.13}$$

An easy calculation yields

$$h \sum_{i=0}^{N} R_h(U_h^\varepsilon)|_i \,\phi(x_i) = \varepsilon \sum_{i=1}^{N} \int_{u_{i-1}}^{u_i} \lambda(s)\, ds \cdot \frac{\phi(x_i)-\phi(x_{i-1})}{h}. \tag{5.14}$$

Since $0 \leqq \lambda(s) \leqq |a(s)|\rho/2 \leqq C\rho$ holds, the right hand side of (5.14) is bounded by

$$Ch \sum_{i=1}^{N} |u_i - u_{i-1}| \frac{|\phi(x_i)-\phi(x_{i-1})|}{h}.$$

Using further Theorem 3, we get (5.13). This completes the proof.

The equation (5.13) implies that our scheme is near E-O scheme, though the former and the latter are of different type. From the viewpoint of computational task, E-O scheme is superior to our scheme. Because integral terms in E-O scheme are simpler than in our scheme. However, if we add the condition $|a(z)| \geqq \sigma > 0$ for $|z| \leqq C_0$ to H1 and H2, we can prove that the solution of our scheme converges uniformly in ε to the correct solution of (1.1) with $O(h)$ in the L^1 norm. This will be shown in the forthcoming paper which is in preparation.

6. Numerical Experiments

In this section, we shall present some numerical computations. Test problems have already been used as numerical examples in [1, 2] and [4]. For solving (1.2), we adopted an iteration method

$$u^{(\nu+1)} = G_h(u^{(\nu)}), \qquad \nu = 0, 1, 2, \ldots,$$

where the operator G_h was defined by (3.2). A parameter k contained in G_h was chosen to satisfy the condition (3.1). As the initial guess $u^{(0)}$, we chose a straight line between the boundary values. We performed all computations with $N=8$, 16, 32 and 64 for $\varepsilon = 10^{-6}$.

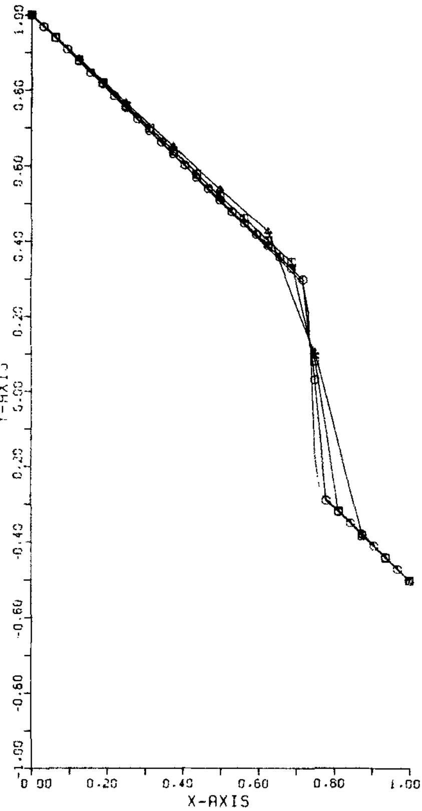
Figures 1 to 4 show computed solutions of the problem

$$\varepsilon y'' - \tfrac{1}{2}(y^2)' - y = 0, \qquad 0 < x < 1,$$
$$y(0) = A, \qquad y(1) = B,$$

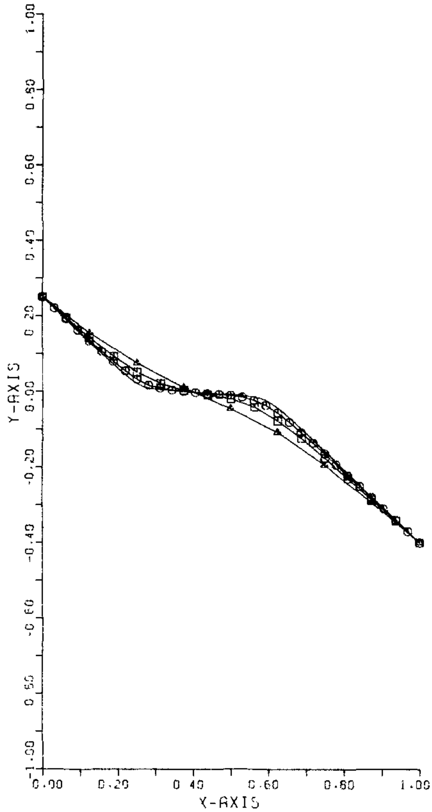
where A and B are given in the table below:

