Numer. Math. 48, 99–118 (1986)

Numerische Mathematik

On Spline Galerkin Methods for Singular Integral Equations with Piecewise Continuous Coefficients

S. Prößdorf and A. Rathsfeld

Institut für Mathematik der Akademie der Wissenschaften der DDR, Mohrenstraße 39, DDR-1086 Berlin

Summary. This article analizes the convergence of the Galerkin method with polynomial splines on arbitrary meshes for systems of singular integral equations with piecewise continuous coefficients in L^2 on closed or open Ljapunov curves. It is proved that this method converges if and, for scalar equations and equidistant partitions, only if the integral operator is strongly elliptic (in some generalized sense). Using the complete asymptotics of the solution, we provide error estimates for equidistant and for special nonuniform partitions.

Subject Classifications: AMS (MOS): 65R20, 45E05; CR: 61.9.

Introduction

Singular integral equations arising from numerous boundary value problems in aerodynamics, elasticity and thermoelasticity, electromagnetics, vibration theory and many other fields of engineering have the form

$$c(t)\,x(t)+\frac{d(t)}{\pi i}\int_{\Gamma}\frac{x(\tau)}{\tau-t}\,d\tau+\int_{\Gamma}K(t,\tau)\,x(\tau)\,d\tau=y(t),\qquad t\in\Gamma, \tag{0.1}$$

see e.g. Anderssen [2], Kupradse [11], Michlin, Prößdorf [12], Muskhelishvili [13] and Parton, Perlin [15]. In this equation Γ is a closed or open plane curve, c, d, K and y are given functions (in general matrix – or vector-valued and discontinuous), x is the unknown function and the first integral is defined as a Cauchy principal value.

In many practical computations with Eq. (0.1) one uses spline approximations for the unknown function x on Γ. The two most popular discretization schemes are the Galerkin and collocation methods. If Γ is a closed curve and c and d are continuous functions, then convergence and error estimates for Galerkin and collocation methods using smooth polynomial splines follow from recent studies by Arnold, Saranen, Stephan and Wendland in [4, 23, 27], by Nedelec in [14] and by Elschner, Schmidt and the authors in [8, 16, 17, 19, 21, 22, 24, 25, 26] (see also the surveys just given by Elschner, Prößdorf [18] and by Wendland [29]). Convergence results on the spline collocation method in the space L^2 for Eq. (0.1) with piecewise continuous coefficients c and d have most likely been first obtained by the authors in [19]. They were then generalized by Schmidt in [25]. For Γ a finite interval, convergence and error analysis for Galerkin's method with (smooth as well as weighted) splines has recently been developed by Elschner [9] for the case of continuous c and d such that $\operatorname{Re}\{c(t) \pm d(t)\} > 0$ on Γ. The special case where K is real-valued, $c \equiv 1$ and d is purely imaginary has earlier been considered by Thomas [28].

It is the aim of this paper to give a convergence and error analysis for the spline Galerkin method in the space L^2 if c and d are piecewise continuous matrix functions and Eq. (0.1) is strongly elliptic (in the sense of the subsequent definition). For sake of brevity, all results of this paper are proved for the case $K \equiv 0$, since they easily extend to the complete equation (0.1). (see, e.g. [10]).

We now introduce several notational conventions. $\mathbb{C}^m (m \in \mathbb{N})$ denotes m-dimensional complex Euclidean space with the usual scalar product

$$[x, y] := \sum_{i=1}^{m} x_i \bar{y}_i, \qquad \forall\, x, y \in \mathbb{C}^m,$$
$$x = (x_1, \ldots, x_m)^T, \qquad y = (y_1, \ldots, y_m)^T.$$

$\mathbb{C}^{m \times m}$ stands for the set of all complex-valued $m \times m$ matrices.

Let Γ be a closed or open oriented plane Ljapunov curve having a regular parametrization

$$\Gamma := \{t = \gamma(s) \colon s \in [0, 1]\}, \qquad \gamma \colon [0, 1] \to \mathbb{C};$$

for Γ closed, we assume $\gamma(0) = \gamma(1)$. By $L^2(\Gamma, \mathbb{C}^m)$ we denote the Hilbert space of all square Lebesgue-integrable $\mathbb{C}^m$-valued functions on Γ with scalar product

$$(f, g) := \int_\Gamma [f(t), g(t)]\, |dt|, \qquad \forall\, f, g \in L^2(\Gamma, \mathbb{C}^m).$$

The symbol $PC(\Gamma, \mathbb{C}^{m \times m})$ designates the space of all $\mathbb{C}^{m \times m}$-valued functions a on Γ which are piecewise continuous in the following sense: for each $t \in \Gamma$ the finite limits $a(t \pm 0) := \lim\limits_{\substack{\tau \to t \pm 0 \\ \tau \in \Gamma}} a(\tau)$ (with respect to the orientation of Γ) exist and a is discontinuous at most at a finite number of points $t \in \Gamma$. $C(\Gamma, \mathbb{C}^{m \times m})$ is the subspace of all continuous $\mathbb{C}^{m \times m}$-valued functions on Γ.

In $L^2(\Gamma, \mathbb{C}^m)$ we consider singular integral operators of the form

$$A = aP_\Gamma + bQ_\Gamma \tag{0.2}$$

with coefficients $a, b \in PC(\Gamma, \mathbb{C}^{m \times m})$. Here P_Γ and Q_Γ denote the operators

$$P_\Gamma := 1/2(I + S_\Gamma), \qquad Q_\Gamma := 1/2(I - S_\Gamma),$$

I the indentity operator and S_Γ the Cauchy singular operator

$$(S_\Gamma x)(t) := 1/\pi i \int_\Gamma \frac{x(\tau)}{\tau - t}\, d\tau (t \in \Gamma).$$

That $A \in \mathscr{L}(L^2(\Gamma, \mathbb{C}^m))$ (see [12]) is well established and the operator defined by the left-hand side of Eq. (0.1) with $K \equiv 0$ has the form (0.2), where $a = c + d$ and $b = c - d$.

We call operator $A = aP_\Gamma + bQ_\Gamma$ strongly elliptic[1], if there exist a compact operator $T \in \mathscr{L}(L^2(\Gamma, \mathbb{C}^m))$ and an invertible function $\Theta \in PC(\Gamma, \mathbb{C}^{m \times m})$, discontinuous at most at the points of discontinuity of a and b, such that $A = \Theta(A_0 + T)$, where A_0 has a positive real part - i.e.

$$\operatorname{Re}(A_0 f, f) \geqq \varepsilon(f, f), \quad \forall f \in L^2(\Gamma, \mathbb{C}^m)$$

with $\varepsilon(\text{const}) > 0$.

In [20] we have shown that A is strongly elliptic if and only if

$$\forall t \in \Gamma \colon \exists\, b(t \pm 0)^{-1} \quad \text{and} \quad \exists\, C \in \mathbb{C}^{m \times m} \colon \operatorname{Re} C > 0,\ \operatorname{Re} C(b^{-1}a)(t \pm 0) > 0. \tag{B}$$

(If $\Gamma = (\alpha, \beta)$ is an open curve, then condition (B) need only be satisfied from the right or from the left, according as we consider α or β respectively.) In Lemma 1.2 further conditions equivalent to (B) for special cases are formulated.

Now let $\{t_0 = \gamma(s_0),\ t_1 = \gamma(s_1), \ldots, t_N = \gamma(s_N)\} \subseteq \Gamma\ (0 = s_0 < s_1 < \ldots s_N = 1)$ be a given set of points such that a and b are continuous on $\Gamma \setminus \{t_0, \ldots, t_N\}$. A sequence $\{\Delta_k\}_{k \in \mathbb{N}}$ $(\Delta_k = \{\sigma_0^k, \ldots, \sigma_{n_k}^k\},\ 0 = \sigma_0^k < \sigma_1^k < \ldots \sigma_{n_k}^k = 1)$ of partitions of $[0, 1]$ is called admissible if $\{s_0, \ldots, s_N\} \subseteq \Delta_k (k \in \mathbb{N})$ and if

$$h_{\Delta_k} := \max\{\sigma_{i+1}^k - \sigma_i^k \mid i = 0, 1, \ldots, n_k - 1\} \to 0 \quad (k \to \infty).$$

Let $PS_d(\Delta_k, \mathbb{C})$ $(d \in \mathbb{N})$ denote the space of all $\varphi \in PC(\Gamma, \mathbb{C})$ such that $\varphi \circ \gamma$ is $(d - 1)$ times continuously differentiable on $[0, 1] \setminus \{s_0, \ldots, s_N\}$ and the restriction of $\varphi \circ \gamma$ to $[\sigma_i^k, \sigma_{i+1}^k]$ is a polynomial of degree not exceeding d - i.e. the functions $\varphi \in PS_d(\Delta_k, \mathbb{C})$ are splines with maximal smoothness at $\sigma_i^k \in \Delta_k \setminus \{s_0, \ldots, s_N\}$ and with no smoothness condition at $s_i (i = 0, \ldots, N)$. $PS_d(\Delta_k, \mathbb{C}^m)$ stands for the space of vector-valued functions $f = (f_1, \ldots, f_m) \in PC(\Gamma, \mathbb{C}^m)$ having components $f_j \in PS_d(\Delta_k, \mathbb{C})$. Obviously, $PS_0(\Delta_k, \mathbb{C}^m)$ is the space of step functions.

