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A Rapid Generalized Method of Bisection for Solving Systems of Non-linear Equations

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Summary. A rapid Generalized Method of Bisection for solving Systems of Non-linear Equations is presented in this paper, based on the non-zero value of the topological degree. Further, while the method does not compute the topological degree, it takes care of keeping its non-zero value during the bisections and thus results in a fast bisection algorithm.

Subject Classifications: AMS(MOS): 65H10; CR: G1.5.

1. Introduction

The author of [4] in his paper gave an efficient degree computation method for a generalized method of bisection (GMB), for the solution of non-linear systems of equations of the form:

$$F^n = \theta^n \equiv (0, 0, \ldots, 0), \tag{1}$$

with

$$F^n = (f_1, f_2, \ldots, f_n) \colon \bar{\mathscr{D}} \subset \mathbb{R}^n \to \mathbb{R}^n,$$

where: $\mathscr{D}$ is an n-dimensional region in $\mathbb{R}^n$ and $\bar{\mathscr{D}}$ its closure.

In the above paper, the author, in fact, improved F. Stenger's [6] method for the calculation of the topological degree $d(F^n, \mathscr{D}, \theta^n)$ of F^n at θ^n relative to $\mathscr{D}$. And, then, by repeated computation of the topological degree he produced an algorithm for the solution of non-linear systems of equations in a way that is a generalization of the method of bisection.

In this paper we further improve the algorithm by introducing the concept of an admissible-polygon through which we remove all calculations concerning topological degree, which as is known, is quite a time-consuming procedure and in fact a weak point of the algorithm given in [4]. In addition, we produce a priori error bounds of the method of bisection, which, by the way, was not given in [4]. Lastly, we deal with three typical well known test cases and give the detailed algorithm for solving non-linear systems.

2. The Admissible n-polygon

Notation 2.1. In this paper we shall usually use the following sets:

$$\mathscr{V}=\{1,2,3,\dots,2^n\},\qquad \mathscr{V}_1=\{1,2,3,\dots,2^{n-1}\},\qquad \mathscr{E}=\{1,2,\dots,n\}$$

and superscripts to denote dimension and subscripts for indexing.

In the development of our analysis the concept of the admissible n-polygon will play a fundamental role. To define it we first introduce the tool of the n-complete matrix.

Definition 2.2. An *n-complete matrix* $\mathscr{M}_n=(C_{ij})$, $i\in\mathscr{V}$, $j\in\mathscr{E}$ is a $2^n\times n$ matrix with elements in each row the respective coordinates of the vectors of the Cartesian product:

$$\prod_{i=1}^{n}(-1,1)=(-1,1)\times(-1,1)\times\dots\times(-1,1).$$

Obviously there are many matrices $\mathscr{M}_n$ and in our analysis we can use any one of them. Yet we are going to choose the one whose selection appears to be more natural than the others, since it will be closely related with the sequence of the natural numbers $1,2,3,\dots,2^n$. More specifically, let B_i^n be the n-digit binary forms of the numbers $i-1$, $i\in\mathscr{V}$. Then, we formulate a $2^n\times n$ matrix $\mathscr{M}_n^*$, with entries in row i, $i\in\mathscr{V}$, the digits of B_i^n. Finally, in the matrix $\mathscr{M}_n^*$, we replace each zero element by -1 and thus we come up with a new matrix $\mathscr{M}_n$, which we call the *n-complete matrix.* By construction, it is obvious that the n-complete matrix $\mathscr{M}_n$ depends only on the dimensionality of the problem, and so, for example, when $n=1,2$ we have respectively:

$$n=1\quad \begin{matrix} B_1^1=0\\ B_2^1=1\end{matrix}\;\to\mathscr{M}_1^*=\begin{bmatrix}B_1^1\\ B_2^1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\;\to\mathscr{M}_1=\begin{bmatrix}-1\\1\end{bmatrix},$$

$$n=2\quad \begin{matrix} B_1^2=00\\ B_2^2=01\\ B_3^2=10\\ B_4^2=11\end{matrix}\;\to\mathscr{M}_2^*=\begin{bmatrix}B_1^2\\ B_2^2\\ B_3^2\\ B_4^2\end{bmatrix}=\begin{bmatrix}0&0\\0&1\\1&0\\1&1\end{bmatrix}\to\mathscr{M}_2=\begin{bmatrix}-1&-1\\-1&1\\1&-1\\1&1\end{bmatrix}.$$

Definition 2.3. Let $\mathbf{P}^n$ be an n-polygon [3, 4] with 2^n vertices. Suppose further that $F^n=(f_1,f_2,\dots,f_n)$: $\mathbf{P}^n\subset\mathbb{R}^n\to\mathbb{R}^n$. Then we define the *vector of signs of F^n relative to a vertex X_k*, $\mathscr{S}(F^n,X_k)$, by:

$$\mathscr{S}(F^n,X_k)=(\operatorname{sgn}f_1(X_k),\operatorname{sgn}f_2(X_k),\dots,\operatorname{sgn}f_n(X_k)),$$

and the *matrix of signs of F^n relative to the vertices X_k*, $k\in\mathscr{V}$ by:

$$\mathscr{S}=[\mathscr{S}(F^n,X_1)^{\mathrm{T}},\mathscr{S}(F^n,X_2)^{\mathrm{T}},\dots,\mathscr{S}(F^n,X_m)^{\mathrm{T}},\dots,\mathscr{S}(F^n,X_{2^n})^{\mathrm{T}}]^{\mathrm{T}};$$

of course, the matrix of signs $\mathscr{S}$ is a $2^n\times n$ matrix such that its m-th row is the vector $\mathscr{S}(F^n,X_m)$.

Finally, the function $\operatorname{sgn}(t)$, $t \in \mathbb{R}$ is the known sign function with values:

$$\operatorname{sgn}(t) = \begin{cases} 1, & \text{if } t > 0 \\ 0, & \text{if } t = 0 \\ -1, & \text{if } t < 0. \end{cases}$$

Now, we can proceed with the definition of an admissible n-polygon.

Definition 2.4. An n-polygon with 2^n vertices, $\mathbf{P}^n \subset \mathbb{R}^n$ is called an *admissible n-polygon relative to* $F^n = (f_1, f_2, \ldots, f_n)$: $\mathbf{P}^n \to \mathbb{R}^n$, iff the matrix of signs of F^n relative to its vertices is identical with $\mathcal{M}_n$.

In an admissible n-polygon we define the following concepts.

Definition 2.5. A *proper 1-simplex* is a 1-simplex $\langle X_p, X_q \rangle$ with extreme points the vertices X_p and X_q of an admissible n-polygon $\mathbf{P}^n$ relative to F^n: $\mathbf{P}^n \to \mathbb{R}^n$, iff the corresponding coordinates of the vectors $\mathcal{S}(F^n, X_p)$ and $\mathcal{S}(F^n, X_q)$ differ from each other only in one case.

