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Stream Vectors in Three Dimensional Aerodynamics

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Summary. This work deals with the decomposition of a vector field $\mathbf{u}$ into $\mathbf{u}=\nabla\times\psi+\nabla\phi$. Non homogeneous boundary conditions on ψ or ϕ are investigated; applications to the computation of inviscid flows are given; finally a conforming finite element implementation is studied and tested.

Subject Classifications: AMS(MOS): 65N30; CR: G1.8.

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0. Introduction

It is a well known result of analysis (De Rham [1]) that any solenoidal vector field $\mathbf{u}$ ($\nabla\cdot\mathbf{u}=0$) is the curl of a stream vector ψ. This property is used extensively in two dimensional fluid mechanics; there ψ being perpendicular to the plane of fluid motion, on simply connected domains, it is uniquely defined from $\mathbf{u}$ and while $\mathbf{u}$ has two non trivial components, ψ has only one.

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In three dimensions the situation is more complex because ψ is not uniquely defined from $\mathbf{u}$ unless additional constrained are assigned to ψ. Bernardi [2] and Dominguez [3] have studied sufficient conditions on ψ to have unicity for each $\mathbf{u}$, when the flow field is tangent to the boundaries, i.e. $\mathbf{u}\cdot\mathbf{n}=0$; the main idea is to assume that ψ also is solenoidal. But it is worth noting that such decompositions do not reduce the complexity of the problem of finding $\mathbf{u}$ since one would trade $\mathbf{u}$ unknown with $\nabla\cdot\mathbf{u}=0$ for ψ unknown with $\nabla\cdot\psi=0$.

Recently, however, a renewed interest for stream vectors ψ can be seen in aeronautical engineering (Lacor and Hirsch [4], Sokhey [5], Papaillou [6], Amara [7]) because it appeared that ψ could be a correction to isentropic potential calculations. The basic idea is as follows: flows around airplanes, for example, at transonic speeds, are well approximated by (H, γ given):

$$\nabla\cdot[(H-\tfrac{1}{2}|\nabla\phi|^2)^{1/\gamma-1}\nabla\phi]=0, \tag{1}$$

$$\mathbf{u}=\nabla\phi; \tag{2}$$

however when strong shocks develop in the flow $\nabla\times\mathbf{u}$ is no longer small and (2) fails; thus in [4] for instance, (2) is replaced by

$$\mathbf{u}=\nabla\phi+\nabla\times\psi \tag{3}$$

and one derives additional equations for ψ from the general equations of fluid motion. But is is difficult to find a good set of boundary conditions for ψ; this problem is related to the unicity of the decomposition (3).

The purpose of the present work is to study the decomposition (3) when $\mathbf{u}$ is a given 3-d vector field and ψ is solenoidal. We shall start with the homogeneous case $\mathbf{u}\cdot\mathbf{n}=0$, thus recalling essentially the results of Bernardi [2], Dominguez [3, 8]. Then the case $\mathbf{u}\cdot\mathbf{n}\neq 0$ is investigated and it will be obvious that existence and unicity of ϕ and ψ in (3) is possible if one solves first a Laplace-Beltrami problem on the boundaries.

In Sect. 2 some applications to aerodynamics are given, the ideas of Lacor and Hirsch [4], Sokhey [5], Amara [7] are extended to general stationary inviscid flows.

Finally, in Sect. 3, a P^1 conforming finite element discretization of the continuous problem is given. Existence and unicity of the decomposition is shown; error estimates are obtained on polyhedral domains for the homogeneous cases. The implementation of the method is studied and some numerical results on simple 3-d geometries are presented.

This paper is intended for numerical analysts and does not assume any knowledge of fluid mechanics.

Notations

Ω is a bounded open subset of R^3, usually assumed simply connected with C^2 boundary
Γ boundary of Ω, $\mathbf{n}$ its normal
Γ_i simply connected components of Γ
$H^1(\Omega)=\{w\in L^2(\Omega)\colon \nabla w\in L^2(\Omega)^3\}$
$H^1(\Omega)^n=\{\mathbf{v}\colon v_i\in H^1(\Omega),\ i=1\ldots n\}$

$H^1(\Omega)/R = H^1(\Omega)$ quotiented by the constants
$H(\nabla\cdot, \Omega) = \{\mathbf{v} \in L^2(\Omega)^3 : \nabla\cdot\mathbf{v} \in L^2(\Omega)\}$
$H(\nabla\times, \Omega) = \{\mathbf{v} \in L^2(\Omega)^3 : \nabla\times\mathbf{v} \in L^2(\Omega)^3\}$
$H(\Delta, \Omega) = \{w \in H^1(\Omega) : \Delta w \in L^2(\Omega)\}$
$(\mathbf{a}, \mathbf{b}) = \int_\Omega \mathbf{a}\cdot\mathbf{b}\,dx$
$|a|_0 = (\mathbf{a}, \mathbf{a})^{1/2}$
$(\mathbf{u}\otimes\mathbf{v})_{ij} = u_i v_j$
$\boldsymbol{\psi}, \mathbf{v}, \boldsymbol{\xi}, \mathbf{n}, \mathbf{t}, \mathbf{u}$: vector valued functions of $\mathbf{x}\in\Omega$
ϕ, w, q: scalar valued functions of $\mathbf{x}\in\Omega$

1. The Continuous Problem

Let Ω be a bounded open set of R^3 with boundary Γ.

Let $\mathbf{u}$ be a given 3-d vector field on Ω. The basic problem is to find ϕ and $\boldsymbol{\psi}$ such that

$$\mathbf{u}(x) = \nabla\phi(x) + \nabla\times\boldsymbol{\psi}(x) \qquad \forall x\in\Omega. \tag{4}$$

By taking the divergence of (4) we get

$$\Delta\phi = \nabla\cdot\mathbf{u}. \tag{5}$$

If we add one of the two Neumann conditions

$$\left.\frac{\partial\phi}{\partial n}\right|_\Gamma = 0 \quad \text{or} \quad \left.\frac{\partial\phi}{\partial n}\right|_\Gamma = \mathbf{u}\cdot\mathbf{n} \tag{6}$$

then ϕ is unique up to a constant and only $\boldsymbol{\psi}$ remains to be found.

Taking the curl of (4) yields

$$\nabla\times\nabla\times\boldsymbol{\psi} = \nabla\times\mathbf{u} \tag{7}$$

while (6) and (4) bring:

$$\mathbf{n}\cdot\nabla\times\boldsymbol{\psi}|_\Gamma = \mathbf{u}\cdot\mathbf{n} \quad \text{or} \quad 0. \tag{8}$$

However (7) and (8) do not define $\boldsymbol{\psi}$ even up to a constant. It is known that

$$\nabla\times\nabla\times\boldsymbol{\psi} = -\Delta\boldsymbol{\psi} + \nabla\nabla\cdot\boldsymbol{\psi} \qquad \forall\boldsymbol{\psi} \tag{9}$$

so $\nabla\times\nabla\times$ is a strongly elliptic operator only on the space of solenoidal vector fields: thus Bernardi [2] adds

$$\nabla\cdot\boldsymbol{\psi} = 0; \tag{10}$$

but even then (8) is a non standard boundary condition for $\nabla\times\nabla\times$; the natural one (see the variational formulation) is

$$\boldsymbol{\psi}\times\mathbf{n}|_\Gamma = \mathbf{g}. \tag{11}$$

The connection between (8) and (11) is given by the following lemma.

Lemma 1. *Assume Γ Lipschitz continuous, then*

$$\int_\Gamma (\mathbf{v}\times\mathbf{n})\cdot\nabla w\,d\gamma=-\int_\Gamma(\nabla\times\mathbf{v})\cdot\mathbf{n}\,w\,d\gamma\qquad\forall w\in H^1(\Omega)\ \forall\mathbf{v}\in H^1(\Omega)^3\tag{12}$$

Proof. Use Green's formulae:

$$(\nabla\times\mathbf{v},\psi)=(\nabla\times\psi,\mathbf{v})+\int_\Gamma(\mathbf{v}\times\mathbf{n})\cdot\psi\,d\gamma$$

$$(\nabla\cdot\mathbf{v},\phi)=-(\nabla\phi,\mathbf{v})+\int_\Gamma\mathbf{v}\cdot\mathbf{n}\phi\,d\gamma$$

on

$$\int_\Omega\mathbf{v}\cdot\nabla\times(\nabla w)\,d\mathbf{x}\quad\text{and}\quad\int_\Omega w\nabla\cdot(\nabla\times\mathbf{v})\,d\mathbf{x}.$$

See Dominguez [8, Theorem 2.1] for details. □

Formula (12) yields the following implication

$$\psi\times\mathbf{n}|_\Gamma=0\Rightarrow\mathbf{n}\cdot\nabla\times\psi|_\Gamma=0;\tag{13}$$

so decomposition (4) will be easier when $\mathbf{u}\cdot\mathbf{n}=0$ (see (7)) or when the second boundary condition in (6) is used; this will be referred as "homogeneous boundary conditions".

1.1. Homogeneous Boundary Conditions on ψ

We are now in a position to state the first result on decomposition (4); it is a straightforward generalization of the result of Bernardi [2] extended to non simply connected domain by Dominguez [8].

Theorem 1. *Let Ω be a bounded open set of R^3 simply connected, with boundary Γ of class C^2. Let $\mathbf{u}$ be a given 3-d vector field of $L^2(\Omega)^3$.*

Let ϕ be the unique solution of

$$(\nabla\phi,\nabla w)=(\mathbf{u},\nabla w)\qquad\forall w\in H^1(\Omega)/R;\ \phi\in H^1(\Omega)/R.\tag{14}$$

Let ψ be the unique solution of

$$(\nabla\times\psi,\nabla\times\mathbf{v})+(\nabla\cdot\psi,\nabla\cdot\mathbf{v})=(\mathbf{u},\nabla\times\mathbf{v})\qquad\forall\mathbf{v}\in V;\ \psi\in V\tag{15}$$

where

$$V=\{\mathbf{v}\in H^1(\Omega)^3:\ \mathbf{v}\times\mathbf{n}|_\Gamma=\mathbf{0},\ \int_{\Gamma_i}\mathbf{v}\cdot\mathbf{n}\,d\gamma=0\ \ \forall\Gamma_i \text{ connected component of } \Gamma\}\tag{16}$$

then

$$\mathbf{u}=\nabla\phi+\nabla\times\psi\qquad\text{in }\Omega\tag{17}$$

and

$$\nabla\cdot\psi=0.\tag{18}$$

Proof. Since the proof is somewhat long, some of the technical points have been put in Annexe A. So first we check that (14) is a well posed Neumann problem (in fact (5)+(6b)) because $\mathbf{u}$ is in $L^2(\Omega)^3$. Thus ϕ is unique in $H^1(\Omega)/R$. Now let us study the

a) Well Posedness of (15)