The Galerkin method for the equation $Ax = y$ may now be formulated as follows: Find $x_{\Delta_k} \in PS_d(\Delta_k, \mathbb{C}^m)$ satisfying

$$(Ax_{\Delta_k}, \varphi) = (y, \varphi) \quad (\forall \varphi \in PS_d(\Delta_k, \mathbb{C}^m)). \tag{0.3}$$

We say that the Galerkin method with respect to the admissible sequence $\{\Delta_k\}_{k \in \mathbb{N}}$ is convergent for the operator A, if (0.3) is uniquely solvable for sufficiently large k and if x_{Δ_k} converges to $x = A^{-1}y$ in $L^2(\Gamma, \mathbb{C}^m)$ for any $y \in L^2(\Gamma, \mathbb{C}^m)$.

We proceed in this paper as follows. In Sect. 1 we examine the Galerkin method (0.3) with respect to any admissible sequence $\{\Delta_k\}_{k \in \mathbb{N}}$ for the operator A defined by (0.2). By means of the discrete commutator property, given in Lemma 1.1, it is proved that this method is convergent in $L^2(\Gamma, \mathbb{C}^m)$ if A is strongly elliptic and invertible. In the special case $\Gamma = (0, 1)$, $m = 1$ and $a, b \in C$

---

[1] See Stephan and Wendland [27], who introduced this concept in case of smooth functions Θ, a, b

[0, 1], Theorem 1.1 yields a result of Elschner ([9], Theorem 3.1). In Sect. 2 we show that strong ellipticity of A is necessary for the Galerkin method to be convergent, provided the partitions are equidistant and $m=1$. For continuous coefficients a and b and a closed curve Γ, this fact was established in [24] by means of a different technique. As a corollary to our proof, which is based on certain localization techniques and on the method of associated operators (see [19]), we obtain that the strong ellipticity of A is sufficient for the stability of the collocation method with piecewise linear splines. Section 3 deals with estimating the error of the Galerkin method (0.3) by means of utilizing the complete asymptotics of the solutions of (0.1) which has been established in [9]. We show for instance, that an asymptotic order of convergence of $O(n_k^{-d-1})$ can be achieved on special nonuniform partitions. This fact was obtained in [28] and [9] by using weighted continuous splines on nonuniform meshes for continuous coefficients a and b and $\Gamma=(0, 1)$.

All results of Sects. 1 and 2 are valid for singular integral operators of the form $P_\Gamma a+Q_\Gamma b$; one need only to take the adjoints of the operators $a^* P_\Gamma + b^* Q_\Gamma$.

1. The Convergence of the Galerkin Method for Strongly Elliptic Operators

Let P_{Δ_k} denote the orthogonal projection of $L^2(\Gamma, \mathbb{C}^m)$ onto $PS_d(\Delta_k, \mathbb{C}^m)$. Then (0.3) is equivalent to

$$P_{\Delta_k} A P_{\Delta_k} x_{\Delta_k} = P_{\Delta_k} y.$$

Lemma 1.1. *If $\{\Delta_k\}$ is an admissible sequence of partitions and $f \in PC(\Gamma, \mathbb{C}^{m\times m})$ is continuous on $\Gamma \setminus \{t_0, t_1, \ldots, t_N\}$, then*

$$\|(I-P_{\Delta_k}) f P_{\Delta_k}\| \to 0 (k\to\infty)$$
$$\|P_{\Delta_k} f (I-P_{\Delta_k})\| \to 0 (k\to\infty).$$

Proof. We only need to prove $\|(I-P_{\Delta_k}) f P_{\Delta_k}\| \to 0$, since the other assertion follows by taking the operator adjoints.

$$f=(f_{ij})_{i,j=1}^m \in PC(\Gamma, \mathbb{C}^{m\times m})$$

implies

$$(I-P_{\Delta_k}) f P_{\Delta_k} = ((I-P_{\Delta_k}) f_{ij} P_{\Delta_k})_{i,j=1}^m,$$

where P_{Δ_k} in the entries of the last matrix are the orthogonal projections of $L^2(\Gamma, \mathbb{C})$ onto $PS_d(\Delta_k, \mathbb{C})$. Therefore, we have to prove $\|(I-P_{\Delta_k}) f P_{\Delta_k}\| \to 0$ for the case $m=1$ only.

Now we introduce a new scalar product in $L^2(\Gamma, \mathbb{C})$ by

$$(f, g)_1 := \int_0^1 (f\circ\gamma)(s)(\bar{g}\circ\gamma)(s)\, ds.$$

Clearly, the norms $\|f\|=(f,f)^{1/2}$ and $\|f\|_1=(f,f)_1^{1/2}$ are equivalent, and thus it suffices to show $\|(I-P_{\Delta_k})fP_{\Delta_k}\|_1\to 0$, where $\|\cdot\|_1$ denotes the operator norm corresponding to the norm $\|\cdot\|_1$ in $L^2(\Gamma,\mathbb{C})$. Let $\tilde{P}_{\Delta_k}$ be the orthogonal projection of $L^2(\Gamma,\mathbb{C})$ onto $PS_d(\Delta_k,\mathbb{C})$ corresponding to $(\cdot,\cdot)_1$. Since $\sup_k\|P_{\Delta_k}\|_1<\infty$, we obtain (see [1])

$$\begin{aligned}\|(I-P_{\Delta_k})fP_{\Delta_k}g\|_1&=\|fP_{\Delta_k}g-P_{\Delta_k}(fP_{\Delta_k}g)\|_1\\&\leqq C\|(fP_{\Delta_k}g)-\tilde{P}_{\Delta_k}(fP_{\Delta_k}g)\|_1\\&=C\|(I-\tilde{P}_{\Delta_k})f\tilde{P}_{\Delta_k}P_{\Delta_k}g\|_1,\end{aligned}$$

where C is a positive constant. Consequently, we only have to show $\|(I-\tilde{P}_{\Delta_k})f\tilde{P}_{\Delta_k}\|_1\to 0$. Hence, we may assume $\Gamma=[0,1]$.

Let $f^i(i=0,\dots,N-1)$ be the function vanishing on $[0,1]\setminus[s_i,s_{i+1}]$ and equal to f on $[s_i,s_{i+1}]$, and $P^i_{\Delta_k}$ be the orthogonal projection onto the subspace of all functions $f\in PS_d(\Delta_k,\mathbb{C})$ vanishing on $[0,1]\setminus[s_i,s_{i+1}]$. Thus

$$(I-P_{\Delta_k})fP_{\Delta_k}=\sum_{i=0}^{N-1}(I-P^i_{\Delta_k})f^iP^i_{\Delta_k}$$

and we only need to demonstrate $\|(I-P^i_{\Delta_k})f^iP^i_{\Delta_k}\|\to 0$ $(i=0,\dots,N-1)$. We may therefore assume: $f\in C([0,1],\mathbb{C})$ and P_{Δ_k} is the orthogonal projection of $L^2([0,1],\mathbb{C})$ onto $S_d(\Delta_k,\mathbb{C})$. $S_d(\Delta_k,\mathbb{C})$ is the subspace of all $f\in L^2([0,1],\mathbb{C})$, which are $(d-1)$ times continuously differentiable and coincide with a polynomial of degree not exceeding d on every interval $[\sigma_i^k,\sigma_{i+1}^k]$ $(i=0,\dots,n_k-1)$.

We set

$$\sigma^k_{-r}:=-1/2r(\sigma_1^k-\sigma_0^k)\qquad(r=1,2,\dots,d)$$

and

$$\sigma^k_{n_k+r}:=1+1/2r(\sigma^k_{n_k}-\sigma^k_{n_k-1})\qquad(r=1,2,\dots,d).$$

A base $\{\psi_i\}_{i=-d}^{n_k}$ of $S_d(\Delta_k,\mathbb{C})$ can now be given as follows (see [6]):

$$g(s,t):=(s-t)_+^d$$

$$\psi_i(t):=\sqrt{(d+1)}\sqrt{(\sigma^k_{i+d+1}-\sigma^k_i)}\,g(\sigma_i^k,\dots,\sigma^k_{i+d+1};t)\qquad(t\in[0,1])$$

wherein $g(\sigma_i^k,\dots,\sigma^k_{i+d+1};t)$ represents the divided difference of $g(s,t)$ at $\sigma_i^k,\sigma_{i+1}^k,\dots,\sigma^k_{i+d+1}$. It is well known that $\psi_i\geqq 0$ and ψ_i has support $[\sigma_i^k,\sigma^k_{i+d+1}]$ (see [6]). Moreover, there exists a positive constant D such that

$$D^{-1}\left\|\sum_{i=-d}^{n_k-1}\xi_i\psi_i\right\|\leqq\left(\sum_{i=-d}^{n_k-1}|\xi_i|^2\right)^{1/2}\leqq D\left\|\sum_{i=-d}^{n_k-1}\xi_i\psi_i\right\|\tag{1.1}$$

(see [7]). We obtain

$$\left\|f\sum_{i=-d}^{n_k-1}\xi_i\psi_i-P_{\Delta_k}f\sum_{i=-d}^{n_k-1}\xi_i\psi_i\right\|\leqq\left\|f\sum_{i=-d}^{n_k-1}\xi_i\psi_i-\sum_{i=-d}^{n_k-1}f(\sigma_i^k)\,\xi_i\psi_i\right\|,\tag{1.2}$$

where $f(\sigma_i^k):=f(0)$ $(i=-d,-d+1,\dots,-1)$. For $t\in[\sigma_j^k,\sigma_{j+1}^k]$,

$$\begin{aligned}\left|f(t)\sum_{i=-d}^{n_k-1}\xi_i\psi_i(t)-\sum_{i=-d}^{n_k-1}f(\sigma_i^k)\,\xi_i\psi_i(t)\right|&=\left|\sum_{i=j-d}^{j+1}(f(t)-f(\sigma_i^k))\,\xi_i\psi_i(t)\right|\\&\leqq\omega(f,dh_{\Delta_k})\sum_{i=j-d}^{j+1}|\xi_i|\,|\psi_i(t)|\\&\leqq\omega(f,dh_{\Delta_k})\left|\sum_{i=-d}^{n_k-1}|\xi_i|\,\psi_i(t)\right|\end{aligned}\tag{1.3}$$

with $\omega(f,\delta):=\sup\{|f(t_1)-f(t_2)|\,|\,t_1,t_2\in[0,1],\ |t_1-t_2|<\delta\}$ (modulus of continuity). (1.1) and (1.3) yield

$$\begin{aligned}\left\|f\sum_{i=-d}^{n_k-1}\xi_i\psi_i-\sum_{i=-d}^{n_k-1}f(\sigma_i^k)\,\xi_i\psi_i\right\|&\leqq\omega(f,dh_{\Delta_k})\left\|\sum_{i=-d}^{n_k-1}|\xi_i|\,\psi_i\right\|\\&\leqq\omega(f,dh_{\Delta_k})\,D^2\left\|\sum_{i=-d}^{n_k-1}\xi_i\psi_i\right\|\end{aligned}$$

and (1.2) implies

$$\left\|(I-P_{\Delta_k})\,f\sum_{i=-d}^{n_k-1}\xi_i\psi_i\right\|\leqq D^2\,\omega(f,dh_{\Delta_k})\left\|\sum_{i=-d}^{n_k-1}\xi_i\psi_i\right\|.$$

Since $h_{\Delta_k}\to 0$, the proof is completed.