Fig.	1	2	3	4
A	1.0	0.25	0.1	−1.0
B	−0.5	−0.4	−2.0	0.5

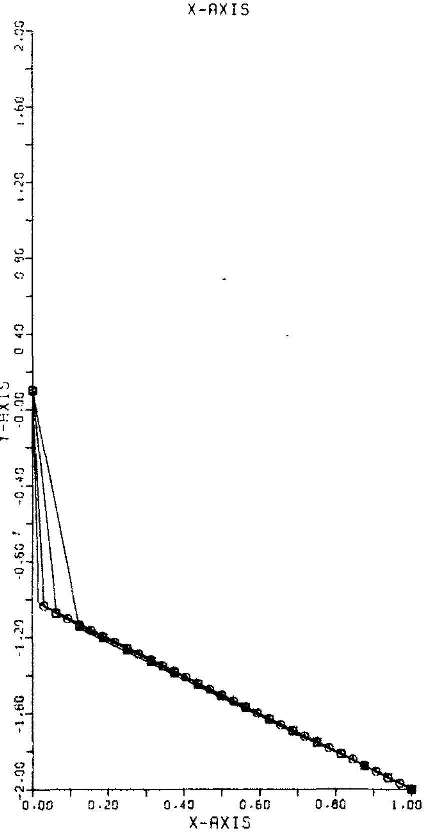
Figure 1 shows that numerical solutions have shock layers at $x=0.75$. The true solution also exhibits such a phenomenon at the same point. Figure 2 shows that shock layers vanish, but corner layers occur near the two points $x=0.25$


