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On the Entropy of the Geodesic Flow in Manifolds Without Conjugate Points

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Introduction

Let M be a compact Riemannian manifold and $\tilde{M}$ its universal covering. In [7] Manning introduced the volume growth rate λ of M defined by:

$$\lambda = \lim_{r \to +\infty} \frac{1}{r} \log \operatorname{vol} B_r(p)$$

where $p \in \tilde{M}$ and $B_r(p)$ denotes the ball with center p and radius r. He proved that this limit exists and is independent of p. In fact the relation between the growth of the function $\operatorname{vol} B_r(p)$, the curvature of M and its fundamental group had already been considered by several authors (Milnor [10], Margulis [9], Dinaburg [2]). The constant λ and $\pi_1(M)$ can be related as follows: Let N be a fundamental domain of $\tilde{M}$ and let $G = \{g \in \pi_1(M) \mid gN \cap N \neq \emptyset\}$. Milnor proved in [10] that G is a generator of $\pi_1(M)$. Define the growth rate α of $\pi_1(M)$ with respect to the generator G as $\lim_{k \to +\infty} k^{-1} \log N(k)$ where $N(k)$ is the number of elements of $\pi_1(M)$ which can be written in the form $\prod_{i=1}^{k} g_i$, $g_i \in G$, $1 \leqq i \leqq k$. This limit always exists and depends on G. However, the fact that α is zero or positive is independent of the generator [10]. In the second case we say that $\pi_1(M)$ has exponential growth. Then, if d_1 and d_2 denote the minimum and maximum distances between the sets N and $\bigcup \{gN \mid gN \cap N = \emptyset\}$, the following inequality holds:

$$\frac{\alpha}{d_2} \leqq \lambda \leqq \frac{\alpha}{d_1}.$$

These inequalities were essentially proved by Dinaburg [2] and Milnor [10] respectively. In particular $\lambda > 0$ if and only if $\pi_1(M)$ has exponential growth. In [7] Manning related the dynamics of the geodesic flow φ: $R \times TM_1 \to TM_1$ to λ in the following way: If $h_{\text{top}}(\varphi)$ denotes the topological entropy of φ:

$$h_{\text{top}}(\varphi) \geqq \lambda \tag{1}$$

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and

$$h_{\text{top}}(\varphi)=\lambda \tag{2}$$

if the manifold has non positive sectional curvature. In fact his proof holds without any change for the more general case of manifolds without focal points and, with some alterations, it can be extended to Riemannian manifolds whose geodesic flow is Anosov. It is then natural to ask whether (2) holds for manifolds without conjugate points. This class of manifolds obviously contains the manifolds without focal points and, by a result of Klingenberg [6], manifolds whose geodesic flow is Anosov. The main objective of this work is to give a positive answer to this question under some mild regularity assumptions on the Riemannian structure:

Theorem I. $h_{\text{top}}(\varphi)=\lambda$ *if M has no conjugate points and its Riemannian metric is Hölder C^3.*

Manning's proof of (2) uses the hypothesis on the curvature of M only to insure that if γ_i: $[a,b]\to\tilde{M}$, $i=1,2$ are geodesic arcs, then:

$$d(\gamma_1(t),\gamma_2(t))\leqq d(\gamma_1(a),\gamma_2(a))+d(\gamma_1(b),\gamma_2(b)). \tag{3}$$

This property is also true for manifolds without focal points (see [5] for a proof). When the geodesic flow is Anosov it is not difficult to prove, using the results of Eberlein [3] on the behaviour of Jacobi vector fields for these manifolds, that (3) holds if we multiply its second member by a certain uniersal constant independent of the geodesic arcs. This modified version of (3) is enough to prove (2) applying the same method used by Manning. As far as we know no simple modification of (3) holds for manifolds without conjugate points, but it can be expected that the essential geometric meaning of (3), namely that geodesic arcs do not spread apart more than their endpoints, is in some weaker form true. Our proof of Theorem I follows a different method. We shall use Przytycki's inequality for the topological entropy [14]. Let N be a compact boundaryless manifold and let φ: $R\times N\to N$ be a Hölder C^1 flow. Przytycki's inequality states that:

$$h_{\text{top}}(\varphi)\leqq \lim_{t\to+\infty}\frac{1}{t}\log\int_N \|(D_x\varphi_t)^\wedge\|\,d\lambda_0(x)$$