Definition 2.6. A polygon which is constructed by 2^{n-1} vertices of an admissible n-polygon $\mathbf{P}^n$ relative to F^n: $\mathbf{P}^n \to \mathbb{R}^n$ will be called the r-th *side* of $\mathbf{P}^n$, and will be denoted by Π_r, iff for all vertices X_k, $k \in \mathcal{V}_1$ the corresponding vectors $\mathcal{S}(F^n, X_k)$ have their r-th coordinate equal to each other. Moreover, if this common r-th element is -1 (or 1) then the Π_r will be called *negative* (or *positive*) r-th *side* and will be denoted by Π_{r-} (or Π_{r+}).

The following two lemmas will be useful in our analysis.

Lemma 2.7. *In each admissible n-polygon $\mathbf{P}^n$ relative to $F^n = (f_1, f_2, \ldots, f_n)$: $\mathbf{P}^n \to \mathbb{R}^n$, there are n positive and n negative sides.*

Proof. According to Definition 2.6, a side should be created by 2^{n-1} vertices X_k, $k \in \mathcal{V}_1$ such that all the vectors $\mathcal{S}(F^n, X_k)$, $k \in \mathcal{V}_1$ have a common coordinate which, of course, can be the 1-st, 2-nd, ..., n-th coordinate, which again can be either -1 or 1 (see Def. 2.2). Consequently, there will be $2n$ sides, n negatives and n positives. □

Lemma 2.8. *Each side Π_r^{n-1} of an admissible n-polygon $\mathbf{P}^n$ relative to $F^n = (f_1, f_2, \ldots, f_n)$: $\mathbf{P}^n \to \mathbb{R}^n$ is itself an admissible $(n-1)$-polygon relative to $F_r^{n-1} = (f_1, f_2, \ldots, f_{r-1}, f_{r+1}, \ldots, f_n)$: $\Pi_r^{n-1} \to \mathbb{R}^{n-1}$.*

Proof. According to Definition 2.6, the $(n-1)$-polygon Π_r^{n-1} has 2^{n-1} vertices y_k, $k \in \mathcal{V}_1$. In addition, the coordinates of the vectors $\mathcal{S}(F_r^{n-1}, y_k)$ for all $k \in \mathcal{V}_1$ are easily derived from the corresponding to vertices y_k rows of $\mathcal{M}_n$, if we delete the elements of the r-th column. Finally, we can easily check that these coordinates are identical with the corresponding elements of the $(n-1)$-complete matrix $\mathcal{M}_{n-1}$. Consequently according to Definition 2.4, the side Π_r^{n-1} is indeed an admissible $(n-1)$-polygon. □

Theorem 2.9. *Let $\langle X_i, i \in \mathcal{V} \rangle$ be the ordered set of vertices of an admissible n-polygon $\mathbf{P}^n$ relative to continuous $F^n = (f_1, f_2, \ldots, f_n)$: $\mathbf{P}^n \subset \mathbb{R}^n \to \mathbb{R}^n$ and such that $\theta^n = (0, 0, \ldots, 0) \notin F^n(b(P^n))$, ($b(\mathbf{P}^n)$ is the boundary of $\mathbf{P}^n$). Suppose that $\{\Pi_i\}_{i=1}^{2n}$ is* *the set of its sides and* $\mathscr{Q}=\{\mathbf{S}_{i,t}^{n-1},\ i=1,2,\dots,2n,\ t=1,2,\dots,t_i\}$ *is a finite set of* $(n-1)$*-simplexes* $\mathbf{S}_{i,t}^{n-1}$*, on the boundary of* P^n *such as:*

(i) *the boundary of* $\mathbf{P}^n$, $b(\mathbf{P}^n)$, *is given by:*

$$b(\mathbf{P}^n)=\sum_{i=1}^{2n}\sum_{t=1}^{t_i}\mathbf{S}_{i,t}^{n-1},$$

(ii) *they have disjoint interiors,*

(iii) *they make* $b(\mathbf{P}^n)$ *sufficiently refined relative to* $sgn(F^n)$ *and*

(iv) *the extreme points of each* $\mathbf{S}_{i,t}^{n-1}$ *are vertices of* $\boldsymbol{\Pi}_i$*. Then, the topological degree of* F^n *at* θ^n *relative to* $\mathbf{P}^n$, $d(F^n,\mathbf{P}^n,\theta^n)$, *is equal to* ± 1.

Proof. According to the Recursion Formula [6, 4] the following relationship:

$$d(F^n,\mathbf{P}^n,\theta^n)=\sum_{k_1\in\mathscr{I}_1} d(F_1^{n-1},\mathbf{S}_{k_1}^{n-1},\theta^{n-1}), \tag{2}$$

holds, where: $F_1^{n-1}=(f_2,f_3,\dots,f_n)$: $b(\mathbf{P}^n)\to\mathbb{R}^{n-1}$ and $\mathscr{I}_1$ is the set of indices such as $k_1\in\mathscr{I}_1$ iff $f_1>0$ on $\mathbf{S}_{k_1}^{n-1}\in\mathscr{Q}$.

Moreover, from Definition 2.6 it is apparent that the side $\boldsymbol{\Pi}_{1+}$ is the only side of $\mathbf{P}^n$ where $f_1>0$ on it; therefore, all $\mathbf{S}_{k_1}^{n-1}$, $k_1\in\mathscr{I}_1$ are constructed by vertices on $\boldsymbol{\Pi}_{1+}$. On the other hand, we have,

$$\mathscr{S}(F^n,X_{2^n})=(1,1,\dots,1,1) \quad\text{and}\quad \mathscr{S}(F^n,X_{2^n-1})=(1,1,\dots,1,-1);$$

consequently, the last and the last but one vertices of $\mathbf{P}^n$, X_{2^n} and X_{2^n-1}, belong to $\boldsymbol{\Pi}_{1+}$. Lastly, according to assumption (ii) of the theorem there should be only one $(n-1)$-simplex, say $\mathbf{S}_{k_1^*}^{n-1}$, which includes both X_{2^n} and X_{2^n-1}. Then, by means of the Parity-Theorem [4], we obtain:

$$d(F_1^{n-1},\mathbf{S}_{k_1}^{n-1},\theta^{n-1})=\sum_{t=1}^{m_{k_1}}\operatorname{Par}(\mathscr{R}(\mathbf{S}_{k_1,t}^{n-2}\ F_1^{n-1})), \tag{3}$$

where: $b(\mathbf{S}_{k_1}^{n-1})=\sum_{t=1}^{m_{k_1}}\mathbf{S}_{k_1,t}^{n-2}$.