Remark. Lemma 1.1 was independently given in [16, 17] and in [3] for the case of smooth functions f. In these papers however, the authors considered arbitrary Sobolev norms instead of L^2-norms and consequently the partitions were assumed to be regular. (See also [25], where a detailed proof of the aforementioned result is given for the case of Sobolev norms.) The idea of the proof presented above is based on the method of derivation of Lemma 4.1 in [21].

Corollary 1.1. *If $\{\Delta_k\}$ is admissible, then P_{Δ_k} converges strongly to I* – i.e. $P_{\Delta_k}\to I$.

Proof. Consider $g=(g_1,g_2,\ldots,g_m)\in C(\Gamma,\mathbb{C}^m)$ and set

$$f:=(g_i\,\delta_{ij})_{i,j=1}^m\in C(\Gamma,\mathbb{C}^{m\times m}),\ h=(1,\ldots,1)\in C(\Gamma,\mathbb{C}^m).$$

Lemma 1.1 implies $(I-P_{\Delta_k})\,g=(I-P_{\Delta_k})\,fP_{\Delta_k}h\to 0$ and further $P_{\Delta_k}\to I$, since $\sup\|P_{\Delta_k}\|<\infty$.

It is now rather simple to prove

Theorem 1.1. *If $\{\Delta_k\}$ is an arbitrary admissible sequence of partitions and $A=aP_\Gamma+bQ_\Gamma\in\mathscr{L}(L^2(\Gamma,\mathbb{C}^m))$ is invertible and strongly elliptic, then the Galerkin method with respect to the sequence $\{\Delta_k\}$ is convergent for A.*

Proof. Since $P_{\Delta_k}\to I$, the Galerkin method is convergent if and only if the approximate operators $P_{\Delta_k}AP_{\Delta_k}$ are stable – i.e.

$$\|P_{\Delta_k}AP_{\Delta_k}\varphi\|\geqq C\|\varphi\|\qquad(\forall\,\varphi\in PS_d(\Delta_k,\mathbb{C}^m),\ \forall\,k\geqq k_0),$$

where C is a positive constant and $k_0\in\mathbb{N}$ (see, e.g. [10]). The strongly elliptic operator A admits the representation $A=\Theta A_0+T$, where $\Theta\in PC(\Gamma,\mathbb{C}^{m\times m})$ is continuous on $\Gamma\setminus\{t_0, \ldots, t_N\}$, $\operatorname{Re} A_0 > 0$ and T is compact. Lemma 1.1 yields

$$P_{\Delta_k}\Theta^{-1}P_{\Delta_k}P_{\Delta_k}\Theta P_{\Delta_k} = -P_{\Delta_k}\Theta^{-1}(I-P_{\Delta_k})\Theta P_{\Delta_k} + P_{\Delta_k},$$
$$\|P_{\Delta_k}\Theta^{-1}(I-P_{\Delta_k})\Theta P_{\Delta_k}\| \to 0.$$

Thus $P_{\Delta_k}\Theta P_{\Delta_k}$ are stable. Furthermore $\operatorname{Re} A_0 > 0$ implies that the operators $P_{\Delta_k}A_0P_{\Delta_k}$ are also stable. The stability of the operators

$$P_{\Delta_k}AP_{\Delta_k} = (P_{\Delta_k}\Theta P_{\Delta_k})(P_{\Delta_k}A_0P_{\Delta_k}) + P_{\Delta_k}\Theta(I-P_{\Delta_k})A_0P_{\Delta_k} + P_{\Delta_k}TP_{\Delta_k}$$

follows from Lemma 1.1 and the fact that T is compact (see, e.g. [10]).

In the following two cases, condition (B) may be easily verified.

Lemma 1.2. [22, 20].

a) *If* $m=1$, *then* (B) *holds at* $\tau\in\Gamma$ *if and only if* $b(\tau\pm 0)\neq 0$ *and*

$$\forall\,\mu\in[0,1]: \{\mu(b^{-1}a)(\tau+0) + (1-\mu)(b^{-1}a)(\tau-0)\} \notin (-\infty, 0]. \tag{1.4}$$

b) *If* a *and* b *are continuous at* $\tau\in\Gamma$, *then condition* (B) *holds at* τ *if and only if*

$$\forall\,\mu\in[0,1]: \det(\mu a(\tau) + (1-\mu)\,b(\tau)) \neq 0.$$

2. A Necessary Condition for the Convergence of the Galerkin method. A Collocation Method

In this section we assume all partitions to be equidistant – i.e. $\sigma_{i+1}^k - \sigma_i^k = \sigma_{j+1}^k - \sigma_j^k$ ($\forall\, i, j = 0, \ldots, n_k - 1$). Actually, there exist admissible sequences of equidistant partitions In particular, we can choose for a given set $\{t_0, \ldots, t_N\} \subseteq \Gamma$ a regular parametrization γ such that $\gamma(i/N) = t_i$ and thus get an admissible sequence of equidistant partitions

$$\Delta_k := \{\sigma_0^k := 0, \sigma_1^k := 1/kN, \sigma_2^k := 2/kN, \ldots, \sigma_{kN}^k = 1\}.$$

Now let us consider the Galerkin method (0.3) for the singular operator A.

Theorem 2.1. *Let* $m=1$ *and* $a, b \in PC(\Gamma, \mathbb{C})$. *If* $\{\Delta_k\}$ *is an admissible sequence of equidistant partitions and the Galerkin method with respect to this sequence is convergent for* $A = aP_\Gamma + bQ_\Gamma \in \mathscr{L}(L^2(\Gamma, \mathbb{C}))$, *then* A *is strongly elliptic.*

Before we prove Theorem 2.1, we introduce the same notations as in [19] for the collocation method, and we construct to this end a base of $PS_d(\Delta_k, \mathbb{C}^m)(m \geqq 1)$. On every interval (s_i, s_{i+1}) $(i=0, \ldots, N-1)$ the elements of this base differ from the functions ψ_i of Sect. 1 by at most a constant factor, however, because our partitions are equidistant, we can use the representation of [5], Sect. 4.2.

Let $\sigma_i = \sigma_i^k = i/n$ $(i=0, \ldots, n)$, $n = n_k$ and $t_i = \gamma(s_i)$, $s_i = k_i/n$ $(i=0, \ldots, N)$. We denote the characteristic functions of $[s_i, s_{i+1}]$ and $[-1, 0]$ by χ_i and Π respectively and further, Π_{d+1} stands for the $(d+1)$-fold convolution of Π. Thus the space $S_d(\Delta_k, \mathbb{C})$ (see Sect. 1) has the base

$$\tilde{\psi}_j(s) := \Pi_{d+1}(ns - j) \qquad (j = 1, \ldots, n+d).$$

For $t=\gamma(s)$ set

$$
\begin{aligned}
&\varphi_1(t):=(\chi_0\tilde{\psi}_1)(s),\ \varphi_2(t):=(\chi_0\tilde{\psi}_2)(s),\ \dots,\ \varphi_{k_1+d}(t):=(\chi_0\tilde{\psi}_{k_1+d})(s),\\
&\varphi_{k_1+d+1}(t):=(\chi_1\tilde{\psi}_{k_1+1})(s),\\
&\varphi_{k_1+d+2}(t):=(\chi_1\tilde{\psi}_{k_1+2})(s),\ \dots,\ \varphi_{k_2+2d}(t):=(\chi_1\tilde{\psi}_{k_2+d})(s),\\
&\vdots\\
&\varphi_{k_j+jd+1}(t):=(\chi_j\tilde{\psi}_{k_j+1})(s),\ \varphi_{k_j+jd+2}(t):=(\chi_j\tilde{\psi}_{k_j+2})(s),\ \dots,\\
&\varphi_{k_{j+1}+(j+1)d}(t):=(\chi_j\tilde{\psi}_{k_{j+1}+d})(s),\\
&\vdots\\
&\varphi_{k_{N-1}+(N-1)d+1}(t):=(\chi_{N-1}\tilde{\psi}_{k_{N-1}+1})(s),\\
&\varphi_{k_{N-1}+(N-1)d+2}(t):=(\chi_{N-1}\tilde{\psi}_{k_{N-1}+2})(s),\ \dots,\\
&\varphi_{n+Nd}(t):=(\chi_{N-1}\tilde{\psi}_{n+d})(s).
\end{aligned}
\tag{2.1}
$$

Every function $\varphi\in PS_d(\Delta_k, \mathbb{C}^m)$ admits a representation

$$\varphi=\sum_{i=1}^{n+Nd}\xi_i\varphi_i, \qquad \xi_i\in\mathbb{C}^m$$

and there exist suitable positive constants D_1, D_2 such that out of (1.1) follows

$$D_1\left\|\sum_{i=1}^{n+Nd}\xi_i\varphi_i\right\|\leqq n^{-1/2}\left(\sum_{i=1}^{n+Nd}|\xi_i|^2\right)^{1/2}\leqq D_2\left\|\sum_{i=1}^{n+Nd}\xi_i\varphi_i\right\| \tag{2.2}$$

with $|\xi_i|:=[\xi_i, \xi_i]^{1/2}$.