where

$$(D_x\varphi_t)^\wedge: (T_xN)^\wedge\to(T_{\varphi_t(x)}N)^\wedge$$

denotes the linear map induced by $D_x\varphi_t$ on the exterior algebra of the tangent space T_xN and λ_0 is the Lebesgue measure on N. If $N=TM_1$ and φ is the geodesic flow we shall prove that:

$$\lim_{t\to+\infty}\frac{1}{t}\log\int_{TM_1}\|(D_x\varphi_t)^\wedge\|\,d\lambda_0(x)\leqq\lambda$$

which together with (1) implies:

$$h_{\text{top}}(\varphi)=\lim_{t\to+\infty}\frac{1}{t}\log\int_N\|(D_x\varphi_t)^\wedge\|\,d\lambda_0(x)=\lambda.$$

The limit in Przytycki's inequality had already been considered by Shub and Sacksteder [16] in connection with the Entropy Conjecture and has interesting geometrical connotations.

We now consider the problem of finding estimates for the metric entropies $h_\mu(\varphi)$ where μ is a φ-invariant probability on TM_1. Let us recall the main results of Oseledec theory [11] which are fundamental for this kind of problems. Consider a C^1 flow φ: $\mathbb{R} \times N \to N$, where N is a compact boundaryless manifold. It follows from Oseledec's results that there exists a total probability Borel set $\Lambda \subset N$ (i.e. such that $\mu(\Lambda^c)=0$ for any φ-invariant probability μ on N) such that at each $x \in \Lambda$ there exists a unique splitting $T_x N = E^u(x) \oplus E^c(x) \oplus E^s(x)$ satisfying:

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|(D_x \varphi_t) v\| = 0 \qquad v \in E^c(x),$$

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|(D_x \varphi_t) v\| < 0 \qquad v \in E^s(x),$$

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|(D_x \varphi_t) v\| > 0 \qquad v \in E^u(x).$$

Moreover the limit:

$$\chi(x) = \lim_{t\to\pm\infty} \frac{1}{t} \log |\det (D_x \varphi_t)/S| \tag{4}$$

exists and is independent of S, for any $x \in \Lambda$ and any subspace $E^u(x) \subset S \subset E^u(x) \oplus E^c(x)$. The importance of $\chi(x)$ is that if μ is a φ-invariant probability measure on N (Ruelle [15]):

$$h_\mu(\varphi) \leqq \int_N \chi \, d\mu \tag{5}$$

and if φ is Hölder C^1 and μ is absolutely continuous with respect to Lebesgue measure (Pesin [12]):

$$h_\mu(\varphi) = \int_N \chi \, d\mu \tag{6}$$

We shall try to estimate the metric entropies of the geodesic flow using these formulas. If $(p,v) \in TM_1$, denote by $E(p,v)$ the subspace of $T_p M$ orthogonal to v and by $K(p,v)$: $E(p,v) \hookleftarrow$ the linear map defined by $K(p,v) w = R_p(v,w) v$ where R_p is the curvature tensor of M at p. The Ricatti equation of M is:

$$\dot{U} + U^2 + K = 0. \tag{7}$$

A solution of the Ricatti equation is a function that to each $(p,v) \in TM_1$ assigns a self adjoint linear map $U(p,v)$: $E(p,v) \hookleftarrow$ such that if τ_t: $T_p M \to T_{\gamma(t)} M$ denotes the parallel transport along the geodesic with initial condition (p,v) then the limit

$$\dot{U}(p,v) = \lim_{t\to 0} \frac{1}{t} (\tau_t^{-1} \circ U(\varphi_t(p,v)) \circ \tau_t - U(p,v))$$

exists for all $(p,v) \in TM_1$ and satisfies:

$$\dot{U}(p,v)+U^2(p,v)+K(p,v)=0.$$

The results of Green in [4] can be reformulated as saying that if M has no conjugate points (7) has measurable solutions and any solution is bounded. There is a close relationship between solutions of (7) and invariant subbundles of the tangent flow $T\varphi\colon R \times T(TM_1) \to T(TM_1)$, as we shall explain in Sect. I.

The function χ and the solutions of the Ricatti equation are connected by the following result:

Theorem II. *If M has no conjugate points there exists a measurable solution U^+ of the Ricatti equation such that the equality:*

$$\chi(x)=\lim_{T\to+\infty}\frac{1}{T}\int_0^T \operatorname{tr}(U^+\circ\varphi_t(x))\,dt$$

holds for a total probability set of points x in TM_1.