Furthermore, we observe that the range simplex $\mathscr{R}(\mathbf{S}_{k_1^*,t}^{n-2},F_1^{n-1})$, with

$$b(\mathbf{S}_{k_1^*}^{n-1})=\sum_{t=1}^{m_{k_1^*}}\mathbf{S}_{k_1^*,t}^{n-2},$$

is the only one which is usable; consequently, combining (2) and (3) we have:

$$d(F^n,\mathbf{P}^n,\theta^n)=d(F_1^{n-1},\mathbf{S}_{k_1^*}^{n-1},\theta^{n-1}). \tag{4}$$

Finally, by repeated use of the above procedure we eventually get:

$$d(F^n,\mathbf{P}^n,\theta^n)=d(F_1^{n-1},\mathbf{S}_{k_1^*}^{n-1},\theta^{n-1})=\dots=d(F_{n-2}^2,\mathbf{S}_{k_{n-2}^*}^2,\theta^2), \tag{5}$$

where the three vertices of the 2-simplex $\mathbf{S}_{k_{n-2}^*}^2$, namely X_{2^n}, X_{2^n-1} and X_r, are such that the functions values satisfy: $f_i>0$, $i=1(1)(n-2)$. Also the $f_{n-1}<0$ on

X_r, because if $f_{n-1}>0$ on X_r then we easily get:

$$\mathscr{S}(F^n, X_r) \equiv \mathscr{S}(F^n, X_{2^n}) \quad \text{or} \quad \mathscr{S}(F^n, X_r) \equiv \mathscr{S}(F^n, X_{2^n-1}),$$

which means that two rows of the n-complete matrix are identical, but this is absurd. Consequently $f_{n-1}<0$ on X_r; therefore either $\mathscr{S}(F^2_{n-2}, X_r)=(-1, 1)$ or $\mathscr{S}(F^2_{n-2}, X_r)=(-1, -1)$. Moreover, from (5) we have:

$$d(F^2_{n-2}, \mathbf{S}^2_{k\bar{n}-2}, \theta^2) = \sum_{i=1}^{3} \operatorname{Par}(\mathscr{R}(\mathbf{S}^1_{k\bar{n}-2,i}, F^2_{n-2})).$$

So, either

$$d(F^2_{n-2}, \mathbf{S}^2_{k\bar{n}-2}, \theta^2) = \operatorname{Par}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = 1 \quad \text{or} \quad d(F^2_{n-2}, \mathbf{S}^2_{k\bar{n}-2}, \theta^2) = \operatorname{Par}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -1. \tag{6}$$

Combining (5), (6) we get: $d(F^n, \mathbf{P}^n, \theta^n) = \pm 1$ which proves the theorem. □

Next, we are going to set up an algorithm for creating admissible n-polygons $\mathbf{P}^n$ relative to F^n: $\mathbf{P}^n \to \mathbb{R}^n$, which, in turn, will be used for the initialization of our rapid generalized method of bisection. To do this, we need a way of locating admissible polygons, as well as the means of specifying, for each vertex, its associated proper 1-simplexes, that are to be subdivided by the process of bisection. The following lemma will facilitate the whole procedure.

Lemma 2.10. *Suppose that X_i, $i \in \mathscr{V}$, are the vertices of an admissible n-polygon $\mathbf{P}^n$ relative to F^n: $\mathbf{P}^n \to \mathbb{R}^n$. Then* (a) *for each vertex X_i there are exactly n other vertices X_k such as the* 1*-simplexes $\langle X_i, X_k \rangle$, $k \in \mathscr{V}$ are proper* 1*-simplexes, and* (b) *the subindexes k are given by $k = i - 2^{n-j} C_{ij}$, $i \in \mathscr{V}$, $j \in \mathscr{E}$ and C_{ij} are the elements of the n-complete matrix $\mathscr{M}_n$.*

Proof. According to Definition 2.5 the 1-simplex $\langle X_i, X_k \rangle$ will be a proper simplex iff the corresponding coordinates of the n-vectors $\mathscr{S}(F^n, X_i)$ and $\mathscr{S}(F^n, X_k)$ differ from each other in only one case, which of course, can be either the 1-st, or the 2-nd, ..., or the n-th coordinate, that proves part (a) of the lemma. Next, for tackling the other part of the lemma we take the i-th and the k-th rows of the n-complete matrix $\mathscr{M}_n$, with $k = i - 2^{n-j} C_{ij}$, $i \in \mathscr{V}$, $j \in \mathscr{E}$ and we are going to prove that the corresponding elements of the rows differ from each other, only in the j-th column. Indeed, by definition, we easily get $i - k = 2^{n-j} C_{ij}$, or $(i-1)-(k-1) = 2^{n-j} C_{ij}$, with $C_{ij} = \pm 1$. Therefore, the difference of the orders $(i-1)$ and $(k-1)$ will be an integer power of 2, multiplied by C_{ij}, so the n-digit binary expressions of $(i-1)$ and $(k-1)$ will differ from each other in only one digit of the same order, in fact, in the j-th most significant digit, for $j \in \mathscr{E}$. Consequently, on the basis of the correspondence we have set up among the elements of $\mathscr{M}_n$ and the binary digits, we conclude that the rows i and k of $\mathscr{M}_n$ will be different only in the elements of the j-th column, for $j \in \mathscr{E}$, which proves part (b) of the lemma. □

3. Constructing Admissible n-polygons

The procedure for constructing admissible n-polygons is based on the following three algorithms:

(a) *The algorithm for setting up the n-complete Matrix* $\mathcal{M}_n$ [see 8].

(b) *The algorithm for computing an initial n-polygon,* with vertices whose coordinates are given by the corresponding elements of the rows of the $2^n \times n$ matrix $\mathcal{R}$ defined by:

$$\mathcal{R} = \mathcal{G} + \mathcal{M}_n^* \mathcal{B},$$

where:

$$\mathcal{G} = (g_{ij}), \quad g_{ij} = X_j^{in}, \quad i \in \mathcal{V},\ j \in \mathcal{E},$$
$$\mathcal{B} = (b_{ij}), \quad b_{ij} = \delta_i^j h_j, \quad i, j \in \mathcal{E},$$

$X^{in} = (X_1^{in}, X_2^{in}, \ldots, X_n^{in}) \in \mathbb{R}^n$ arbitrary initial point, $(h_1, h_2, \ldots, h_n)$ arbitrary stepsizes in each direction, $\mathcal{M}_n^*$ the associated matrix of the n-complete matrix $\mathcal{M}_n$ (see Def. 2.2), and δ_i^j the well known Kronecker's delta.

(c) *The algorithm for applying a modified version of the bisection method in* $\mathbb{R}$ that we have produced and is very suitable in this case.

Next, we give a detailed description of all the steps of the algorithm for the construction of admissible n-polygons, while the algorithm itself follows.