The space V, defined by (16), is a closed subspace of $H^1(\Omega)^3$ (continuity of traces) and all integrals in (15) exist. The bilinear form in $\boldsymbol{\psi}$ and $\mathbf{v}$ and the linear form in $\mathbf{v}$, in (15), are obviously continuous on V. From the variational form of (9):

$$(\nabla\boldsymbol{\psi}, \nabla\mathbf{v}) = (\nabla\times\boldsymbol{\psi}, \nabla\times\mathbf{v}) + (\nabla\cdot\boldsymbol{\psi}, \nabla\cdot\mathbf{v}) + \int_\Gamma 2\frac{\boldsymbol{\psi}\cdot\mathbf{v}}{R}\,d\gamma \qquad \forall\mathbf{v}, \boldsymbol{\psi}\in V \tag{19}$$

where R is the mean radius of curvature of Γ, we get

$$|\nabla\boldsymbol{\psi}|_0^2 \leqq |\nabla\times\boldsymbol{\psi}|_0^2 + |\nabla\cdot\boldsymbol{\psi}|_0^2 \qquad \text{if } \Omega \text{ is convex } (R>0). \tag{20}$$

It is shown in Annexe A that $\boldsymbol{\psi}\to|\nabla\boldsymbol{\psi}|_0$ is an equivalent norm on V (Lemma A1) so (15) is a strongly elliptic problem and $\boldsymbol{\psi}$ is unique (Lions and Magenes [9], for example). When Ω is not convex the strong ellipticity is shown by Lemma A2.

b) $\nabla\cdot\boldsymbol{\psi}=0$

Let us take $\mathbf{v}=\nabla w$ in (15). To be in V we must have

$$w\in H^2(\Omega), \qquad \nabla w\times\mathbf{n}|_\Gamma=\mathbf{0}, \qquad \int_{\Gamma_i}\frac{\partial w}{\partial n}\,d\gamma=0 \quad \forall i \tag{21}$$

condition (21.b) will be replaced by

$$w|_{\Gamma_i}=K_i \qquad \text{(constant)}, \tag{22}$$

with such w, (15) reduces to

$$(\nabla\cdot\boldsymbol{\psi}, \Delta w)=0 \qquad \forall w \text{ satisfying (21), (22)}. \tag{23}$$

Therefore to get $\nabla\cdot\boldsymbol{\psi}=0$ it remains only to show that

$$\forall f\in L^2(\Omega) \ \text{ with } \int_\Omega f\,dx=0 \quad \exists w\in H^2(\Omega) \ \text{ with } \begin{cases} -\Delta w=f \\ w \text{ constant on each } \Gamma_i \\ \int_{\Gamma_i}\frac{\partial w}{\partial n}\,d\gamma=0 \quad \forall i. \end{cases} \tag{24}$$

This may be done as follows:

Define w^0 and θ^i by

$$-\Delta w^0=f \qquad w^0|_\Gamma=0, \tag{25}$$

$$-\Delta\theta^i=0 \qquad \theta^i|_{\Gamma_j}=\delta_{ij}. \tag{26}$$

Let

$$w = w^0 + \sum_j K_j \theta^j. \tag{27}$$

Then

$$\int_{\Gamma_i} \frac{\partial w}{\partial n} d\gamma - \int_{\Gamma_i} \frac{\partial w^0}{\partial n} d\gamma = \sum_j K_j \int_{\Gamma_i} \frac{\partial \theta^j}{\partial n} d\gamma \tag{28}$$

and hence

$$-\int_{\Gamma_i} \frac{\partial w^0}{\partial n} d\gamma = \sum_j K_j \int_{\Gamma} \theta^i \frac{\partial \theta^j}{\partial n} d\gamma = \sum_j K_j (\nabla \theta^i, \nabla \theta^j). \tag{29}$$

So $\{K_j\}_j$ is the solution of a linear system; Lemma A3 shows that the system in K is positive definite, up to a constant, so w is defined uniquely, up to a constant.

c) Proof of (17)

Let

$$\boldsymbol{\chi} = \nabla \phi + \nabla \times \psi - \mathbf{u}. \tag{30}$$

Then by (15) and (18) we have

$$\nabla \times \boldsymbol{\chi} = \nabla \times \nabla \times \psi - \nabla \times \mathbf{u} = 0 \quad \text{in } \Omega. \tag{31}$$

Thus there exists ξ (De Rham [1]) such that

$$\boldsymbol{\chi} = \nabla \xi \quad \text{in } \mathscr{D}'(\Omega). \tag{32}$$

But by (30) and (14) we have

$$\Delta \xi = \nabla \cdot \boldsymbol{\chi} = \Delta \phi - \nabla \cdot \mathbf{u} = 0 \tag{33}$$

and similarly

$$\left.\frac{\partial \xi}{\partial n}\right|_{\Gamma} = \boldsymbol{\chi} \cdot \mathbf{n} = \frac{\partial \phi}{\partial n} - \mathbf{u} \cdot \mathbf{n} = 0 \tag{34}$$

so ξ is constant and $\boldsymbol{\chi} = \mathbf{0}$. □

The result can be extended to non simply connected domains Ω (Dominguez [8, Corollary 3.2]). Then

$$\mathbf{u} = \nabla \times \psi + \nabla \phi \tag{35}$$

where ϕ is found by a problem similar to (14):

$$\begin{aligned} &\Delta \phi = \nabla \cdot \mathbf{u} \quad \text{in } \Omega - \Sigma \\ &\left.\frac{\partial \phi}{\partial n}\right|_{\Gamma} = \mathbf{u} \cdot \mathbf{n} \\ &[\phi]_{\Sigma_j} = \lambda_j, \quad \left[\frac{\partial \phi}{\partial n}\right]_{\Sigma_j} = 0 \quad ([\;]_{\Sigma_j} = \text{jump of, through } \Sigma_j) \end{aligned} \tag{36}$$

where $\{\Sigma_j\}$ is a set of surfaces linking Γ_i so that $\Omega - \Sigma$ is simply connected and where $\{\lambda_j\}$ is a set of constants.

In turn ψ is determined by (15) *and*

$$\int_{\Gamma_i} \psi \cdot \mathbf{n}\, d\gamma = \mu_i, \quad \text{(constants)}. \tag{37}$$

The decomposition is no longer unique. It is unique when μ_i are given; λ_j are adjusted so that $\nabla \times \psi + \nabla \phi = \mathbf{u}$ on Σ. The difficulty with such decomposition in non simply connected domains comes from the fact that

$$\nabla \times \mathbf{v} = 0 \quad \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega \quad \mathbf{v} \cdot \mathbf{n}|_\Gamma = 0, \tag{38}$$

does not imply $\mathbf{v} = 0$ (we used this in c)), (Foias and Temam [10]).

Remark 1. From (19) we see that (15) is interpreted by the mixed Dirichlet-Fréchet problem:

$$\begin{gathered} -\Delta \psi = \nabla \times \mathbf{u} \quad \text{in } \Omega \\ \frac{\partial \psi}{\partial n} \cdot \mathbf{n} - 2\frac{\psi \cdot \mathbf{n}}{R} = 0 \quad \text{on } \Gamma \\ \psi \times n|_\Gamma = 0. \end{gathered} \tag{39}$$

Remark 2. We need $\Gamma \in C^2$ to proove strong ellipticity of (15). It seems that this property also holds if Γ is piecewise C^2 only.

1.2. Non Homogeneous Boundary Condition on ψ

We shall see in Sect. 2 that it is not always feasible to have non homogeneous conditions on ϕ. Thus in this paragraph we wish to investigate the possibility of having ϕ defined by

$$\begin{gathered} -\Delta \phi = \nabla \cdot \mathbf{u} \quad \text{in } \Omega \\ \left.\frac{\partial \phi}{\partial n}\right|_\Gamma = 0 \end{gathered} \tag{40}$$

and

$$\mathbf{u} = \nabla \phi + \nabla \times \psi. \tag{41}$$

As we pointed out earlier the difficulty is now to find g such that

$$\psi \times \mathbf{n}|_\Gamma = g \Rightarrow \mathbf{n} \cdot \nabla \times \psi|_\Gamma = \mathbf{u} \cdot \mathbf{n}. \tag{42}$$

Theorem 2. *Let Ω be a bounded open set of R^3 simply connected with boundary Γ C^2-regular. Let $\mathbf{u} \in H(\nabla \cdot, \Omega)$ such that*

$$\int_{\Gamma_i} \mathbf{u} \cdot \mathbf{n}\, d\gamma = 0 \quad \textit{for all connected components } \Gamma_i \textit{ of } \Gamma. \tag{43}$$

If ϕ is the unique solution of

$$(\nabla\phi, \nabla w)=(\nabla\cdot\mathbf{u}, w) \quad \forall w\in H^1(\Omega)/R;\ \phi\in H^1(\Omega)/R \tag{44}$$

and if $\boldsymbol{\psi}$ is the unique solution of

$$\begin{aligned}&(\nabla\times\boldsymbol{\psi}, \nabla\times\mathbf{v})+(\nabla\cdot\boldsymbol{\psi}, \nabla\cdot\mathbf{v})=(\mathbf{u}, \nabla\times\mathbf{v}) \quad \forall \mathbf{v}\in V \ (\textit{defined in } (16))\\ &\boldsymbol{\psi}\in H^1(\Omega)^3, \quad \boldsymbol{\psi}\times\mathbf{n}|_\Gamma=\nabla q(s_1, s_2), \quad \int_{\Gamma_i}\boldsymbol{\psi}\cdot\mathbf{n}\,d\gamma=0 \quad \forall i\end{aligned} \tag{45}$$

where $q|_\Gamma$ is the unique solution of the Laplace-Beltrami equation

$$\int_\Gamma\left(\frac{\partial q}{\partial s_1}\frac{\partial w}{\partial s_1}+\frac{\partial q}{\partial s_2}\frac{\partial w}{\partial s_2}\right)d\gamma=-\int_\Gamma\mathbf{u}\cdot\mathbf{n}\,w\,d\gamma \quad \forall w\in H^1(\Gamma)/R;\ q\in H^1(\Gamma)/R \tag{46}$$

($\{s_1 s_2\}$ is a set of orthogonal local coordinate system on Γ), then

$$\mathbf{u}=\nabla\phi+\nabla\times\boldsymbol{\psi}, \tag{47}$$

$$\nabla\cdot\boldsymbol{\psi}=0. \tag{48}$$

Proof. The proof is similar to that of Theorem 1 except for the presence of (46). As $\mathbf{u}\in H(\nabla\cdot, \Omega)$, $\mathbf{u}\cdot\mathbf{n}$ is in $H^{-\frac{1}{2}}(\Gamma)$; problem (46) is well posed, and $\nabla q|_\Gamma$ is in $L^2(\Gamma)$ at least. Therefore ϕ and $\boldsymbol{\psi}$ are well defined by (44)–(45).

