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Asymptotic Error Expansion and Richardson Extrapolation for Linear Finite Elements *

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Summary. The elliptic Ritz projection with linear finite elements is shown to admit asymptotic error expansions on certain uniform meshes. This justifies the application of Richardson extrapolation for increasing the accuracy.

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1. Introduction

The Richardson extrapolation to the limit is a common way of increasing the accuracy of low order finite difference schemes applied to ordinary as well as to partial differential equations. Its success depends on the presence of asymptotic error expansions of the type

$$u_h(z) = u(z) + \sum_{k=1}^{n} h^{2k} e^{(k)}(z) + o(h^{2n}), \tag{1.1}$$

in mesh points z, where the coefficients $e^{(k)}(z)$ are independent of the mesh size parameter h. For elliptic equations in more than one dimension the derivation of such expansions usually relies on some kind of discrete maximum principle satisfied by the difference operator; see, e.g., [13, 1], and the literature cited there. Recently, it has been shown in [6, 7], and [8] how one can obtain error expansions for the Ritz projection method applied to the Dirichlet problem

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = b \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

on a convex polygonal domain $\Omega \subset \mathbb{R}^2$. For linear finite elements on a uniform triangulation there holds

$$u_h(z) = u(z) + h^2 e^{(1)}(z) + R_h(z; u), \tag{1.3}$$

in nodal points z, where $R_h(z; u) = O\left(h^3 \ln \frac{1}{h}\right)$, provided the solution u is sufficiently smooth. The proof uses finite element techniques and, therefore, also applies to some cases where no strict maximum principle is available.

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In the present paper we improve and extend the basic result (1.3) in several respects. The remainder term is shown to be globally of the order $O(h^{\min[4,2+\pi/\omega]})$, where $\omega \in [\pi/3, \pi)$ is the maximum interior angle of $\partial\Omega$. In the interior of Ω the order is at least $O(h^4)$. In order to allow for more flexibility in approximating general polygonal domains, we also consider triangulations which are only piecewise uniform with respect to some macro-triangulation. Then, the expansion (1.3) remains valid with a remainder term of the order $O\left(h^4 \ln\frac{1}{h}\right)$, in the nodal points having positive distance from the vertices of the macro-triangles. Finally, in the case of a curved boundary we show that extrapolation may increase the order of the scheme to $O\left(h^3 \ln\frac{1}{h}\right)$, at least in the interior of Ω. These theoretical results are confirmed by some numerical tests which are reported in Sect. 6.

The key to our results is an exact representation formula for the error $(u-u_h)(z)$ given in [7]; since this source is not generally available the argument is repeated in Sect. 2 in full detail. The estimates of the remainder terms are based on sharp L^1-error estimates for discrete Green functions which are proven in the Appendix by using the methods of [4], and [9].

The presence of the asymptotic expansion (1.3) justifies the use of Richardson extrapolation to the limit $h=0$, for increasing the second-order accuracy of linear finite elements at least to order three, or four. For example, let T_h be a (locally) uniform triangulation of a smoothly bounded domain and let the triangulation $T_{h/2}$ be generated from T_h by dividing each triangle as usual into four congruent subtriangles. Then the known error behavior of linear finite elements,

$$u_h(z)=u(z)+O\left(h^2 \ln\frac{1}{h}\right),$$

for $z \in \Omega$, may be improved to

$$\frac{1}{3}(4u_{h/2}-u_h)(z)=u(z)+O\left(h^3 \ln\frac{1}{h}\right),$$

for interior nodal points z belonging to T_h. A similar result can also be obtained for the approximation of the gradient ∇u.

In the following, $L^p(\Omega)$, $1 \leq p \leq \infty$, and $H^m(\Omega)$, $H_0^m(\Omega)$, $W^{m,p}(\Omega)$, $m \in \mathbb{N}$, are the usual Lebesgue and Sobolev spaces, respectively. The symbol c is used for a generic positive constant which may vary with the context but is always independent of the mesh size h.

2. Error Representation on a General Triangulation

We consider the model Dirichlet problem

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u=b \quad \text{on} \quad \partial\Omega, \tag{2.1}$$

where Ω is a bounded Lipschitz domain in $\mathbb{R}^2$, and f and b are smooth functions. For the following, the solution of (2.1) is assumed to be sufficiently smooth, say $u \in C^{2n}(\bar{\Omega})$, for some $n \geq 1$.

Let $T_h = \{K\}$ be a finite regular triangulation of Ω of width h with all its boundary vertices on $\partial\Omega$. Here, the term "regular" means that the triangles $K \in T_h$ meet only in entire common sides or in vertices, and that each $K \in T_h$ contains a circle of radius ch and is contained in a circle of radius $\frac{1}{c}h$, with a constant c independent of K and h. Below, we shall mostly consider "uniform" triangulations, generated by a set of three direction vectors. Corresponding to T_h, we define the following finite element spaces

$$S_h = \{v_h \in C(\Omega_h) \mid v_h \text{ linear on each } K \in T_h\},$$
$$S_h^0 = \{v_h \in S_h \mid v_h \equiv 0 \text{ on } \partial\Omega_h\}, \qquad \Omega_h = \cup\{K \in T_h\}.$$

The Ritz projection $u_h \in S_h$ of the solution u of (2.1) is determined by the conditions $u_h(P) = b(P)$, for nodal points $P \in \partial\Omega$, and

$$\int_{\Omega_h} \nabla u_h \nabla \varphi_h dx = \int_{\Omega_h} f \varphi_h dx, \quad \text{for all} \quad \varphi_h \in S_h^0. \tag{2.2}$$

We shall compare u_h with the piecewise linear interpolant $i_h u \in S_h$ of u. To this end, for any fixed $z \in \Omega_h$, let $g_h^z \in S_h^0$ be the discrete Green function defined by

$$\int_{\Omega_h} \nabla \varphi_h \nabla g_h^z dx = \varphi_h(z), \quad \text{for all} \quad \varphi_h \in S_h^0. \tag{2.3}$$

In the following we suppress the subscript z. Since $u_h - i_h u \in S_h^0$, there holds

$$(u_h - i_h u)(z) = \int_{\Omega_h} \nabla(u_h - i_h u) \nabla g_h dx = \int_{\Omega_h} \nabla(u - i_h u) \nabla g_h dx = \sum_{K \in T_h} I_k, \tag{2.4}$$

where

$$I_k = \int_K \nabla(u - i_h u) \cdot \nabla g_h dx = \int_{\partial K} (u - i_h u) n \cdot \nabla g_h ds.$$

(If $K \not\subset \bar{\Omega}$, u is thought to be smoothly extended to all of $\mathbb{R}^2$.)

Let $K \in T_h$ be an arbitrary fixed triangle. We use the notation A for its area, and $P_i (i = 1, 2, 3)$ for its vertices in counter-clockwise ordering.

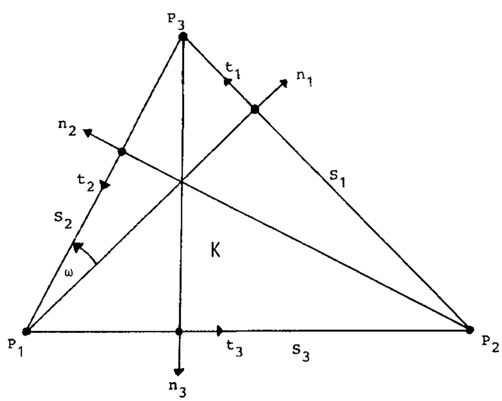

Fig. 1. S_i side of K opposite to P_i, h_i length of S_i, n_i outer normal unit vector along S_i, t_i tangent unit vector along S_i, D_i directional derivative along S_i, N_i nodal basis function, $N_i(P_j) = \delta_{ij}$

Lemma 1. *On the triangle K, there holds*

$$\sum_{i=1}^{3} \nabla N_i \equiv 0, \tag{2.5}$$

$$\nabla N_i = -\frac{h_i}{2A} n_i \quad (i=1,2,3), \tag{2.6}$$

$$t_{i+1} \cdot n_i = -\frac{2A}{h_i h_{i+1}}, \qquad t_i \cdot n_{i+1} = \frac{2A}{h_i h_{i+1}} \quad (i=1,2,3), \tag{2.7}$$

where the index $i+1$ *is used* mod(3).

Proof. (2.5) follows from $\sum_{i=1}^{3} N_i \equiv 1$ on K. Since $N_i(P_j) = \delta_{ij}$, there holds $D_i N_i \equiv 0$ and, consequently, $\nabla N_i \equiv -H_i^{-1} n_i$, where H_i denotes the height of P_i over S_i. Hence, observing $A = \frac{1}{2} H_i h_i$, we obtain (2.6). Finally, in view of the relations $H_1 = h_2 \cos\omega$ and $\cos\omega = t_2 \cdot n_1$, there holds $A = \frac{1}{2} h_1 h_2 t_2 \cdot n_1$, which implies the first identity in (2.7) for $i=1$. The others follow similarly. □

We split I_K as follows

$$I_K = \sum_{i=1}^{3} I_K^{(i)}, \qquad I_K^{(i)} = \int_{S_i} (u - i_h u) n_i \cdot \nabla g_h ds.$$

On S_1, we have, using (2.5) and (2.6),

$$\begin{aligned} n_1 \cdot \nabla g_h &\equiv \sum_{i=1}^{3} g_h(P_i) n_1 \cdot \nabla N_i \\ &\equiv \{g_h(P_1) - g_h(P_2)\} n_1 \cdot \nabla N_1 + \{g_h(P_3) - g_h(P_2)\} n_1 \cdot \nabla N_3 \\ &\equiv -h_3 D_3 g_h n_1 \cdot \nabla N_1 + h_1 D_1 g_h n_1 \cdot \nabla N_3 \\ &\equiv \frac{h_1 h_3}{2A} \{D_3 g_h - n_1 \cdot n_3 D_1 g_h\}. \end{aligned} \tag{2.8}$$

Further, by the Euler-MacLaurin formula, there holds

$$\int_{S_1} (u - i_h u) ds = \sum_{k=1}^{n-1} \beta_k h_1^{2k} \int_{S_1} D_1^{2k} u ds + h_1^{2n} \int_{S_1} \beta_n(s) D_1^{2n} u ds, \tag{2.9}$$

where $\beta_k (k=1, \ldots, n-1)$ are certain constants independent of h_1, and $\beta_n \in C_0^1(S_1) \cap C^2(S_1)$. Combining (2.8) and (2.9), we obtain the identities

$$\begin{aligned} I_K^{(1)} &= n_1 \cdot \nabla g_h \int_{S_1} (u - i_h u) ds \\ &= \frac{h_1 h_3}{2A} \{D_3 g_h - n_1 \cdot n_3 D_1 g_h\} \sum_{k=1}^{n-1} \beta_k h_1^{2k} \int_{S_1} D_1^{2k} u ds \\ &\quad + \frac{h_1 h_3}{2A} \{D_3 g_h - n_1 \cdot n_3 D_1 g_h\} h_1^{2n} \int_{S_1} \beta_n(s) D_1^{2n} u ds. \end{aligned}$$

The following considerations concern the case $n \geqq 2$.