We now fix $\tau=\gamma(\sigma)\in\Gamma$ and derive the subsequent necessary condition at point τ (see Lemma 2.2). Using this condition, we shall obtain that condition (B) is necessary if $m=1$. Moreover, for $m\geqq 1$ and $d=0$, we get a stability condition for a certain collocation method.

Inequality (2.2) allows us to identify the operators of $\mathscr{L}(PS_d(\Delta_k, \mathbb{C}^m))$ with matrices over finite dimensional Euclidean spaces. For this purpose, we introduce associated operators analogously to [19]. Let $l^2(\mathbb{C}^m)$ be the Hilbert space of the following sequences:

$$l^2(\mathbb{C}^m):=\{\xi:=\{\xi_s\}_{s=-\infty}^{\infty}\mid \xi_s\in\mathbb{C}^m,\ \|\xi\|<\infty\},$$

$$\|\xi\|:=(\xi,\xi)^{1/2}, \qquad (\xi,\eta):=\sum_{s=-\infty}^{\infty}[\xi_s,\eta_s].$$

We introduce projections P, Q, $P_k\in\mathscr{L}(l^2(\mathbb{C}^m))$ by

$$P\xi:=\tilde{\xi}, \qquad \tilde{\xi}_s:=\begin{cases}\xi_s & \text{if } s\geqq 0\\ 0 & \text{if } s<0,\end{cases}$$
$$Q:=I-P.$$

For an $i=i(\sigma,k)$ ($i\in\{0,1,\dots,n\}$) and a $j=j(\sigma,k)$ ($j\in\{0,\dots,N\}$), we have $\sigma\in[i/n, (i+1)/n)$ and $k_j\leqq i<k_{j+1}$. We set

$$P_k\xi:=\tilde{\xi}, \qquad \tilde{\xi}_s:=\begin{cases}\xi_s & \text{if } -(i+jd)\leqq s\leqq(n-i)+(N-j)d-1\\ 0 & \text{otherwise}\end{cases}$$

and define

$$E_k\colon \operatorname{im} P_k \to PS_d(\Delta_k, \mathbb{C}^m),$$

$$E_k \xi := \sqrt{n}\{\xi_{-(i+jd)}\varphi_1 + \xi_{-(i+jd)+1}\varphi_2 + \dots + \xi_0\varphi_{i+jd+1} + \xi_1\varphi_{i+jd+2} + \dots + \xi_{(n-i)+(N-j)d-1}\varphi_{n+Nd}\}.$$

(Thus P_k, E_k are chosen in such a manner that $E_k^{-1}\varphi_{i+jd+1} = \{\delta_{i,0}\}_{i=-\infty}^{\infty}$; however, φ_{i+jd+1} is an element of the base of $PS_d(\Delta_k, \mathbb{C}^m)$, the support of which contains $\tau = \gamma(\sigma)$.) To every operator $B_k \in \mathscr{L}(PS_d(\Delta_k, \mathbb{C}^m))$ we associate the operator $\tilde{B}_k := E_k^{-1} B_k E_k \in \mathscr{L}(\operatorname{im} P_k)$, which can be identified with $\tilde{B}_k P_k \in \mathscr{L}(l^2(\mathbb{C}^m))$.

If the Galerkin method converges, then the operators

$$A_k := P_{\Delta_k} A P_{\Delta_k} \in \mathscr{L}(PS_d(\Delta_k, \mathbb{C}^m))$$

are stable. By (2.2) the same fact holds for the sequences $\{\tilde{A}_k\}$ and $\{\tilde{A}_k^*\}$, where $\tilde{A}_k^*$ denotes the adjoint operator in $\mathscr{L}(l^2(\mathbb{C}^m))$. Hence the strong limit of $\{\tilde{A}_k\}$, the adjoint of which is the strong limit of $\{\tilde{A}_k^*\}$, is invertible.

Now we examine the limit of $\{\tilde{A}_k\}$. The convergence of $\{\tilde{A}_k^*\}$ is derivable in the same way. We set

$$c := 1/2(a+b), \quad d := 1/2(a-b), \quad c_k := P_{\Delta_k} c P_{\Delta_k}, \quad d_k := P_{\Delta_k} d P_{\Delta_k}, \quad S_k := P_{\Delta_k} S_\Gamma P_{\Delta_k}.$$

Then, by Lemma 1.1, $\{\tilde{A}_k\}$ has the same limit as $\{\tilde{B}_k := \tilde{c}_k + \tilde{d}_k \tilde{S}_k\}$.

Lemma 2.1. (i) *For* $\sigma \in (0, 1)$ *the following statements are valid:*

a) $\{\tilde{c}_k\}$ *converges strongly to* $(c(\tau+0)P + c(\tau-0)Q + T_1)$, *where* $T_1 \in \mathscr{L}(l^2(\mathbb{C}^m))$ *is compact.*

b) $\{\tilde{S}_k\}$ *converges strongly to* $(C(\rho) + T_2)$, *where* $T_2 \in \mathscr{L}(l^2(\mathbb{C}^m))$ *is compact and* $C(\rho)$ *denotes the operator of convolution generated by the following function (see* [19]*):*

$$\rho(s) := -\frac{\sum_{r\in\mathbb{Z}} \frac{\operatorname{sign}(r+1/2)}{(s+2r\pi)^{2(d+1)}}}{\sum_{r\in\mathbb{Z}} \frac{1}{(s+2r\pi)^{2(d+1)}}} \quad (s \in [0, 2\pi]).$$

(ii) *For* $\sigma = 0$ *or* $\sigma = 1$ *the following statements are valid:*

c) $\{\tilde{c}_k\}$ *converges strongly to* $(c(\tau)P + T_1)$ *or* $(c(\tau)Q + T_1)$, *where* $T_1 \in \mathscr{L}(\operatorname{im} P)$ *and* $T_1 \in \mathscr{L}(\operatorname{im} Q)$ *are compact respectively.*

d) $\{\tilde{S}_k\}$ *converges strongly to* $(PC(\rho)P + T_2)$ *or* $(QC(\rho)Q + T_2)$, *where* $T_2 \in \mathscr{L}(\operatorname{im} P)$ *and* $T_2 \in \mathscr{L}(\operatorname{im} Q)$ *are compact respectively.*

Proof. We prove (i). Assertion (ii) is demonstrated analogously.

Let $\sigma \in (0, 1)$ and denote by $w(c, h)$ the modulus of continuity of $c \circ \gamma$:

$$w(c, h) := \sup\{|c(\gamma(x)) - c(\gamma(y))| \mid x, y \in [0, 1], |x-y| \leq h, s_j \notin [x, y] (j = 1, \dots, N)\}.$$

Further, G_k is the following matrix, which will be identified with the corresponding operators in $\operatorname{im} P_k$ or $l^2(\mathbb{C}^m)$:

$$G_k := (n(\varphi_{s+i+jd+1}, \varphi_{r+i+jd+1}))_{r,s=-(i+jd)}^{(n-i)+(N-j)d-1}.$$

Writing down the matrix of c_k corresponding to the base $\{\varphi_r\}$, we obtain

$$\tilde{c}_k = G_k^{-1}(n(c\varphi_{r+i+jd+1}, \varphi_{s+i+jd+1}))_{s,r=-(i+jd)}^{(n-i)+(N-j)d-1}.$$

An easy computation yields (choose any $\sigma'_{r+i+jd+1} \in \operatorname{supp} \varphi_{r+i+jd+1}$)

$$\tilde{c}_k = G_k^{-1}(\delta_{r,s}\, c(\sigma'_{r+i+jd+1})))_{s,r=-(i+jd)}^{(n-i)+(N-j)d-1}\, G_k + O(w(c, 1/n)) \tag{2.3}$$

where $O(w(c, 1/n))$ denotes a matrix, whose operator norm is majorized by a constant multiple of $w(c, 1/n)$. This term thus becomes negligible in our further considerations. Inequality (2.2) leads to

$$D_2^{-2}(\sum |\xi_s|^2) \leqq n(\sum \xi_s \varphi_s, \sum \xi_r \varphi_r) \leqq D_1^{-2}(\sum |\xi_s|^2)$$
$$D_2^{-2} \|\{\xi_s\}\|^2 \leqq (\{\xi_s\}, G_k\{\xi_s\}) \leqq D_1^{-2} \|\{\xi_s\}\|^2.$$

Hence, G_k is invertible and $\{G_k\}$, $\{G_k^{-1}\}$ are uniformly bounded.