Then, by (5) and the ergodic theorem:

$$h_\mu(\varphi) \leqq \int_{TM_1} \operatorname{tr} U^+\, d\mu \tag{8}$$

and if the Riemannian metric is Hölder C^3 and λ_0 denotes the Lebesgue measure of TM_1, by (6):

$$h_{\lambda_0}(\varphi)=\int_{TM_1} \operatorname{tr} U^+\, d\lambda_0. \tag{9}$$

Using (8) and (9) some interesting estimates for the metric entropies of φ can be obtained. Consider the space of bounded measurable functions that to each $(p,v)\in TM_1$ associate a self adjoint linear map $A(p,v)$ of $E(p,v)$ and identify two such functions which coincide μ-a.e. In this space we can define an inner product:

$$\langle U, V\rangle=\int_{TM_1} \operatorname{tr} U V d\mu.$$

Then Cauchy-Schwarz's inequality implies:

$$\int_{TM_1} \operatorname{tr} U V \leqq \Big(\int_{TM_1} \operatorname{tr} U^2\, d\mu\Big)^{1/2}\Big(\int_{TM} \operatorname{tr} V^2\, d\mu\Big)^{1/2}$$

and equality holds if and only if U is a scalar multiple of V. Then (8) implies

$$\begin{aligned} h_\mu(\varphi) &\leqq \Big(\int_{TM_1} \operatorname{tr} U^{+2}\, d\mu\Big)^{1/2}\Big(\int_{TM_1} \operatorname{tr} I^2\, d\mu\Big)^{1/2} \\ &=(n-1)^{1/2}\Big(\int_{TM_1} \operatorname{tr} U^{+2}\, d\mu\Big)^{1/2} \end{aligned}$$

where $n=\dim M$. The φ-invariance of μ implies

$$\int_{TM_1} \operatorname{tr} \dot{U}^+\, d\mu=0.$$

Then, integration of (7) with respect to μ yields

$$\int_{TM_1} \operatorname{tr} U^{+2}\, d\mu = -\int_{TM_1} \operatorname{tr} K\, d\mu.$$

But $(n-1)^{-1} \operatorname{tr} K(\theta)$ is the Ricci tensor $\operatorname{Ric}(\theta)$. Hence, we have proved the following Corollary:

Corollary II.1. *If M has no conjugate points:*

$$h_\mu(\varphi) \leqq (n-1)\Big(-\int_{TM_1} \operatorname{Ric}\, d\mu\Big)^{1/2}$$

for any φ-invariant probability μ on TM_1. If the equality holds, the sectional curvatures at points of the support of μ are constant.

When μ is the Lebesgue measure λ_0 on TM_1 and λ_1 is the Lebesgue measure on M (normalized in order to have $\lambda_0(TM_1)=\lambda_1(M)=1$) we have the following result:

Corollary II.2. *If M has no conjugate points and $S(p)$ denotes the scalar curvature at $p\in M$:*

$$h_{\lambda_0}(\varphi) \leqq (n-1)\Big(-\int_{TM_1} S\, d\lambda_1\Big)^{1/2}$$

and the equality holds if and only if M has constant sectional curvatures.

To prove this corollary denote by λ_2 the Lebesgue measure on the $(n-1)$-dimensional sphere of R^n and recall the equality:

$$\int_{S^{n-1}} \langle A x, x\rangle\, d\lambda_2(x) = \frac{1}{n}\lambda_2(S^{n-1}) \operatorname{tr} A. \tag{10}$$

Moreover, if $(p,v)\in TM_1$ and $\{x_1,\ldots,x_{n-1}\}$ is an orthonormal basis of $E(p,v)$:

$$\operatorname{Ric}(p,v) = \frac{1}{n-1}\sum_{i=1}^{n-1} \langle R_p(v,x_i)\, v, x_i\rangle.$$

Then if $A\colon T_pM \hookleftarrow$ is defined by

$$A v = \frac{1}{n-1}\sum_{i=1}^{n-1} R_p(x_i, v)\, x_i$$

we have:

$$\operatorname{Ric}(p,v) = \langle A v, v\rangle$$

and then by (10) and setting $x_n = v$:

$$\int_{S^{n-1}} \operatorname{Ric}(p,v)\, d\lambda_2(v) = \frac{1}{n}\lambda_2(S^{n-1}) \frac{1}{n-1}\sum_{j=1}^{n}\sum_{i=1}^{n-1} \langle R_p(x_j,x_i)\, x_i, x_j\rangle$$
$$= \lambda_2(S^{n-1})\, S(p),$$

Then:

$$\int_{TM_1} \mathrm{Ric}\, d\lambda_0 = \frac{1}{\lambda_2(S^{n-1})} \int_M \Big(\int_{S^{n-1}} \mathrm{Ric}(p,v)\, d\lambda_2(v)\Big)\, d\lambda_1(p) = \int_M S\, d\lambda_1 .$$

The proof is now completed using Corollary II.1.