Y-AXIS
X-AXIS
1.00
0.80
0.60
0.40
0.20
0.00
-0.20
-0.40
-0.60
-0.80
-1.00
0.00
0.20
0.40
0.60
0.80
1.00

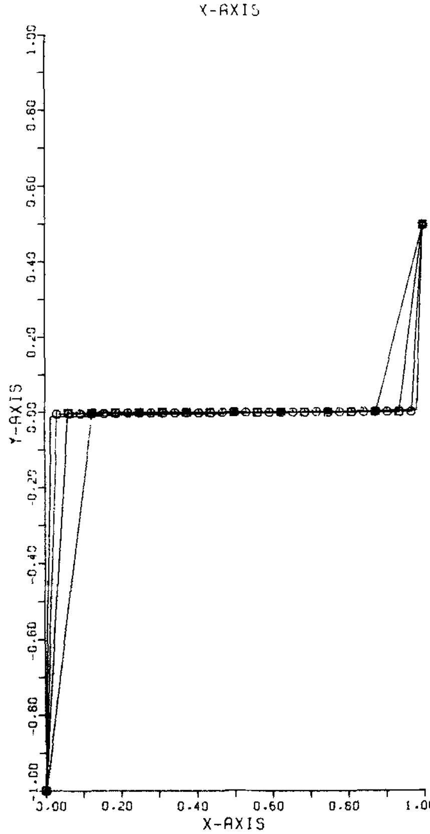
1


Y-AXIS
X-AXIS
1.00
0.80
0.60
0.40
0.20
0.00
-0.20
-0.40
-0.60
-1.00
0.00
0.20
0.40
0.60
0.80
1.00

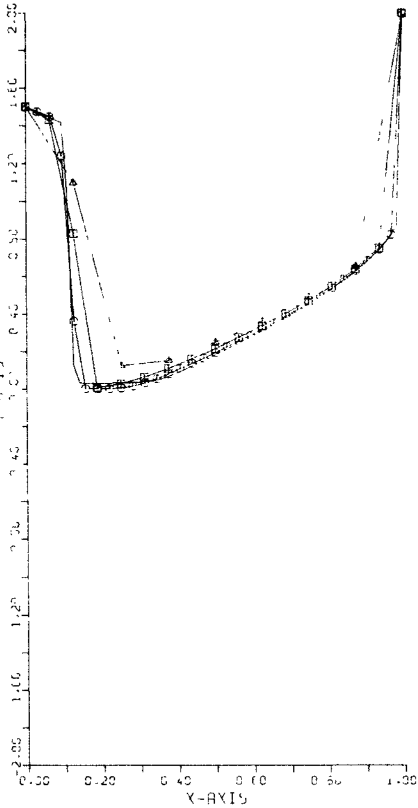
2


Y-AXIS
X-AXIS
2.00
1.60
1.20
0.40
0.00
-0.40
-1.20
-1.60
-2.00
0.00
0.20
0.40
0.60
0.80
1.00

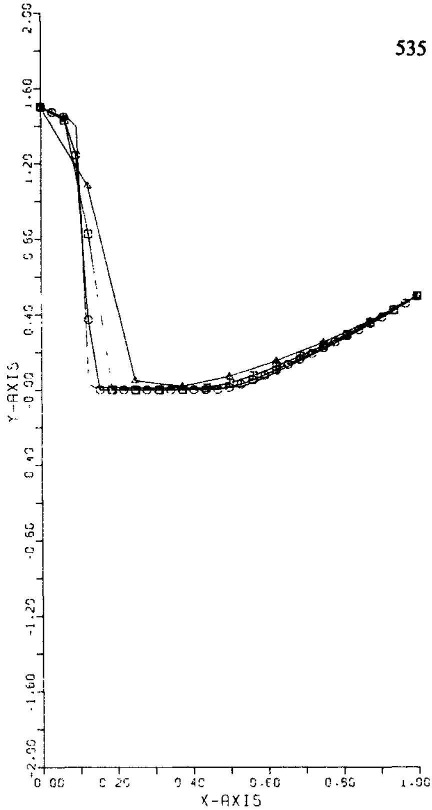
3


Y-AXIS
X-AXIS
1.00
0.80
0.60
0.40
0.20
0.00
-0.20
-0.40
-0.60
-0.80
-1.00
0.20
0.40
0.60
0.80
1.00

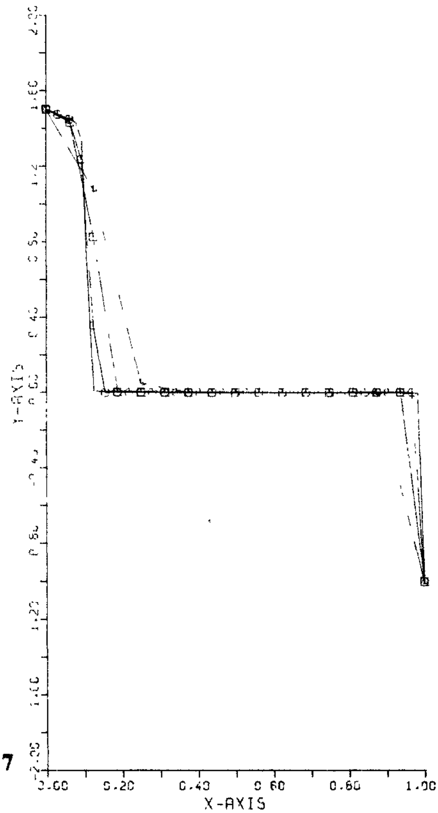
4


Y-AXIS
X-AXIS
2.00
1.60
1.20
0.80
0.40
0.00
-0.40
-0.80
-1.20
-1.60
-2.00
0.00
0.20
0.40
0.60
0.80
1.00

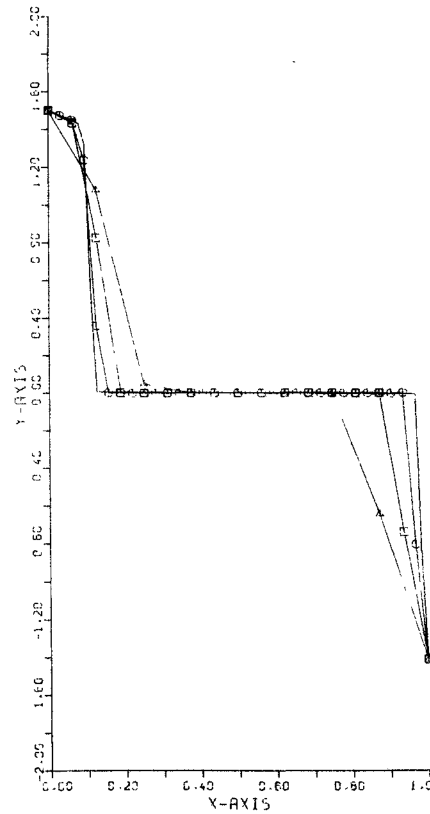
5


Y-AXIS
X-AXIS
2.00
1.60
1.20
0.80
0.40
0.00
-0.40
-0.80
-1.20
-1.60
-2.00
0.00
0.20
0.40
0.60