One shows that $\nabla\cdot\boldsymbol{\psi}=0$ exactly as in Theorem 1. Then let

$$\boldsymbol{\chi}=\nabla\phi+\nabla\times\boldsymbol{\psi}-\mathbf{u}. \tag{49}$$

By (45) $\nabla\times\boldsymbol{\chi}=0$ so for some ξ we have $\boldsymbol{\chi}=\nabla\xi$ and by (49) and (44):

$$\Delta\xi=0. \tag{50}$$

Now from (49) and (45) respectively

$$\int_\Gamma\mathbf{n}\cdot\nabla\times\boldsymbol{\psi}\,w\,d\gamma=\int_\Gamma\left(\frac{\partial\xi}{\partial n}+\mathbf{u}\cdot\mathbf{n}\right)w\,d\gamma, \tag{51}$$

$$-\int_\Gamma(\boldsymbol{\psi}\times\mathbf{n})\cdot\nabla w\,d\gamma=-\int_\Gamma\nabla q\cdot\nabla w\,d\gamma=\int_\Gamma\mathbf{u}\cdot\mathbf{n}\,w\,d\gamma \tag{52}$$

where the last equality is due to (46), so finally by Lemma 1

$$\int_\Gamma\frac{\partial\xi}{\partial n}w\,d\gamma=0 \quad \forall w. \tag{53}$$

Thus ξ is constant and $\boldsymbol{\chi}=0$; this shows (47).

2. Applications

2.1. Wings and Airplanes at Small Mach Numbers

Figure 1 shows a wing W inside a domain of boundary S which is assumed big enough to approximate infinity. The flow is assumed incompressible and irro-

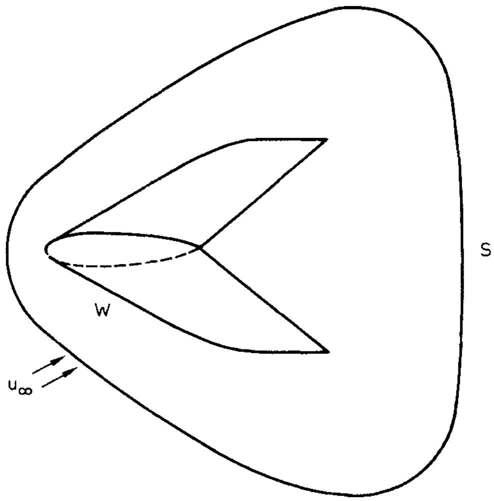


Fig. 1. Typical geometry of a three dimensional swept wing in an enclosure which approximates infinity. Notice that the domain is simply connected but the boundary is not smooth (edges at the trailing edges)

tational (no lift):

$$\nabla \cdot \mathbf{u}=0 \qquad \nabla \times \mathbf{u}=0. \tag{54}$$

Usually Theorem 1 is used to compute $\mathbf{u}$ as $\nabla\phi$. An alternative is to use Theorem 2.

$$\mathbf{u}=\nabla\phi+\nabla\times\boldsymbol{\psi} \tag{55}$$

but (44) and (54) give

$$\phi=\text{constant}. \tag{56}$$

The flow is assumed uniform at infinity and tangential to W so

$$\mathbf{u}\cdot\mathbf{n}|_S=\mathbf{u}_\infty\cdot\mathbf{n} \qquad \mathbf{u}\cdot\mathbf{n}|_W=0. \tag{57}$$

It is easy to check that if R the radius of S is constant, the solution of (46) is

$$q|_S=-\tfrac{1}{2}\mathbf{u}_\infty\cdot\mathbf{n}R^2, \qquad q|_W=0. \tag{58}$$

Indeed if t_1, t_2 are two orthogonal tangent vectors on S

$$\frac{\partial q}{\partial s_i}=+\tfrac{1}{2}\mathbf{u}_\infty\cdot\frac{\mathbf{t}_i}{R}R^2; \quad \frac{\partial^2 q}{\partial s_i^2}=+\tfrac{1}{2}\mathbf{u}_\infty\cdot\mathbf{n} \qquad i=1,2. \tag{59}$$

Therefore the flow is determined by solving (45) only. If S is not a sphere a semi explicit solution of the form

$$q=(\alpha'\mathbf{t}+\beta'\mathbf{n}+\gamma'\mathbf{t}\times\mathbf{n})\cdot\mathbf{u}_\infty \tag{60}$$

can be found also where α', β', γ' are solutions of PDEs in $s_1 s_2$ involving the radius of curvatures and the torsion. However a direct solution of (45) is probably easier, as we shall show in Sect. 3.

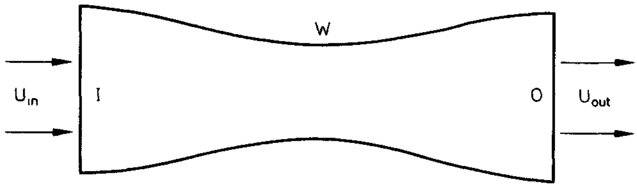


Fig. 2. Typical geometry of an axisymetric nozzle; the figure shows a two dimensional section. As in Fig. 2 the domain is simply connected but not smooth

There is also an alternative: since u is uniform at infinity, ψ must be asymptotic to

$$\psi=\begin{pmatrix}0\\-\alpha x_3\\(1-\alpha)x_2\end{pmatrix}u_{\infty 1}+\begin{pmatrix}(1-\beta)x_3\\0\\-\beta x_1\end{pmatrix}u_{\infty 2}+\begin{pmatrix}-\gamma x_2\\(1-\gamma)x_1\\0\end{pmatrix}u_{\infty 3} \tag{61}$$

where α, β, γ are any constants. From (61), $\psi\times\mathbf{n}|_\Gamma$ can be computed and this value can be used in (45) instead of ∇q. While we would still have

$$\mathbf{u}=\nabla\times\psi \qquad \nabla\cdot\psi=0 \tag{62}$$

there is no garantee that $\mathbf{n}\cdot\nabla\times\psi=\mathbf{u}_\infty\cdot\mathbf{n}$ on Γ_∞; this will be true asymptotically only.

2.2. *Nozzles at low Mach Numbers*

Figure 2 shows the geometry of a nozzle. Typical boundary conditions are

$$\begin{aligned}\mathbf{u}\cdot\mathbf{n}&=\mathbf{u}_i\cdot\mathbf{n} && \text{at intake boundary } I\\ &=\mathbf{u}_0\cdot\mathbf{n} && \text{at exit boundary } 0\\ &=0 && \text{on wall boundary } W\end{aligned} \tag{63}$$

where $\mathbf{u}_i$ and $\mathbf{u}_0$ are such that

$$\int_\Gamma \mathbf{u}\cdot\mathbf{n}\,d\gamma=0. \tag{64}$$

As for wings the flow is obtained by $\nabla\times\psi$ where ψ is the solution of (45). However (46) must be solved on Γ and Γ is not C^2!

In the proof of Theorem 2, the regularity of Γ is needed only to prove well-posedness of (45) and (46). Notice that (46) can also be written as

$$\int_\Gamma\left(\nabla q\cdot\nabla w-\frac{\partial q}{\partial n}\frac{\partial w}{\partial n}\right)d\gamma=-\int_\Gamma \mathbf{u}\cdot\mathbf{n}\,w\,d\gamma \tag{65}$$

where q is extended inside Ω and ∇ is the 3-d gradient. Thus Theorem 2 also holds for Γ piecewise C^2 only, say, but more investigations are needed to show that ψ and q are unique. The problem of corners will pop up also in Sect. 3.

2.3. *Entropy Corrections for Potential Compressible Flows*

The stationary Euler equations for ideal gas are

$$\nabla\cdot(\rho\mathbf{u})=0, \tag{66}$$

$$\nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u})+\nabla p=0, \tag{67}$$

$$\nabla\cdot\rho\mathbf{u}\left(\frac{p}{\rho}\frac{\gamma}{\gamma-1}+\frac{u^2}{2}\right)=0, \tag{68}$$

$$p=\rho^\gamma S \tag{69}$$

where $\mathbf{u}$, ρ, p, S are the velocity density pressure and entropy of the flow.

Because (66) holds, Eq. (68) gives the enthalpy in the flow from its value at infinity:

$$H(\mathbf{x})=\frac{p}{\rho}\frac{\gamma}{\gamma-1}+\frac{u^2}{2}=H_\infty(\mathbf{z}(\mathbf{x})) \tag{70}$$

where H_∞ is the (given) enthalpy on Γ and $\mathbf{z}(\mathbf{x})$ is the upstream intersection of Γ with the stream line that passes at x. Expanding (67) yields

$$-\rho\mathbf{u}\times\omega+\rho\nabla\frac{u^2}{2}+\nabla\rho^\gamma S=0 \tag{71}$$

where ω is the vorticity

$$\omega=\nabla\times\mathbf{u}. \tag{72}$$

Resolving the vectorial product in (71) and using (70) to evaluate u^2 gives an equation for ω in terms of an unknown function λ:

$$\omega=\lambda\rho\mathbf{u}-\frac{\mathbf{u}}{u^2}\times\left(\nabla H-\frac{\rho^{\gamma-1}}{\gamma-1}\nabla S\right)+[\omega]_\Sigma \tag{73}$$

where $[\omega]$ denotes the jump of the tangential component of ω across Σ. It must be remembered that (72) implies $\nabla\cdot\omega=0$ so

$$\rho\mathbf{u}\nabla\lambda=\nabla\cdot\left[\frac{\mathbf{u}}{u^2}\times\left(\nabla H-\frac{\rho^{\gamma-1}}{\gamma-1}\nabla S\right)\right] \tag{74}$$

and finally by taking the scalar product of (71) with $\mathbf{u}$, an equation for S is found:

$$\mathbf{u}\cdot\nabla S=0. \tag{75}$$

Notice that (75) is not valid at shocks because (71) makes no sense at shocks.

Let us investigate the system (66), (73), (74), (75), (70) when $\mathbf{u}$ is decomposed by Theorem 1 into

$$\mathbf{u}=\nabla\phi+\nabla\times\psi \tag{76}$$

with ϕ and ψ solutions of (14) and (15). If H_∞ is constant, if there are no shocks and if S is constant at infinity, then by (75) and (70) S and H are constant and by (73) and (74) $\omega=0$ if $\rho\omega\cdot\mathbf{u}$ is zero at infinity. So by (15) $\psi=0$ and it is easy to see that (70) and (76) reduce to the transonic equation (1).

Therefore one may attempt to compute a correction to the transonic equation when the conditions stated above are not met (shocks, S or H non constant at infinity, $\omega \cdot \mathbf{u} \neq 0$ at infinity). One may proceed as follows:

0. Compute a first guess for the velocity $\mathbf{u}$ and density ρ by the transonic Eq. (1).

1. Compute S by (75), the Rankine-Hugoniot conditions at shocks, and its value at infinity upstream (see [15] for numerical solutions of such equations).