First, we set up the n-complete matrix $\mathcal{M}_n$; with the help of Lemma 2.10 we can find the orders of vertices of an admissible n-polygon, as well as the positions of the extreme points of its proper 1-simplexes. Then, we take an initial arbitrary point $X^{in} \in \mathbb{R}^n$ and arbitrary stepsizes $(h_1, h_2, \ldots, h_n)$ in each coordinate direction and on the basis of them, we construct an initial n-polygon. Afterwards, we locate the positions of those pairs of vertices which are going to be extreme points of proper 1-simplexes, of the above admissible n-polygon. Then, for each one of the above pairs, say the (i, j) pair, we take the corresponding vertices of the initial n-polygon, that is the pair (V_i, V_j), which by construction is such that their corresponding coordinates differ from each other only in one case. Next we compute the points of intersection of F^n and $\langle V_i, V_j \rangle$ and by suitable perturbation of the intersection of F^n and $\langle V_i, V_j \rangle$ we change the initial n-polygon in such a way that an admissible n-polygon may eventually emerge; if not, then we continue with another pair and the process goes on.

Example 3.1. (see F. Stenger, p. 36). Suppose $n=2$, $F^2 = (X^2 - 4Y, Y^2 - 2X + 4Y)$, $(X_1^{in}, X_2^{in}) = (-2, -0.25)$, $(h_1, h_2) = (4, 0.5)$ the arbitrary stepsizes, and $\varepsilon = 10^{-2}$ the desired accuracy.

We first find the 2-complete matrix $\mathcal{M}_2$:

$$\mathcal{M}_2 = \begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

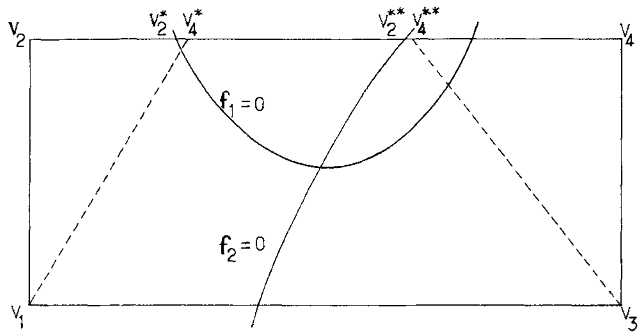

Fig. 1

Next we compute matrix $\mathscr{R}$ by $\mathscr{R}=\mathscr{G}+\mathscr{M}_2^*\mathscr{B}$, with:

$$\mathscr{G}=\begin{bmatrix}-2 & -0.25\\ -2 & -0.25\\ -2 & -0.25\\ -2 & -0.25\end{bmatrix}, \quad \mathscr{M}_2^*=\begin{bmatrix}0 & 0\\ 0 & 1\\ 1 & 0\\ 1 & 1\end{bmatrix}, \quad \mathscr{B}=\begin{bmatrix}4 & 0\\ 0 & 0.5\end{bmatrix}.$$

Therefore the vertices of the initial 2-polygon are:

$$V_1=(-2,\,-0.25), \qquad V_2=(-2,\,0.25), \qquad V_3=(2,\,-0.25), \qquad V_4=(2,\,0.25).$$

Next, we check if the initial 2-polygon is an admissible 2-polygon; to do this, we form the matrix $\mathscr{S}$ of signs of F^2 relative to the vertices V_1, V_2, V_3 and V_4:

$$\mathscr{S}=\begin{bmatrix}\operatorname{sgn} f_1(V_1) & \operatorname{sgn} f_2(V_1)\\ \operatorname{sgn} f_1(V_2) & \operatorname{sgn} f_2(V_2)\\ \operatorname{sgn} f_1(V_3) & \operatorname{sgn} f_2(V_3)\\ \operatorname{sgn} f_1(V_4) & \operatorname{sgn} f_2(V_4)\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & 1\\ 1 & -1\\ 1 & -1\end{bmatrix},$$

and compare it with $\mathscr{M}_2$. Since they are different the initial 2-polygon is not an admissible one. Therefore, we should find suitable points, V^*, in $\mathbb{R}^2$ such that their $\mathscr{S}(F^2, V^*)$ produce those rows of $\mathscr{M}_2$ which are missing in $\mathscr{S}$. Points V^* should lie on the 1-simplexes whose extreme points are those vertices of the initial 2-polygon which correspond to the positions of the pairs given by:

$$Z_1=(1,2), \qquad Z_2=(1,3), \qquad Z_3=(2,4), \qquad Z_4(3,4).$$

So, the above 1-simplexes are:

$$\mathbf{S}_1^1=\langle V_1, V_2\rangle, \quad \mathbf{S}_2^1=\langle V_1, V_3\rangle, \quad \mathbf{S}_3^1=\langle V_2, V_4\rangle, \quad \mathbf{S}_4^1=\langle V_3, V_4\rangle. \qquad (7)$$

On the other hand, points V^* should also lie in neighborhoods of the points of intersection of F^2 with the 1-simplexes $\mathbf{S}_1^1$, $\mathbf{S}_2^1$, $\mathbf{S}_3^1$, $\mathbf{S}_4^1$. Thus, for the first 1-simplex $\langle V_1, V_2\rangle$ and functions f_1 and f_2 we find that there is no intersection point, while for the intersection of the 1-simplex $\langle V_1, V_3\rangle$ and F^2 we find a point of intersection which is proven to be of no use. Finally, for the computation of intersections of $F^2=(f_1, f_2)$ and $\langle V_2, V_4\rangle$, we observe that the second coordinate of the vertices V_2 and V_4 remains constant; therefore, we should consider the function f_1 as function of its 1-st variable only and so we solve the equation:

$$f_1(X, 0.25) \equiv X^2-1=0. \tag{8}$$

The solution X^* of (8) is recommended to be computed in the interval $[-2, 2)$ with accuracy $\varepsilon=10^{-2}$ and by the following modified bisection method [see 8]:

$$X_{n+1}=X_n+\operatorname{sgn}(f_1(X_0))\operatorname{sgn}(f_1(X_n))h/2^{n+1}, \qquad n=0, 1, \ldots$$

with $X_0=-2$ and the interval length h equal to 4.

So, we find $X^*=-1$ and consequently points V^* are given by:

$$V_2^*=(-1-\varepsilon^*, 0.25) \quad \text{and} \quad V_4^*=(-1+\varepsilon^*, 0.25), \tag{9}$$

with $\varepsilon^*=0.1>\varepsilon$.

Next, we test whether points V_2^* and V_4^* are vertices of the admissible 2-polygon under construction. To do this we form the vectors:

$$\mathscr{S}(F^2, V_2^*)=(\operatorname{sgn} f_1(-1.1, 0.25), \operatorname{sgn} f_2(-1.1, 0.25))=(1, 1)$$

and

$$\mathscr{S}(F^2, V_4^*)=(\operatorname{sgn} f_1(-0.9, 0.25), \operatorname{sgn} f_2(-0.9, 0.25))=(-1, 1)$$

and check if any of them coincide with any row of $\mathscr{M}_2$ which is not already present in $\mathscr{S}$; indeed, we find that only $\mathscr{S}(F^2, V_4^*)$ coincides with the second row of $\mathscr{M}_2$ (which does not exist in $\mathscr{S}$).