Lemma 2. *For $v \in C^1(K)$, there holds*

$$h_1 \int_{S_3} v\,ds - h_3 \int_{S_1} v\,ds = \frac{h_1 h_2 h_3}{2A} \int_K D_2 v\,dx. \tag{2.11}$$

Proof. By the theorem of Gauss, there holds

$$\int_K D_2 v\,dx = \int_{\partial K} t_2 \cdot n v\,ds = \int_{S_1} t_2 \cdot n_1 v\,ds + \int_{S_3} t_2 \cdot n_3 v\,ds,$$

and hence, observing (2.7),

$$\int_K D_2 v\,dx = -\frac{2A}{h_1 h_2} \int_{S_1} v\,ds + \frac{2A}{h_2 h_3} \int_{S_3} v\,ds. \quad \square$$

We apply (2.11) to the integrals $\int_{S_1} D^{2k} u\,ds$ in (2.10), to obtain

$$\begin{aligned} I_K^{(1)} = {} & \frac{h_1 h_3}{2A} D_3 g_h \sum_{k=1}^{n-1} \beta_k h_1^{2k} \left\{ \frac{h_1}{h_3} \int_{S_3} D_1^{2k} u\,ds - \frac{h_1 h_2}{2A} \int_K D_2 D_1^{2k} u\,dx \right\} \\ & - \frac{h_1 h_3}{2A} n_1 \cdot n_3 D_1 g_h \sum_{k=1}^{n-1} \beta_k h_1^{2k} \int_{S_1} D_1^{2k} u\,ds \\ & + \frac{h_1 h_3}{2A} \{ D_3 g_h - n_1 \cdot n_3 D_1 g_h \} h_1^{2n} \int_{S_1} \beta_n(s) D_1^{2n} u\,ds \\ = {} & \sum_{k=1}^{n-1} \beta_k \left\{ \frac{h_1^{2k+2}}{2A} \int_{S_3} D_3 g_h D_1^{2k} u\,ds - \frac{h_1^{2k+1} h_3}{2A} n_1 \cdot n_3 \int_{S_1} D_1 g_h D_1^{2k} u\,ds \right. \\ & \left. - \frac{h_1^{2k+2} h_2 h_3}{4A^2} \int_K D_3 g_h D_2 D_1^{2k} u\,dx \right\} \\ & + \frac{h_1^{2n+1} h_3}{2A} \int_{S_1} \beta_n(s) \{ D_3 g_h - n_1 \cdot n_3 D_1 g_h \} D_1^{2n} u\,ds. \end{aligned}$$

For the area integrals there holds, in view of (2.7),

$$\begin{aligned} \int_K D_3 g_h D_2 D_1^{2k} u\,dx = {} & -\int_K g_h D_3 D_2 D_1^{2k} u\,dx + \int_{S_1} t_3 \cdot n_1 g_h D_2 D_1^{2k} u\,ds \\ & + \int_{S_2} t_3 \cdot n_2 g_h D_2 D_1^{2k} u\,ds \\ = {} & -\int_K g_h D_3 D_2 D_1^{2k} u\,dx + \frac{2A}{h_1 h_3} \int_{S_1} g_h D_2 D_1^{2k} u\,ds \\ & - \frac{2A}{h_2 h_3} \int_{S_2} g_h D_2 D_1^{2k} u\,ds. \end{aligned}$$

Inserting this into the above equation for $I_K^{(1)}$ eventually gives us the identity (valid for $n \geqq 1$)

$$
\begin{aligned}
I_K^{(1)} = \sum_{k=1}^{n-1} \beta_k \frac{h_1^{2k+1}}{2A} \Big\{ & h_1 \int_{S_3} D_3 g_h D_1^{2k} u\, ds - h_3 n_1 \cdot n_3 \int_{S_1} D_1 g_h D_1^{2k} u\, ds \\
& - h_2 \int_{S_1} g_h D_2 D_1^{2k} u\, ds + h_1 \int_{S_2} g_h D_2 D_1^{2k} u\, ds \Big\} \\
& + \sum_{k=1}^{n-1} \beta_k \frac{h_1^{2k+2} h_2 h_3}{4A^2} \int_K g_h D_3 D_2 D_1^{2k} u\, dx \\
& + \frac{h_1^{2n+1} h_3}{2A} \int_{S_1} \beta_n(s) \{ D_3 g_h - n_1 \cdot n_3 D_1 g_h \} D_1^{2n} u\, ds .
\end{aligned}
$$

The corresponding identities for $I_K^{(2)}$ and $I_K^{(3)}$ are obtained from (2.12) by shifting the indices 1, 2, 3→2, 3, 1, and 1, 2, 3→3, 1, 2, respectively. The representation

$$
u_h(z) = i_h u(z) + \sum_{K \in T_h} \sum_{i=1}^{3} I_K^{(i)} \tag{2.13}
$$

contains all the information we will need about the behavior of the discretization error $u_h - u$.

3. Error Expansion on a Uniform Triangulation

In the following, let Ω be a convex polygonal domain and T_h a uniform triangulation generated by the tangential unit vectors t_1, t_2, and t_3. We shall evaluate the error representation (2.13) for this special situation. To this end, let h be some mesh size parameter with $0 < h \leqq h_0 < 1$, and let $A = \alpha h^2$ and $h_i = \lambda_i h$ (i=1,2,3).

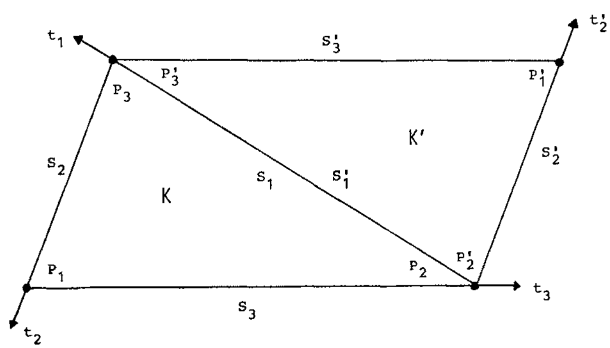

Fig. 2

If the identity (2.12) for $I_K^{(1)}$ and its analogues for $I_K^{(2)}$, $I_K^{(3)}$ are summed up for all triangles $K \in T_h$, the following simplifications occur.

(i) All line integrals in the first sum of the type

$$\int_{S_i} D_i g_h \ldots ds, \qquad \int_{S_i} g_h \ldots ds$$

over interior sides S_i are cancelled, since $D_i = -D_i'$. The remaining boundary integrals also vanish since $g_h \equiv 0$ on $\partial\Omega$.

(ii) The area integrals combine to

$$\Sigma_h^{(n)}(z; u) = \sum_{k=1}^{n-1} h^{2k} e_h^{(k)}(z; u),$$

where

$$e_h^{(k)}(z; u) = \frac{\lambda_1 \lambda_2 \lambda_3}{4\alpha^2} \beta_k \int_\Omega g_h^z \left[D_1 D_2 D_3 \sum_{i=1}^{3} \lambda_i^{2k+1} D_i^{2k-1} \right] u dx.$$

(iii) The remainder terms add up to

$$R_h^{(n)}(z; u) = h^{2n} \sum_{i=1}^{3} \frac{\lambda_i^{2n+1} \lambda_{i+2}}{2\alpha} \Phi_i(z; u),$$

where

$$\Phi_i(z; u) = \sum_{K \in T_h} \int_{S_i} \beta_n(s) \{ D_{i+2} g_h^z - n_i \cdot n_{i+2} D_i g_h^z \} D_i^{2n} u ds,$$

and the indices are used mod(3). Taking again into account that $D_i = -D_i'$, and that $g_h \equiv 0$ on $\partial\Omega$, the sums $\Phi_i(z; u)$ reduce to

$$\Phi_i(z; u) = \sum_{K \in T_h} \int_{S_i} \beta_n D_{i+2} g_h^z D_i^{2n} u ds.$$

Let P_m denote the space of all polynomials of degree less or equal m. From the above observations we immediately obtain the following result.

Lemma 3. *If $u \in P_{2n+1}$, then there holds*

$$u_h(z) = i_h u(z) + \sum_{k=1}^{n-1} h^{2k} e_h^{(k)}(z; u), \quad \text{for} \quad z \in \Omega. \tag{3.1}$$

As a particular consequence of Lemma 3 we see that on a three-directional triangulation the Ritz projection of a cubic polynomial coincides with its interpolant.

Next, we state the basic result of this section.

Theorem 1. *If $u \in C^{2n+\varepsilon}(\bar{\Omega})$, for some $\varepsilon \geq 0$. Then, there holds*

$$u_h(z) = i_h u(z) + \sum_{k=1}^{n-1} h^{2k} e_h^{(k)}(z; u) + R_h^{(n)}(z; u), \tag{3.2}$$

where the remainder term is uniformly of the order $O\left(h^{2n} \ln \frac{1}{h}\right)$ if $\varepsilon = 0$, and $O(h^{2n})$ if $\varepsilon > 0$.