0.80
1.00

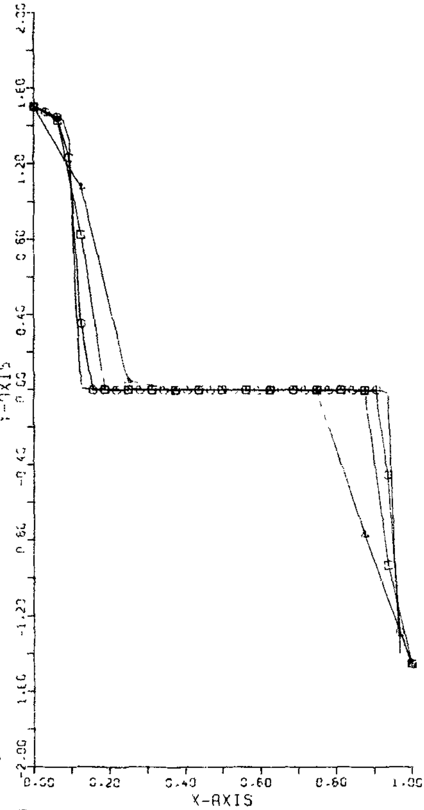
6


Y-AXIS
X-AXIS
2.00
1.60
1.20
0.80
0.40
0.00
-0.40
-0.80
-1.20
-1.60
-2.00
0.00
0.20
0.40
0.60
0.80
1.00

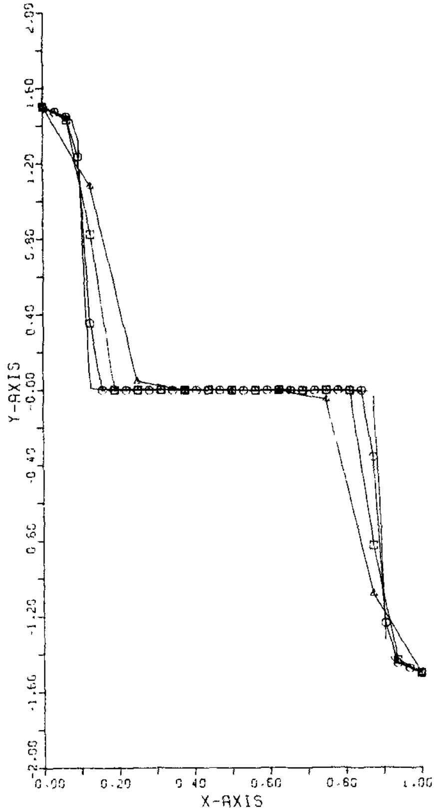
7


Y-AXIS
X-AXIS
2.00
1.60
1.20
0.80
0.40
0.00
-0.40
-0.80
-1.20
-1.60
-2.00
0.00
0.20
0.40
0.60
0.80
1.00

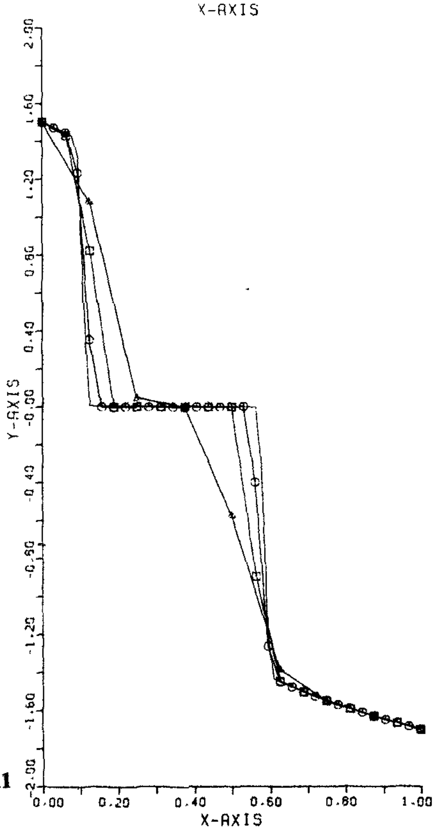
8


Y-AXIS
X-AXIS
0.00
0.20
0.40
0.60
0.80
1.00
-2.00
-1.60
-1.20
-0.80
-0.40
0.00
0.40
0.80
1.20
1.60
2.00

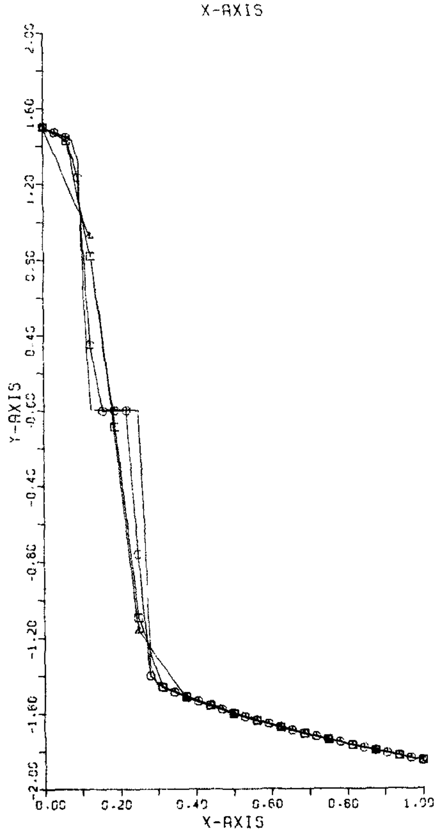
9


Y-AXIS
X-AXIS

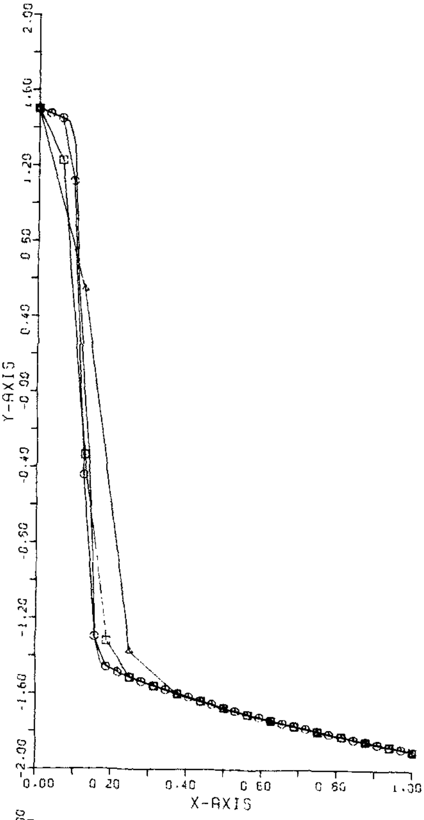
10


Y-AXIS
X-AXIS
0.00
0.20
0.40
0.60
0.80
1.00
-2.00
-1.60
-1.20
-0.80
-0.40
0.00
0.40
0.80
1.20
1.60
2.00

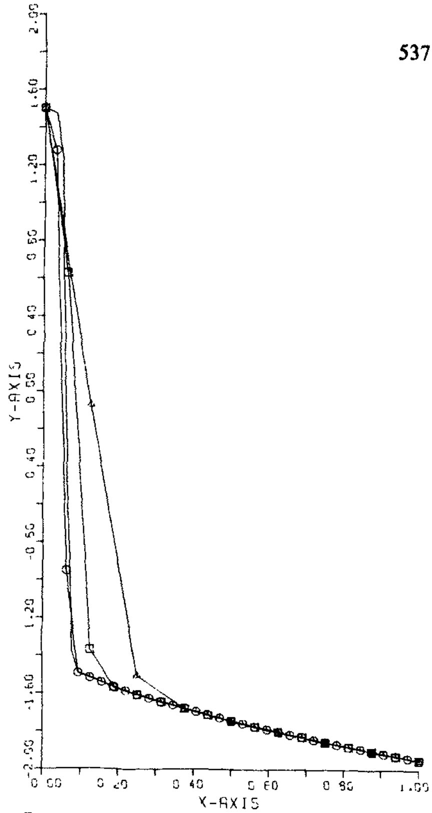
11


Y-AXIS
X-AXIS

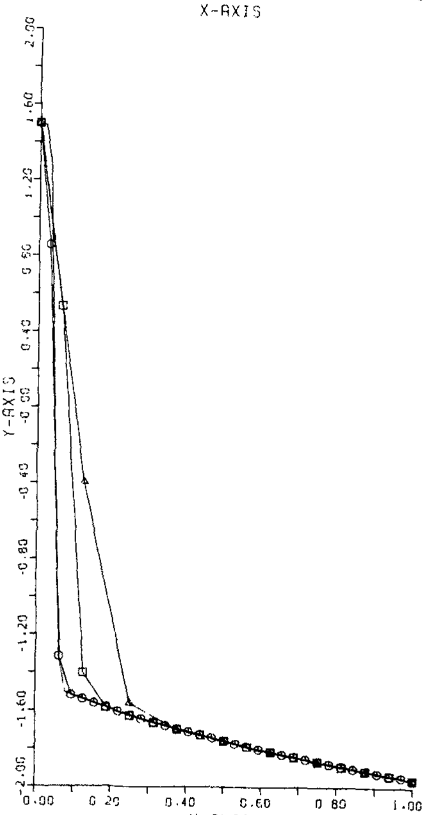
12


Y-AXIS
2.00
1.60
1.20
0.80
0.40
0.00
-0.40