2. Compute H from its value at infinity upstream by (69), (70).

3. Compute ω by (73)–(74) and $\rho\omega \cdot \mathbf{u}$ at infinity upstream.

4. Compute ψ from ω by (15) (the right hand side must be integrated by part).

5. Compute a new ϕ by solving (66)–(70), i.e.

$$\nabla \cdot \left\{ \left[(H - \tfrac{1}{2} |\nabla\phi + \nabla \times \psi|^2) \frac{1}{S} \right]^{1/\gamma - 1} (\nabla\phi + \nabla \times \psi) \right\} = 0$$
$$\rho \left. \frac{\partial\phi}{\partial n} \right|_{\Gamma} = (\rho\mathbf{u})_\infty \cdot \mathbf{n} \tag{77}$$

(see [13, 14] for solution methods for (77).

6. Set
$$\mathbf{u} = \nabla\phi + \nabla \times \psi, \tag{78}$$
$$\rho = \left[\frac{\gamma - 1}{\gamma S} (H - \tfrac{1}{2} u^2) \right]^{1/\gamma - 1}. \tag{79}$$

and start new loop with Step 1.

A proof of convergence of this iterative process is of course difficult. However preliminary tests by Lacor and Hirsch [4], Sokhey [5] and the authors [11] indicate that the method works; it is a nice alternative to the direct solution of (66)–(69) by time marching methods (see [12] for example).

Following Amara [7] one could also use (66) to write

$$\rho\mathbf{u} = \nabla \times \psi'. \tag{80}$$

Since $\rho\mathbf{u} \cdot \mathbf{n} \neq 0$, one would have to use Theorem 2 in a similar iterative process:

0. Compute ρ, $\mathbf{u}$ by solving the transonic equation.

1. Compute ψ' by (45)–(46) in Theorem 2, with u replaced by ρu.

2. Compute S, H, ω as in Steps 1, 2, 3 above.

3. Compute a new ψ' by solving the non linear problem

$$\nabla \times \rho^{-1}(|\nabla \times \psi'|) \nabla \times \psi' = \omega \tag{81}$$

where $\rho(v)$ is the solution of (see (79))

$$\rho^{\gamma - 1} = \frac{\gamma - 1}{\gamma S} \left(H - \frac{1}{2} \frac{v^2}{\rho^2} \right) \tag{82}$$

and start a new loop with Step 1.

Although this method is conceptually more simple, it is difficult to implement because of the non uniqueness of the map $v \to \rho(v)$ defined by (82). Also Theorem 2 must be adapted to (81) so that ψ' is unique. However it is likely to handle better non zero $\rho \boldsymbol{\omega} \cdot \mathbf{u}$ or S at infinity. Wings with lift can also be treated in this fashion since it corresponds to $\rho \boldsymbol{\omega} \cdot \mathbf{u} \neq 0$ at the trailing edge of W.

3. Finite Element Approximation

The decomposition of Theorems 1 and 2 involve a Neumann problem for ϕ, a mixed problem for ψ and a Laplace-Beltrami problem for q. Since this is a 3-d problem we shall use the simplest finite element: tetrahedra with piecewise affine functions.

Discretization of ϕ by this element offers no difficulty (see Ciarlet [16], for example).

3.1. Approximation of ψ

Let ψ be the solution of (u, ψ_Γ given)

$$\begin{aligned} &(\nabla \times \psi, \nabla \times \mathbf{v}) + (\nabla \cdot \psi, \nabla \cdot \mathbf{v}) = (\nabla \times \mathbf{u}, \mathbf{v}) \qquad \forall \mathbf{v} \in V \\ &\psi - \psi_\Gamma \in V = \{\mathbf{v} \in H^1(\Omega)^3 \colon \mathbf{v} \times \mathbf{n}|_\Gamma = 0, \ \int_{\Gamma_i} \mathbf{v} \cdot \mathbf{n}\, d\gamma = 0\} \end{aligned} \tag{83}$$

where ψ_Γ is such that

$$\int_{\Gamma_i} \psi_\Gamma \cdot \mathbf{n}\, d\gamma = 0 \qquad \forall \Gamma_i \text{ connected component of } \Gamma.$$

To discretize (83) one proceeds as follows:

Assume Ω_h is polygonal and close to Ω and let $\mathscr{C}_h$ be a triangulation of Ω in the sense that it is a collection of non overlapping tetrahedra $\{T_k\}$ which cover Ω_h and whose vertices never lie in the middle of a face or edge of another tetrahedra.

Then define

$$V_h = \{\mathbf{v} \in C^0(\bar{\Omega})^3 \colon \mathbf{v}|_{T_k} \text{ is affine; } \mathbf{v}(q^j) \times \mathbf{n}_h^j = 0 \ \forall j, q^j \in \Gamma \ \int_{\Gamma_i} \mathbf{v} \cdot \mathbf{n}\, d\gamma = 0\} \tag{84}$$

where $\{q^j\}$ are the *vertices* of $\mathscr{C}_h$, $\mathbf{n}_h^j$ is an approximation of the normal of Ω at q^j:

$$\mathbf{n}_h^j \simeq \mathbf{n}(\mathbf{q}^j). \tag{85}$$

Now let ψ_h be the solution of

$$\begin{aligned} &(\nabla \times \psi_h, \nabla \times \mathbf{v}_h) + (\nabla \cdot \psi_h, \nabla \cdot \mathbf{v}_h) = (\nabla \times \mathbf{u}, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in V_h \\ &\psi_h - \psi_{\Gamma_h} \in V_h \end{aligned} \tag{86}$$

where ψ_{Γ_h} is the continuous piecewise affine vector field which coincides with ψ_Γ on the vertices of Γ_h.

A basis for V_h may be constructed from the usual basis $\{w^i\}$ of the P^1 element:

let $w^i(x)$ be continuous, piecewise affine on $\mathscr{C}_h$, and such that

$$w^i(q^j)=\delta_{ij}; \tag{87}$$

let $\{e^l\}_{1,2,3}$ be an orthonormal basis of R^3; assume for clarity that the first M vertices are internal; define

$$\begin{aligned} \mathbf{v}^{3(i-1)+l}(x)&=w^i(x)\,\mathbf{e}^l, \quad i=1\dots M,\ (q^i\notin\Gamma_h),\ l=1,2,3\\ \mathbf{v}^{j+2M}(x)&=\bar{w}^j(x)\,n_h^j, \quad j>M,\ (q^i\in\Gamma_h). \end{aligned} \tag{88}$$

where

$$\bar{w}^j(x)=w^j(x)-\Big(\sum_{q^l\in\Gamma_{i(j)}} w^l(x)\Big)\int_{\Gamma_{i(j)}} w^j(x)\,\mathbf{n}_h^j\cdot\mathbf{n}(x)\,d\gamma\Big/\int_{\Gamma_{i(j)}} \mathbf{n}_h^j\cdot\mathbf{n}(x)\,d\gamma \tag{89}$$

($\Gamma_{i(j)}$ = connected component of Γ_h which contains q^j).

Obviously $\mathbf{v}^k$ belongs to V_h; $\{\mathbf{v}^k\}$ spans V_h; an independent subfamily is obtained if one $\mathbf{v}^j$ per Γ_j is removed. Then on this basis (86) is a linear system

$$\mathsf{A}\,\boldsymbol{\Psi}=\mathbf{b} \tag{90}$$

where

$$A_{ij}=(\nabla\times\mathbf{v}^i,\nabla\times\mathbf{v}^j)+(\nabla\cdot\mathbf{v}^i,\nabla\cdot\mathbf{v}^j), \tag{91}$$

$$b_i=(\nabla\times\mathbf{u},\mathbf{v}^i)-(\nabla\times\psi_{\Gamma_h},\nabla\times\mathbf{v}^i)-(\nabla\cdot\psi_{\Gamma_h},\nabla\cdot\mathbf{v}^i). \tag{92}$$

A is positive definite because

$${}^t\boldsymbol{\Psi}\,\mathsf{A}\,\boldsymbol{\Psi}=|\nabla\times\psi_h|_0^2+|\nabla\cdot\psi_h|_0^2=0 \Rightarrow \psi_h=0 \quad \text{by Lemma A4.} \tag{93}$$

Therefore (86) *is a well defined computable problem.*

Now let us estimate $\|\psi-\psi_h\|$. In all generality one ought to study the problem with $\Omega_h\neq\Omega$ or with isoparametric elements. But this leads to technical difficulties which would require a separated paper (see Verfurth [17], for example). Thus we shall assume $\Omega_h=\Omega$, polygonal but still retain $n_h\neq n$ in (86) to see how accurately the normal has to be approximated when $\Omega_h\neq\Omega$.

Proposition 1. *Assume Ω polyhedral with boundary Γ and normal n. Let ψ be the solution of* (83) *and ψ_h the solution of* (86) *with $\psi_\Gamma=\psi|_\Gamma$. Assume $\nabla\times\psi$ in $L^\infty(\Omega)$, let $\{\mathscr{C}_h\}_h$ be a family of triangulations with all angles greater than $\alpha>0$ and smaller than $\beta<\pi$ for all h, the largest length of the edges of the tetrahedra. Assume $\mathbf{n}_h^i$ is such that*

$$\Big(\int_\Gamma\sum_i|\mathbf{n}-\mathbf{n}_h^i|^2\,w^i(x)\,d\gamma\Big)^{1/2}\leqq Ch^{3/2} \tag{94}$$

then for h small enough

$$|\nabla\times(\psi-\psi_h)|_0\leqq C\|\psi\|_{2,\Omega}h;\quad |\nabla\cdot\psi_h|_0\leqq C\|\psi\|_{2,\Omega}h. \tag{95}$$

Proof. Notice first that it is not unreasonable to assume that ψ is in $H^2(\Omega)^3$ even though Ω is polyhedral. For example if $\bar{\psi}$ is the solution of (83) on $\bar{\Omega}\supset\Omega$, $\bar{\Omega}$ smooth and $\psi_\Gamma=\bar{\psi}$ on Γ then $\psi=\bar{\psi}$ is the solution of (83) in Ω, and $\bar{\psi}$ is smooth.