Another easy corollary of Theorem II is the following lower bound for $h_{\lambda_0}(\varphi)$ when the curvature of M is negative:

Corollary II.3. *If M is Hölder C^3 and all its sectional curvatures are negative:*

$$h_{\lambda_0}(\varphi) \geqq (n-1)^{1/2} \int_{TM_1} (-\mathrm{Ric})^{1/2}\, d\lambda_0 .$$

Moreover the equality holds if and only if the sectional curvature is constant.

This inequality was proved in the 2-dimensional case by Manning [8] and its proof is based on essentially the same method. P. Sarnak also obtained weaker forms of Corollaries II.2 and II.3 for manifolds of negative curvature [17]. We shall use the following stronger version of Theorem II: when M has negative curvature then there exists a measurable solution U of the Ricatti equation satisfying the statement of the theorem and such that there exists $c>0$ such that $\langle U(\theta)x, x\rangle \geqq c\,\|x\|^2$ for every $\theta \in TM_1$, $x \in E(\theta)$. In Sect. II, after proving Theorem II we shall show that the solution of the Ricatti equation constructed in its proof satisfies this extra condition if M has negative curvature. In particular it follows that the function det U^+ is bounded away from zero and then log det U^+ is integrable. Moreover

$$\begin{aligned} \frac{d}{dt}(\log \det U^+(\varphi_t(\theta)))|_{t=0} &= \mathrm{tr}\, \dot{U}(\theta)\, U^{-1}(\theta) \\ &= -\mathrm{tr}\, U^+(\theta) - \mathrm{tr}\, K(\theta)\, U^{+^{-1}}(\theta). \end{aligned}$$

Integrating with respect to λ_0 we obtain, by the φ-invariance of λ_0:

$$\int_{TM_1} \mathrm{tr}\, U^+\, d\lambda_0 = -\int_{TM_1} \mathrm{tr}\, K U^{+^{-1}}\, d\lambda_0 .$$

Now observe that if A and B are positive self adjoint maps of a Hilbert space:

$$\mathrm{tr}\, AB \leqq (\mathrm{tr}\, A^2)^{1/2} (\mathrm{tr}\, B^2)^{1/2} \leqq \mathrm{tr}\, A\, \mathrm{tr}\, B. \tag{11}$$

In particular $\mathrm{tr}\, AB = \mathrm{tr}\, A\, \mathrm{tr}\, B$ implies that A is a scalar multiple of B. From this property and Cauchy-Schwarz's inequality we obtain:

$$\begin{aligned} \int_{TM_1} (-\mathrm{tr}\, K)^{1/2}\, d\lambda_0 &= \int_{TM_1} (-\mathrm{tr}\, K U^{+^{-1}} U^+)^{1/2}\, d\lambda_0 \\ &\leqq \int_{TM_1} (\mathrm{tr}(-K U^{+^{-1}}))^{1/2} (\mathrm{tr}\, U^+)^{1/2}\, d\lambda_0 \\ &\leqq \Big(\int_{TM_1} \mathrm{tr}(-K U^{+^{-1}})\, d\lambda_0 \Big)^{1/2} \Big(\int_{TM_1} \mathrm{tr}\, U^+\, d\lambda_0 \Big)^{1/2} \\ &= \int_{TM_1} \mathrm{tr}\, U^+\, d\lambda_0 = h_{\lambda_0}(\varphi). \end{aligned} \tag{12}$$

Replacing $\operatorname{tr} K$ by $(n-1)\,\mathrm{Ric}$ the inequality of the corollary is proved. If the equality holds we must have:

$$\operatorname{tr}(-K U^{+^{-1}} U^{+}) = \operatorname{tr}(-K U^{+^{-1}}) \operatorname{tr} U^{+} \quad \text{a.e.}$$

Then there exists a measurable function λ: $TM_1 \to R$ such that $KU^{+^{-1}} = \lambda U^{+}$ a.e. But if the first and last terms of (12) coincide we must also have that $\operatorname{tr}(-KU^{+^{-1}})$ is a constant scalar multiple of $\operatorname{tr} U^{+}$. Since $-KU^{+^{-1}} = -\lambda U^{+}$ this implies that λ is constant curvature. The converse property is well known.