-0.80
-1.20
-1.60
-2.00
X-AXIS
0.00
0.20
0.40
0.60
0.80
1.00

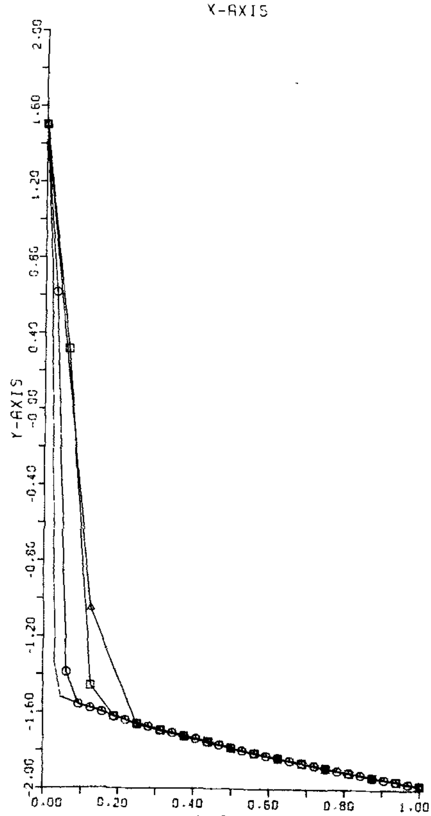
13


Y-AXIS
2.00
1.60
1.20
0.80
0.40
0.00
-0.40
-0.80
-1.20
-1.60
-2.00
X-AXIS
0.00
0.20
0.40
0.60
0.80
1.00

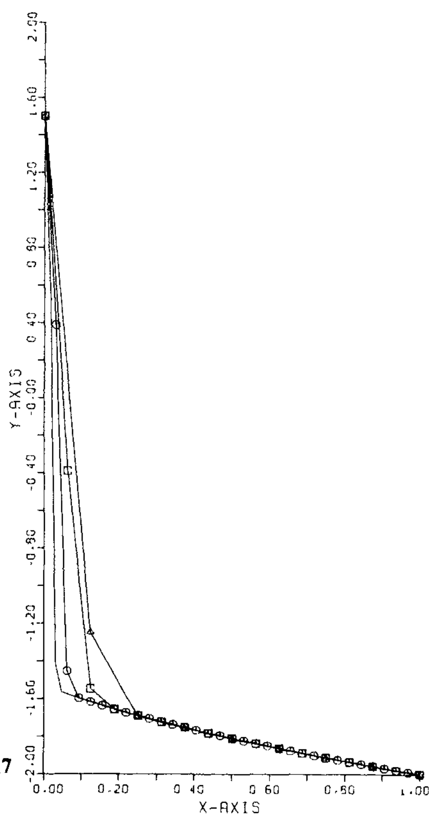
14


Y-AXIS
2.00
1.60
1.20
0.80
0.40
0.00
-0.40
-0.80
-1.20
-1.60
-2.00
X-AXIS
0.00
0.20
0.40
0.60
0.80
1.00

15


Y-AXIS
2.00
1.60
1.20
0.80
0.40
0.00
-0.40
-0.80
-1.20
-1.60
-2.00
X-AXIS
0.00
0.20
0.40
0.60
0.80
1.00

16



17

and $x=0.6$. In Fig. 3, we can observe a boundary layer phenomenon at the left end point. Numerical solutions in Fig. 4 have boundary layers at both end points.

Next, we solved the problem

$$\varepsilon y'' - \tfrac{1}{4}(y^2(y^2-2))' - y = 0, \qquad 0<x<1,$$
$$y(0)=1.5, \qquad y(1)=B$$