From (83) we see that

$$-\Delta\psi=\nabla\times\mathbf{u} \tag{96}$$

but

$$\int_\Omega(-\Delta\psi)\mathbf{v}\,dx=\int_\Omega(\nabla\times\psi\cdot\nabla\times\mathbf{v}+\nabla\cdot\psi\nabla\cdot\mathbf{v})\,dx$$
$$+\int_\Gamma((\nabla\times\psi)\cdot(\mathbf{v}\times\mathbf{n})-\mathbf{v}\cdot\mathbf{n}\nabla\cdot\psi)\,d\gamma \tag{97}$$

so if we set

$$a(\psi,\mathbf{v})=(\nabla\times\psi,\nabla\times\mathbf{v})+(\nabla\cdot\psi,\nabla\cdot\mathbf{v}) \tag{98}$$

and remember that $\nabla\cdot\psi=0$, we find that

$$a(\psi,\mathbf{v})=(\nabla\times\mathbf{u},\mathbf{v})+\int_\Gamma(\nabla\times\psi)\cdot(\mathbf{n}\times\mathbf{v})\,d\gamma \qquad \forall\,\mathbf{v}\in H^1(\Omega)^3. \tag{99}$$

By definition of ψ_h

$$a(\psi_h,\mathbf{v}_h)=(\nabla\times\mathbf{u},\mathbf{v}_h) \qquad \forall\,\mathbf{v}_h\in V_h, \tag{100}$$

therefore

$$a(\psi-\psi_h,\psi-\psi_h)=a(\psi-\chi_h,\psi-\chi_h)+2a(\psi-\psi_h,\chi_h-\psi_h)-a(\psi_h-\chi_h,\psi_h-\chi_h)$$
$$\leqq a(\psi-\chi_h,\psi-\chi_h)+\int_\Gamma 2\mathbf{n}\times(\chi_h-\psi_h)\cdot\nabla\times\psi\,d\gamma$$
$$\forall\,\chi_h \text{ such that } \chi_h-\psi_h\in V_h. \tag{101}$$

Let us take

$$\chi_h(x)=\sum\psi(q^i)\,w^i(\mathbf{x}). \tag{102}$$

Since $\psi=\psi_\Gamma$ on Γ, $\chi_h-\psi_{\Gamma_h}=0$ on Γ so $\chi_h-\psi_h\in V_h$.

Then

$$a(\psi-\chi_h,\psi-\chi_h)^{1/2}\leqq C\|\psi\|_{2,\Omega}h. \tag{103}$$

To find an upper bound for

$$I=\left|\int_\Gamma\mathbf{n}\times(\chi_h-\psi_h)\nabla\times\psi\,d\gamma\right| \tag{104}$$

we notice that $\chi_h-\psi_h\in V_h$ implies

$$\chi_h(\mathbf{q}^i)-\psi_h(\mathbf{q}^i)=(\chi_h(q^i)-\psi_h(q^i))\cdot\mathbf{n}_h^i\mathbf{n}_h^i \qquad \forall\,\mathbf{q}^i \text{ vertex of } \Gamma. \tag{105}$$

But

$$\mathbf{n}\times\mathbf{n}_h^i=-(\mathbf{n}_h^i-\mathbf{n})\times\mathbf{n} \tag{106}$$

so

$$I \leqq \|\nabla \times \psi\|_{\infty} \int_{\Gamma} |\sum_{i} (\chi_h(q^i) - \psi_h(q^i)) \cdot \mathbf{n}_h^i(\mathbf{n}_h^i - \mathbf{n}) \times \mathbf{n}\, w^i|\, d\gamma$$
$$\leqq \|\nabla \times \psi\|_{\infty} (\int_{\Gamma} \sum_{i} |\chi_h(q^i) - \psi_h(q^i)|^2 w_i\, d\gamma)^{1/2} (\int_{\Gamma} \sum_{i} |\mathbf{n}_h^i - \mathbf{n}|^2 w^i\, d\gamma)^{1/2} \tag{107}$$

and by Lemma A5 and (94).

$$\leqq C \|\nabla \times \psi\|_{\infty} |\nabla(\chi_h - \psi_h)|_{0,\Omega} h^{3/2} \tag{108}$$

By Lemma A6:

$$|\nabla(\chi_h - \psi_h)|_{0,\Omega}^2 \leqq C a(\chi_h - \psi_h, \chi_h - \psi_h) \tag{109}$$

so

$$I \leqq C h a(\chi_h - \psi_h, \chi_h - \psi_h)^{1/2}$$
$$\leqq C h (a(\chi_h - \psi, \chi_h - \psi)^{1/2} + a(\psi_h - \psi, \psi_h - \psi)^{1/2}). \tag{110}$$

Let us put the pieces together; first (110) in (101) yields

$$X^2 - ChX \leqq Y^2 + ChY \tag{111}$$

where

$$X = a(\psi_h - \psi, \psi_h - \psi)^{1/2}, \qquad Y = a(\chi_h - \psi, \chi_h - \psi)^{1/2}. \tag{112}$$

So

$$X \leqq \frac{C}{2} h + \left[\left(\frac{C}{2} h \right)^2 + Y^2 + ChY \right]^{1/2}$$
$$\leqq Ch + Y + \sqrt{ChY} \leqq 2Ch + 2Y. \tag{113}$$

Now (103) is an estimate for Y which completes the proof. Notice that the hypothesis $\psi|_{\Gamma} = \psi_{\Gamma}$ appeared only in (102); if it does not hold (102) can be modified and the result still be true but the proof is substantially longer. □

Important Remark. When Ω has edges it is not possible to verify (94) unless the triangulation is refined in the direction perpendicular to the edge so that the mean size of the tetrahedra is $h^{3/2}$ in that direction. On a general polyhedral domain, $\mathbf{n}_h^i$ may be defined as

$$\mathbf{n}_h^i = \sum_{T_k \supset \{q^i\}} \mathbf{n}_{T_k} \text{ area } (T_k) / \sum_{T_k \supset \{q^i\}} \text{ area } (T_k). \tag{114}$$

Then (94) is satisfied when $\mathbf{n}$ is the normal of a smooth domain Ω which Ω_h approximates (all vertices of Γ_h are on Γ). Furthermore it has the following interesting property.

Proposition 2. *Let* ψ_h *be the solution of* (86) *with* $\mathbf{n}_h^i$ *given by* (114) *and* $\psi_{\Gamma} = \{0, -x_3, 0\}$. *Then* $\psi_h = \{0, -x_3, 0\}$.

Proof. With $\psi_h = -x_3 \mathbf{e}^2$ we have

$$(\nabla \times \psi_h, \nabla \times \mathbf{v}_h) + (\nabla \cdot \psi_h, \nabla \cdot \mathbf{v}_h) = \int_\Gamma (\mathbf{e}^1 \times \mathbf{n}) \cdot \mathbf{v}_h \, d\gamma$$

$$= \sum_{q^i \in \Gamma} \mathbf{v}_h^i \sum_{T_K \supset \{q^i\}} \mathbf{e}^1 \times \mathbf{n}_{T_K} \frac{\text{area}(T_k)}{3} = \sum_{q^i \in \Gamma} \mathbf{v}_h^i (\mathbf{e}^1 \times \mathbf{n}_h^i) = 0 \tag{115}$$

because $\mathbf{v}_h^i$ is parallel to $\mathbf{n}_h^i$, being in V_h. □

This last property is quite important for external problem like wing computations where S approximates infinity. Near S, h is not expected to be small but ψ_h is close to $\{0, -x_3, 0\}$ which is an exact solution of the discrete problem.

3.2. Approximation of q

To approximate q we rewrite (46) as

$$\int_\Gamma \left(\nabla q \nabla w - \frac{\partial q}{\partial n} \frac{\partial w}{\partial n}\right) d\gamma = -\int_\Gamma \mathbf{u} \cdot \mathbf{n} \, w \, d\gamma \qquad \forall w \in H^{3/2}(\Omega)/R; \; q \in H^{3/2}(\Omega)/R. \tag{116}$$

In this form q is not uniquely defined on Ω but its trace on Γ is unique. Then consider Ω_h a polygonal approximation of Ω, $\mathscr{C}_h$ a triangulation of Ω made of tetrahedra (and Γ_h and $\mathscr{S}_h$ the boundary of Ω_h and its triangulation

$$\mathscr{S}_h = \{T_k \cap \Gamma_h : T_k \in \mathscr{C}_h\}. \tag{117}$$

Problem (116) is approximated by

$$\begin{aligned} &\int_{\Gamma_h} \left(\nabla q_h \cdot \nabla w_h - \frac{\partial q_h}{\partial n} \cdot \frac{\partial w_h}{\partial n}\right) d\gamma = -\int_{\Gamma_h} \mathbf{u} \cdot \mathbf{n} \, w_h \, d\gamma \qquad \forall w_h \in H_h \\ &q_h \in H_h = \{w_h \in C^0(\Omega_h) : w_h|_{T_k} \text{ is affine}; \; \int_{\Gamma_h} w_h \, d\gamma = 0\}. \end{aligned} \tag{118}$$

Note however that $\psi_h = \nabla q_h$ can only be applied in a mean sense since ψ_h is piecewise affine. More complicate elements for ψ_h or q_h may be used also (Nedelec [18]).

A basis for H_h is constructed as usual

$$\tilde{w}^i(x) = w^i(x) - \int_{\Gamma_h} w^i \, d\gamma \Big/ \int_{\Gamma_h} d\gamma \qquad \mathbf{q}^i \in \Gamma_h. \tag{119}$$

Then (118) is the symetric linear system.

$$\mathbf{B}\mathbf{q} = \mathbf{c}, \tag{120}$$

$$B_{ij} = \int_{\Gamma_h} \left(\nabla \tilde{w}^i \cdot \nabla \tilde{w}^j - \frac{\partial \tilde{w}^i}{\partial n} \frac{\partial \tilde{w}^j}{\partial n}\right) d\gamma, \tag{121}$$

$$c_j = \int_{\Gamma_h} -\mathbf{u} \cdot \mathbf{n} \, \tilde{w}^j \, d\gamma \tag{122}$$

and $\mathbf{B}$ is positive definite. Indeed

$$\mathbf{q}^t \mathbf{B} \mathbf{q}=0 \Rightarrow q|_{T_k \cap \Gamma_h} \text{ constant} \Rightarrow q=0 \tag{123}$$

since it has zero mean.

Error estimates are obtained as for the standard Laplace equation. If $b(\ ,\ )$ is the bilinear form in (116), if $\Omega=\Omega_h$ then

$$\begin{aligned} b(q-q_h, q-q_h) &\leqq b(q-\pi_h q, q-\pi_h q) \\ &\leqq |\nabla(q-\pi_h q)|^2_{0,\Gamma} \\ &\leqq Ch^2 \|q\|^2_{2,\Gamma}. \end{aligned} \tag{124}$$

So if $\nabla_s=(\partial/\partial s_1, \partial/\partial s_2)$ then

$$|\nabla_s(q-q_h)| \leqq Ch \|q\|_{2,\Gamma}. \tag{125}$$

However for polygonal domains q will not be in $H^2(\Gamma)$; in fact q is not even C^1.

3.3. Numerical Tests

Problem (86) was tested on the wing problem when W is a portion of wing or a sphere, with $\nabla \times \mathbf{u}=\mathbf{0}$ and $\psi_\Gamma=\{0, -x_3, 0\}$ on S and $\mathbf{0}$ on W. Then (118) was solved with $\mathbf{u}=\{1,0,0\}$ and (86) was solved again with ∇q and compared with the first tests.