We didn't find any significant lower estimates for metric entropies but we conjecture that for manifolds without conjugate points either $h_{\lambda_0}(\varphi) > 0$ or M is flat. Replacing $h_{\lambda_0}(\varphi)$ by $h_{\mathrm{top}}(\varphi)$ we obtain a weaker conjecture that is by Theorem I equivalent to the question of whether the fundamental group of a non flat manifold without conjugate points has exponential growth. For non flat manifolds without focal points it is known that $h_{\lambda_0}(\varphi) > 0$ (Pesin [13]). In particular $\lambda = h_{\mathrm{top}}(\varphi) \geqq h_{\lambda_0}(\varphi) > 0$ and the fundamental group has exponential growth. This property had already been proved by Avez [1].

I. Proof of Theorem I

Suppose that γ: $\mathbb{R} \to M$ is a geodesic arc parametrized by arc length and that for each $t \in \mathbb{R}$ we give a linear map $V(t)$: $E(\gamma(0), \dot\gamma(0)) \to E(\gamma(t), \dot\gamma(t))$. We define the derivative of V as the linear map:

$$\left(\frac{DV}{Dt}\right)(t)\colon E(\gamma(0), \dot\gamma(0)) \to E(\gamma(t), \dot\gamma(t))$$

defined by:

$$\left(\frac{DV}{Dt}\right)(t)\, w = \frac{D(V(t)\, w)}{Dt}.$$

For each $\theta \in TM_1$ let $Y_\theta(t)$: $E(\theta) \to E(\varphi_t(\theta))$, $t \in \mathbb{R}$, be the solution of the Jacobi equation:

$$\left(\frac{D^2 Y_\theta}{Dt^2}\right)(t) = -K(\varphi_t(\theta))\, Y_\theta(t) \tag{1}$$

with initial condition:

$$Y_\theta(0) = 0,$$

$$\left(\frac{DY_\theta}{Dt}\right)(0) = I.$$

Then for all $\theta \in TM_1$ and $v \in E(\theta)$, the Jacobi perpendicular vector field J along the geodesic determined by θ with initial conditions $J(0) = 0$, $(DJ/Dt)(0) = v$ is given by $J(t) = Y_\theta(t)\, v$. It is well known that for all $p \in M$ the derivative of the exponential map $\exp_p$: $T_p M \to M$ is given by the expression:

$$(D_{tv} \exp_p)\, w = \frac{1}{t}\, Y_{(p,v)}(t)\, w \tag{2}$$

where $t>0$, $v\in T_pM$ has $\|v\|=1$ and $w\in E(p,v)$. Then:

$$\det(D_{tv}\exp_p)=\frac{1}{t^{n-1}}\det Y_{(p,v)}(t). \tag{3}$$

Moreover, if $S_p=\{w\in T_pM \mid \|w\|=1\}$, it follows from (3) that

$$\operatorname{vol} B_r(p)=\int_{S_r}\left(\int_0^r |\det Y_{(p,v)}(t)|\,dt\right)dv. \tag{4}$$

If $\theta\in TM_1$ and $t\neq 0$ define the linear map $U_\theta(t)$: $E(\varphi_t(\theta))\hookleftarrow$ by:

$$U_\theta(t)=\left(\frac{DY_\theta}{Dt}\right)(t)\,Y_\theta(t)^{-1}.$$

Since M has no conjugate points $Y_\theta(t)$ is an isomorphism for all $\theta\in TM_1$ and $t\neq 0$. Hence $Y_\theta(t)^{-1}$ exists and $U_\theta(t)$ is well defined. Moreover $U_\theta(t)$ is self adjoint. A proof of this elementary fact can be found in Green [4] or directly checked by the reader, observing that the derivatives of the functions of t $\langle U_\theta^{-1}(t)v,w\rangle$ and $\langle v, U_\theta(t)^{-1}w\rangle$ coincide and these functions converge to 0 if $t\to 0$. Green also proved in [4] that there exists $A>0$ such that

$$\|U_\theta(t)\|\leqq A \tag{5}$$

for all $\theta\in TM_1$ and $t\geqq 1$. To prove Theorem I we shall need the following lemmas. The first one is a slightly improved version of Manning's proof of the existence of the growth rate λ.