Now, to find the intersection of f_2 with $\langle V_2, V_4\rangle$ we follow the same procedure and find successively: the equation: $f_2(X, 0.25)=-2X+1.0625$, the point of intersection: $(X^*, X_2^{in})=(0.53, 0.25)$, the points: $V_2^{**}=(0.43, 0.25)$ and $V_4^{**}=(0.63, 0.25)$, the vectors: $\mathscr{S}(F^2, V_2^{**})=(-1, 1)$ and $\mathscr{S}(F^2, V_4^{**})=(-1, -1)$. In this case, only the vector $\mathscr{S}(F^2, V_4^{**})$ coincides with the 1-st row of $\mathscr{M}_2$ (which is not present in S).

Now, we observe that all the missing rows of $\mathscr{M}_2$ have appeared; therefore, the process is concluded and an admissible 2-polygon is the one with vertices V_4^{**}, V_4^*, V_3 and V_1; so:

$$\mathbf{P}^2=\langle V_4^{**}, V_4^*, V_3, V_1\rangle.$$

Note that any three of the vertices of $\mathbf{P}^2$ do not belong to the same 1-simplex.

Algorithm 3.2 *(Creating an admissible n-polygon)*

1. *Input* ε, n.
2. *Compute the n-complete matrix* $\mathscr{M}_n=(C_{ij})$.
3. *Compute initial n-polygon and store its vertices in matrix* $\mathscr{R}=(r_{ij})$.
4. *Set all entries of a* $2^n \times n$ *matrix* $\mathscr{A}=(a_{ij})$ *equal to zero.*
5. *Compute for each row* R_i *of* $\mathscr{R}$ *the vector* $\mathscr{S}(F^n, R_i)$ *and compare it with the rows of* $\mathscr{M}_n$; *if it is identical with any row of* $\mathscr{M}_n$ *then store* R_i *in the corresponding row of* $\mathscr{A}$, *otherwise continue with the next row of* $\mathscr{R}$.

6. *If there is any row of* $\mathscr{A}$ *with elements equal to zero, go to next step; otherwise go to step* 23.
7. *Find the orders of the extreme points for all* $n2^{n-1}$ *proper* 1-*simplexes* $S_\beta = (S_\beta^1, S_\beta^2)$.
8. $j \leftarrow 0,\ k \leftarrow 0,\ u \leftarrow 0,\ v \leftarrow 0,\ l \leftarrow 0.$
9. $j \leftarrow j+1,\ p \leftarrow S_j^1,\ q \leftarrow S_j^2.$
10. *For all* $i = 1(1)n$ *execute steps* 11 *and* 12.
11. *If* $r_{p,i} \neq r_{q,i}$ *and* $r_{p,d} = r_{q,d}$ $\forall d \in \{1, 2, \ldots, \hat{i}, \ldots, n\}$ *then continue, otherwise return to step* 10.
12. $h \leftarrow |r_{p,i} - r_{q,i}|$
    $t \leftarrow \min\{|r_{p,i}|, |r_{q,i}|\}$
    $X_d \leftarrow r_{p,d}$, *for all* $d \in \{1, 2, \ldots, \hat{i}, \ldots, n\}$
    $X_i \leftarrow t$.
13. *For all* $m = 1(1)n$ *execute:*
    Find a zero of $f_m(X_1, X_2, \ldots, X_i, \ldots, X_n)$ *within the interval* $(t, t+h)$ *and store it in* Z_m; *if there is no zero put* $Z_m \leftarrow 0$.
14. $k \leftarrow k+1$; *if* $k = n$ *go to step* 20, *otherwise continue.*
15. $u \leftarrow u+1$; *if* $u = n$ *go to step* 18, *otherwise continue.*
16. $v \leftarrow v+1$; *if* $v = n$ *go to step* 15, *otherwise continue.*
17. *If* $Z_u = 0$ *then* $\varepsilon_1 \leftarrow 0$, *else* $\varepsilon_1 \leftarrow \varepsilon$ *and compute* $B_{k,u,v} \leftarrow Z_u + \varepsilon_1$ *and* $B^*_{k,u,v} \leftarrow Z_u - \varepsilon_1$.
18. $l \leftarrow l+1$, *if* $l = n$ *go to step* 14, *otherwise continue.*
19. $y_{k,j,l} \leftarrow B_{k,k,l}$ *store it in* $\mathscr{Y}$
    $y^*_{k,j,l} \leftarrow B^*_{k,k,l}$ *store it in* $\mathscr{Y}^*$.
20. *If* $j \neq n2^{n-1}$ *go to step* 9, *otherwise continue.*
21. *Compute for each row* Y_i *of* $\mathscr{Y}$ *the vector* $\mathscr{S}(F^n, Y_i)$ *and compare it with the rows of* $\mathscr{M}_n$, *if it is identical with any row of* $\mathscr{M}_n$ *then store* Y_i *in the corresponding row of* $\mathscr{A}$, *else try again with the next row of* $\mathscr{Y}$ *until exhaustion.*
22. *If there is any row of* $\mathscr{A}$ *with elements equal to zero put* $Y \leftarrow Y^*$ *and go to step* 21; *otherwise continue.*
23. *Output* $\mathscr{A}$.

4. The Fast Generalized Method of Bisection

The fast generalized method of bisection that we are going to develop in this section is based on the topological degree theory and especially on the following properties: If $\mathbf{P}^n$ is an n-dimensional polygon and $F^n\colon \mathbf{P}^n \to \mathbb{R}^n$ is continuous, with $F^n(X) \neq \theta^n$ $\forall X \in b(\mathbf{P}^n)$, then, there is at least one root of the system $F^n = \theta^n$ within $\mathbf{P}^n$, provided that $d(F^n, \mathbf{P}^n, \theta^n) \neq 0$; note that if the topological degree is equal to zero then no conclusions can be drawn because more information is needed [1, 2, 5]. So, although the topological degree plays a considerable role in the existence theory yet its value does not constitute a precise count of all the roots of the system. Furthermore, in a root finding algorithm based on degree theory the exact computation of the topological degree can be avoided, since the keeping of its non-zero value is sufficient to secure the location of the root. That is exactly what we do in our algorithm; we delete all computing concerning topological degree and take care to preserve its non-zero value during the iterations, thus gaining tremendously in speed over previous attempts.

The description of our algorithm goes as follows:

Suppose $\mathbf{P}^n$ is an admissible n-polygon relative to F^n: $\mathbf{P}^n \to \mathbb{R}^n$, and let $b(P^n)$ be sufficiently refined relative to $\operatorname{sgn}(F^n)$ [3, 4, 6, 7]; then Theorem 2.9 indicates that $\mathbf{P}^n$ includes at least one root of the system, so we shall construct our bisection algorithm in such a way that the new refined n-polygon $\mathbf{P}^n_*$ to be an admissible one. To do this we bisect all proper 1-simplexes of $\mathbf{P}^n$ in the following way.