Now let $u, v \in \mathbb{Z}$, $d < |u|$, $|v| < \min\{|i-k_s| - d \mid s=0, \ldots, N;\ k_s \neq i\}$, $u_1 = u+d$, $v_1 = v+d$ for $u, v < 0$ and $i = k_j$. For $u, v > 0$ or $i \neq k_j$, we set $u_1 = u$, $v_1 = v$. If $i \neq k_j$ or $u > 0$, then

$$\varphi_{u+i+jd+1} = \varphi_{k_j+jd+(i-k_j+u+1)} = \chi_j \tilde{\psi}_{k_j+(i-k_j+u+1)} = \tilde{\psi}_{i+u_1+1}.$$

If $i = k_j$ and $u < 0$, then

$$\begin{aligned}\varphi_{u+i+jd+1} &= \varphi_{k_{j-1}+(j-1)d+k_j-k_{j-1}+u+d+1} \\ &= \chi_{j-1} \tilde{\psi}_{k_{j-1}+(i-k_{j-1}+u+d+1)} = \tilde{\psi}_{i+u_1+1},\end{aligned}$$

therefore,

$$\begin{aligned}(G_k)_{u,v} &= n(\varphi_{v+i+jd+1}, \varphi_{u+i+jd+1}) \\ &= n \int_0^1 \tilde{\psi}_{i+u_1+1}(s)\, \tilde{\psi}_{i+v_1+1}(s)\, |\gamma'(s)|\, ds \\ &= \int_{\mathbb{R}} \Pi_{d+1}(s-(i+u_1+1))\, \Pi_{d+1}(s-(i+v_1+1))\, |\gamma'(s/n)|\, ds.\end{aligned}$$

Consequently, $(G_k)_{u,v} = 0$ for $u \cdot v < 0$ and by holding u and v fixed, we obtain

$$\begin{aligned}\lim_{k\to\infty} (G_k)_{u,v} &= |\gamma'(\sigma)| \int_{\mathbb{R}} \Pi_{d+1}(s-(u_1-v_1))\, \Pi_{d+1}(s)\, ds \\ &= |\gamma'(\sigma)| \int_{\mathbb{R}} e^{-i(u_1-v_1)s} |F\Pi_{d+1}(s)|^2\, ds,\end{aligned} \tag{2.5}$$

where $F\Pi_{d+1}$ denotes the Fourier transform of Π_{d+1}. This means

$$F\Pi_{d+1} = (\sqrt{2\pi})^d (F\Pi)^{d+1},$$

$$(F\Pi)(s) = \sqrt{2/\pi}\, \frac{\sin s/2}{s}\, e^{-i \cdot s/2},$$

$$|F\Pi_{d+1}(s)|^2 = 1/2\pi \left(\frac{\sin s/2}{s/2}\right)^{2(d+1)}.$$

Substituting the last equation in (2.5) yields

$$\lim_{k\to\infty}(G_k)_{u,v}=1/2\pi\int_0^{2\pi} e^{-i(u_1-v_1)s}\zeta(s)\,ds,$$

$$\zeta(s):=4^{(d+1)}|\gamma'(\sigma)|(\sin s/2)^{2(d+1)}\sum_{r\in\mathbb{Z}}\frac{1}{(s+2r\pi)^{2(d+1)}},$$

$$\lim_{k\to\infty}(G_k)_{u,v}=0 \qquad (\text{for } u\cdot v<0).$$

For $|u|$, $|v|\leqq d$, we compute $\lim(G_k)_{u,v}$ in an analogous way. Thereby we have in Eq. (2.5) the product of Π_{d+1} with some characteristic function instead of Π_{d+1}. Since only $(2d+1)$ diagonals of G_k have non-zero elements, one easily concludes that G_k converges strongly to $(C(\zeta)+T_3)$, where $T_3\in\mathscr{L}(l^2(\mathbb{C}^m))$ is a suitable operator of finite range. The appearance of operator T_3 is explained by the fact that (2.5) does not hold for $i=k_j$ and $|u|$, $|v|<d$; however $T_3=0$ if $i\neq k_j$. Out of $\sup_k\|G_k^{-1}\|<\infty$ and $G_k\to(C(\zeta)+T_3)$ follows that G_k^{-1} converges strongly to $(C(\zeta^{-1})+T_4)$, where $T_4\in\mathscr{L}(l^2(\mathbb{C}^m))$ is compact. Now a) is an easy consequence of (2.3).

For $d<|u|$, $|v|<\min\{|i-k_s|-d\,|\,s=0,\dots,N;\,k_s\neq i\}$ we obtain analogously to (2.3) and (2.5)

$$D_k:=(n(S_\Gamma\varphi_{r+i+jd+1},\varphi_{s+i+jd+1}))_{s,r=-(i+jd)}^{(n-i)+(N-j)d-1},\qquad \tilde{S}_k=G_k^{-1}D_k$$

$$\begin{aligned}\lim_{k\to\infty}(D_k)_{u,v}&=|\gamma'(\sigma)|\int_{\mathbb{R}}(S_{\mathbb{R}}\Pi_{d+1}(\cdot-(v_1-u_1))(s)\Pi_{d+1}(s)\,ds\\ &=-|\gamma'(\sigma)|\int_{\mathbb{R}}e^{-i(u_1-v_1)s}|F\Pi_{d+1}(s)|^2\operatorname{sign}s\,ds\\ &=1/2\pi\int_0^{2\pi}e^{-i(u_1-v_1)s}\vartheta(s)\,ds,\end{aligned}\tag{2.6}$$

where

$$\vartheta(s):=-4^{(d+1)}|\gamma'(\sigma)|(\sin s/2)^{2(d+1)}\sum_{r\in\mathbb{Z}}\frac{\operatorname{sign}(r+1/2)}{(s+2r\pi)^{2(d+1)}}.$$

The techniques of the proofs of Lemma 3.1 and Lemma 2.3 in [19] show that $D_k\to C(\vartheta)$ for $i\neq k_j$ (i.e., $\tau\notin\{t_0,\dots,t_N\}$) and

$$D_k\to(PC(\vartheta)P+QC(\vartheta)Q+PC(\vartheta_1)Q+QC(\vartheta_2)P+T_5)\qquad\text{for } i=k_j,$$

where $\vartheta_1(s):=e^{ids}\vartheta(s)$, $\vartheta_2(s)=e^{-ids}\vartheta(s)$. The finite range operator T_5 appears here, since (2.6) does not hold for $i=k_j$ and $|u|$, $|v|<d$. For $i=k_j$ and $|u|$, $|v|<d$ we have a formula analogous to (2.6), where again a product of Π_{d+1} with some characteristic function appears instead of Π_{d+1}. Now $PC(\vartheta_1)Q=PC(\vartheta)Q+PC(\vartheta_3)Q$, where

$$\vartheta_3(s)=(e^{ids}-1)\,\vartheta(s).$$

Since ϑ_3 is a 2π-periodic and continuous function on $\mathbb{R}$, the operator $PC(\vartheta_3)Q$ is compact. Consequently, we obtain $D_k\to(C(\vartheta)+T_6)$, where $T_6\in\mathscr{L}(l^2(\mathbb{C}^m))$ is compact. In any case, we have derived $S_k = G_k^{-1} D_k \to (C(\rho) + T_2)$ for a suitable compact $T_2 \in \mathcal{L}(l^2(\mathbb{C}^m))$, and the proof of Lemma 2.1 is complete.

After defining

$$\begin{aligned}
A^\tau &:= P(c(\tau+0)+d(\tau+0)\,C(\rho))+Q(c(\tau-0)+d(\tau-0)\,C(\rho)) \quad (\tau\in\Gamma),\\
B^\tau &:= P(c(\tau+0)+d(\tau+0)\,C(\rho))\,P \quad (\tau=\gamma(0)),\\
C^\tau &:= Q(c(\tau-0)+d(\tau-0)\,C(\rho))\,Q \quad (\tau=\gamma(1)),
\end{aligned}$$

Lemma 2.1 yields $\tilde{A}_k \to (A^\tau + T)$ for $\sigma \in (0, 1)$, $\tilde{A}_k \to (B^\tau + T)$ for $\sigma = 0$ and $\tilde{A}_k \to (C^\tau + T)$ for $\sigma = 1$, where T is compact. If further $\tau \notin \{t_0, \ldots, t_N\}$ or $d = 0$, then the proof of Lemma 2.1 yields $T = 0$. The limit of $\{\tilde{A}_k\}$ established above and the considerations preceeding Lemma 2.1 lead to

Lemma 2.2 (i) *Let Γ be a closed curve. If the Galerkin method with respect to an admissible sequence of equidistant partitions $\{\Delta_k\}$ converges for the operator A, then the operators $A^\tau (\tau \in \Gamma)$ are Fredholm operators with index* 0.

(ii) *Let Γ be open. If the Galerkin method with respect to an admissible sequence of equidistant partitions $\{\Delta_k\}$ converges for the operator A, then the operators $A^{\gamma(\sigma)} (\sigma \in (0, 1))$, $B^{\gamma(0)}$, $C^{\gamma(1)}$ are Fredholm operators with index* 0.

(iii) *If in addition $m = 1$ or $d = 0$, then the operators A^τ, B^τ, C^τ of* (i), (ii) *are invertible.*

Proof of Theorem 2.1. Let $m = 1$. If the Galerkin method converges, then Lemma 2.2 implies that operators $A^\tau (\tau \in \Gamma)$ or $A^{\gamma(\sigma)}$ $(\sigma \in (0, 1))$, $B^{\gamma(0)}$, $C^{\gamma(1)}$ are invertible. Now one shows analogously to the proof of Corollary 4.2 in [19] that (1.4) is satisfied. Thus the strong ellipticity follows via Lemma 1.2.