Lemma I.1. *For every $\varepsilon>0$ there exists $C_\varepsilon>0$ such that:*

$$\operatorname{vol} B_r(p)\leqq C_\varepsilon \exp(\lambda+\varepsilon)r$$

for every $r>0$ and $p\in\tilde{M}$.

Proof. Choose a fundamental domain N of $\tilde{M}$. Let $a=\operatorname{diam}(\tilde{N})$. Then $B_r(q)\subset B_{r+a}(p)$ for every p and q in N and all $r>0$. Therefore $\operatorname{vol} B_r(q)\leqq \operatorname{vol} B_{r+a}(q)$. Fix $p\in N$ and choose $r_0>0$ such that $\operatorname{vol} B_r(p)\leqq \exp(\lambda+\varepsilon)r$ if $r\geqq r_0$. Then if $q\in N$:

$$\operatorname{vol} B_r(q)\leqq \operatorname{vol} B_{r+a}(p)\leqq \exp(\lambda+\varepsilon)a\cdot\exp(\lambda+\varepsilon)r. \tag{6}$$

Since for every $q\in\tilde{M}$ there exists $q'\in N$ with $\operatorname{vol} B_r(q)=\operatorname{vol} B_r(q')$ for all $r>0$, the inequality (6) holds for all $q\in\tilde{M}$ and every $r\geqq r_0$. Putting $C_\varepsilon=\max\{\exp(\lambda+\varepsilon)a, \max\{\exp(-(\lambda+\varepsilon)r)\operatorname{vol} B_r(q)\mid 0<r\leqq r_0, q\in\tilde{M}\}\}$ the lemma is proved.

The second lemma is much more delicate and we shall give its proof only after completing the demonstration of Theorem I:

Lemma I.2. *There exists $C>0$ such that:*

$$\|(D_\theta\varphi_t)^\wedge\|\leqq C\,|\det Y_\theta(t)|$$

for every $\theta\in TM_1$ and t with $|t|\geqq 1$.

Then, by Przytycki's inequality and Lemma I.2:

$$h(\varphi) \leqq \limsup_{t \to +\infty} \frac{1}{t} \log \int_{TM_1} \|(D_x \varphi_t)^\wedge\| \, d\lambda_0(x)$$

$$\leqq \limsup_{t \to +\infty} \frac{1}{t} \log \int_{TM_1} |\det Y_x(t)| \, d\lambda_0(x).$$

Since $\det Y_x(t) \neq 0$ for all $x \in TM_1$ and $t > 0$ it follows that $\det Y_x(t)$ has constant sign. Then the definition of $U_x(t)$ implies:

$$\frac{d}{dt} |\det Y_x(t)| = \frac{d}{dt} (\det Y_x(t)) = \operatorname{tr} U_x(t) \det Y_x(t)$$

$$= \operatorname{tr} U_x(t) \, |\det Y_x(t)|$$

and

$$h(\varphi) \leqq \limsup_{t \to +\infty} \frac{1}{t} \log \int_{TM_1} (|\det Y_x(1)|$$

$$+ \int_1^t \operatorname{tr} U_x(s) \, |\det Y_x(s)| \, ds) \, d\lambda_0(x).$$

By (5):

$$h(\varphi) \leqq \limsup_{t \to +\infty} \frac{1}{t} \log \int_{TM_1} (n-1) A \int_1^t |\det Y_x(s)| \, ds) \, d\lambda_0(x)$$

$$= \limsup_{t \to +\infty} \frac{1}{t} \log \int_{TM_1} \left(\int_1^t |\det Y_x(s)| \, ds \right) d\lambda_0(x)$$

$$= \limsup_{t \to +\infty} \frac{1}{t} \log \int_M \int_{S_p} \int_1^t |\det Y_{(p,v)}(s)| \, ds \, dv \, d\lambda_1(p).$$

By (4):

$$h(\varphi) \leqq \limsup_{t \to +\infty} \frac{1}{t} \log \int_M (\operatorname{vol} B_t(p) - \operatorname{vol} B_1(p)) \, d\lambda_1(p)$$

$$= \limsup_{t \to +\infty} \frac{1}{t} \log \int_M \operatorname{vol} B_t(p) \, d\lambda_1(p).$$

Given $\varepsilon > 0$ and taken $C_\varepsilon > 0$ with the property of Lemma I.1 we obtain:

$$h(\varphi) \leqq \limsup_{t \to +\infty} \frac{1}{t} \log \int_M C_\varepsilon \exp(\lambda + \varepsilon) t \, d\lambda_1 = \lambda + \varepsilon.$$

Since ε is arbitrarily small the theorem is proved.

