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A Note on Comparison Theorems for Nonnegative Matrices

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Summary. In a recent paper, [4], Csordas and Varga have unified and extended earlier theorems, of Varga in [10] and Woźnicki in [11], on the comparison of the asymptotic rates of convergence of two iteration matrices induced by two regular splittings. The main purpose of this note is to show a connection between the Csordas-Varga paper and a paper by Beauwens, [1], in which a comparison theorem is developed for the asymptotic rate of convergence of two nonnegative iteration matrices induced by two splittings which are not necessarily regular. Monotonic norms already used in [1] play an important role in our work here.

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Introduction

The comparison of the asymptotic rates of convergence of nonnegative iteration matrices induced by two splittings of the same matrix has arisen in several recent papers. For example, in a paper by Csordas and Varga, [4, Theorem 2], a result is presented that unifies and extends earlier theorems by Varga, [10, Theorem 3.15], and Woźnicki, [11, Theorem 13], in which the iteration matrices are induced by two regular splittings of the same monotone matrix. Another recent result is due to Beauwens, [1, Theorem 2.3]. Beauwens' theorem compares the asymptotic convergence rates of two splittings of the same matrix, but under milder conditions than matrix monotonicity. The primary purpose of this note is to derive a link between the result of Csordas and Varga and the theorem due to Beauwens. In the proof of his theorem, Beauwens uses monotonic norms in the comparison of the spectral radii of certain nonnegative matrices. This idea provides an auxiliary purpose for this note as we further highlight the use of such norms in proving comparison results in the absence of an entrywise ordering between the matrices whose spectral radii we wish to compare.

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For the convenience of the reader we shall now briefly explain some of the terminology used in the preceding paragraph. First, an $n \times n$ matrix A is called *monotone* if for any n-vector x, $Ax \geqq 0$ implies $x \geqq 0$, where "$\geqq$" denotes the usual entrywise ordering. A celebrated result due to Collatz, [3], is that an $n \times n$ matrix A is monotone if and only if A is invertible and $A^{-1} \geqq 0$.

Second, the splitting of an $n \times n$ matrix A into

$$A = M - N, \quad \det(M) \neq 0 \tag{1}$$

is said to be *regular* if $M^{-1} \geqq 0$ and $N \geqq 0$ (see Varga, [10], Definition 3.5). Regular splittings play an important role in the problem of solving the $n \times n$ system of linear equations $Ax = b$. One numerical method for solving this problem can be obtained from (1) by performing the iterations

$$x_{i+1} = M^{-1} N x_i + M^{-1} b \qquad i = 0, 1, 2, \ldots. \tag{2}$$

It is well known that the scheme given by (2) converges to the solution $x = A^{-1} b$ from any initial vector x_0 if and only if the *spectral radius*

$$\rho(M^{-1} N) := \max\{|\lambda| : \det(\lambda I - M^{-1} N) = 0\} < 1.$$

In this case the *asymptotic convergence rate* of (2) is given by

$$R_\infty(M^{-1} N) := -\ln[\rho(M^{-1} N)].$$

Theoretically, an accepted rule for preferring one iteration scheme over another is to select the scheme which yields the larger asymptotic convergence rate. In practice, however, additional considerations such as sparsity and conditioning may also play an important part in our choice as to which iterative method should be adopted in any specific situation.

Let A be a nonsingular matrix of order n and consider the splittings of A into

$$A = M_1 - N_1 = M_2 - N_2, \tag{3}$$

where both M_1 and M_2 are nonsingular. For convenience we shall employ letters of the alphabet to represent the following conditions on these splittings:

V: (Varga [10], Theorem 3.5. See also [9]) *The splittings of A in* (3) *are regular and*

$$N_2 \geqq N_1. \tag{4}$$

W: (Woźnicki [11], Theorem 13) *The splittings of A in* (3) *are regular and*

$$M_1^{-1} \geqq M_2^{-1}. \tag{5}$$

C.V.: (Csordas and Varga [4], Theorem 2) *The splittings of A in* (3) *are regular and there exists an index $j \geqq 1$ such that*

$$(A^{-1} N_1)^j A^{-1} \leqq (A^{-1} N_2)^j A^{-1}. \tag{6}$$

B: (Beauwens [1], Theorem 2.3) *The splittings of A in* (3) *satisfy $M_1^{-1} N_1 \geqq 0$ and $M_2^{-1} N_2 \geqq 0$ and*

$$(A^{-1} N_1)^2 \leqq (A^{-1} N_2)(A^{-1} N_1). \tag{7}$$

The next condition may be thought of as a generalized Beauwens/Csordas-Varga condition:

G.B.C.V.: *The splittings of A in* (3) *satisfy the inequalities $M_1^{-1}N_1 \geqq 0$ and $M_2^{-1}N_2 \geqq 0$ and there exist indices $j \geqq 1$ and $l \geqq 0$ such that*

$$(A^{-1}N_1)^{j+l} \leqq (A^{-1}N_2)^j (A^{-1}N_1)^l. \tag{8}$$

Observation 1. Let (3) *represent two splittings of the $n \times n$ nonsingular matrix A such that $A^{-1}N_1 \geqq 0$ and $A^{-1}N_2 \geqq 0$. The following tree of implications represents the relative strength of the properties introduced above:*

$$\begin{array}{ccccccc} V & \Rightarrow & W & \Rightarrow & C.V. & \Rightarrow & G.B.C.V. \\ & & & & & B \nearrow & \end{array}$$

Furthermore, between none of the properties does the reverse implication hold.

Proof. Verification that $V \Rightarrow W$, but that in general $W \not\Rightarrow V$ may be found in Woźnicki, [11]. Next, $W \Rightarrow C.V.$, but $C.V. \not\Rightarrow W$ is in Csordas and Varga, [4]. The implications $C.V. \Rightarrow G.B.C.V.$ and $B \Rightarrow G.B.C.V.$ are immediate. The fact that $B \not\Rightarrow C.V.$ follows from the literature. With regard to this, we mention in particular examples due to Ortega and Rheinboldt in their work on weak regular splittings. (Note: A *weak regular splitting* is a splitting of A into $A = M - N$ with $\det(M) \neq 0$, $M^{-1} \geqq 0$ and $M^{-1}N \geqq 0$.) These examples can be found in [6] and [7].

Continuing, we next give an example illustrating that $G.B.C.V. \not\Rightarrow B$. Let

$$\begin{aligned} A &= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}^{-1} \\ &= \underset{M_1}{\begin{bmatrix} 3 & -3/2 \\ -1 & 4 \end{bmatrix}} - \underset{N_1}{\begin{bmatrix} 1 & -1/2 \\ 0 & 2 \end{bmatrix}} \\ &= \underset{M_2}{\begin{bmatrix} 4 & -1/2 \\ -1 & 2 \end{bmatrix}} - \underset{N_2}{\begin{bmatrix} 2 & 1/2 \\ 0 & 0 \end{bmatrix}}. \end{aligned} \tag{9}$$

Then

$$M_1^{-1}N_1 = (2/21)\begin{bmatrix} 4 & 1 \\ 1 & 11/2 \end{bmatrix} \quad \text{and} \quad M_2^{-1}N_2 = (2/15)\begin{bmatrix} 4 & 1 \\ 2 & 1/2 \end{bmatrix},$$

$$\rho(M_1^{-1}N_1) = 4/7 < 3/5 = \rho(M_2^{-1}N_2),$$

and

$$A^{-1}N_1 = (1/3)\begin{bmatrix} 2 & 1 \\ 1 & 7/2 \end{bmatrix} \quad \text{and} \quad A^{-1}N_2 = (1/3)\begin{bmatrix} 4 & 1 \\ 2 & 1/2 \end{bmatrix}.$$

But,

$$(A^{-1}N_1)^2 \nleqq (A^{-1}N_2)(A^{-1}N_1). \tag{10}$$

We shall postpone the proof that there exist indices $j \geqq 1$ and $l \geqq 0$ such that (8) is satisfied to remark (ii) following the proof of Theorem 2. □

The following lemma contains a very *slight* addition to the statement of Theorem 3.13 in Varga, [10]. Its purpose is to give the minimum requirements under which our Theorem 1 can be stated. We shall omit its proof as it is essentially given in Varga, [10, pp. 88–89].

Lemma 1. (Varga [10], Theorem 3.13) *Let $A=M-N$ be a splitting for the $n\times n$ nonsingular matrix A such that M is invertible and $M^{-1}N\geqq 0$. Then $\rho(M^{-1}N)<1$ if and only if $A^{-1}N\geqq 0$.*

We are now ready to state one of the main results of this note.