Finally, we consider the collocation method described in [19]. Assume Γ closed and $\Delta_k := \{0 = \sigma_0, \sigma_1 = 1/k, \sigma_2 = 2/k, \ldots, \sigma_k = 1\}$ $(k = n)$. We seek a piecewise linear approximation x_{Δ_k} of the solution x of $Ax = y$ satisfying

$$(Ax_{\Delta_k})(\gamma(j/k)) = y(\gamma(j/k))(j = 0, \ldots, k-1). \tag{2.7}$$

Now define

$$\psi_j^{(k)}(\gamma(s)) := \begin{cases} \dfrac{\gamma(s)-\gamma((j-1)/k)}{\gamma(j/k)-\gamma((j-1)/k)} & \text{if } s\in[(j-1)/k, j/k] \\ \dfrac{\gamma((j+1)/k)-\gamma(s)}{\gamma((j+1)/k)-\gamma(j/k)} & \text{if } s\in[j/k, (j+1)/k] \\ 0 & \text{otherwise} \end{cases}$$

and $K_k f := \sum_{j=0}^{k-1} f(\gamma(j/k))\,\psi_j^{(k)}$. Obviously, $\{\psi_j^{(k)}\}_{j=0}^{k-1}$ is a base of the space $\tilde{S}_1(\Delta_k, \mathbb{C}^m)$ of all piecewise linear (linear in $t = \gamma(s)$) functions subordinate to Δ_k and K_k is the interpolation projection onto this subspace. Equations (2.7) are equivalent to

$$K_k A x_{\Delta_k} = K_k y.$$

In [19] we proved that if A is invertible and the operators $K_k A \,|\, \tilde{S}_1(\Delta_k, \mathbb{C}^m)$ are stable, then $x_{\Delta_k} (k \to \infty)$ converges to the solution x of $Ax = y$ for all continuous right-hand sides y. We also gave a necessary and sufficient condition for stability of the aforementioned convergence scheme (see Theorem 4.1 in [19]). We now state

Corollary 2.1. *Let* $a, b \in PC(\Gamma, \mathbb{C}^{m\times m})$. *If the invertible operator* $A = aP_\Gamma + bQ_\Gamma \in \mathscr{L}(L^2(\Gamma, \mathbb{C}^m))$ *is strongly elliptic, then the approximate operators* $K_k A \mid \tilde{S}_1(\Delta_k, \mathbb{C}^m)$ *of the collocation method* (2.7) *are stable.*

Proof. By Theorem 1.1, the Galerkin method with respect to any admissible sequence of equidistant partitions converges for A. Therefore, Lemma 2.2 implies the invertibility of the operators $A^\tau(d=0)$. However, operators $A^\tau(d=0)$ coincide with the associated operators A_τ of Theorem 4.1 in [19]. Theorem 4.1 of [19] yields Corollary 2.1.

3. The Asymptotical Order of Convergence

For smooth curve Γ, let $H^s(\Gamma)$ ($s \in \mathbb{R}$, $s \geqq 0$) be the usual Sobolev space of order s on Γ. By $H^s_{t_0,\ldots,t_N}(\Gamma)$ we denote the sum $H^s[t_0, t_1] \oplus \ldots \oplus H^s[t_{N-1}, t_N]$, where $[t_i, t_{i+1}]$ is the arc on Γ with the end points t_i, t_{i+1}.

For ε an arbitrary positive number satisfying $\varepsilon < \max_i (s_i - s_{i-1})$ ($i = 1, \ldots, N$), we choose the partitions Δ_k such that

$$
\begin{aligned}
\Delta_k \cap (s_i - \varepsilon, s_i + \varepsilon) &= \left\{ s_i \pm \left(\frac{j}{N_k}\right)^{\beta_i} \,\middle|\, j \in \mathbb{N} \right\} \cap (s_i - \varepsilon, s_i + \varepsilon), \qquad i = 1, \ldots, N-1 \\
\Delta_k \cap [0, \varepsilon) &= \left\{ \left(\frac{j}{N_k}\right)^{\beta_0} \,\middle|\, j \in \mathbb{N} \right\} \cap [0, \varepsilon) \\
\Delta_k \cap (1-\varepsilon, 1] &= \left\{ 1 - \left(\frac{j}{N_k}\right)^{\beta_N} \,\middle|\, j \in \mathbb{N} \right\} \cap (1-\varepsilon, 1],
\end{aligned}
\tag{3.1}
$$

where β_i ($\beta_i \geqq 1$) is a given real number and $N_k \in \mathbb{N}$. In addition assume $N_k \to \infty$ ($k \to \infty$) and the existence of a positive constant C with

$$
h_{\Delta_k} \leqq C 1/N_k, \qquad C^{-1} N_k \leqq n_k \leqq C N_k. \tag{3.2}
$$

If $\beta_i = 1$ ($i = 0, \ldots, N$), then equidistant partitions satisfy (3.1) and (3.2).

For functions $a, b \in PC(\Gamma, \mathbb{C})$ continuous on $\Gamma \setminus \{t_0, \ldots, t_N\}$, we set

$$
\kappa_j := \frac{1}{2\pi i} \ln \left(\frac{a(t_j+0)\, b(t_j-0)}{a(t_j-0)\, b(t_j+0)} \right),
$$

where ln denotes the continuous branch of the logarithm in $\mathbb{C} \setminus (-\infty, 0]$ which takes real values on the positive real axis. Further, if Γ is open, set $a(t_0-0) = b(t_0-0) = a(t_N+0) = b(t_N+0) = 1$.

Theorem 3.1. *Let* $\{\Delta_k\}$ *satisfy* (3.1) *and* (3.2), $a, b \in H^{r+2}_{t_0,\ldots,t_N}(\Gamma)$ ($r \in \mathbb{N}$), $1/2 + r + \operatorname{Re} \kappa_i < s < 3/2 + r + \operatorname{Re} \kappa_i$ ($i = 0, \ldots, N$), $A = aP_\Gamma + bQ_\Gamma$ *be invertible in* $L^2(\Gamma, \mathbb{C})$ *and strongly elliptic. If* $y \in H^s_{t_0,\ldots,t_N}(\Gamma)$, $x = A^{-1} y$ *and* x_{Δ_k} *is the solution of* (0.3), *then*

$$\|x_{\Delta_k}-x\|_{L^2(\Gamma,\mathbb{C})}\leqq C_0(1/n_k)^\mu(\ln n_k)^\nu,$$

where C_0 is a constant and

$$\mu:=\min\{s, d+1, \beta_i(\operatorname{Re}\kappa_i+1/2)\,|\,i=0,\dots,N\},$$
$$\nu:=\begin{cases}1/2 & \text{if } s\geqq d+1 \text{ and } \min\{\beta_i(\operatorname{Re}\kappa_i+1/2), i=0,\dots,N\}=d+1\\ 0 & \text{otherwise.}\end{cases}$$

For simplicity sake, we shall prove Theorem 3.1 for the following special case only (see Theorem 3.3). Let $\Gamma=(0,1)$, $a, b\in C[0,1]$. Thus instead of the "critical" points $\{t_i\,|\,i=0,\dots,N\}$ we have merely the "critical" points $\{t_0=0, t_N=1\}$. We now follow the ideas developed in [9]. First we determine the asymptotic behavior of $x=A^{-1}y$ at 0 and 1. From the proofs in [9], Sect. 2, we conclude

Theorem 3.2. *Assume $a, b\in H^{r+2}(0,1)$ $(r\in\mathbb{N})$, $A=aP_{(0,1)}+bQ_{(0,1)}$ invertible in $L^2((0,1),\mathbb{C})$ and $y\in H^s(0,1)$, where $\operatorname{Re}\kappa_i+1/2+r<s<\operatorname{Re}\kappa_i+3/2+r$ $(i=0,1)$. Then the function $x=A^{-1}y$ has the representation*

$$x(t)=x_0(t)+t^{\kappa_0}\sum_{m=0}^{r}\sum_{s=0}^{m}\alpha_{m,s}t^m(\ln t)^s$$
$$+(1-t)^{\kappa_1}\sum_{m=0}^{r}\sum_{s=0}^{m}\beta_{m,s}(1-t)^m(\ln(1-t))^s,$$

where $x_0\in H^s(0,1)$ and $\alpha_{m,s}$, $\beta_{m,s}\in\mathbb{C}$.

If x_{Δ_k} is the solution of (0.3) and $x=A^{-1}y$, then

$$\|x-x_{\Delta_k}\|\leqq C\|x-P_{\Delta_k}x\|. \tag{3.3}$$

(See, e.g., [10].) Thus the proof of Theorem 3.1 for the immediately preceeding special case requires only estimating $\|(I-P_{\Delta_k})x\|$.

Lemma 3.1 (see, e.g., [9]). *If $x_0\in H^s(0,1)$, then there exists a constant C such that*

$$\|(I-P_{\Delta_k})x_0\|_{L^2(\Gamma,\mathbb{C})}\leqq C(1/n_k)^{\min(s,d+1)}.$$

For further estimates, we need the quasi-interpolant Q_{Δ_k} introduced by de Boor [7],

$$Q_{\Delta_k}g:=\sum_{i=-d}^{n_k-1}\lambda_i(g)\,\psi_i\sqrt{\frac{\sigma_{i+d+1}-\sigma_i}{d+1}},$$

where ψ_i is defined as in Sect. 1 and

$$\lambda_i(g):=\sum_{j=0}^{d}(-1)^{d-j}\omega_i^{(d-j)}(\tau_i)\,g^{(j)}(\tau_i)\qquad(i=-d,\dots,n_k-1),$$
$$\omega_i(t):=(t-\sigma_{i+1})\dots(t-\sigma_{i+d})\cdot 1/d!,$$
$$\tau_i\in(\sigma_i,\sigma_{i+d+1})\cap[1/2\sigma_1, 1]. \tag{3.4}$$

The operator Q_{Δ_k} is a projection onto $S_d(\Delta_k,\mathbb{C})$.