Proof. In view of the foregoing discussion, it remains to estimate the quantities

$$\Phi_i(z; u) = \sum_{K\in T_h} \int_{S_i} \beta_n D_{i+2} g_h D_i^{2n} u\, ds.$$

To this end, we use the regularized Green function $\tilde{g} = \tilde{g}^z \in H_0^1(\Omega) \cap H^2(\Omega)$ defined in the Appendix. In view of the relation

$$\int_\Omega \nabla \varphi_h \cdot \nabla \tilde{g}\, dx = \varphi_h(z), \quad \text{for all} \quad \varphi_h \in S_h^0,$$

the function $g_h \in S_h^0$ may be considered as the Ritz projection of $\tilde{g}$.

For $\tilde{g}$ we have the estimate (valid on a general regular triangulation)

$$\|\nabla(\tilde{g} - g_h)\|_{L^1} + h\|\nabla^2 \tilde{g}\|_{L^1} = O\left(h \ln \frac{1}{h}\right), \tag{3.3}$$

uniformly for $z \in \Omega$; see [4] and the Appendix. Since $\tilde{g} \in H^2(\Omega)$, on a uniform triangulation there holds

$$\sum_{K\in T_h} \int_{S_i} \beta_n D_{i+2} \tilde{g} D_i^{2n} u\, ds = \sum_{S_i \subset \partial\Omega_h} \int_{S_i} \beta_n D_{i+2} \tilde{g} D_i^{2n} u\, ds. \tag{3.4}$$

In view of (3.4), we conclude that, for $u \in C^{2n}(\bar{\Omega})$,

$$|\Phi_i(z; u)| \leqq c \sum_{K\in T_h} \int_{\partial K} |\nabla(\tilde{g} - g_h)|\, ds + c \int_{\partial\Omega_h} |\nabla \tilde{g}|\, ds. \tag{3.5}$$

Hence, by a trace theorem,

$$|\Phi_i(z; u)| \leqq c\{h^{-1} \|\nabla(\tilde{g} - g_h)\|_{L^1} + \|\nabla^2 \tilde{g}\|_{L^1}\}. \tag{3.6}$$

This clearly implies the desired representation (3.2), for $\varepsilon = 0$.

Next, let $u \in C^{2n+\varepsilon}(\bar{\Omega})$, for some $\varepsilon > 0$. In order to remove the logarithm, we have to treat the remainder terms $\Phi_i(z; u)$ more carefully. Let $p \in P_{2n}$ be the $2n$-th order Taylor polynomial of u, at the point z, satisfying

$$|D_i^{2n}(u - p)(x)| \leqq c|x - z|^\varepsilon, \quad \text{for} \quad x \in \Omega. \tag{3.7}$$

In view of Lemma 3, there holds

$$u_h(x) = i_h u(x) + \sum_{k=1}^{n-1} h^{2k} e_h^{(k)}(x; u) + R_h^{(n)}(x; u - p). \tag{3.8}$$

We shall use the weight function

$$\sigma(x) = \sigma_z(x) = (|x - z|^2 + \kappa^2 h^2)^{1/2},$$

where the parameter $\kappa \geqq 1$ is chosen sufficiently large such that

$$\max_{K\in T_h} \left\{ \max_{x\in K} \sigma(x) \Big/ \min_{x\in K} \sigma(x) \right\} \leqq c, \tag{3.9}$$

uniformly for $z \in \Omega$; for further properties of $\sigma(x)$ see the Appendix. Then, in view of (3.4) and (3.7), there holds

$$|\Phi_i(z; u-p)| \leqq c \sum_{K \in T_h} \int_{\partial K} \sigma^\varepsilon |\nabla(\tilde{g}-g_h)| ds + c \int_{\partial \Omega_h} \sigma^\varepsilon |\nabla \tilde{g}| ds. \tag{3.10}$$

Using again a trace theorem and observing (3.9), we conclude from (3.10) that

$$|\Phi_i(z; u-p)| \leqq c\{h^{-1} \|\sigma^\varepsilon \nabla(\tilde{g}-g_h)\|_{L^1} + \|\sigma^\varepsilon \nabla^2 \tilde{g}\|_{L^1} + \varepsilon \|\sigma^{\varepsilon-1} \nabla \tilde{g}\|_{L^1}\}. \tag{3.11}$$

Then, observing that $\|\sigma^{\varepsilon-2}\|_{L^1} \leqq \frac{c}{\varepsilon}$, the assertion follows from the estimates

$$\|\sigma^{1+\varepsilon/2} \nabla(\tilde{g}-g_h)\|_{L^2} \leqq ch, \tag{3.12}$$

$$\|\sigma^{\varepsilon/2} \nabla \tilde{g}\|_{L^2} + \|\sigma^{1+\varepsilon/2} \nabla^2 \tilde{g}\|_{L^2} \leqq c, \tag{3.13}$$

which are proven in the Appendix. □

Next, we want to derive asymptotic expansions for the error $u_h - u$, with coefficients $e^{(k)}$ independent of h. To this end, we shall need more information about the regularity to be expected for the solution u of the Dirichlet problem (1.2) on a polygonal domain. Since Ω is convex, $u \in H^2(\Omega)$. Further, u is smooth on interior domains $\Omega_0 \subset\subset \Omega$. At the corner points z_j the regularity may be reduced depending on the corresponding interior angles ω_j. Let (r_j, θ_j) denote local polar coordinates at z_j. Then, u admits a representation of the form (see [12])

$$u = \sum_j \gamma_j s_j + \tilde{u}, \tag{3.14}$$

where $s_j(r_j, \theta_j) = r_j^{\pi/\omega_j} \sin\left(\frac{\pi}{\omega_j}\theta_j\right)$, for $\omega_j \neq \frac{\pi}{2}$, and $\tilde{u} \in C^{2+\varepsilon}(\bar{\Omega})$ with some $\varepsilon > 0$. In the special case $\omega_j = \frac{\pi}{2}$, one has

$$s_j(r_j, \theta_j) = r_j^2 \left(\ln r_j \sin 2\theta_j + \theta_j \cos 2\theta_j + \frac{\pi}{2} \sin^2 \theta_j\right),$$

near the corner points z_j.

On a general regular triangulation of Ω, there hold the convergence estimates

$$\|u - u_h\|_{L^\infty} = \begin{cases} O(h^{1+\alpha}), & \text{for} \quad 0 \leqq \alpha < 1, \\ O\left(h^2 \ln \frac{1}{h}\right), & \text{for} \quad \alpha = 1, \end{cases} \tag{3.15}$$

for $u \in C^{1+\alpha}(\bar{\Omega})$; this may be proven, for instance, by the methods of [4, 9]. If the triangulation is uniform, we obtain the following

Corollary 2. *If $u \in C^{2+\varepsilon}(\bar{\Omega})$, for some $\varepsilon > 0$, then there holds*

$$\|u - u_h\|_{L^\infty} = O(h^2). \tag{3.16}$$

This directly follows from Theorem 1, with $n=1$, and the interpolation estimate

$$\|u-i_h u\|_{L^\infty}=O(h^2), \tag{3.17}$$

for $u\in W^{2,\infty}(\Omega)$. For the weak solution of (1.2), there holds

$$\|u-u_h\|_{L^\infty(\Omega_0)}=O(h^2), \tag{3.18}$$

on any subdomain $\Omega_0\subset\Omega$ having positive distance from the corner points of $\partial\Omega$. This local result also follows from the proof of Theorem 1, if one multiplies the Taylor polynomial p of u at the point $z\in\Omega_0$ by some smooth cut-off function χ, separating Ω_0 from the corner points of $\partial\Omega$, and observes that $u\in H^2(\Omega)\cap C^{2+\varepsilon}(\overline{\Omega\cap\operatorname{supp}(\chi)})$.

For a uniform triangulation with rectangular triangles the above logarithm free error estimate has been proven in [5], and earlier by finite difference methods in [2]. The estimate (3.18) even holds true if the triangulation is only required to be uniform in some subdomain containing Ω_0. As a further by-product of Theorem 1 we obtain the following super convergence result.

Corollary 3. *If $u\in C^{4+\varepsilon}(\bar\Omega)$, for some $\varepsilon>0$, and if the uniform triangulation consists of equi-lateral triangles, then there holds*

$$(u-u_h)(z)=O(h^4), \tag{3.19}$$

uniformly in nodal points $z\in\Omega$.

On the equi-lateral triangulation we have

$$\lambda_1=\lambda_2=\lambda_3,\qquad D_1+D_2+D_3=0,$$

and consequently, $e_h^{(1)}\equiv 0$. Then, the result (3.19) follows from (3.2), with $n=2$. In this special situation the finite element scheme may be interpreted as a fourth-order Hermitian finite difference approximation on the equi-lateral triangular mesh; see [3; Chap. V].

For the following, we denote by $g=g^z$ the Green function of problem (1.2) corresponding to a point $z\in\Omega$, and set

$$w^{(k)}(x;u)=\frac{\lambda_1\lambda_2\lambda_3}{4\alpha^2}\beta_k D_1D_2D_3\sum_{i=1}^{3}\lambda_i^{2k+1}D_i^{2k-1}u(x).$$

Lemma 4. *The coefficients $e_h^{(k)}$ in the representation (3.2) are the Ritz projections of the functions*

$$e^{(k)}(z;u)=\int_\Omega w^{(k)}(x;u)g^z(x)\,dx,$$

which are the solutions of the Dirichlet problem

$$-\Delta v=w^{(k)}\quad in\quad \Omega,\qquad v=0\quad on\quad \partial\Omega. \tag{3.20}$$

Proof. The assertion is an easy consequence of the symmetry of the Green functions, $g^z(x)=g^x(z)$ and $g_h^z(x)=g_h^x(z)$. □

Combining Lemma 4 with Theorem 1, we obtain the following basic expansion result.

Theorem 2. *Let $u \in C^{4+\varepsilon}(\bar{\Omega})$, for some $\varepsilon > 0$. Then, there holds*

$$u_h(z) = u(z) + h^2 e^{(1)}(z; u) + O(h^4) \tag{3.21}$$

in nodal points $z \in \Omega$ having positive distance from the corner points of $\partial\Omega$. If the maximum interior angle ω of $\partial\Omega$ differs from $\frac{\pi}{2}$, then

$$u_h(z) = u(z) + h^2 e^{(1)}(z; u) + O(h^{\min[4, 2+\pi/\omega]}) \tag{3.22}$$

holds uniformly in nodal points $z \in \Omega$.