Let $\langle X_i, X_j \rangle$ be a proper 1-simplex of $\mathbf{P}^n$ where:

$$X_i = (X_{i1}, X_{i2}, \ldots, X_{in}) \quad \text{and} \quad X_j = (X_{j1}, X_{j2}, \ldots, X_{jn});$$

then we define the point:

$$B = ((X_{i1} + X_{j1})/2, (X_{i2} + X_{j2})/2, \ldots, (X_{in} + X_{jn})/2).$$

Next we distinguish the following three cases:

(a) If the vectors $\mathscr{S}(F^n, B)$ and $\mathscr{S}(F^n, X_i)$ are identical then we replace X_i by B and the process continues with the next proper 1-simplex.

(b) If the vectors $\mathscr{S}(F^n, B)$ and $\mathscr{S}(F^n, X_j)$ are identical then we replace X_j by B and the process continues with the next proper 1-simplex.

(c) Otherwise, the process continues with the next proper 1-simplex.

Lemma 4.1. *Let* $\mathbf{P}^n$ *be an admissible n-polygon relative to* F^n: $\mathbf{P}^n \to \mathbb{R}^n$. *Suppose further that the new bisection method is applied to one proper* 1*-simplex of* $\mathbf{P}^n$. *Then the new n-polygon* $\mathbf{P}^n_*$ *is an admissible one.*

Proof. Let $\langle X_i, X_j \rangle$ be the proper 1-simplex which is going to be bisected; then according to the bisection procedure we described before we take the midpoint $B = (X_i + X_j)/2$ and, of course, either any of the two end-points will be replaced by B or no change will take place. In the latter case the lemma is obvious, while in the former case, let us further suppose that vertex X_i is replaced by B, then the vectors $\mathscr{S}(F^n, X_i)$ and $\mathscr{S}(F^n, B)$ will be identical and therefore the matrix $\mathscr{S}_*$ of signs of F^n relative to the vertices of $\mathbf{P}^n_*$ will coincide with the matrix $\mathscr{S}$ of signs of F^n relative to the vertices of $\mathbf{P}^n$ and consequently since by hypothesis $\mathbf{P}^n$ is an admissible n-polygon, so is $\mathbf{P}^n_*$. □

Remark 4.2. From the above lemma it is apparent that the refined n-polygon $\mathbf{P}^n_*$ includes also at least one root of the system (see Theorem 2.9).

Definition 4.3. Let $\langle X_i, i \in \mathscr{V} \rangle$ be the ordered set of all vertices of an admissible n-polygon $\mathbf{P}^n$ relative to F^n: $\mathbf{P}^n \to \mathbb{R}^n$. Then a *diagonal* D of $\mathbf{P}^n$ is the 1-simplex, say $\langle X_k, X_m \rangle$, such that all the components of the vectors $\mathscr{S}(F^n, X_k)$ and $\mathscr{S}(F^n, X_m)$ are different from each other.

Lemma 4.4. *Let* $\langle X_i, i \in \mathscr{V} \rangle$ *be as in Definition 4.3. Then* (a) *for each vertex* X_k *there is exactly one other vertex* X_m *such as the* 1-*simplex* $\langle X_k, X_m \rangle$ *is a diagonal of* $\mathbf{P}^n$, *and* (b) *the subindex* m, $\forall k \in \mathscr{V}_1$ *is given by* $m = 2^n + 1 - k$.

Proof. According to Definition 4.3 all the components of the vectors $\mathscr{S}(F^n, X_k)$ and $\mathscr{S}(F^n, X_m)$ are different from each other; consequently the k-th and m-th rows of $\mathscr{M}_n$ must differ from each other in all entries; therefore $\langle X_k, X_m \rangle$ is a diagonal of $\mathbf{P}^n$. Moreover, since the values of the entries can be either 1 or -1 the above diagonal is unique. Next, to prove part (b) of the lemma, we observe that the given relationship can be written as:

$$(k-1)+(m-1)=2^n-1,$$

or, if we make use of the n-digit binary forms,

$$B_k^n + B_m^n = 111\ldots1,$$

which means that the two rows of $\mathscr{M}_n$ differ from each other in all corresponding components; thus the lemma is proven. □

Definition 4.5. Let $\langle X_i, i \in \mathscr{V} \rangle$ be as in Definition 4.3. We define as an *estimate of the solution of the system* $F^n = \theta^n$, the midpoint of a diagonal of $\mathbf{P}^n$.

Definition 4.6. Let $\langle X_i, i \in \mathscr{V} \rangle$ be as in Definition 4.3. We define as the *diameter*, $\Delta(\mathbf{P}^n)$, of $\mathbf{P}^n$, the length of the longest proper 1-simplex of $\mathbf{P}^n$ (distances are measured in Euclidean norms), while the length of its longest 1-simplex is defined as the *mesh* of $\mathbf{P}^n$, $m(\mathbf{P}^n)$.

Now, we are in a position to determine the minimum number of bisections that are required by the method in order that the new n-polygon $\mathbf{P}_*^n$, which is obtained at the end is such that its diameter $\Delta(\mathbf{P}_*^n)$ is less than a predetermined number ε.

Lemma 4.7. *Suppose that* $\mathbf{P}^n$ *is an admissible n-polygon. Then, the minimum number* v *of bisections of the proper* 1-*simplexes of* $\mathbf{P}^n$ *which are required in obtaining a* $\mathbf{P}_*^n$ *and such that* $\Delta(\mathbf{P}_*^n) \leqq \varepsilon$ *is given by:*

$$v \geqq \log(\Delta(\mathbf{P}^n) \cdot \varepsilon^{-1})/\log 2.$$

Proof. Let $\langle X_i, X_j \rangle$ be such that $\Delta(\mathbf{P}^n) = \|X_i - X_j\|_2$. Then we bisect any proper 1-simplex of $\mathbf{P}^n$ obtaining thus a refined admissible n-polygon $\mathbf{P}_1^n$. We claim that the diameters of $\mathbf{P}^n$ and $\mathbf{P}_1^n$ satisfy the relationship:

$$\frac{\Delta(\mathbf{P}^n)}{2} \leqq \Delta(\mathbf{P}_1^n). \tag{10}$$

To do this we distinguish the following two cases:

I) Suppose that after the bisection the initial proper 1-simplex $\langle X_i, X_j \rangle$ of $\mathbf{P}^n$ is unchanged in $\mathbf{P}_1^n$. Then $\Delta(\mathbf{P}_1^n)$ will be either greater than or equal to $\Delta(\mathbf{P}^n)$, which, of course verifies (10).