whose solution exihibits a variety of phenomena corresponding to the values of B (see [2]). To see whether the solutions of our scheme reproduce such phenomena or not, we ran our scheme with B below:

Fig.	5	6	7	8	9	10	11	12	13	14	15	16	17
B	2.0	0.5	−1.0	−1.41	−1.45	−1.50	−1.70	−1.85	−1.90	−1.95	−1.96	−1.98	−2.00

From numerical results shown in Figs. 5 to 17, we can read the locations where boundary, corner and shock layers appear. We can also observe how shock points change with the values of B.

The symbols "△", "□", "○" and "—" in each figure denote numerical solutions for $h=1/8$, $h=1/16$, $h=1/32$ and $h=1/64$, respectively.

References

1. Abrahamsson, L., Osher, S.: Monotone difference schemes for singular perturbation problems. SIAM J. Numer. Anal. **19**, 979-992 (1982)
2. Lorenz, J.: Nonlinear boundary value problems with turning points and properties of difference schemes. In: Theory and Applications of Singular Perturbations (W. Eckhaus, E.M. de Jager, eds.), Proceedings of a Conference Held in Oberwolfach, August 16-22, 1981. Lect. Notes Math. **942**, p. 150. Berlin-Heidelberg-New York: Springer 1982
3. Niijima, K.: A uniformly convergent difference scheme for a semilinear singular perturbation problem. Numer. Math. **43**, 175-198 (1984)
4. Osher, S.: Nonlinear singular perturbation problems and one sided difference schemes. SIAM J. Numer. Anal. **18**, 129-144 (1981)
5. Rall, L.B.: Computational Solution of Nonlinear Operator Equations. New York: John Wiley and Sons 1969
6. Sanders, R.: On convergence of monotone finite difference schemes with variable spatial differencing. Math. Comput. **40**, 91-106 (1983)

Received March 1, 1984/October 25, 1984