Proof. By Theorem 1, we have

$$\begin{aligned} u_h(z) &= i_h u(z) + h^2 e_h^{(1)}(z; u) + O(h^4) \\ &= i_h u(z) + h^2 e^{(1)}(z; u) + h^2 \{e_h^{(1)}(z; u) - e^{(1)}(z; u)\} + O(h^4). \end{aligned}$$

By Lemma 4, $e_h^{(1)}$ is the Ritz projection of the function $e^{(1)}$ satisfying

$$-\Delta e^{(1)} = w^{(1)} \quad \text{in} \quad \Omega, \qquad e^{(1)} = 0 \quad \text{on} \quad \partial\Omega.$$

In view of the representation (3.14), we may conclude that $e^{(1)} \in C^{1+\alpha}(\bar{\Omega})$ with $\alpha = \frac{\pi}{\omega} - 1$, for $\frac{\pi}{2} < \omega < \pi$, and $e^{(1)} \in C^{2+\varepsilon}(\bar{\Omega})$ with some $\varepsilon > 0$, for $\frac{\pi}{3} \leqq \omega < \frac{\pi}{2}$.

Hence, by the error estimates (3.15) and (3.16),

$$\|e^{(1)}(\cdot; u) - e_h^{(1)}(\cdot; u)\|_{L^\infty} = O(h^{\min[2, \pi/\omega]}),$$

which implies (3.22). The local expansion (3.21) follows in the same way by the local error estimate (3.18). □

The case $\omega = \frac{\pi}{2}$ is somewhat exceptional since here the regularity of $e^{(1)}$ is determined by local compatibility of the data. If

$$\frac{\lambda_1 \lambda_2 \lambda_3}{4\alpha^2} \beta_1 D_1 D_2 D_3 \sum_{i=1}^{3} \lambda_i^3 D_i u(P_j) = 0 \tag{3.23}$$

in the corner points, then $e^{(1)} \in C^{2+\varepsilon}(\bar{\Omega})$, and (3.22) carries over to that case; otherwise the remainder term is of the reduced order $O\left(h^4 \ln \frac{1}{h}\right)$. Condition (3.23) may always be satisfied by local modification of the data f, b.

A more careful analysis of the difference $e_h^{(1)} - e^{(1)}$ yields an extended expansion of the form

$$u_h(z) = u(z) + h^2 e^{(1)}(z; u) + h^4 \tilde{e}^{(2)}(z; u) + O(h^{\min[6, 4+\pi/\omega]}), \tag{3.24}$$

in *interior* nodal points z, where

$$\tilde{e}^{(2)}(z;u)=e^{(2)}(z;u)+e^{(1)}(z;e^{(1)}(\cdot;u)).$$

We skip the very technical details.

Finally, we like to emphasize the fact that the proofs of our expansion results do *not* rely on any kind of discrete maximum principle, in contrast to most of the related results for finite difference schemes. In fact, the triangulations T_h are allowed to consist of triangles with obtuse angles in which case the discrete analogues of the model problem (1.2) are not of monotone type. This implies that, by a simple coordinate transformation, our results in a certain sense carry over to the case of a more general elliptic operator

$$A=-\sum_{i,j=1}^{2} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j},$$

with constant coefficients, which is not necessarily separable.

4. Error Expansion on a Piecewise Uniform Triangulation

Let the polygonal domain Ω be subdivided into a finite number of macro-triangles $\Omega^{(j)}$, and let $T_h^{(j)}$ be uniform triangulations of the $\Omega^{(j)}$, such that $T_h=\bigcup_j T_h^{(j)}$ is a regular triangulation of Ω.

Summing the identity (2.12) for $I_K^{(1)}$, and its analogues for $I_K^{(2)}, I_K^{(3)}$, separately for $K\in T_h^{(j)}$, we obtain the following.

(i) The line integrals of the type

$$\int_{S_i} D_i g_h \ldots ds, \qquad \int_{S_i} g_h \ldots ds,$$

over interior sides S_i in $\Omega^{(j)}$ are calcelled. The contributions from sides $S_i \subset \partial\Omega^{(j)}$ add up to

$$\begin{aligned} L_h^{(k,j)}(z;u)=\frac{\beta_k}{2\alpha}\sum_{i=1}^{3}\Big\{&\lambda_{i+1}^{2k+2}\int_{\Gamma_i} D_i g_h D_{i+1}^{2k} u\,ds \\ &-\lambda_i^{2k+1}\lambda_{i+2} n_i\cdot n_{i+2}\int_{\Gamma_i} D_i g_h D_i^{2k} u\,ds \\ &-\lambda_i^{2k+1}\lambda_{i+1}\int_{\Gamma_i} g_h D_{i+1} D_i^{2k} u\,ds+\lambda_{i+2}^{2k+2}\int_{\Gamma_i} g_h D_i D_{i+2}^{2k} u\,ds\Big\}, \end{aligned}$$

where $\lambda_i=\lambda_i^{(j)}, \alpha=\alpha^{(j)}, D_i=D_i^{(j)}$ are the characteristic quantities of the triangulation $T_h^{(j)}$, and $\Gamma_i=\Gamma_i^{(j)}$ is the part of $\partial\Omega^{(j)}$ corresponding to the direction $t_i^{(j)}$.

(ii) The area integrals and the remainder terms combine to

$$\Sigma_h^{(n,j)}(z;u)=\sum_{k=1}^{n-1} h^{2k} e_h^{(k,j)}(z;u)$$

and

$$R_h^{(n,j)}(z;u) = h^{2n} \sum_{i=1}^{3} \frac{\lambda_i^{2n+1}\lambda_{i+2}}{2\alpha} \Phi_i^{(j)}(z;u),$$

where the quantities $e_h^{(k,j)}(z;u)$ and $\Phi_i^{(j)}(z;u)$ are of the same form as $e_h^{(k)}(z;u)$ and $\Phi_i(z;u)$, respectively, but now corresponding to the triangulation $T_h^{(j)}$ of $\Omega^{(j)}$.

Using the above notation, we introduce the functions

$$e_h^{(k)}(z;u) = \sum_j \{e_h^{(k,j)}(z;u) + L_h^{(k,j)}(z;u)\},$$

and

$$R_h^{(n)}(z;u) = \sum_j R_h^{(n,j)}(z;u).$$

Then, there holds the error representation

$$u_h(z) = i_h u(z) + \sum_{k=1}^{n-1} h^{2k} e_h^{(k)}(z;u) + R_h^{(n)}(z;u). \tag{4.1}$$

Analogously to the preceding section we define the continuous coefficients $e^{(k)}(z;u)$, $e^{(k,j)}(z;u)$, and $L^{(k,j)}(z;u)$ by replacing g_h by g in the terms $e_h^{(k,j)}(z;u)$, and $L_h^{(k,j)}(z;u)$. With this notation we have the following.

Theorem 3. *Let $u \in C^{4+\varepsilon}(\bar{\Omega})$, for some $\varepsilon > 0$. Then there holds*

$$u_h(z) = u(z) + h^2 e^{(1)}(z;u) + O\left(h^4 \ln\frac{1}{h}\right), \tag{4.2}$$

in nodal points $z \in \Omega$ having positive distance from the vertices of the macro-triangulation.

We note that the expansion (4.2) even holds in nodal points on the macro-edges where the triangulation is not uniform.

Proof. From the proof of Theorem 1 we directly see that the remainder term in (4.1) is again of the order

$$R_h^{(2)}(z;u) = O\left(h^4 \ln\frac{1}{h}\right) \tag{4.3}$$

uniformly for $z \in \Omega$, if $u \in C^4(\bar{\Omega})$. Indeed, the only difference to the case of a globally uniform triangulation is that now, on the right hand side in (3.4), the sum has to be extended over all edges $S_i \in \bigcup_j \partial\Omega^{(j)}$.

In view of (4.3), it remains to estimate the difference $e^{(1)} - e_h^{(1)}$. First, we consider the area integral $v_h = \sum_j e_h^{(1,j)}(\cdot;u) \in S_h$ which is again the Ritz projection of the function $v = \sum_j e^{(1,j)}(\cdot;u)$. Clearly, v is the solution of the Dirichlet problem (3.20) with a right hand side $w \in L^2(\Omega)$ being piecewise C^ε, $w \in C^\varepsilon(\overline{\Omega^{(j)}})$. From this one may infer that $v \in H^2(\Omega)$ and piecewise $C^{2+\varepsilon}$, $v \in C^{2+\varepsilon}(\overline{\Omega^{(j)} \setminus B^{(j)}})$, where $B^{(j)}$ denotes some neighborhood of the vertices of $\Omega^{(j)}$. Since the triangulation T_h matches the macro-edges, we obtain, by a standard argument, that

$$\sum_j \{e^{(1,j)}(z;u) - e_h^{(1,j)}(z;u)\} = O\left(h^2 \ln \frac{1}{h}\right), \tag{4.4}$$

for points $z \in \Omega \setminus B^{(j)}$.

Next, we compare the terms $L_h^{(1,j)}(z;u)$ and $L^{(1,j)}(z;u)$. That means that we have to consider line integrals over macro-edges Γ_i of the type

$$\int_{\Gamma_i} D_i(g_h - g) D_i^2 u \, ds \quad \text{and} \quad \int_{\Gamma_i} (g_h - g) D_{i+1} D_i^2 u \, ds .$$

Let us fix some macro-edge Γ_i with end points P, Q. Then, integration by parts yields

$$\int_{\Gamma_i} D_i(g_h - g) D_i^2 u \, ds = -\int_{\Gamma_i} (g_h - g) D_i^3 u \, ds + (g_h - g) D_i^2 u|_P^Q .$$

Hence, it remains to estimate quantities of the form

$$\int_{\Gamma} (g - g_h) \psi \, ds \quad \text{and} \quad (g - g_h)(P),$$

where Γ is a macro-edge, P an *interior* macro-vertex, and ψ stands for the trace of a $C^{1+\varepsilon}$-function.