Theorem 1. *Suppose that A is an $n\times n$ nonsingular matrix and that* (3) *represents two splittings of A satisfying the G.B.C.V. conditions. Then*

$$0<R_\infty(M_2^{-1}N_2)\leqq R_\infty(M_1^{-1}N_1). \tag{11}$$

Proof. Notice first that the splittings in (3) satisfy the conditions of Lemma 1 and hence we have that both $\rho(M_1^{-1}N_1)$ and $\rho(M_2^{-1}N_2)$ are less than unity. Without loss of generality we shall assume $\rho(M_1^{-1}N_1)>0$, since there is nothing to prove otherwise. Under this assumption we have that $M_1^{-1}N_1\neq 0$ so that $A^{-1}N_1$ must also be a nonzero matrix since

$$M_1^{-1}N_1=(I+A^{-1}N_1)^{-1}A^{-1}N_1.$$

This fact will be used implicitly later in the proof.

In order to show $R_\infty(M_2^{-1}N_2)\leqq R_\infty(M_1^{-1}N_1)$ we first note from (8) that for any index $k\geqq 1$,

$$\begin{aligned}(A^{-1}N_1)^{kj+l}&=(A^{-1}N_1)^{j+l}(A^{-1}N_1)^{(k-1)j}\\&\leqq(A^{-1}N_2)^j(A^{-1}N_1)^l(A^{-1}N_1)^{(k-1)j}\\&=(A^{-1}N_2)^j(A^{-1}N_1)^{j+l}(A^{-1}N_1)^{(k-2)j}\\&\leqq\ldots\leqq(A^{-1}N_2)^{kj}(A^{-1}N_1)^l\end{aligned} \tag{12}$$

Now let u be an arbitrary n-vector whose entries are positive. Then the functional

$$\|x\|_u=\inf_{\alpha>0}\{-\alpha u\leqq x\leqq \alpha u\}$$

is a monotonic norm on R^n which induces the operator norm

$$\|A\|_u=\sup\{\|Ax\|_u/\|x\|_u\}.$$

Moreover, $\|A\|_u=\|Au\|_u$. (See, for example, Housholder [5], and Rheinboldt and Vandergraft [8].) Thus, it readily follows from (12) that

$$\|(A^{-1}N_1)^{kj+l}\|_u\leqq\|(A^{-1}N_2)^{kj}\|_u\|(A^{-1}N_1)^l\|_u. \tag{13}$$

Consider the following sequences of numbers:

(i) $\{\|(A^{-1}N_1)^{kj+l}\|_u^{1/(kj)}\}_{j=1}^\infty$,

(ii) $\{\|(A^{-1}N_2)^{kj}\|_u^{1/(kj)}\}_{j=1}^\infty$,

and

(iii) $\{\|(A^{-1}N_1)^{l}\|_u^{1/(kj)}\}_{j=1}^\infty$.

Sequence (iii) has a limit equal to 1, while sequences (i) and (ii) are subsequences of convergent sequences (e.g., Young [12], Corollary 3.7.3). Hence, by the calculus of limits,

$$\rho(A^{-1}N_1) \leqq \rho(A^{-1}N_2). \tag{14}$$

But now (11) follows from (14) in accordance with the spectral inequalities developed in Sect. 3.6 of Varga's book, [10]. □

Given the inequality in (11) between the asymptotic rates of convergence for two iterative schemes, one is often interested in when the inequality can be made strict. For two regular splittings or A in (3) which satisfy either condition V or condition W, the respective authors of [10] and [11] give conditions in their works to ensure a strict inequality between the asymptotic convergence rates. Conditions ensuring strict inequality between the asymptotic rates of convergence for two iteration matrices induced by splittings of A in (3) which satisfy the *G.B.C.V.* requirements are harder to determine. There are two reasons for this difficulty:

(i) Our proof of the inequality in (11) results from a limiting argument in which strict inequalities may not be preserved.

(ii) Although the matrices $A^{-1}N$ and $M^{-1}N$ appearing in our theorem are nonnegative, we do not assume that A^{-1}, M^{-1} or N are positive, or even nonnegative, thus allowing for cancellations to occur.

Therefore, it remains an open question as to what additional and appropriate conditions should be required from *G.B.C.V.* splittings to obtain strict inequality in (11).