Now we shall prove I.2. For its proof, as well as for the proof of Theorem II that we shall give in the next section, the crucial step is the following lemma:

Lemma I.3. *There exists $B > 0$ such that:*

$$\|Y_\theta(t) v\| \geqq B \|v\|$$

for all $\theta \in TM_1$, $|t| \geqq 1$ and $v \in E(\theta)$.

Proof. Define $K_0 = \sup\{\|K(\theta)\| \mid \theta \in TM_1\}$ and $A' = 3^{-1}K_0 + A + 1$, where A is given by (5). Fix a C^∞ function $\psi: R \hookleftarrow$ such that $\psi(0) = 1$ and $\psi(t) = 0$ for $|t| \geqq 1$. Define:

$$C = \int_{-1}^{1} (|\psi''\psi| + K_0\psi^2)\, dt$$

and choose B satisfying

$$0 < B < (A'C)^{-\frac{1}{2}}. \tag{7}$$

Suppose that $\|Y_\theta(T)v\| < B\|v\|$ for some $\theta \in TM_1$, $v \in E_\theta$ and $|T| > 1$. Assume that $T > 1$ (the other case follows applying the same method). Let $\gamma: R \to M$ be the geodesic with initial condition θ and J the perpendicular Jacobi vector field with initial conditions $J(0) = 0$, $(DJ/Dt)(0) = v$. Then $J(t) = Y_\theta(t)v$ and $\|J(T)\| < B\|v\|$. On the space Ω_T of continuous piecewise C^2 perpendicular vector fields along the arc $\gamma/[-1, T+1]$ that vanish at the endpoints define the index form:

$$I_T(V, W) = -\int_{-1}^{T+1} \left\langle \frac{D^2 V}{Dt^2}(t) + K(\varphi_t(\theta))\, V, W \right\rangle - \sum_i \left\langle \Delta_i\left(\frac{DV}{Dt}\right), W(t_i) \right\rangle$$

where $\Delta_i(DV/Dt)$ denotes the jump of the function DV/Dt at the discontinuity t_i. Following Klingenberg [6] define an element $\tilde{J} \in \Omega_T$ by $\tilde{J}(t) = 0$ for $-1 \leqq t \leqq 0$, $\tilde{J}(t) = J(t)$ for $0 \leqq t \leqq T$ and $\tilde{J}(t) = (T+1-t)\,\tau_T^t J(T)$ for $T \leqq t \leqq T+1$, where $\tau_T^t: T_{\gamma(T)}M \to T_{\gamma(t)}M$ denotes the parallel transport along γ. Then:

$$I_T(\tilde{J}, \tilde{J}) = -\int_T^{T+1} (T+1-t)^2 \langle K(\varphi_t(\theta))\, \tau_T^t J(T) \rangle\, dt$$
$$+ \left\langle \left(\frac{DJ}{Dt}\right)(T), J(T) \right\rangle + \|J(T)\|^2.$$

But $(DJ/Dt)(T) = U_\theta(T)\,J(T)$. Then, since $T \geqq 1$, (5) implies

$$\left| \left\langle \left(\frac{DJ}{dt}\right)(T), J(T) \right\rangle \right| \leqq A\|J(T)\|^2 \leqq AB^2.$$

Then:

$$I_T(\tilde{J}, \tilde{J}) \leqq \tfrac{1}{3} K_0 \|J(T)\|^2 + (A+1)\, B^2 \leqq A'B^2.$$

Define $Z \in \Omega_T$ by $Z(t) = \psi(t)\,\tau_0^t((DJ/Dt)(0))$ for $|t| \leqq 1$ and $Z(t) = 0$ otherwise. It is easy to verify that:

$$I_T(\tilde{J}, Z) = -\|(DJ/Dt)(0)\|^2 = -1,$$

$$I_T(Z, Z) = -\int_{-1}^{1} (\psi''(t)\,\psi(t) + \psi^2(t)\, \langle K(\varphi_t(\theta))\, \tau_0^t(DJ/Dt)(0),$$
$$\tau_0^t(DJ/Dt)(0) \rangle)\, dt \leqq \left(\int_{-1}^{1} |\psi''\psi|\, dt + K_0 \int_{-1}^{1} \psi^2\, dt \right) = C,$$

Then, if $\lambda \in \mathbb{R}$:

$$I_T(Z-\lambda \tilde{J}, Z-\lambda \tilde{J}) \leqq C+2\lambda+\lambda^2 A' B^2.$$

Since $4-4A'CB^2>0$ (because B was chosen satisfying (7)) this polynomial in λ has two real roots. Therefore $I_T(Z-\lambda \tilde{J},\ Z-\lambda \tilde{J})<0$ for some value of λ, thus proving the existence of conjugate points in the arc $\gamma/[-1, T+1]$.