In the Appendix, Lemma A3, we prove the local estimate

$$(g - g_h)(P) = O\left(h^2 \ln \frac{1}{h}\right), \tag{4.5}$$

for points $P \in \Omega$, having positive distance from z, and from the corner points of $\partial\Omega$. Further, we prove the "negative norm" estimate

$$\int_{\Gamma} (g - g_h) \psi \, ds = O\left(h^2 \ln \frac{1}{h}\right), \tag{4.6}$$

for any straight line $\Gamma \subset \Omega$ consisting of edges of the triangulation, where ψ is the trace of a $C^{1+\varepsilon}$-function, and $\operatorname{dist}(z, \{P, Q\}) > 0$. Combining the estimates (4.6) and (4.5) implies that

$$L^{(1,j)}(z;u) - L_h^{(1,j)}(z;u) = O\left(h^2 \ln \frac{1}{h}\right), \tag{4.7}$$

for points $z \in \Omega \setminus \bigcup_j B^{(j)}$. This completes the proof. $\square$

We note that the expansion (4.2) extends to all nodal points z along the boundary $\partial\Omega$ with the reduced order $O\left(h^3 \ln \frac{1}{h}\right)$ of the remainder term. To see this, we replace the estimate (4.6) by

$$\int_\Gamma (g-g_h)\psi\, ds \leqq c\|\nabla(g-g_h)\|_{L^1}, \tag{4.8}$$

and use a result from [4] (see also the Appendix),

$$\|\nabla(g-g_h)\|_{L^1} = O\left(h \ln\frac{1}{h}\right), \tag{4.9}$$

which holds uniformly for $z \in \Omega$, on a general regular triangulation. In the estimate (4.4), we remain with the order $O(h)$, uniformly for $z \in \Omega$, since $v \in H^2(\Omega)$. This together with (4.3) clearly implies the desired result.

As a by-product of Theorem 3 we obtain some information concerning the best possible order of convergence of linear finite elements. Let the macro-triangulation have only one interior vertex, P, and, using the notation of Sect. 2, let the edges be numbered such that $P = P_3^{(j)}$ for every $\Omega^{(j)}$ containing P. Then, from the proof of Theorem 3, we see that

$$u_h(z) = u(z) + h^2 g_h^z(P) \sum_j E_j(P) + O(h^2), \tag{4.10}$$

where

$$E_j(P) = \frac{1}{2\alpha}\left[(\lambda_2^4 D_2^2 u - \lambda_1^3 \lambda_3 n_1 \cdot n_3 D_1^2 u) - (\lambda_3^4 D_3^2 u - \lambda_2^3 \lambda_1 n_2 \cdot n_1 D_2^2 u)\right](P).$$

Consequently, if

$$\sum_j E_j(P) \neq 0,$$

the error in the nodal point $z = P$ behaves asymptotically not better than

$$u_h(P) = u(P) + O\left(h^2 \ln\frac{1}{h}\right), \tag{4.11}$$

since $g_h^P(P) \geqq c \ln\frac{1}{h}$. This proves that, in general, the Ritz projection onto linear finite elements does not allow for a better convergence estimate than (3.15), even if u is arbitrarily smooth; for a further discussion of this point we refer to [9] and [5].

5. Error Expansion on a Smoothly Bounded Domain

Let us finally consider the error representation on a domain Ω with a smooth boundary $\partial\Omega$. In general it is no longer possible to subdivide Ω by a piecewise uniform triangulation. Instead, let Ω_h^0 be an interior polygonal domain satisfying $\max_{x \in \partial\Omega} \operatorname{dist}(x, \Omega_h^0) = O(h)$, and let the triangulation T_h be uniform in Ω_h^0. On $B_h = \Omega_h \setminus \Omega_h^0$, the triangulation is only assumed to be regular in the usual sense.

Analogously to the preceding sections we have to discuss the terms in the error representation formula (2.13).

(i) As noted in Sect. 3, all line integrals in (2.13) not belonging to the remainder are cancelled on edges interior to Ω_h^0. Therefore, we have the contribution

$$L_h^{(k)}(z;u) = \sum_j{}' L_h^{(k,j)}(z;u),$$

where $L_h^{(k,j)}$ is defined as in Sect. 4, but now the index j refers to every triangle separately. The sum $\sum'$ extends over all triangles containing the edges $\Gamma_i \subset B_h \cup \partial\Omega_h^0$.

(ii) The sum over the area integrals in (2.13) results in an integral over Ω_h,

$$\begin{aligned}\tilde{e}_h^{(k)}(z;u) &= \beta_k \sum_{K\in T_h} \frac{\lambda_1\lambda_2\lambda_3}{4\alpha^2} \int_K g_h D_1 D_2 D_3 \sum_{i=1}^{3} \lambda_i^{2k+1} D_i^{2k-1} u\,dx \\ &= \int_{\Omega_h} g_h D_k(u)\,dx .\end{aligned}$$

Since the characteristic quantities D_i, λ_i, and α will be different for every $K \in B_h$, the functions $D_k(u)$ are only piecewise smooth near the boundary.

If we again define the function $e^{(k)}(z;u)$ by replacing g_h by g in the coefficient

$$e_h^{(k)}(z;u) = \tilde{e}_h^{(k)}(z;u) + L_h^{(k)}(z;u),$$

we obtain the following local error expansion.

Theorem 4. *For $u \in C^4(\bar{\Omega})$, there holds*

$$u_h(z) = u(z) + h^2 e^{(1)}(z;u) + O\left(h^3 \ln\frac{1}{h}\right) \tag{5.1}$$

uniformly in interior nodal points $z \in \Omega_0 \subset\subset \Omega$.

Proof. From the proof of Theorem 1 we see again that the remainder term $R_h^{(2)}(z;u)$ is of the order

$$R_h^{(2)}(z;u) = O\left(h^4 \ln\frac{1}{h}\right), \tag{5.2}$$

uniformly for $z \in \Omega_h$. In the present case of a locally uniform triangulation the sum on the right hand side in (3.4) has to be extended over all edges $\Gamma_i \subset B_h = \Omega_h \backslash \Omega_h^0$. Since B_h has width $O(h)$, we may conclude that

$$\sum_{S_i \subset B_h} \int_{S_i} |\nabla\tilde{g}|\,ds \leqq c\left\{h^{-1} \int_{B_h} |\nabla\tilde{g}|\,dx + \int_{B_h} |\nabla^2\tilde{g}|\,dx\right\} \leqq c\|\nabla^2\tilde{g}\|_{L^1},$$

which again leads to the crucial estimate (3.6).

Hence, it remains to estimate the difference $e^{(1)}(z;u) - e_h^{(1)}(z;u)$. For the area integrals we again use the fact that $e_h^{(1)}(z;u)$ is the Ritz projection of

$$\int_\Omega g D_1(u)\,dx,$$

which is the solution of the problem

$$-\Delta v = D_1(u) \quad \text{in} \quad \Omega, \qquad v=0 \quad \text{on} \quad \partial\Omega.$$

As noted above, $D_1(u)$ is merely bounded. Hence, we may use a result from [4], Satz 2,

$$\|u-u_h\|_{L^\infty} \leqq ch^2\left(\ln\frac{1}{h}\right)^2 \|\Delta u\|_{L^\infty},$$

to obtain

$$\|\tilde{e}_h^{(1)}(\cdot\,;u)-\tilde{e}^{(1)}(\cdot\,;u)\|_{L^\infty} \leqq \left\|\tilde{e}_h^{(1)}(\cdot\,;u)-\int_\Omega gD_1(u)dx\right\|_{L^\infty} + O\left(h^2\ln\frac{1}{h}\right) = O\left(h^2\left(\ln\frac{1}{h}\right)^2\right). \tag{5.3}$$

Next, we consider the line integrals collected in $L^{(1)}(z;u)-L_h^{(1)}(z;u)$. Let P_i, Q_i denote the endpoints of the line segments Γ_i. Using integration by parts, we obtain

$$L^{(1)}(z;u)-L_h^{(1)}(z;u) = \sum_{\Gamma_i}{}' \int_{\Gamma_i} (g-g_h)D^3u\,ds + \sum_{\Gamma_i}{}'(g-g_h)D^2u|_{P_i}^{Q_i}. \tag{5.4}$$

In $\sum_{\Gamma_i}'$ all contributions over the edges in $B_h \cup \partial\Omega_h^0$ are added up, and $D^k u$ contains appropriate combinations of k-th order directional derivatives of u. The first term on the right hand side again admits a global estimate. By a trace theorem we have

$$\left|\sum_{\Gamma_i}{}' \int_{\Gamma_i} (g-g_h)D^3u\,ds\right| \leqq c\int_{B_h} |\nabla(g-g_h)|\,dx = O\left(h\ln\frac{1}{h}\right), \tag{5.5}$$

uniformly for $z\in\Omega$; for the L^1-estimate for the Green function see the Appendix. For the last term in the identity (5.4) we can only prove a local result. Using the pointwise estimate

$$|(g^z-g_h^z)(x)| = O\left(h^2\ln\frac{1}{h}\right), \quad \text{for} \quad |x-z|>c,$$

proven in the Appendix, we obtain

$$\sum_{\Gamma_i}{}'(g-g_h)D^2u|_{P_i}^{Q_i} = O\left(h^2\ln\frac{1}{h}\right)\operatorname{card}\{P_i\in B_h\} = O\left(h\ln\frac{1}{h}\right). \tag{5.6}$$

The combination of the error contributions (5.3), (5.5), and (5.6) gives us the desired result. □

We note that, of course, the definition of the error coefficient $e^{(1)}(z;u)$ depends on the triangulation T_h through the geometry of the boundary B_h which may change with the mesh size h. In order to use the expansion (5.1) for Richardson extrapolation, we construct a refinement $T_{h/2}$ of T_h by, first, subdividing each triangle $K\in T_h$ into four congruent subtriangles and, then, projecting the new nodal points on $\partial\Omega_h$ onto the curved boundary $\partial\Omega$. For this special refinement, we have

$$\tfrac{1}{3}\{4u_{h/2}(z)-u_h(z)\}=u(z)+\frac{h^2}{3}\{e^{(1)}(z;u,T_{h/2})-e^{(1)}(z;u,T_h)\}$$
$$+O\left(h^3\ln\frac{1}{h}\right). \tag{5.7}$$