Both linear systems (90) and (120) were solved by the conjugate gradient method. Since (90) is big its special structure was used for storage. Indeed from (19) (which holds if ψ *or* v is zero on Γ) we see that A_{ij} is also, for some suitable indices

$$A_{ij}=(\nabla w^l, \nabla w^k)\mathbf{e}^n \cdot \mathbf{e}^m \qquad \text{if } q^i, q^j \notin \Gamma. \tag{126}$$

Hence A has the following structure

$$A=\begin{matrix} D & 0 & 0 & B^1 \\ 0 & D & 0 & B^2 \\ 0 & 0 & D & B^3 \\ B^1 & B^2 & B^3 & E \end{matrix} \tag{127}$$

where the last row and column corresponds to boundary indices and where

$$D_{ij}=(\nabla w^i, \nabla w^j); \qquad B^k_{ij}=D_{ij} n^j_{hk}; \qquad E_{ij}=\mathbf{n}^i_h \cdot \mathbf{n}^j_h D_{ij}+\mathbf{n}^i_h \times \mathbf{n}^j_h \cdot \int_\Omega \nabla w^i \times \nabla w^j \, dx. \tag{128}$$

The first geometry is a section of NACA 0012 cylindrical wing discretized by 468 vertices (337 on the boundary) and 1,548 elements. Thus the linear system is 730×730. Typically 30 iterations of conjugate gradients are sufficient to reduce the gradient to 10^{-8} times its initial value.

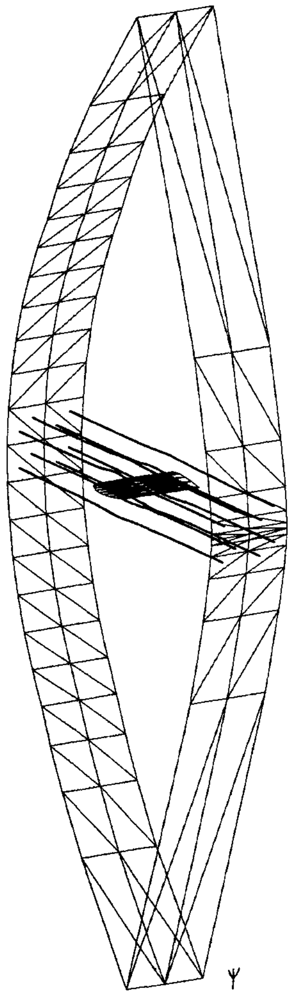


Fig. 3. Domain of computation and stream lines of a flow around a section of wing, computed from the rotational of the stream vector

Note that the domain is not simply connected but since the flow is symmetric with respect to the horizontal plane, calculations could have been performed on half the domain (which is simply connected) so the theory is still valid. Figure 3 shows some stream lines (lines parallel to u) and Fig. 4 shows the stream lines on a vertical plane tangent to the flow.

In Figs. 5 and 6 the geometry has been deformed; the wing is no longer cylindrical so it is a real 3-D flow (still symmetric). Figure 5 shows some stream lines computed by ψ solution of (86); it is to be compared with Fig. 6 which shows the same stream lines when the velocity is computed as the gradient of the potential ϕ (decomposition of Theorem 1, $\psi=0$). On an element near the

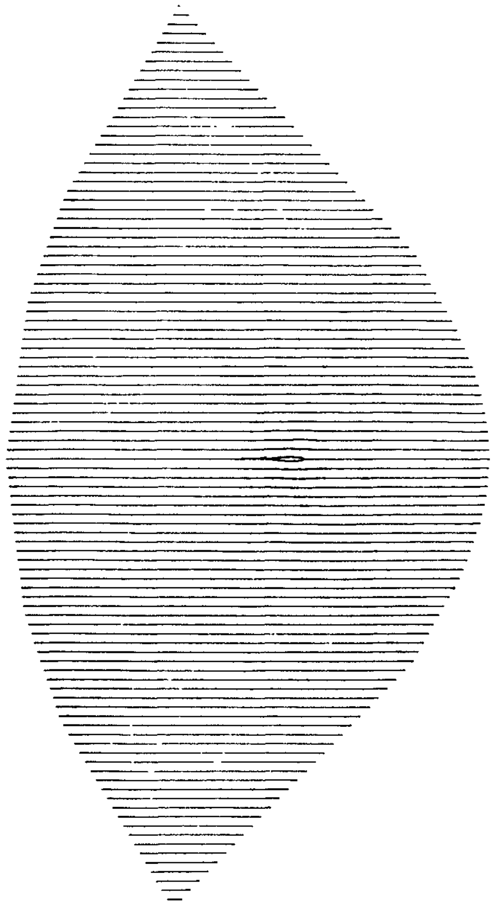


Fig. 4. Stream lines in the vertical plane of symetry of the domain of Fig. 3 for the flow computed from the rotational of the stream vector

wing for instance, we obtained

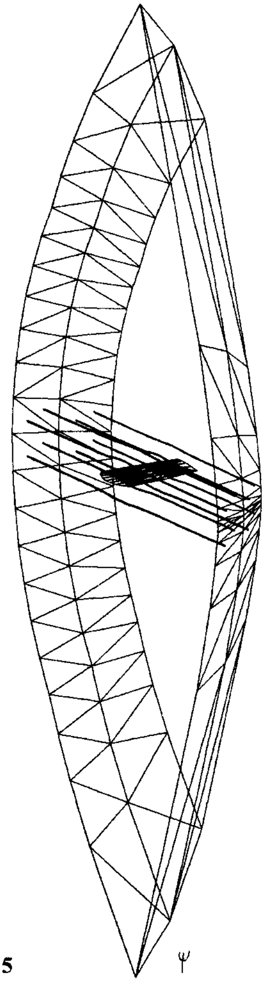
$$\nabla \times \psi = (1.152, -0.014, -0.050), \qquad \nabla\phi = (1.109, 0.0008, -0.001). \qquad (129)$$

Computing time for ψ was 48″ and for ϕ 28″ on an IBM 3081.

The second geometry is made of two concentric spheres S and W with 552 vertices, 184 on the boundaries and 2,300 elements. The linear system (90) is 1,288 × 1,288. The Laplace-Beltrami problem (118) is 184 × 184, 7 iterations decreased the gradient by a factor of 10^8.

Figure 7 shows some stream lines projected on the plane $z=0$, calculated from ψ solution of (86) with $\psi_\Gamma = \{0, -x_3, 0\}$ (Fig. 7a) or from ϕ solution of (14) (with $\nabla \cdot \mathbf{u} = 0$) (Fig. 7b) or from ψ solution of (86) with $\psi_\Gamma = \nabla q$ (Fig. 7c).

Figure 8 is the same case as Fig. 7.c but shown in 3-*d*.



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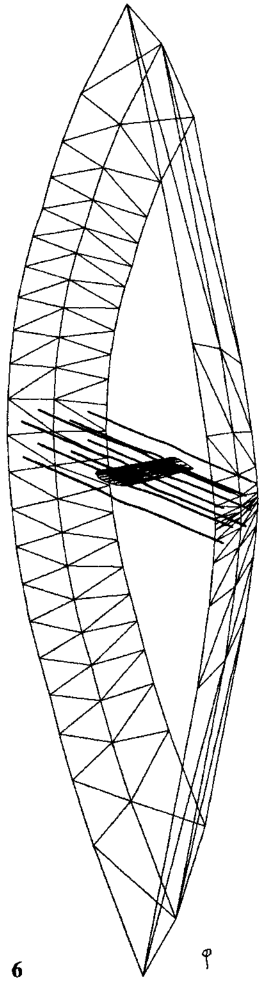

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Fig. 5. Stream lines of a real $3d$ flow around a section of wing computed from the rotational of the stream vector (to be compared with Fig. 6)

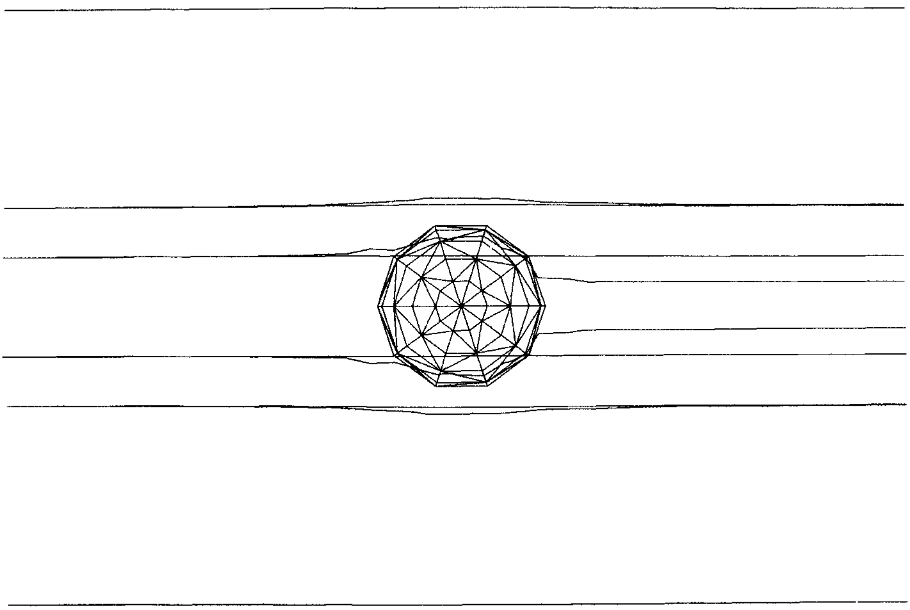
Fig. 6. Stream lines of the same flow as in Fig. 5 but computed by the gradient of the potential of the flow

One last check was made to test the effect of the Laplace Beltrami problem. We took

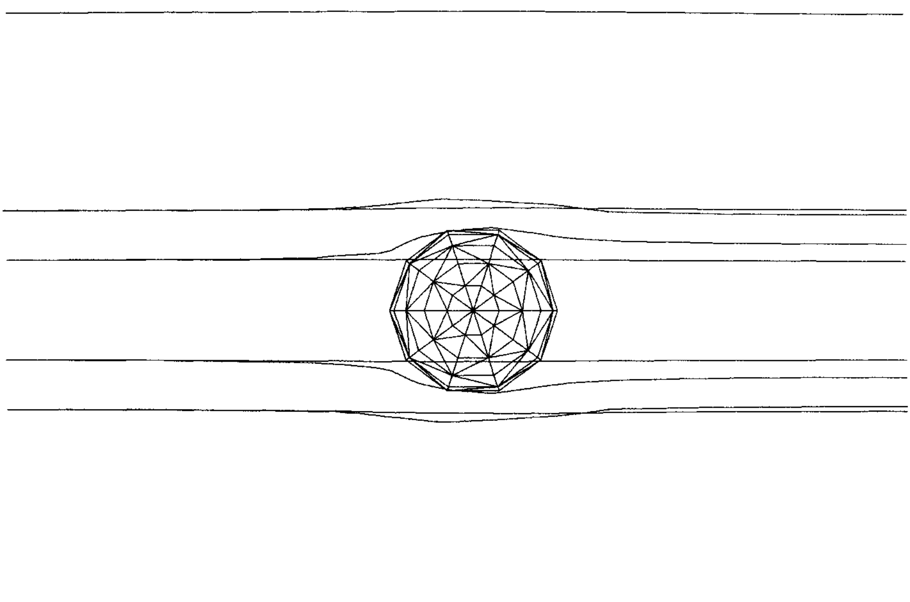
$$u \cdot n = n_1 \quad \text{on } S \textit{ and } W \tag{130}$$

so that the continuous problem has an analytical solution $\nabla \times \psi = (1, 0, 0)$. Figure 9 shows some stream lines computed from $\nabla \times \psi_h$, ψ_h solution of (86) with $\psi_\Gamma = \nabla q$ and q solution of (118).