Lemma 3.2. *For any $(d+1)$-times continuously differentiable function f on $(0, 1)$, the estimate*

$$\int_{\sigma_i}^{\sigma_{i+1}} |f-Q_{\Delta_k} f|^2 \leqq C(\sigma_{i+d+1}-\sigma_{i-d-1})^{2(d+1)+1} \max_{t\in[\sigma_{i-d-1},\, \sigma_{i+d+1}]} |f^{(d+1)}(t)|^2 \quad (i=0, \ldots, n_k-1) \tag{3.5}$$

holds, where C is a constant independent of f and n_k.

Proof. For $t\in[\sigma_i, \sigma_{i+1}]$, we obtain

$$\left| f(t)-\sum_{j=0}^{d} f^{(j)}(\sigma_i) \frac{(t-\sigma_i)^j}{j!}\right| \leqq C(\sigma_{i+1}-\sigma_i)^{(d+1)} \max_{t\in[\sigma_i,\, \sigma_{i+1}]} |f^{(d+1)}(t)|$$

in particular,

$$\int_{\sigma_i}^{\sigma_{i+1}} \left| f(t)-\sum_{j=0}^{d} f^{(j)}(\sigma_i) \frac{(t-\sigma_i)^j}{j!}\right|^2 dt \leqq C(\sigma_{i+1}-\sigma_i)^{2(d+1)+1} \max_{t\in[\sigma_i,\, \sigma_{i+1}]} |f^{(d+1)}(t)|^2. \tag{3.6}$$

On the other hand,

$$\begin{aligned}\left|\sum_{j=0}^{d} f^{(j)}(\sigma_i) \frac{(t-\sigma_i)^j}{j!} - Q_{\Delta_k} f(t)\right| &= \left| Q_{\Delta_k}\left(\sum_{j=0}^{d} f^{(j)}(\sigma_i) \frac{(\cdot-\sigma_i)^j}{j!} - f\right)(t)\right| \\ &= \left|\sum_{s=i-d}^{i+1} \lambda_s\left(\sum_{j=0}^{d} f^{(j)}(\sigma_i) \frac{(\cdot-\sigma_i)^j}{j!} - f\right) \sqrt{\frac{(\sigma_{s+d+1}-\sigma_s)}{d+1}}\, \psi_s(t)\right|\end{aligned}$$

and (1.1) yields

$$\begin{aligned}&\int_{\sigma_i}^{\sigma_{i+1}} \left|\sum_{j=0}^{d} f^{(j)}(\sigma_i) \frac{(t-\sigma_i)^j}{j!} - Q_{\Delta_k} f(t)\right|^2 dt \\ &\leqq C \sum_{s=i-d}^{i+1} (\sigma_{i+d+1}-\sigma_{i-d-1}) \left|\lambda_s\left(\sum_{j=0}^{d} f^{(j)}(\sigma_i) \frac{(\cdot-\sigma_i)^j}{j!} - f\right)\right|^2.\end{aligned} \tag{3.7}$$

Setting $g(t)=f(t)-\sum_{j=0}^{d} f^{(j)}(\sigma_i) \frac{(t-\sigma_i)^j}{j!}$, we obtain

$$|g^{(j)}(\tau_s)| \leqq C(\sigma_{i+d+1}-\sigma_{i-d-1})^{d+1-j} \max_{t\in[\sigma_{i-d-1},\, \sigma_{i+d+1}]} |f^{(d+1)}(t)|$$

and

$$|\omega_s^{(d-j)}(\tau_s)| \leqq C(\sigma_{i+d+1}-\sigma_{i-d-1})^j \quad (s=i-d, \ldots, i+1).$$

Therefore, by means of (3.4) we have

$$|\lambda_s(g)| \leqq C(\sigma_{i+d+1}-\sigma_{i-d-1})^{d+1} \max_{t\in[\sigma_{i-d-1},\, \sigma_{i+d+1}]} |f^{(d+1)}(t)|.$$

From the last inequality and (3.7) we obtain

$$\int_{\sigma_i}^{\sigma_{i+1}} \left| \sum_{j=0}^{d} f^{(j)}(\sigma_j) \frac{(t-\sigma_i)^j}{j!} - Q_{\Delta_k} f(t) \right|^2 dt$$
$$\leqq C(\sigma_{i+d+1} - \sigma_{i-d-1})^{2(d+1)+1} \max_{t \in [\sigma_{i-d-1}, \sigma_{i+d+1}]} |f^{(d+1)}(t)|^2$$

which together with (3.6) implies (3.5).

Lemma 3.3. *Let* $\{\Delta_k\}$ *satisfy* (3.1) *and* (3.2) *and* $f(t) := t^{\alpha + i\beta}(\ln t)^r$ (α, $\beta \in \mathbb{R}$, $\alpha > -1/2$, $r \in \mathbb{N}$). *Then there exists a constant* C *such that*

$$\|(I - P_{\Delta_k}) f\|_{L^2((0,1),\mathbb{C})} \leqq \|(I - Q_{\Delta_k}) f\|_{L^2((0,1),\mathbb{C})}$$
$$\leqq C \begin{cases} \left(\dfrac{1}{n_k}\right)^{\beta_0(\alpha+\frac{1}{2})} (\ln n_k)^r & \text{if } \beta_0(\alpha+\frac{1}{2}) < d+1 \\ \left(\dfrac{1}{n_k}\right)^{d+1} (\ln n_k)^{2r+\frac{1}{2}} & \text{if } \beta_0(\alpha+\frac{1}{2}) = d+1 \\ \left(\dfrac{1}{n_k}\right)^{d+1} (\ln n_k)^{2r} & \text{if } \beta_0(\alpha+\frac{1}{2}) > d+1 \end{cases} \tag{3.8}$$

Proof. Considering Lemma 3.2, $\sup_{t \in [\varepsilon, 1]} |f^{(d+1)}(t)| < \infty$ and $|\sigma_{i+d+1} - \sigma_{i-d-1}| \leqq C\, 1/n_k$ we can easily see that

$$\int_{\varepsilon}^{1} |f - Q_{\Delta_k} f|^2 \leqq C\, 1/n_k^{2(d+1)}.$$

Consequently, we may assume without loss of generality $\varepsilon = 1$ (see (3.1)) – i.e., $\sigma_i = \sigma_i^k = (i/n_k)^{\beta_0}$ $(i = 0, 1, \ldots, n_k)$.

Let $i \geqq 2(d+1)$. This implies

$$(\sigma_{i+d+1} - \sigma_{i-d-1}) \leqq C (i/n_k)^{\beta_0 - 1}\, 1/n_k,$$
$$\max_{t \in [\sigma_{i-d-1}, \sigma_{i+d+1}]} |f^{(d+1)}(t)| \leqq C (i/n_k)^{\beta_0(\alpha-d-1)} (\ln i)^r (\ln n_k)^r + C$$

and Lemma 3.2 yields

$$\int_{\sigma_{2(d+1)}}^{1} |f - Q_{\Delta_k} f|^2 \leqq C \sum_{i=2(d+1)}^{n_k} (1/n_k)^{2(d+1)+1} (i/n_k)^{(\beta_0-1)(2(d+1)+1)}$$
$$\cdot \{(i/n_k)^{2\beta_0(\alpha-d-1)} (\ln i)^{2r} (\ln n_k)^{2r} + 1\},$$

$$\left(\int_{\sigma_{2(d+1)}}^{1} |f - Q_{\Delta_k} f|^2 \right)^{1/2} \leqq C \begin{cases} (1/n_k)^{\beta_0(\alpha+1/2)} (\ln n_k)^r & \text{if } \beta_0(\alpha+1/2) < d+1 \\ (1/n_k)^{d+1} (\ln n_k)^{2r+1/2} & \text{if } \beta_0(\alpha+1/2) = d+1 \\ (1/n_k)^{d+1} (\ln n_k)^{2r} & \text{if } \beta_0(\alpha+1/2) > d+1 \end{cases} \tag{3.9}$$

For $i \leqq 2(d+1)$, we have

$$|\omega_s^{(d-j)}(\tau_s)| \leqq C (1/n_k)^{\beta_0 \cdot j}, \quad |f^{(j)}(\tau_s)| \leqq C (1/n_k)^{\beta_0(\alpha-j)} (\ln n_k)^r.$$

Now, (3.4) and (1.1) together yield

$$\int_0^{\sigma_{2(d+1)}} |Q_{\Delta_k} f|^2 \leqq (\sigma_{3(d+1)} - \sigma_0) \sum_{s=0}^{2(d+1)} |\lambda_s(f)|^2 \leqq C(1/n_k)^{\beta_0(2\alpha+1)} (\ln n_k)^{2r},$$

$$\left(\int_0^{\sigma_{2(d+1)}} |Q_{\Delta_k} f|^2\right)^{1/2} \leqq C(1/n_k)^{\beta_0(\alpha+1/2)} (\ln n_k)^r, \tag{3.10}$$

whereas

$$\left(\int_0^{\sigma_{2(d+1)}} |f|^2\right)^{1/2} \leqq C(1/n_k)^{\beta_0(\alpha+1/2)} (\ln n_k)^r. \tag{3.11}$$

Finally, (3.8) follows from (3.9), (3.10) and (3.11).

An immediate consequence of Theorem 3.2, (3.3), Lemma 3.1 and Lemma 3.3 is the following special case of Theorem 3.1.