The difference of the two error coefficients vanishes up to the contributions from the boundary strips $B_{h/2}$ and B_h, respectively. There are three different groups of terms which remain to be estimated:

1. area integrals of the form

$$\sum_{K\in T'_{h/2}}\frac{\lambda_1^4\lambda_2\lambda_3}{4\alpha^2}\int_K gD_1^4D_2D_3u\,dx-\sum_{K\in T'_h}\frac{\lambda_1^4\lambda_2\lambda_3}{4\alpha^2}\int_K gD_1^4D_2D_3u\,dx,$$

where $T'_h=\{K\in T_h,\ K\subset B_h\}$,

2. line integrals of the form

$$\sum_{\Gamma_i\in S'_{h/2}}\frac{\lambda_{i+1}^4}{2\alpha}\int_{\Gamma_i} D_igD_{i+1}^2u\,ds-\sum_{\Gamma_i\in S_h}\frac{\lambda_{i+1}^4}{2\alpha}\int_{\Gamma_i} D_igD_{i+1}^2u\,ds,$$

where $S_h=\{\Gamma_i,\ \Gamma_i\subset B_h\backslash\partial\Omega_h\}$, $S'_{h/2}=\{\Gamma_i\in S_{h/2},\ \Gamma_i\subset\cup\{\Gamma\in S_h\}\}$,

3. line integrals of the form

$$\sum_{\Gamma_i\in S_{h/2}\backslash S'_{h/2}}\frac{\lambda_{i+1}^4}{2\alpha}\int_{\Gamma_i} D_igD_{i+1}^2u\,ds.$$

In the boundary strip $B_{h/2}$ two adjacent triangles do not necessarily form an accurate parallelogram. However, integrals of the form

$$\int_{\Gamma_i} D_ig\{(t_{i+1}\cdot\nabla)^2u-(t'_{i+1}\cdot\nabla)^2u\}\,ds$$

over edges $\Gamma_i\in S_{h/2}\backslash S'_{h/2}$ are of order $O(h^2)$, since (see Fig. 2) $t_{i+1}+t'_{i+1}=O(h)$, by the construction of $T_{h/2}$. By arguments of this type one easily sees that

$$e^{(1)}(z;u,T_{h/2})-e^{(1)}(z;u,T_h)=O(h),$$

which, in view of (5.7), yields the extrapolation formula

$$\tfrac{1}{3}\{4u_{h/2}(z)-u_h(z)\}=u(z)+O\left(h^3\ln\frac{1}{h}\right), \tag{5.8}$$

in interior nodal points z belonging to T_h.

The fact that we cannot obtain the expansion in Theorem 4 uniformly for all nodal points $z\in\Omega$ resembles the well known result for finite difference schemes that only a boundary approximation of sufficiently high degree will lead to a global expansion; see, e.g., [13].

6. Numerical Tests

For verifying the theoretical results of this paper, we have solved the model problem (1.2) on several triangular domains, and on the unit square using various piecewise or locally uniform triangulations.

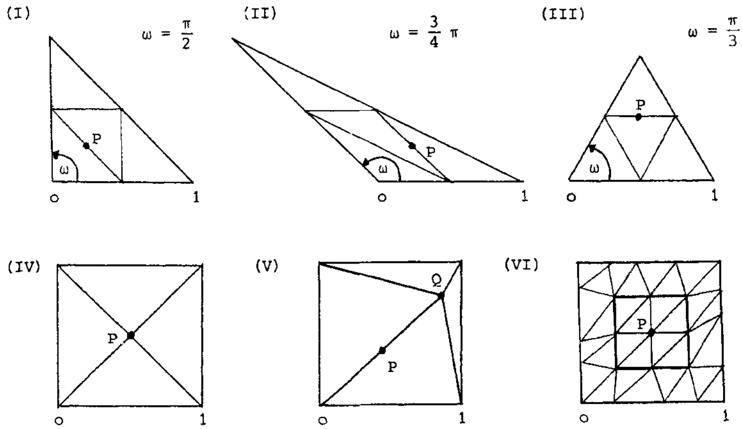

Fig. 3

The above figures show the coarsest triangulation, T_0, corresponding to the mesh size $h=h_0$. From T_0, the refined triangulations T_i, for $i \geqq 1$, are constructed by successive decomposition of each triangle into four congruent subtriangles of size $h_i=2^{-1}h$. In the case (VI) the triangulations T_i are kept uniform up to a boundary strip of width h_i. The data f, b have been chosen such that the solution u is always a polynomial. Although our numerical results cannot be considered as exhaustive, they are certainly representative for the case of a smooth solution. The errors $e_i=u-u_{h_i}$, for $i \geqq 0$, and the extrapolated errors

$$\bar{e}_i=\tfrac{1}{3}(4e_{i+1}-e_i), \qquad \bar{\bar{e}}_i=\tfrac{1}{45}(64e_{i+2}-20e_{i+1}+e_i),$$

have been evaluated at the indicated points P. The following tables show the corresponding error quantities $\varepsilon_i=|e_i|$, $\bar{\varepsilon}_i=|\bar{e}_i|$, $\bar{\bar{\varepsilon}}_i=|\bar{\bar{e}}_i|$, and the approximate orders of convergence m_i, $\bar{m}_i$, $\bar{\bar{m}}_i$, which are calculated according to the formula $m_i=(\ln\varepsilon_i-\ln\varepsilon_{i+1})/\ln 2$. The theoretically predicted orders of convergence are listed as m_∞. We note that the error behavior shown in the Tables 1–6 is representative for all nodal points belonging to the coarser meshes.

Table 1. $u(x,y)=\frac{16}{9}(x^2-x+1)(y^2-y+1)$, $P=(1/4,1/4)$

i	ε_i	m_i	$\bar{\varepsilon}_i$	$\bar{m}_i$	$\bar{\bar{\varepsilon}}_i$	$\bar{\bar{m}}_i$
0	2.0 (−3)	1.962	1.7 (−5)	3.776	1.9 (−7)	5.532
1	5.1 (−4)	1.989	1.3 (−6)	3.930	4.2 (−9)	5.837
2	1.3 (−4)	1.997	8.3 (−8)	3.981	7.3 (−11)	5.952
3	3.2 (−5)	1.999	5.3 (−9)	3.995	1.2 (−12)	
4	8.0 (−6)	2.000	3.3 (−10)			
5	2.0 (−6)	$m_\infty=2$		$m_\infty=4$		$m_\infty=6$

Table 2. $u(x,y)=16(x^2-x)(y^2-y)$, $P=(1/4,1/4)$

i	ε_i	m_i	$\bar{\varepsilon}_i$	$\bar{m}_i$	$\bar{\bar{\varepsilon}}_i$	$\bar{\bar{m}}_i$
0	6.5 (−2)	1.895	1.6 (−3)	3.154	8.7 (−5)	4.076
1	1.7 (−2)	1.955	1.8 (−4)	3.493	5.1 (−6)	4.762
2	4.5 (−3)	1.984	1.6 (−5)	3.768	1.9 (−7)	5.142
3	1.1 (−3)	1.995	1.2 (−6)	3.906	5.4 (−9)	
4	2.8 (−4)	1.999	8.0 (−8)			
5	7.1 (−5)	$m_\infty=2$		$m_\infty=4$		$m_\infty=5.\bar{3}$

Table 3. $u(x,y)=16^2(x^2-x)^2(y^2-y)^2$, $P=(1/2,\sqrt{3}/4)$

i	ε_i	m_i
0	8.3 (−4)	4.037
1	5.1 (−5)	4.008
2	3.2 (−6)	3.996
3	2.0 (−7)	3.971
4	1.3 (−8)	3.887
5	8.5 (−10)	$m_\infty=4$

Table 4. $u(x,y)=16(x^2-x)(y^2-y)$, $P=(1/2,\ 1/2)$

i	ε_i	m_i	ε_i/h_i^2	$\varepsilon_i/(h_i^2 \ln 1/h_i)$
1	7.7 (−2)	0.913	0.309	0.445
2	4.1 (−2)	1.432	0.656	0.473
3	1.5 (−2)	1.610	0.972	0.465
4	5.0 (−3)	1.698	1.274	0.460
5	1.5 (−3)	1.752	1.571	0.453
6	4.6 (−4)		1.866	0.449

Table 5. $u(x,y)=16(x^2-x)(y^2-y)$, $P=(7/16,3/8)$, $Q=(7/8,3/4)$

i	ε_i	m_i	$\bar{\varepsilon}_i$	$\bar{m}_i$	$\bar{\bar{\varepsilon}}_i$	$\bar{\bar{m}}_i$
1	4.5 (−2)	1.907	1.0 (−3)	3.679	1.7 (−5)	5.323
2	1.2 (−2)	1.972	7.8 (−5)	3.889	4.2 (−7)	6.065
3	3.1 (−3)	1.993	5.3 (−6)	3.975	6.2 (−9)	6.283
4	7.7 (−4)	1.998	3.4 (−7)	3.995	8.0 (−11)	
5	1.9 (−4)	2.000	2.1 (−8)			
6	4.8 (−5)	$m_\infty=2$		$m_\infty\sim 4$		$m_\infty\sim ?$

Table 6. $u(x,y)=16(x^2-x)(y^2-y)$, $P=(1/2,1/2)$

i	ε_i	m_i	$\bar{\varepsilon}_i$	$\bar{m}_i$
0	4.0 (−2)	1.891	1.0 (−3)	3.917
1	1.1 (−2)	1.974	6.9 (−5)	2.649
2	3.1 (−3)	1.985	1.1 (−5)	2.948
3	7.7 (−4)	1.992	1.4 (−6)	2.666
4	1.9 (−4)	1.995	2.2 (−7)	$m_\infty\sim 3$

The tests I and II confirm the local $O(h^4)$ result in Theorem 2 and also indicate that an extended expansion of the form (3.24) may be valid. In test III, we see the $O(h^4)$ super convergence of the error e_i which has been predicted in Corollary 3.