In conjunction with the preceding paragraph we wish to make a few observations. First, Csordas and Varga, [4, Theorem 4], obtain a partial converse to their Theorem 2, namely:

Suppose the splittings of the $n \times n$ nonsingular matrix A in (3) *are regular and $A^{-1} > 0$. If*

$$R_\infty(M_1^{-1}N_1) > R_\infty(M_2^{-1}N_2)$$

or equivalently, $\rho(M_1^{-1}N_1) < \rho(M_2^{-1}N_2)$, then for some index $j_0 \geqq 1$,

$$(A^{-1}N_1)^j A^{-1} < (A^{-1}N_2)^j A^{-1}, \quad \forall j \geqq j_0.$$

Reason (ii) in the preceding paragraph suggests that if (3) represents splittings of A, with $A^{-1} > 0$, satisfying $M_1^{-1}N_1 \geqq 0$ and $M_2^{-1}N_2 \geqq 0$ then $\rho(M_1^{-1}N_1) < \rho(M_2^{-1}N_2)$ is insufficient to imply even the weak inequality

$$(A^{-1}N_1)^{j+l} \leqq (A^{-1}N_2)^j (A^{-1}N_1)^l$$

for infinitely many j's and/or l's. (An illustration that this in fact is the case may be seen in the example of remark (iii) following Theorem 2.)

However, a partial converse to our Theorem 1 is possible. For that purpose we require simple results from the theory of matrices and their associated directed graphs. (See, for example, Varga, [10], and Berman and Plemmons, [2].) If B is a matrix of order n, then it has an associated *directed graph*, or digraph,

$$G(B) = [V(B), E(B)],$$

where $V(B)=\{1,2,\ldots,n\}$ denotes the set of vertex labels for the graph and $E(B)$ is the set of directed edges (i,j), where (i,j) is in $E(B)$ if and only if $b_{ij}\neq 0$. $E(B)$ is often called the *graph set* of B.

Suppose B_1 and B_2 are nonnegative matrices of order n. Then $G(B_1B_2)$ is uniquely determined by $G(B_1)$ and $G(B_2)$. In fact, (i,j) is in $E(B_1B_2)$ if and only if there exists a k in $N=\{1,2,\ldots,n\}=V(B_1)=V(B_2)$ such that $(i,k)\in E(B_1)$ and $(k,j)\in E(B_2)$. Simple induction on this idea yields the following: $G(B^p)$ is uniquely determined by $G(B)$. That is, (i,j) is in $E(B^p)$ if and only if there exists an $i-j$ path of length p (possibly containing edge repetition) in $G(B)$.

We are now ready to present our partial converse:

Theorem 2. *Suppose $A=M_1-N_1=M_2-N_2$ are two splittings of a nonsingular matrix A such that M_1 and M_2 are invertible and $M_1^{-1}N_1\geqq 0$ and $M_2^{-1}N_2\geqq 0$. If $\rho(M_1^{-1}N_1)<\rho(M_2^{-1}N_2)$ and the limit*

$$\lim_{j\to\infty}[(A^{-1}N_2/\rho(A^{-1}N_2)]^j=L$$

exists, then for each index k there exists an index j_k such that

$$(A^{-1}N_1)^{j+k}_{st}<[(A^{-1}N_2)^j(A^{-1}N_1)^k]_{st},\quad \forall (s,t)\in E_k \quad \forall j\geqq j_k, \tag{15}$$

where E_k is the graph set of $L(A^{-1}N_1)^k$.

Proof. Given that $\lim_{j\to\infty}[A^{-1}N_2/\rho(A^{-1}N_2)]^j=L$, the discussion prior to the statement of the theorem implies that there exists an index j_k' such that

$$E(L)\subseteq E([A^{-1}N_2/\rho(A^{-1}N_2)]^j),\quad \forall j\geqq j_k'.$$

Hence,

$$E_k\subseteq E[(A^{-1}N_2)^j(A^{-1}N_1)^k/\rho(A^{-1}N_2)^{j+k}],\quad \forall j\geqq j_k'.$$

Now,

$$\lim_{j\to\infty}[(A^{-1}N_2)^j(A^{-1}N_1)^k/\rho(A^{-1}N_2)^{j+k}]_{st}=[L(A^{-1}N_1)^k/\rho(A^{-1}N_2)^k]_{st},\quad \forall (s,t)\in E_k.$$