II) Suppose that after the bisection the $\langle X_i, X_j \rangle$ changes into $\langle X_i, X_j^* \rangle$ (or equivalently into $\langle X_i^*, X_j \rangle$). This can happen if either $\langle X_i, X_j \rangle$ or a neighbouring proper 1-simplex (say $\langle X_j, X_k\rangle$), is bisected. In the first case we shall have either $\Delta(\mathbf{P}_1^n)=\|X_i-X_j^*\|_2=\frac{\|X_i-X_j\|_2}{2}$ (because of bisection) or $\Delta(\mathbf{P}_1^n)>\|X_i-X_j\|_2$; both verify (10).

In the second case, let $\langle X_j, X_k\rangle$ be the neighbouring proper 1-simplex which is bisected. Then, by using norm properties we have (see Fig. 2):

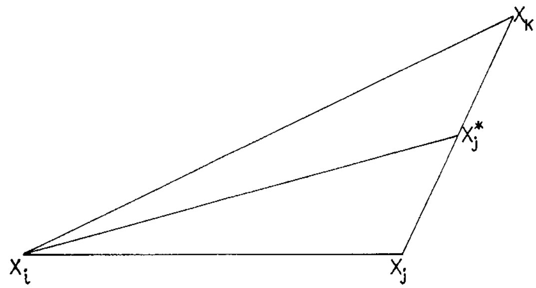

Fig. 2

$$\|X_i-X_j^*\|_2>\|X_i-X_j\|_2-\|X_j-X_j^*\|_2,$$

or

$$\|X_i-X_j^*\|_2>\|X_i-X_j\|_2-\frac{\|X_j-X_k\|_2}{2},$$

or

$$\|X_i-X_j^*\|_2>\|X_i-X_j\|_2-\frac{\|X_i-X_j\|_2}{2} \text{ (since } \|X_i-X_j\|_2>\|X_j-X_k\|_2),$$

or

$$\|X_i-X_j^*\|_2>\frac{\|X_i-X_j\|_2}{2}.$$

Then, obviously:

$$\Delta(\mathbf{P}_1^n)\geqq\|X_i-X_j^*\|_2>\frac{\|X_i-X_j\|_2}{2}=\frac{\Delta(\mathbf{P}^n)}{2} \quad \text{(by definition)},$$

which proves (10).

The procedure continues with new bisections, giving thus, at the k-th bisection, the relationship:

$$\Delta(\mathbf{P}_k^n)\geqq\frac{\Delta(\mathbf{P}^n)}{2^k}. \tag{11}$$

Finally, if the v-th bisection produces $\mathbf{P}_*^n$ satisfying:

$$\Delta(\mathbf{P}_*^n)\leqq\varepsilon,$$

then from (11) we have:

$$\varepsilon\geqq\Delta(\mathbf{P}_*^n)\geqq\Delta(\mathbf{P}^n)/2^v,$$

or, by taking logarithms:

$$v \geqq \log(\Delta(\mathbf{P}^n) \cdot \varepsilon^{-1})/\log 2,$$

which proves the lemma. □

Furthermore, we can easily prove the following two lemmas [see 8], which provide bounds on the mesh of an admissible n-polygon and on the estimate of the solution.

Lemma 4.8. *Suppose that $\langle X_i, i \in \mathcal{V} \rangle$ is as in Definition 4.3 and that $\Delta(\mathbf{P}^n) \leqq \varepsilon$. Then $m(\mathbf{P}^n) \leqq n\varepsilon$.*

Lemma 4.9. *Let $\mathbf{P}^n$ be an admissible n-polygon relative to continuous F^n, such that $\Delta(\mathbf{P}^n) \leqq \varepsilon$. Suppose that r is the midpoint of the longest diagonal of $\mathbf{P}^n$ and let r_* be the exact solution of the system $F^n = \theta^n$. Then $\|r - r_*\|_2 \leqq \frac{n\varepsilon}{2}$.*

In addition, it is easy to prove [see 8] that the new generalized bisection method (with the same assumptions as in [7]), is also an impartial subdivision method. Moreover, according to Theorem 3.4 of [7], after a finite number of iterations of the new generalized method of bisection applied to the boundary $b(\mathbf{P}^n)$, of an admissible n-polygon $\mathbf{P}^n$ relative to continuous F^n, the boundary $b(\mathbf{P}^n)$ will be sufficiently refined relative to $\operatorname{sgn}(F^n)$.

Finally, we proceed with the algorithm of a generalized bisection method.

Algorithm 4.10. *(A Generalized Method of Bisection)*

1. *Read ε, ε^*.*
2. *Store the vertices X_i, $i \in \mathcal{V}$ of $\mathbf{P}^n$, also store the rows C_i, $i \in \mathcal{V}$ of $\mathcal{M}_n$.*
3. *Find the orders of all proper 1-simplexes (S_p^1, S_p^2), $p = 1(1)n2^{n-1}$.*
4. *For all $p = 1(1)n2^{n-1}$, execute:*
   $i \leftarrow S_p^1$, $j \leftarrow S_p^2$, compute $D_p = \|X_i - X_j\|_2$.
5. *$D \leftarrow \max_p \{D_p\}$, compute $v = [\log(D \cdot \varepsilon^{-1})/\log 2] + 1$.*
6. *$t \leftarrow 0$, $m \leftarrow 0$, $k \leftarrow 0$.*
7. *$t \leftarrow t+1$, if $t = v$ go to step 14, otherwise go to step 8.*
8. *$m \leftarrow m+1$, if $m = n2^{n-1}$ go to step 7, otherwise go to step 9.*
9. *$d \leftarrow 0$, $i \leftarrow S_m^1$, $j \leftarrow S_m^2$, compute $B = (X_i + X_j)/2$, also, compute the functions evaluations $F^n(B)$. If $|f_e(B) \leqq \varepsilon^*|$ for all $e \in \mathcal{E}$ stop with B as the approximate root, otherwise compute the $\mathcal{S}(F^n, B)$.*
10. *$k \leftarrow k+1$, if $k = 2^n$ then go to step 8, else go to next step.*
11. *If $\mathcal{S}(F^n, B) = C_k$ then go to step 12, otherwise return to step 10.*
12. *$g \leftarrow X_k$, $X_k \leftarrow B$, if $k \neq i$ and $k \neq j$, go to step 13, else go to step 8.*
13. *$d \leftarrow d+1$, if $d > 2$ then go to step 8, otherwise $B \leftarrow 2X_k - g$ and go to step 9.*
14. *Find the maximum diagonal and take its midpoints as the approximate root.*

The "*relaxation*" which takes place in steps 12 and 13 is discussed in [8].

5. Numerical Examples

The Algorithms 3.2 and 4.10 were programmed in Fortran IV on a Cromemco system III, and many examples were tried in several dimensions. Our experience is that the algorithms behaved predictably and reliably and the results were quite satisfactory. So, Tables 1, 2 and 3 incorporate the various stages in obtaining solutions of three well known test cases; namely, the ones by Rosenbrock, Brown-Conte and Powell.