Test IV is intended to clarify the question for the optimal order of the pointwise convergence discussed at the end of Sect. 3. Although, the triangulation involves a considerable symmetry, the numerical results clearly show the logarithmic error behavior, $O\left(h^2 \ln\frac{1}{h}\right)$; for this particular configuration there holds $\sum_j E_j(P) = -4\Delta u(P) \neq 0$.

Test V supports the result of Theorem 3 that the expansion (4.2) holds true with a remainder term of the order $O\left(h^4 \ln\frac{1}{h}\right)$, even in nodal points on the macro-edges where the uniformity of the mesh is disturbed.

Finally, test VI stands as a model for the case of a smoothly bounded domain in so far as the uniformity of the mesh is perturbed in a boundary strip of width $O(h)$. Further, it is guaranteed that the presence of the corners of $\partial\Omega$ does not effect the desired order of the error expansion, since u is chosen such that the compatibility condition (3.23) is satisfied. The quantities ε_i, in Table 6, refer to the absolute errors on the meshes T_i which are kept uniform up to a boundary strip of width h_i. The extrapolated quantities $\tilde{\varepsilon}_i$ are computed from ε_i and from the errors ε'_{i+1} on the meshes T'_{i+1} which are obtained from T_i by subdividing each triangle into four congruent subtriangles yielding a boundary strip of width $2h_{i+1}$. Hence, ε'_{i+1} may slightly differ from ε_{i+1} which corresponds to the mesh T_{i+1}. The results listed in Table 6 show that our local $O\left(h^3 \ln\frac{1}{h}\right)$ estimate for the remainder term in the expansion (5.1), Theorem 4, is sharp. Further, a comparison of the orders $\tilde{m}_i$ for a sequence of points approaching the boundary indicates that the expansion cannot be extended up to the boundary.

Appendix

In the following it is generally assumed that the domain $\Omega \subset \mathbb{R}^2$ is polygonal and convex, and that the triangulation $T_h = \{K\}$ of Ω is regular in the sense of Sect. 2 but not necessarily uniform. However, the results presented below remain valid with minor changes in the case of a curved boundary, $\partial\Omega \in C^{2+\varepsilon}$, provided the boundary approximation is of the order $O(h^2)$. Since most of the argument used in this section is fairly standard now in the error analysis of the finite element method, we will suppress some of its technical details.

Corresponding to an arbitrary fixed point $z \in K_z$, $K_z \in T_h$, let $g = g^z \in W^{1,2-\varepsilon}(\Omega)$ be the Green function for problem (1.2) and $g_h = g_h^z \in S_h^0$ its discrete analogue defined by (2.3). Further, we introduce a so-called regularized Green function $\tilde{g} = \tilde{g}^z \in H_0^1(\Omega) \cap H^2(\Omega)$ as the solution of the problem

$$-\Delta \tilde{g} = \tilde{\delta} \quad \text{in} \quad \Omega, \qquad \tilde{g} = 0 \quad \text{on} \quad \partial\Omega. \tag{A.1}$$

Here, $\tilde{\delta}=\tilde{\delta}^z \in C_0^\infty(K_z)$ is an approximation to the Dirac functional in z, satisfying $|\nabla^k \tilde{\delta}| \leqq c h^{-2-k}$, $k \geqq 0$, and (see [9])

$$\int_\Omega \varphi_h \tilde{\delta}\, dx = \varphi_h(z) \quad \text{for all} \quad \varphi_h \in S_h^0 . \tag{A.2}$$

Clearly, g_h may be interpreted as the Ritz projection of $\tilde{g}$. Corresponding to the point z, we define the weight function

$$\sigma(x) = (|x-z|^2 + \kappa^2 h^2)^{1/2}, \qquad \kappa \geqq 1,$$

which approximates the distance function $|x-z|$ and satisfies

$$|\nabla^k \sigma(x)| \leqq c \sigma^{1-k}(x) \leqq c(\kappa h)^{1-k}, \qquad k \geqq 1 . \tag{A.3}$$

The parameter κ may be chosen sufficiently large such that

$$\max_{K \in T_h} \left\{ \max_{x \in K} \sigma(x) \Big/ \min_{x \in K} \sigma(x) \right\} \leqq c, \tag{A.4}$$

uniformly for $z \in \Omega$.

Lemma A1. *For any fixed* $\varepsilon > 0$, *there holds*

$$\|\sigma^{\varepsilon/2} \nabla \tilde{g}\|_{L^2} + \|\sigma^{1+\varepsilon/2} \nabla^2 \tilde{g}\|_{L^2} \leqq c \left(\frac{1}{\varepsilon} + \frac{h^\varepsilon}{\varepsilon^2} \right)^{1/2}, \tag{A.5}$$

$$\|\sigma^{1+\varepsilon/2} \nabla(\tilde{g} - g_h)\|_{L^2} \leqq c \left(\frac{1}{\varepsilon} + \frac{h^\varepsilon}{\varepsilon^2} \right)^{1/2} h, \tag{A.6}$$

uniformly for $z \in \Omega$, *if* κ *is chosen sufficiently large independent of* h *and* ε.

Proof. From [9], formula (2.18), we obtain the result

$$\int_\Omega \sigma^{2+\varepsilon} |\nabla(\tilde{g} - g_h)|^2 dx \leqq c h^2 \int_\Omega \sigma^{2+\varepsilon} |\nabla^2 \tilde{g}|^2 dx, \tag{A.7}$$

for $\varepsilon \geqq 0$, which holds for any function $\tilde{g} \in H_0^1(\Omega) \cap H^2(\Omega)$, if κ is chosen sufficiently large. Moreover, a standard argument leads to the a priori estimate

$$\int_\Omega \{\sigma^\varepsilon |\nabla \tilde{g}|^2 + \sigma^{2+\varepsilon} |\nabla^2 \tilde{g}|^2\} dx \leqq \frac{c}{\varepsilon^2} \int_\Omega \sigma^{2+\varepsilon} |\tilde{\delta}|^2 dx + c\varepsilon^2 \int_\Omega \sigma^{\varepsilon-2} |\tilde{g}|^2 dx, \tag{A.8}$$

for $\varepsilon > 0$; see formula (3.6) in [9]. Then, observing that

$$|\tilde{g}(x)| = \left| \int_\Omega \nabla \tilde{g} \nabla g^x dy \right| = \left| \int_\Omega \tilde{\delta} g^x dy \right| \leqq c\{|\ln \sigma(x)| + 1\}, \tag{A.9}$$

and $|\tilde{\delta}| \leqq c h^{-2}$, we find that

$$\int_\Omega \{\sigma^\varepsilon |\nabla \tilde{g}|^2 + \sigma^{2+\varepsilon} |\nabla^2 \tilde{g}|^2\} dx \leqq \frac{c}{\varepsilon} \left\{ 1 + \frac{h^\varepsilon}{\varepsilon} \right\}. \tag{A.10}$$

This clearly implies the assertion. □

Taking $\varepsilon = \left(\ln\frac{1}{h}\right)^{-1}$ in (A.5) and (A.6), and observing that $\|\sigma^{-2}\|_{L^1} \leqq c\ln\frac{1}{h}$, we may obtain the following estimates, for the limit case $\varepsilon = 0$,

$$\|\nabla\tilde{g}\|_{L^2} + \|\nabla^2\tilde{g}\|_{L^1} = O\left(\ln\frac{1}{h}\right), \tag{A.11}$$

$$\|\nabla(\tilde{g} - g_h)\|_{L^1} = O\left(h\ln\frac{1}{h}\right), \tag{A.12}$$

uniformly for $z \in \Omega$. We note that these results, (A.11) and (A.12), have been proven in [4], Satz B4 and Satz 1, for the case of a smoothly bounded domain.

Next, we will derive the local error estimates for the Green function g^z used in the proof of Theorem 3. These estimates are essentially not new, but for our particular situation no explicit proof is apparent to us in the literature; for related results we refer to [10] and [11].

As a further result from [4], Satz 3, we note that

$$\|\nabla(g - g_h)\|_{L^1} = O\left(h\ln\frac{1}{h}\right), \tag{A.13}$$

uniformly for $z \in \Omega$. In order to prove the local error estimates, we need some technical preliminaries.

In [9] it has been shown that the Ritz projection of $H_0^1(\Omega)$ onto S_h^0 is bounded with respect to the norm of $W^{1,\infty}(\Omega)$, namely

$$\|u_h\|_{W^{1,\infty}} \leqq c\,\|u\|_{W^{1,\infty}}, \tag{A.14}$$

for $u \in H_0^1(\Omega) \cap W^{1,\infty}(\Omega)$, if Ω is a convex polygonal domain. This result remains valid also for a curved boundary $\partial\Omega \in C^{2+\varepsilon}$. In the following we shall use a local version of (A.14) which may be proven by combining the methods of [9] with a suitable localization technique (see, e.g., [11]); for the sake of brevity we skip the fairly standard argument. Note that in the estimate (A.14) and in its local analogue presented below, no factor $\ln\frac{1}{h}$ occurs.

Lemma A2. *Let $\Omega_0 \subset \Omega$ and $\Omega_0^d = \{x \in \Omega \mid \operatorname{dist}(x, \Omega_0) < d\}$, for some $d > 0$. Then, for sufficiently small h, there holds*

$$\|u_h\|_{W^{1,\infty}(\Omega_0)} \leqq c\,\|u\|_{W^{1,\infty}(\Omega_0^d)} + \frac{c}{d}\{\|u\|_{L^2} + h\|u\|_{H^1}\}, \tag{A.15}$$

for $u \in H_0^1(\Omega) \cap W^{1,\infty}(\Omega_0^d)$.