But,

$$\lim_{j\to\infty}[(A^{-1}N_1)^{j+k}/\rho(A^{-1}N_2)^{j+k}]=0$$

since $\rho[A^{-1}N_1/\rho(A^{-1}N_2)]<1$ as $\rho(M_1^{-1}N_1)<\rho(M_2^{-1}N_2)$ implies $\rho(A^{-1}N_1)<\rho(A^{-1}N_2)$, a fact which follows from Varga's results concerning spectral inequalities [10, Sect. 3.6]. Thus, there must exist an index j_k for which (15) holds. □

Remarks. (i) Suppose $A^{-1}N_1$, $A^{-1}N_2\geqq 0$ are of order n, $\rho(A^{-1}N_1)<\rho(A^{-1}N_2)$ but that $A^{-1}N_2/\rho(A^{-1}N_2)$ is not semiconvergent because, although the elementary divisors of $A^{-1}N_2/\rho(A^{-1}N_2)$ corresponding to 1 are linear, it has eigenvalues on the unit circle other than 1. Then, for any $a\in(0,1)$ the matrix

$$(A^{-1}N_2)(a)=(1-a)I+aA^{-1}N_2/\rho(A^{-1}N_2)$$

is semiconvergent and a similar result to Theorem 2 holds for the matrices $(A^{-1}N_2)(a)$ and

$$(A^{-1}N_1)(a)=(1-a)I+aA^{-1}N_1/\rho(A^{-1}N_1).$$

(ii) Theorem 2 now permits us to complete the proof that $G.B.C.V. \not\Rightarrow B$ in Observation 1 since in (9) $\rho(M_1^{-1}N_1)=4/7<3/5=\rho(M_2^{-1}N_2)$, and

$$\lim_{j\to\infty}[A^{-1}N_2/\rho(A^{-1}N_2)]^j=A^{-1}N_2/\rho(A^{-1}N_2)$$

as $[A^{-1}N_2/\rho(A^{-1}N_2)]^2=A^{-1}N_2/\rho(A^{-1}N_2)$. Hence, $E_k=E\{L(A^{-1}N_1)^k\}=\{(1,1),(1,2),(2,1),(2,2)\}$ for all $k=1,2,\ldots$ so that (8) holds for some indices j and l.

(iii) Inequalities similar to those of (15) do not in general hold for pairs of indices $(s,t)\notin E_k$, for some k, as the following example illustrates.

$$A=\begin{bmatrix}-0.2 & 0.5\\ 1 & -1.5\end{bmatrix}=\begin{bmatrix}7.5 & 2.5\\ 5 & 1\end{bmatrix}^{-1}$$

$$=\underset{M_1}{\begin{bmatrix}-0.2 & 0.51\\ 1 & -1.4\end{bmatrix}}-\underset{N_1}{\begin{bmatrix}0 & 0.01\\ 0 & 0.1\end{bmatrix}}$$

$$=\underset{M_2}{\begin{bmatrix}-0.2 & 0.6\\ 1 & -1.8\end{bmatrix}}-\underset{N_2}{\begin{bmatrix}0 & 0.1\\ 0 & -0.3\end{bmatrix}}$$

Here $\rho(M_1^{-1}N_1)=3/23<1/6=\rho(M_2^{-1}N_2)$,

$$A^{-1}N_1=\begin{bmatrix}0 & 0.325\\ 0 & 0.15\end{bmatrix}\quad\text{and}\quad A^{-1}N_2=\begin{bmatrix}0 & 0\\ 0 & 0.2\end{bmatrix},$$

and

$$(A^{-1}N^{1})^{j+l}=\begin{bmatrix}0 & *\\ 0 & *\end{bmatrix}\quad\text{and}\quad (A^{-1}N_2)^j(A^{-1}N_1)^l=\begin{bmatrix}0 & 0\\ 0 & *\end{bmatrix},$$

where "*" denotes a positive entry. Moreover, $E_k=\{(2,2)\}$ for all $k=1,2,\ldots$. However, as is easily seen $[(A^{-1}N_1)^{j+l}]_{12}>[(A^{-1}N_2)^j(A^{-1}N_1)^l]_{12}$ for all $j\geqq 1$ and $l\geqq 0$.

(iv) As a final remark, we mention that Theorem 1 can be generalized by replacing the *G.B.C.V.* condition in (8) by certain norm conditions. It is not clear, however, that one would wish to do this as, in practice, the burden of verifying these conditions may be further complicated.

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