Theorem 3.3. *Let* $a, b \in H^{r+2}(0, 1)$ $(r \in \mathbb{N})$, $A = aP_{(0,1)} + bQ_{(0,1)}$ *be invertible in* $L^2((0, 1), \mathbb{C})$ *and strongly elliptic,* $y \in H^s(0, 1)$, *where* $\operatorname{Re}\kappa_i + 1/2 + r < s < \operatorname{Re}\kappa_i + 3/2 + r$ $(i = 0, 1)$, *and* $\{\Delta_k\}$ *satisfy* (3.2) *and the second and third formula of* (3.1). *If* $x = A^{-1} y$ *and* x_{Δ_k} *denotes the solution of* (0.3), *then*

$$\|x_{\Delta_k} - x\|_{L^2((0,1),\mathbb{C})} \leqq C(1/n_k)^{\mu} (\ln n_k)^{\nu},$$

where C *is a constant and*

$$\mu := \min\{s, d+1, \beta_0(\operatorname{Re}\kappa_0 + 1/2), \beta_1(\operatorname{Re}\kappa_1 + 1/2)\}$$

$$\nu := \begin{cases} 1/2 & \text{if } s \geqq d+1 \quad \text{and} \quad \min\{\beta_0(\operatorname{Re}\kappa_0 + 1/2), \beta_1(\operatorname{Re}\kappa_1 + 1/2)\} = d+1 \\ 0 & \text{otherwise.} \end{cases}$$

Remark. Theorems 3.2 and 3.3 remain true for operators $A = P_{(0,1)} a + Q_{(0,1)} b$ and functions $y \in H^s(0, 1)$ satisfying

$$y^{(k)}(0) = y^{(k)}(1) = 0, \qquad k = 0, 1, \ldots, [s - 1/2].$$

Similarly, Theorem 3.1 can be proved for operators $A = P_\Gamma a + Q_\Gamma b$ and functions $y \in H^s(\Gamma)$.

4. A Numerical Example

Let us consider the equation

$$x(s) - 1/\pi \int_{-1}^{1} \frac{x(t)}{t-s}\, dt = 1 \tag{4.1}$$

which has the exact solution

$$x(s) = \frac{1}{\sqrt{2}} \left(\frac{1-s}{1+s}\right)^{1/4}.$$

Using Galerkin's method we determine a piecewise constant approximate solution for x (i.e. $d=0$).

If we set $a\equiv 1-i$, $b\equiv 1+i$, then the integral operator A defined by the left hand side of (4.1) takes the form (0.2). Obviously, A is strongly elliptic (see Lemma 1.2) and with the notations of Sect. 3 we have

$$\operatorname{Re}\kappa_0=-1/4, \qquad \operatorname{Re}\kappa_1=1/4.$$

Thus, according to Sect. 3 we set

$$\sigma_j:=-1+\left(\frac{j}{n_k/2}\right)^5, \quad \sigma_{n_k-j}:=1-\left(\frac{j}{n_k/2}\right)^2, \quad j=0,\ldots,n_k/2,$$
$$\Delta_k:=\{\sigma_0,\ldots,\sigma_{n_k}\}.$$

By exact computation of system (0.3) we obtain the following results.

Table 1

$n_k=2^k$	$\lVert x_{\Delta_k}-x\rVert_{L^2}$	$\dfrac{\ln\lVert x_{\Delta_k}-x\rVert_{L^2}-\ln\lVert x_{\Delta_{k-1}}-x\rVert_{L^2}}{\ln n_k-\ln n_{k-1}}$
2	0.470	
4	0.337	−0.48
8	0.196	−0.78
16	0.107	−0.87
32	0.057	−0.92
64	0.029	−0.95

If the logarithmic error $\ln\lVert x_{\Delta_k}-x\rVert_{L^2}$ is considered as a function of $\ln n_k$, then the last table shows that the slope of this function tends to -1. This fact confirms Theorem 3.3 which asserts $\lVert x_{\Delta_k}-x\rVert_{L^2}\sim 1/n_k$. In the next table we will compare the Galerkin approximation with the true solution at interior points.

Table 2

$t\in[-1,1]$	$x_{\Delta_2}(t)$	$x_{\Delta_4}(t)$	$x_{\Delta_6}(t)$	$x(t)$
−0.95	1.099	1.696	1.775	1.773
−0.8	1.099	1.308	1.202	1.224
−0.5	1.099	1.034	0.950	0.930
0.5	0.515	0.526	0.535	0.537
0.8	0.354	0.404	0.411	0.408

In his paper [28] Thomas implemented and applied Galerkin's method to the same equation. By augmenting the space of piecewise linear functions on $[-1+\delta_0,\,1-\delta_1]$ with the singular functions $(1+s)^{\kappa_0}$ on $[-1,\,-1+\delta_0]$ and $(1-s)^{\kappa_1}$ on $[1-\delta_1,1]$, he formed a new trial space and obtained a better approximation. This modification could be useful also in calculation with splines of higher order. The reader can readily verify the same asymptotical order of convergence for the modified method.

Acknowledgement. The authors would like to thank A. Pomp for the realization of the computations summarized in the tables of Sect. 4.

References

1. Alexits, G.: Einige Beiträge zur Approximationstheorie. Acta Sci. Math. **26**, 212–224 (1965)
2. Anderssen, R.S., De Hoog, F.R., Lukas, M.A.: The application and numerical solution of intergral equations. Netherlands-Germantown-Maryland, USA: Sijthoff & Noordhoff, Alphen van den Rijn 1980
3. Arnold, D.N., Wendland, W.L.: The convergence of spline collocation for strongly elliptic equations on curves. Numer. Math. **46**, 317-341 (1985)
4. Arnold, D.N., Wendland, W.L.: On the asymptotic convergence of collocation methods. Math. Comput. **41**, 349-381 (1983)
5. Aubin, J.P.: Approximation of elliptic boundary value problems. New York-London-Sidney-Toronto: Wiley-Interscience 1972
6. De Boor, C.: A Practical Guide to Splines. Appl. Math. Sci. New York: Springer 1978
7. De Boor, C.: The quasi-interpolant as a tool in elementary polynomial spline theory. In: Approximation theory (G.G. Lorentz, ed.), pp. 269-276. Proceeding of an International Symposium, Austin 1973. New York-London: Academic Press 1973
8. Elschner, J.: A Galerkin method with finite elements for degenerate one dimensional pseudodifferential equations. Math. Nachr. **111**, 111-126 (1983)
9. Elschner, J.: Galerkin Methods with Splines for Singular Integral Equations over (0, 1). Numer. Math. **43**, 265-281 (1984)
10. Gochberg, I.Z., Feldman, I.A.: Faltungsgleichungen und Projektionsverfahren zu ihrer Lösung. Berlin: Akademie-Verlag 1974
11. Kupradse, W.D.: Randwertaufgaben der Schwingungstheorie und Integralgleichungen. Berlin: VEB-Dt. Verlag der Wissenschaften 1956
12. Michlin, S.G., Prößdorf, S.: Singuläre Integraloperatoren. Berlin: Akademie-Verlag 1980
13. Muskhelishvili, N.I.: Some Basic Problems of the Mathematical Theory of Elasticity. Groningen: Noordhoff 1963
14. Nedelec, J.C.: Integral equations with non integrable kernels, Int. Equations and Operator Theory **5**, 562–572 (1982)
15. Parton, V.Z., Perlin, P.I.: Integral Equations of the Theory of Elasticity. Nauka, Moscow (Russian) 1977
16. Prößdorf, S.: A localization principle in the theory of finite element methods. In: Probleme und Methoden der Mathematischen Physik; 8. Tagung Karl-Marx-Stadt 1983. Teubner-Texte zur Mathematik **63**, 169–177 (Leipzig 1984)
17. Prößdorf, S.: Ein Lokalisierungsprinzip in der Theorie der Splineapproximationen und einige Anwendungen. Math. Nachr. **119**, 239-255 (1984)
18. Prößdorf, S., Elschner, J.: Finite element methods for singular integral equations on an interval. Eng. Anal. **1**, 83–87 (1984)
19. Prößdorf, S., Rathsfeld, A.: A spline collocation method for singular integral equations with piecewise continuous coefficients. Int. Equations and Operator Theory **7**, 537–560 (1984)
20. Prößdorf, S., Rathsfeld, A.: Strongly elliptic singular integral operators with piecewise continuous coefficients. Int. Equations Operator Theory (To appear)
21. Prößdorf, S., Schmidt, G.: A finite element collocation method for singular integral equations. Math. Nachr. **100**, 33-60 (1981)
22. Prößdorf, S., Schmidt, G.: A finite element collocation method for systems of singular integral

equations. In: Complex Analysis and Applications, pp. 428–439. Proceeding of the International Conference on Complex Analysis and Applications, Varna (September 20–27, 1981). Sofia: Publishing House of the Bulgarian Academy of Sciences 1984

23. Saranen, J., Wendland, W.L.: On the asymptotic convergence of collocation methods with spline functions of even degree. Math. Comput. (To appear)
24. Schmidt, G.: The convergence of Galerkin and collocation methods with splines for pseudodifferential equations on closed curves. ZAA, **4**, 371–384 (1984)
25. Schmidt, G.: On ε-collocation for pseudodifferential equations on a closed curve. Math. Nachr. (In press)
26. Schmidt, G.: On spline collocation for singular integral equations. Math. Nachr. **111**, 177–196 (1983)
27. Stephan, E., Wendland, W.L.: Remarks to Galerkin and least squares methods with finite elements for general elliptic problems. Manuscr. Geod. **1**, 93–123 (1976)
28. Thomas, K.S.: Galerkin methods for singular integral equations. Math. Comput. **36**, 193–205 (1981)
29. Wendland, W.L.: On the spline approximation of singular integral equations and one-dimensional pseudo-differential equations on closed curves. In: Numerical solution of singular integral equations (A. Gerasoulis, R. Vichnevetsky, eds.), pp. 113–119. IMACS 1984, Proceeding of an IMACS International Symposium held at Lehigh University Bethlehem, Pennsylvenia, USA (June 21–22, 1984)

Received November 27, 1984/July 11, 1985