In the above tables, "V_{ij}" indicates the j-th component of the i-th vertex of the initial n-polygon, "X_{ij}' indicates the j-th component of the i-th vertex of the admissible n-polygon $\mathbf{P}^n$, "S_{ij}" indicates the j-th component of $\mathscr{S}(F^n, X_i)$, "r" indicates the approximate solution of $F^n = \theta^n$, "$B.C.$" indicates the maximum number of bisections cycles of the proper 1-simplexes of $\mathbf{P}^n$ in order to establish the root with precision "ε", "$N.B.$" indicates the bisections cycles and "$F.E.$" indicates the total number of functions evaluations for the generalized method of bisection.

Table 1. Experiment in two dimensions
$F^2 = (10(X_2 - X_1^2), 1 - X_1)$ [Rosenbrock]

i	V_{i1}	V_{i2}	X_{i1}	X_{i2}	S_{i1}	S_{i2}	r_i
1	−5	−5	1.05	−5	−1	−1	0.9999999
2	−5	4	−2.05	4	−1	1	0.9999999
3	4	−5	1.05	4	1	−1	
4	4	4	−1.95	4	1	1	

$\Delta(\mathbf{P}^n) = 9.518932$ B.C. = 27 F.E. = 98 $\varepsilon = 10^{-7}$ N.B. = 24

Table 2. Experiment in three dimensions
$F^3 = (3X_1 + X_2 + 2X_3^2 - 3, -3X_1 + 5X_2^2 + 2X_1X_3 - 1, 25X_1X_2 + 20X_3 + 12)$ [Brown-Conte]

i	V_{i1}	V_{i2}	V_{i3}	X_{i1}	X_{i2}	X_{i3}	S_{i1}	S_{i2}	S_{i3}	r_i
1	−1	−0.5	−2	1	−0.5	−0.45	−1	−1	−1	0.2900523
2	−1	−0.5	1	0.45	−0.5	1	−1	−1	1	0.6874306
3	−1	1	−2	−1	1	−1.53	−1	1	−1	−0.8492385
4	−1	1	1	−1	1	1	−1	1	1	
5	1	−0.5	−2	1	−0.5	−0.55	1	−1	−1	
6	1	−0.5	1	0.55	−0.5	1	1	−1	1	
7	1	1	−2	−1	1	−1.63	1	1	−1	
8	1	1	1	1	1	1	1	1	1	

$\Delta(\mathbf{P}^n) = 3.30498$ B.C. = 25 F.E. = 297 $\varepsilon = 10^{-7}$ N.B. = 24

Table 3. Experiment in four dimensions
$F^4 = (X_1 + 10X_2, \sqrt{5}(X_3 - X_4), (X_2 - 2X_3)^3, \sqrt{10}(X_1 - X_4))$ [Powell]

i	V_{i1}	V_{i2}	V_{i3}	V_{i4}	X_{i1}	X_{i2}	X_{i3}	X_{i4}	S_{i1}	S_{i2}	S_{i3}	S_{i4}	r_i
1	−0.2	−0.15	−0.16	−0.15	−0.2	−0.15	0.14	0.19	−1	−1	−1	−1	−0.0000001
2	−0.2	−0.15	−0.16	−0.60	0.3	−0.15	0.14	0.19	−1	−1	−1	1	0.0000000
3	−0.2	−0.15	0.14	−0.15	0.3	−0.08	−0.16	0.60	−1	−1	1	−1	0.0000001
4	−0.2	−0.15	0.14	−0.60	0.3	−0.08	−0.16	−0.15	−1	−1	1	1	0.0000001
5	−0.2	0.15	−0.16	−0.15	−0.2	−0.03	0.14	−0.15	−1	1	−1	−1	
6	−0.2	0.15	−0.16	−0.60	0.3	−0.08	0.14	−0.15	−1	1	−1	1	
7	−0.2	0.15	0.14	−0.15	−0.2	−0.15	−0.09	−0.15	−1	1	1	−1	
8	−0.2	0.15	0.14	−0.60	0.3	−0.15	−0.09	−0.15	−1	1	1	1	
9	0.3	−0.15	−0.16	−0.15	−0.2	0.15	0.14	0.19	1	−1	−1	−1	
10	0.3	−0.15	−0.16	−0.60	0.3	0.15	0.14	0.19	1	−1	−1	1	
11	0.3	−0.15	0.14	−0.15	0.3	0.02	−0.16	0.60	1	−1	1	−1	
12	0.3	−0.15	0.14	−0.60	0.3	0.02	−0.16	−0.15	1	−1	1	1	
13	0.3	0.15	−0.16	−0.15	−0.2	0.07	0.14	−0.15	1	1	−1	−1	
14	0.3	0.15	−0.16	−0.60	0.3	0.02	0.14	−0.15	1	1	−1	1	
15	0.3	0.15	0.14	−0.15	−0.2	0.15	−0.09	−0.15	1	1	1	−1	
16	0.3	0.15	0.14	−0.60	0.3	0.15	−0.09	−0.15	1	1	1	1	

$\Delta(\mathbf{P}^n) = 0.912689$ B.C. $= 24$ F.E. $= 553$ $\varepsilon = 10^{-7}$ N.B. $= 18$

In the cases where no roots of F^n laid within the initial n-polygon, the Algorithm 3.2 gave correct results.

6. Conclusions and Assessment

The fast generalized method of bisection we have analysed in this paper compares favourably with other methods of bisection [3, 4, 6]; its great advantage being that since it does not compute topological degrees at all, as the other methods do in each iteration, its speed is quite remarkable. Moreover it keeps the advantages of the other methods; that is it needs in the evaluation of the various functions their signs only to be correct, it can be applied for non-differentiable functions and does not require calculations of derivatives.

Our method has also got the advantages of the traditional bisection method, that is we can know beforehand the number of iterations that are needed for the attainment of the root to a prescribed accuracy; as well the starting estimate of the root has not got to be near the root.

Furthermore, the analysis of the method is such that its generalization to higher dimensions is quite trivial; in fact our algorithm is fully automated and handles the dimensionality of the problem just as a parameter; so the algorithm can be used as a starting procedure for obtaining good approximations of roots while for finer refinements we switch into other methods, for which, as we know, good initial approximations are a condition sine qua non.

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Note Added in Proof

Answering question raised by the first referee we compiled the following comparative table which enriches the numerical evidence of the paper, and for which we are thankful to the referee.

Comparative table for localizing a solution within a region

Method	Dim		Number of F.E.	Comments
	2–	3–		
Kearfott's GBM	12 992	16 2152	Min Max	The numbers shown are produced with respect to the 3 problems examined by the author in [4]
Our RGBM	4 12	8 32	Min Max	The numbers shown refer to any problem and are produced from the easily derived formulae Max F.E. $=(n+1)2^n$, Min F.E. $=2^n$, where n is the dimensionality of the problem