We use this result in deriving local error estimates for the generalized Green function $\tilde{g}^y$ corresponding to some point $y \in \Omega$. Let $\Omega_0 \subset \Omega$ be such that for some fixed $d > 0$, Ω_0^d has positive distance from y and from the corner points z_j of $\partial\Omega$. Then, $\tilde{g}^y \in H^2(\Omega) \cap W^{2,\infty}(\Omega_0^d)$ with norms

$$h\|\tilde{g}^y\|_{H^2} + \|\tilde{g}^y\|_{W^{2,\infty}(\Omega_0^d)} \leqq c, \tag{A.16}$$

uniformly for $y \in \Omega$. Applying the bound (A.15) for the Ritz projection of $i_h\tilde{g}^y - g_h^y \in S_h^0$, we obtain

$$\|\tilde{g}^y - g_h^y\|_{W^{1,\infty}(\Omega_0)} \leqq c_d \|\tilde{g}^y - i_h\tilde{g}^y\|_{W^{1,\infty}(\Omega_0^d)} + c_d\{\|\tilde{g}^y - i_h\tilde{g}^y\|_{L^2} + h\|\tilde{g}^y - i_h\tilde{g}^y\|_{H^1}\},$$

and, consequently, in view of (A.16),

$$\|\tilde{g}^y - g_h^y\|_{W^{1,\infty}(\Omega_0)} \leqq c_d h. \tag{A.17}$$

This result is the key to the following.

Lemma A3. *For the Green function g^z, there holds*

$$(g^z - g_h^z)(y) = O\left(h^2 \ln \frac{1}{h}\right), \tag{A.18}$$

in points $y \in \Omega$ having positive distance from z, and from the corner points of $\partial\Omega$.

Proof. Let $z_j, j = 1, \ldots, m$, be the corner points of $\partial\Omega$. The Green function g^z may be split as follows

$$g^z(x) = s^{(0)}(x) + \sum_{j=1}^{m} \gamma_j s^{(j)}(x), \tag{A.19}$$

where $s^{(0)} - 2\pi \ln \frac{1}{|x-z|} \in C^{2+\varepsilon}(\bar{\Omega})$, and $s^{(j)} = r_j^{\pi/\omega_j} \sin\left(\frac{\pi}{\omega_j}\theta_j\right)\chi_j$ are the "singular" functions introduced in (3.14), multiplied by some cut-off functions χ_j, such that $s^{(j)} \equiv 0, j = 1, \ldots, m$, on $\partial\Omega$. Let $s_h^{(j)}$ be the corresponding Ritz projections. We shall estimate the errors $s^{(0)} - s_h^{(0)}$, and $s^{(j)} - s_h^{(j)}, j = 1, \ldots, m$, separately.

Let $y \in \Omega$ be any point with positive distance from z, and from the corner points z_j. Using the regularized Green function $\tilde{g}^y$, we obtain by a simple calculation that (see [4], or [9])

$$(s - s_h)(y) = \int_\Omega \nabla(s - i_h s)\nabla(\tilde{g}^y - g_h^y)\,dx + O(h^2), \tag{A.20}$$

where the notation s stands for any of the "singular" functions $s^{(0)}$ or $s^{(j)}$. We start with the case $s = s^{(0)}$. Let B_z be a circle with positive radius and center in z, such that $\mathrm{dist}(y, B_z) > 0$, and $\mathrm{dist}(z_j, B_z) > 0$. Then, we have

$$\left|\int_{\Omega\setminus B_z} \nabla(s - i_h^{\sim} s)\nabla(\tilde{g}^y - g_h^y)\,dx\right| \leqq c\|\nabla(s - i_h^{\sim} s)\|_{L^\infty(\Omega\setminus B_z)} \|\nabla(\tilde{g}^y - g_h^y)\|_{L^1},$$

$$\left|\int_{\Omega\cap B_z} \nabla(s - i_h^{\sim} s)\nabla(\tilde{g}^y - g_h^y)\,dx\right| \leqq c\|\nabla(s - i_h^{\sim} s)\|_{L^1} \|\nabla(\tilde{g}^y - g_h^y)\|_{L^\infty(\Omega\cap B_z)},$$

where $i_h^{\sim} s \in S_h^0$ is some modified interpolant taking care of the singularity of $s = s^{(0)}$ at z. For the interpolation error one finds

$$\|\nabla(s - i_h^{\sim} s)\|_{L^\infty(\Omega\setminus B_z)} = O(h), \qquad \|\nabla(s - i_h^{\sim} s)\|_{L^1} = O\left(h \ln \frac{1}{h}\right). \tag{A.21}$$

Further, from (A.17), setting $\Omega_0=\Omega\cap B_z$, and from (A.12), with z replaced by y, it follows that

$$\|\nabla(\tilde{g}^y-g_h^y)\|_{L^\infty(\Omega\cap B_z)}=O(h)\,,\qquad \|\nabla(\tilde{g}^y-g_h^y)\|_{L^1}=O\left(h\ln\frac{1}{h}\right).$$

Combining the above estimates yields

$$(s-s_h)(y)=O\left(h^2\ln\frac{1}{h}\right), \tag{A.22}$$

for $s=s^{(0)}$. Now, let B_j be a circle with positive radius and center in z_j, such that $\operatorname{dist}(y,B_j)>0$. Then, by an analogous argument as used above, we obtain (A.22) for $s=s^{(j)}$. This completes the proof of (A.18). □

The following lemma contains a non-standard negative norm error estimate, for the Green function.

Lemma A4. *Let $\Gamma\subset\Omega$ be a straight line consisting entirely of edges of the triangulation T_h, and let ψ be the trace of a $C^{1+\varepsilon}$-function on Γ, for some $\varepsilon>0$. Then, there holds*

$$\int_\Gamma (g^z-g_h^z)\psi\,ds=O\left(h^2\ln\frac{1}{h}\right), \tag{A.23}$$

provided z has positive distance from the end points of Γ.

Proof. We continue using the notation of the proof of Lemma A3. Let $y\in\Gamma$, not necessarily bounded away from the corner points z_j. Then, for $s=s^{(j)}$, $j=1,\dots,m$, the identity (A.20) takes the form

$$(s-s_h)(y)=\int_\Omega \nabla(s-i_hs)\nabla(\tilde{g}^y-g_h^y)\,dx+ch^2\{|y-z_j|^{\pi/\omega_j-2}+1\}\,. \tag{A.24}$$

With the weight function $\sigma_y(x)=(|x-y|^2+\kappa^2h^2)^{1/2}$ satisfying (A.3) and (A.4), there holds

$$\left|\int_\Omega \nabla(s-i_hs)\nabla(\tilde{g}^y-g_h^y)\,dx\right|\leqq\left(\int_\Omega\sigma_y^{-2}|\nabla(s-i_hs)|^2dx\right)^{1/2}\left(\int_\Omega\sigma_y^2|\nabla(\tilde{g}^y-g_h^y)|^2dx\right)^{1/2}.$$

Using (A.4), we find by a straightforward calculation that

$$\begin{aligned}\left(\int_\Omega\sigma_y^{-2}|\nabla(s-i_hs)|^2dx\right)^{1/2}&\leqq ch\left(\int_\Omega\sigma_y^{-2}|\nabla^2s|^2dx\right)^{1/2}\\&\leqq ch\left(\ln\frac{1}{h}\right)^{1/2}\{|y-z_j|^{\pi/\omega_j-2}+1\},\end{aligned}$$

and by Lemma A2, choosing there $\varepsilon=\left(\ln\frac{1}{h}\right)^{-1}$,

$$\left(\int_\Omega\sigma_y^2|\nabla(\tilde{g}^y-g_h^y)|^2dx\right)^{1/2}=O\left(h\left(\ln\frac{1}{h}\right)^{1/2}\right).$$

We insert the last three estimates into (A.24), and obtain

$$|(s-s_h)(y)| \leqq c h^2 \ln \frac{1}{h} \{|y-z_j|^{\pi/\omega_j-2}+1\}. \tag{A.25}$$

Integrating in (A.25) for $y \in \Gamma$ yields

$$\int_\Gamma |s-s_h|\, ds = O\left(h^2 \ln \frac{1}{h}\right), \tag{A.26}$$

for the "singular" functions $s=s^{(j)}$, $j=1, \ldots, m$.

Next, we set $s=s^{(0)}$. Since z has positive distance from the end points P, Q of Γ, there exist circles B_P, B_Q with positive radii and centers in P and Q, respectively, such that $z \notin B_P \cup B_Q$. Then, we may use the estimate (A.22), for $y \in \Gamma \cap (B_P \cup B_Q)$, to obtain

$$\int_{\Gamma \cap (B_P \cup B_Q)} |s-s_h|\, ds = O\left(h^2 \ln \frac{1}{h}\right). \tag{A.27}$$

In view of (A.27), we can assume without loss of generality that the function ψ in (A.23) vanishes in a neighborhood of the endpoints P, Q. Under this assumption, we shall complete the prove of (A.23) by using a duality argument. Let $v \in H_0^1(\Omega)$ be the solution of the auxiliary problem

$$\int_\Omega \nabla\varphi \nabla v\, dx = \int_\Gamma \varphi\psi\, ds, \quad \text{for all} \quad \varphi \in H_0^1(\Omega). \tag{A.28}$$

Clearly, $v \in H^2(\Omega \backslash \Gamma) \cap W^{2,\infty}(\Omega_0 \backslash \Gamma)$, for any subdomain $\Omega_0 \subset \Omega$ having positive distance from the corner points of $\partial\Omega$. Let $v_h \in S_h^0$ be the Ritz projection of v. Since Γ is aligned with the edges of the triangulation, we have the global L^2-estimate

$$\|v-v_h\|_{L^2} + h\|v-v_h\|_{H^1} = O(h^2), \tag{A.29}$$

and, according to (A.15), the local L^∞-estimate

$$\|\nabla(v-v_h)\|_{L^\infty(\Omega_0)} = O(h). \tag{A.30}$$

One can justify setting $\varphi = s-s_h$ in (A.28), to obtain

$$\int_\Gamma (s-s_h)\psi\, ds = \int_\Omega \nabla(s-s_h)\nabla v\, dx = \int_\Omega \nabla(s-\tilde{i}_h s)\nabla(v-v_h)\, dx, \tag{A.31}$$

where $\tilde{i}_h s \in S_h^0$ is some modified interpolant of s. Then, using (A.29), (A.30), together with the estimates (A.21), for the interpolation error, we arrive at the desired negative norm result

$$\int_\Gamma (s-s_h)\psi\, ds = O\left(h^2 \ln \frac{1}{h}\right),$$

for $s=s^{(0)}$. This completes the proof of (A.23). □

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