Finally Fig. 10 shows some values of $|\nabla \phi_h|^2$ as a function of h when Ω is the volume between two spheres as in Figs. 7–9. Four set of tests were performed

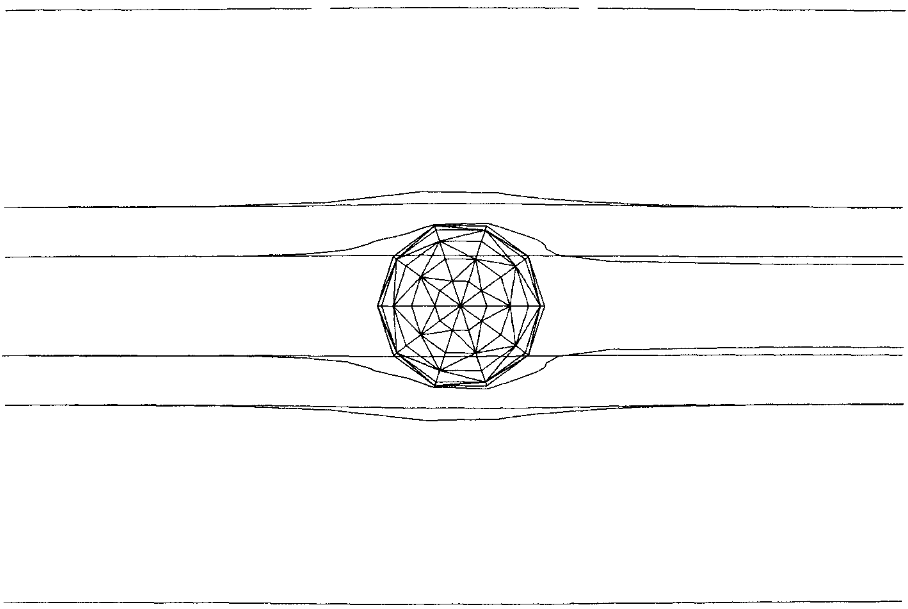


a



b

Fig. 7a–c. Some stream lines of a flow around a sphere. The views are from above **a** the flow is computed with the stream vector given on the boundary and equal to $\{\emptyset, -x_3, \emptyset\}$. **b** The flow is computed as the gradient of the potential. **c** The flow is computed with the stream vector but the boundary conditions are computed by solving the Laplace-Beltrami problem for q



c

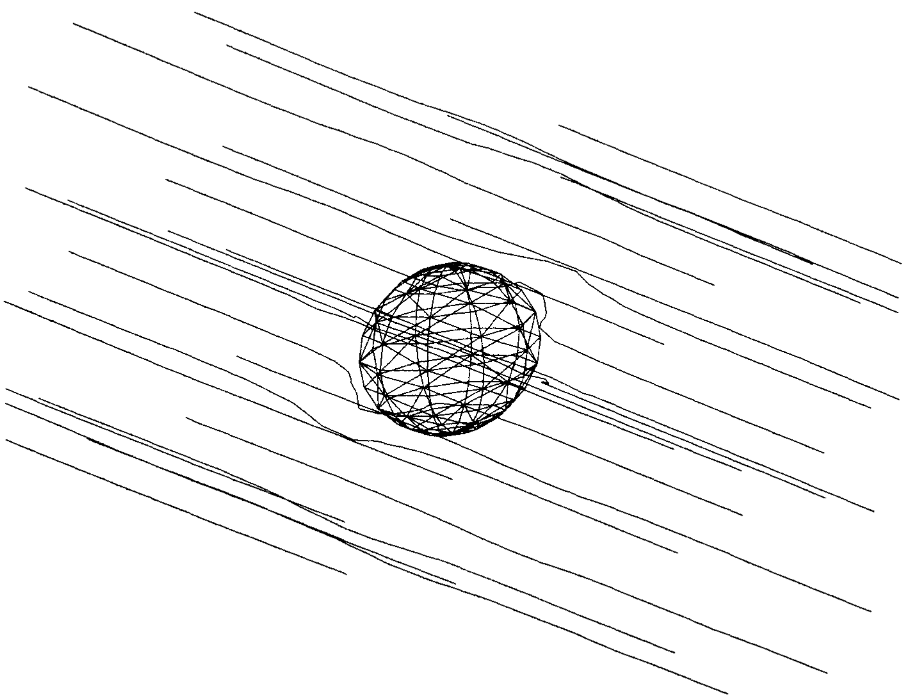


Fig. 8. Same as Fig. 7c but a prospect view

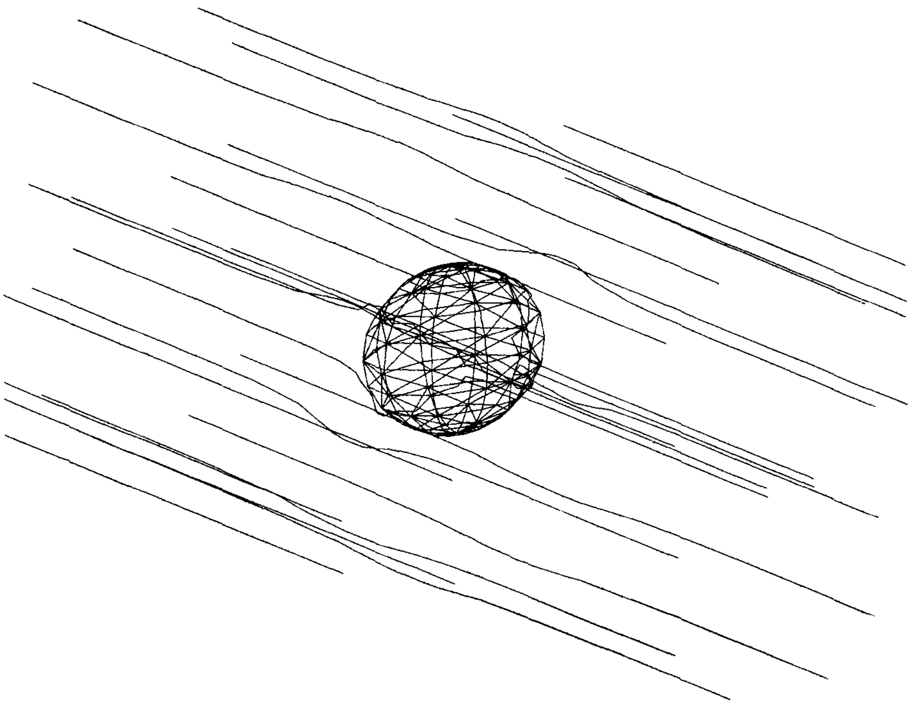


Fig. 9. Stream lines of a flow computed from the stream vector with boundary conditions computed from the Laplace-Beltrami problem for q. The flow is theoretically constant so the stream lines are straight lines

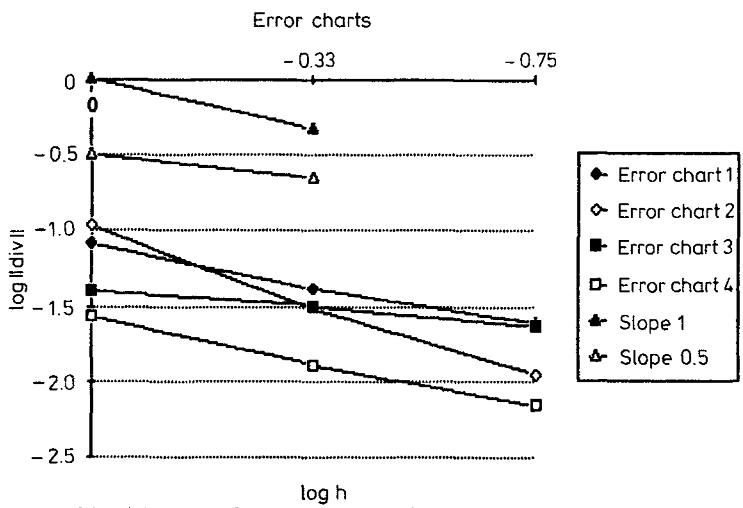


Fig. 10. Plots of $|\nabla \cdot \psi_h|_0$ versus h on a log-log scale when 1) n is exact ψ_d is computed by the Beltrami operator; 2) n is exact but $\psi_d = (1, 0, 0)$; 3) n is replaced by n_h and ψ_d is the solution of the Beltrami problem; 4) n is replaced by n_h and $\psi_d = (1, 0, 0)$. Cases 2 and 4 show an error $O(h)$ while cases 1 and 3 indicate $O(\sqrt{h})$

a) ϕ_h solution of (86) with $u=0$, $\psi_{\Gamma_h}=(1,0,0)$ and $n_h=n$ (error chart 2) or n_h computed by (94) (error chart 4).

b) As above but with ψ_{Γ_h} computed by the Laplace-Beltrami system $n_h=n$ (error chart 1) or n_h by (94) (error chart 3).

Conclusion

We may sumarize by saying that one can use 3-d stream vectors in aeronautic and still be on safe grounds from the point of view of uniqueness and boundary conditions. At first it seems that the most promising applications will be entropy corrections in transonic flows. But even for simple incompressible flows, our numerical tests show that the method is feasible and not so much more expensive than straight potential calculations. In this line on may find that the Kutta-Joukowski condition is easier to apply; however additional developpements are needed for non simply connected domains and non C^2 boundaries. Finally there are other equations of physics where stream vectors (or vector potentials) may be used such as Maxwell's equations.

Appendix A

Lemma A1

$$\mathbf{v} \to |\nabla \mathbf{v}|_{0,\Omega} \quad \textit{is a norm on } V.$$

Proof. We use the Peetre lemma [19, p. 728]:

$$A_i \in \mathscr{L}(E_0, E_i), \qquad A_2 \text{ compact,} \tag{A1}$$

$$\begin{aligned} &\|\mathbf{v}\|_{E_0} \leqq C(\|A_1 \mathbf{v}\|_{E_1} + \|A_2 \mathbf{v}\|_{E_2}) \qquad \forall v \in E_0 \\ &\operatorname{Ker} A_1 = 0 \Rightarrow \mathbf{v} \to \|A_1 \mathbf{v}\|_{E_1} \quad \text{is an equivalent norm on } E_0. \end{aligned} \tag{A2}$$

We take $E_0 = V$, $E_1 = L^2(\Omega)^9$, $E_2 = L^2(\Omega)^3$, A_2 the canonical injection of V into $L^2(\Omega)^3$ and $A_1 = \nabla$, then

$$|\nabla \mathbf{v}|_{0,\Omega} = 0 \Rightarrow \mathbf{v} = \text{constant but } \mathbf{v} \times \mathbf{n}|_\Gamma = 0 \text{ so } \mathbf{v} = 0. \tag{A3}$$

Therefore $|\nabla \mathbf{v}|_{0,\Omega}$ is an equivalent norm on V.

Lemma A2. *Assume Ω bounded, simply connected with C^2 boundary. Then*

$$\mathbf{v} \to (|\nabla \times \mathbf{v}|^2_{0,\Omega} + |\nabla \cdot \mathbf{v}|^2_{0,\Omega})^{1/2} \tag{A4}$$

is an equivalent norm on V.

Proof. From Foias and Temam [10] we know that on V

$$\|\mathbf{v}\|^2_{1,\Omega} \leqq C(|\mathbf{v}|^2_{0,\Omega} + |\nabla \cdot \mathbf{v}|^2_{0,\Omega} + |\nabla \times \mathbf{v}|^2_{0,\Omega})^{1/2}. \tag{A5}$$

By Peetre's lemma again, applied to

$$E_0=\{\mathbf{v}\in H(\nabla\cdot,\Omega)\cap H(\nabla\times,\Omega):\ \mathbf{v}\times\mathbf{n}|_\Gamma=0,\ \int_{\Gamma_i}\mathbf{v}\cdot\mathbf{n}\,d\gamma=0\},\tag{A6}$$

$$E_1=L^2(\Omega)^4,\quad E_2=L^2(\Omega)^3,\quad A_1v=\{\nabla\cdot\mathbf{v},\nabla\times\mathbf{v}\},\quad A_2=I\tag{A7}$$

we find that

$$\mathbf{v}\to(|\nabla\cdot\mathbf{v}|^2_{0,\Omega}+|\nabla\times\mathbf{v}|^2_{0,\Omega})^{1/2}\tag{A8}$$

is an equivalent norm on E_0; that $\operatorname{Ker}A_1=0$ is seen from Lemma A4 but since $V\subset E_0$ we can combine (A5) and (A8) and this gives (A4). □

Lemma A3. *Let $\{\theta^i\}_{1,M}$ be defined by* (26). *Then $(\nabla\theta^i,\nabla\theta^j)$ defines a matrix A of rank $M-1$, which defines a positive definite bilinear form on $\{K:\ \Sigma K_i|\Gamma_i|=0\}$.*

Proof

$$-\Delta\theta^i=0\quad\text{in }\Omega;\qquad\theta^i|_{\Gamma_j}=\delta_{ij}.\tag{A9}$$

Let

$$\phi=\Sigma K_i\theta^i.\tag{A10}$$

Then by Poincaré inequality

$$\begin{aligned}\Sigma K_iK_j(\nabla\theta^i,\nabla\theta^j)=|\nabla\phi|_0^2&=\int_\Gamma\phi\frac{\partial\phi}{\partial n}d\gamma\\&\geqq C\int_\Gamma\left(\phi-\frac{1}{|\Gamma|}\int_\Gamma\phi\,d\gamma\right)^2d\gamma\\&\geqq C\sum_i\left(K_i-\frac{1}{|\Gamma|}\sum_iK_i|\Gamma_i|\right)^2|\Gamma_i|\end{aligned}\tag{A11}$$

so if on the subspace

$$\{K_i:\ \Sigma K_i|\Gamma_i|=0\}\tag{A12}$$

$\{(\nabla\theta^i,\nabla\theta^j)\}_{ij}$ is positive definite. □

Lemma A4

$$\nabla\times\psi=0\quad\nabla\cdot\psi=0\quad\int_{\Gamma_i}\psi\cdot\mathbf{n}\,d\gamma=0,\quad\psi\times\mathbf{n}|_\Gamma=0\Rightarrow\psi=0.\tag{A13}$$

Proof. By (A13.a) there exists ϕ such that $\psi=\nabla\phi$. So

$$\Delta\phi=0\quad\left.\frac{\partial\phi}{\partial s_j}\right|_{\Gamma_i}=0\quad j=1,2\tag{A14}$$

where $s_1 s_2$ are local coordinates on Γ. So ϕ is constant on each Γ_i:

$$\phi|_{\Gamma_i}=K_i.\tag{A15}$$

Now by (A13.c)

$$0=\int_{\Gamma_i}\psi\cdot\mathbf{n}\,d\gamma=\int_\Gamma\frac{\partial\phi}{\partial n}d\gamma=\frac{1}{K_i}\int_{\Gamma_i}\phi\frac{\partial\phi}{\partial n}d\gamma\tag{A16}$$

so

$$0=\int_{\Omega} |\nabla\phi|^2\, d\gamma \tag{A17}$$

and ϕ is constant on Ω; therefore ψ is zero. □

Lemma A5. *Let Ω be polyhedral. Let $\lambda_h \in V_h$; let π_h be the interpolation operator. Then*

$$\Big(\int_{\Gamma} \pi_h \lambda_h^2\, d\gamma\Big)^{1/2} \leqq \frac{C}{\sqrt{h}} |\nabla \lambda_h|_{0,\Omega}. \tag{A18}$$

Proof. If σ_i denotes the area of the support of w^i intersected with Γ, we have

$$\int_{\Gamma} \pi_h \lambda_h^2\, d\gamma = \Sigma\, \lambda_h^2(q^i)\, \sigma_i. \tag{A19}$$

Now

$$\lambda_h^2(q^i) \leqq 2\left(\frac{\lambda_h(q^i)+\lambda_h(q^j)}{2}\right)^2 + 2\left(\frac{\lambda_h(q^i)-\lambda_h(q^j)}{2}\right)^2. \tag{A20}$$

So

$$\begin{aligned}\int_{\Gamma} \pi_h \lambda_h^2\, d\gamma &\leqq C(|\lambda_h|_{0,\Omega}^2 + h^2 |\nabla \lambda_h|_{0,\Gamma}^2)\\ &\leqq C(1+h^2)|\nabla \lambda_h|_{0,\Gamma}^2\end{aligned} \tag{A21}$$

by Poincaré's inequality (or Lemma A2). Finally

$$|\alpha_h|_{0,\Gamma} \leqq \frac{C}{\sqrt{h}} |\alpha_h|_{0,\Omega} \tag{A22}$$

so the Lemma is proved. □

Lemma A6. *If all approximated normals $n_h(q^i)$ are sufficiently near to the continuous normals so that all angles $(n_h(q^i), \nabla w^j)$ are bounded away from zero for all h and all i, j on the same element, then:*

$$|\nabla \psi_h|_{0,\Omega}^2 \leqq C(|\nabla \times \psi_h|^2 + |\nabla \cdot \psi_h|^2) \qquad \forall\, \psi_h \in V_h. \tag{A23}$$

Proof. Referring to the notation of (128) we have

$$|\nabla \psi_h|_{0,\Omega}^2 = \sum_{ij} D_{ij} \psi^i \cdot \psi^j; \tag{A24}$$

but if $\psi_h \in V_h$ then ψ^i is parallel to n_h^i on the boundary so from (127) we see that

$$|\nabla \psi_h|^2 - |\nabla \times \psi_h|^2 - |\nabla \cdot \psi_h|^2 = \sum_{q^i, q^j \in \Gamma} \lambda_i \lambda_j\, n_h^i \times n_h^j \int_{\Omega} \nabla w^i \times \nabla w^j\, dx \tag{A25}$$

where $\{\lambda_i\}$ are defined by

$$\psi_h(q^i) = \lambda_i n_h^i. \tag{A26}$$

Let

$$\psi_h^0(x) = \Sigma\, \lambda_i n_h^i \bar{w}^i(x). \tag{A27}$$

Then (A25) shows that

$$|\nabla\psi_h|^2-|\nabla\times\psi_h|^2-|\nabla\cdot\phi_h|^2=|\nabla\psi_h^0|^2-|\nabla\times\psi_h^0|^2-|\nabla\cdot\psi_h^0|^2. \tag{A28}$$

Now on each boundary tetraedron T with vertices $q^\alpha, q^\beta, q^\gamma, q^\delta\,(q^\delta\notin\Gamma)$

$$\nabla\times\psi_{h|T}^0=r_T \qquad \nabla\cdot\psi_{h|T}^0=d_T \tag{A29}$$

is an overdetermined linear system in $\lambda_\alpha\lambda_\beta\lambda_\gamma$; this $\lambda_\alpha, \lambda_\beta, \lambda_\gamma$ are linear in r_T, d_T or both and so is $\nabla\psi_{h|T}^0$; hence

$$|\nabla\psi_h^0|_{0,T}^2\leqq C(|\nabla\times\psi_h^0|_T^2+|\nabla\cdot\psi_h^0|_T^2). \tag{A30}$$

It remains to show that the linear system (A2) is not degenerated and that C in (A30) is independent of h.

If $n_h^\alpha=n_h^\beta=n_h^\gamma\neq\nabla w^\delta$, $(\delta=\alpha, \beta, \gamma)$, then

$$\nabla\times\psi_h^0=\Sigma\lambda_\alpha n_h^\alpha\times\nabla w^\alpha=n_h^\alpha\times\nabla(\Sigma\lambda_\alpha w^\alpha), \tag{A31}$$

$$\nabla\cdot\psi_h^0=\Sigma\lambda_\alpha n_h^\alpha\cdot\nabla w^\alpha=n_h^\alpha\cdot\nabla(\Sigma\lambda_\alpha w^\alpha) \tag{A32}$$

so

$$\nabla\psi_h^0=n_h^\alpha\times\nabla(\Sigma\lambda_\alpha w^\alpha)=n_h^\alpha\times(n_h^\alpha\times\nabla\times\psi_h^0+n_h^\alpha\nabla\cdot\psi_h^0) \tag{A33}$$

from which we find that

$$|\nabla\psi_h^0|_{0,T}\leqq 1\,(|\nabla\times\psi_h^0|_{0,T}+|\nabla\cdot\psi_h^0|_{0,T}). \tag{A34}$$

When $n_h^\alpha\neq n_h^\beta\neq n_h^\gamma$ then we can use a continuity argument and say that there will be a cone of admissible normals for which (A34) will hold with, say, 1 replaced by 2. This cone is independent of h because (A27)–(A28) yield λ_i proportional to h and linear in r_T and d_T and therefore $\nabla\psi_h^0$ is of order h^0 and linear in r_T and d_T as in (A34).

If T has one or two nodes on the boundary only it is even simpler to prove (A34). For example with one, (A28) implies:

$$\lambda_\alpha n_h^\alpha\times\nabla w^\alpha=r_T \tag{A35}$$

so

$$|\nabla\psi_h^0|=|n_h^\alpha\times\nabla w^\alpha|\,|r_T|/|n_h^\alpha\times\nabla w^\alpha|\leqq|r_T|/|\sin(n_h^\alpha, \nabla w^\alpha)|. \tag{A36}$$

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