Now we are ready to prove I.2. Consider the vector bundle F on TM_1 whose fiber F_θ over 0 is the subspace of $T_\theta(TM_1)$ orthogonal to the vector field associated to the geodesic flow. It is well known that F is invariant under the derivative $D\varphi_t$ of the flow φ_t. Consider also the vector bundle G on TM_1 whose fiber G_θ over θ is $E(\theta)\times E(\theta)$. If $t\in \mathbb{R}$ define the vector bundle isomorphism ψ_t: $G \hookleftarrow$ as follows: if $(v, w)\in G_\theta$ let J be the Jacobi vector field along the geodesic determined by θ, with initial conditions $J(0)=v$, $(DJ/Dt)(0)=w$. Set $\psi_t(v, w)=(J(t), (DJ/Dt)(t))$. The transformations ψ_t define a flow of vector bundle isomorphisms covering φ_t. In [3] Eberlein proved that there exists a vector bundle isomorphism H: $G\to F$, covering the identity and isometric in each fiber, such that

$$H\circ \psi_t=((D\varphi_t)/F)\circ H$$

for all $t\in R$. Therefore:

$$\|(\psi_t/G_\theta)^\wedge\| = \|((D_\theta\varphi_t)/F)^\wedge\|$$

and since:

$$\|((D_\theta\varphi_t)/F)^\wedge\| = \|(D_\theta\varphi_t)^\wedge\|$$

we obtain:

$$\|(\psi_t/G_\theta)^\wedge\| = \|(D_\theta\varphi_t)^\wedge\|.$$

Hence our problem is now to find $C>0$ satisfying

$$\|(\psi_t/G_\theta)^\wedge\| \leqq C\,|\det Y_\theta(t)|$$

for all $|t|\geqq 1$, $\theta\in TM_1$. From now on we shall consider the case $t\geqq 1$. The case $t\leqq -1$ follows applying the same methods. Write graph $U_\theta(t) =\{(u, U_\theta(t)\,u)\in G_{\varphi_t(\theta)}\,|\, u\in E(\varphi_t(\theta))\}$. The definitions of ψ_t and $U_\theta(t)$ imply:

$$\psi_t(\{0\}\times E(\theta))=\text{graph } U_\theta(t) \tag{8}$$

for all $t\in\mathbb{R}$. In particular:

$$\psi_{-t}(\{0\}\times E(\varphi_t(\theta)))=\text{graph } U_{\varphi_t(\theta)}(-t). \tag{9}$$

Consider the splittings:

$$G_\theta=(\{0\}\times E(\theta)) \oplus \text{graph } U_{\varphi_t(\theta)}(-t), \tag{10}$$

$$G_{\varphi_t(\theta)}=(\{0\}\times E(\varphi_t(\theta))) \oplus \text{graph } U_\theta(t). \tag{11}$$

Observe that by (8) and (9) ψ_t/G_θ sends the factors in the first decomposition onto those of the second. Moreover if $u\in E(\varphi_t(\theta))$:

$$\psi_{-t}(0, u)=(Y_{\varphi_t(\theta)}(-t)\,u,\ U_{\varphi_t(\theta)}(-t)\,Y_{\varphi_t(\theta)}(-t)\,u).$$

Then, if $t \geqq 1$:

$$\|\psi_{-t}(0, u)\| \geqq \| Y_{\varphi_t(\theta)}(-t) u\| \geqq B \|u\|.$$

This inequality, together with (9), implies:

$$\|(\psi_t/\text{graph } U_{\varphi_t(\theta)}(-t))\| \leqq B^{-1}. \tag{12}$$

We can always assume $B<1$. Then (12) and the observation above about how ψ_t transforms the splitting (10) in the splitting (11), implies:

$$\|(\psi_t/G_\theta)^\wedge\| \leqq B^{-(n-1)} \|(\psi_t/\{0\} \times E(\theta))^\wedge\|. \tag{13}$$

Moreover, if π_t: $G_{\varphi_t(\theta)} \to E(\varphi_t(\theta))$ is the canonical projection onto the first factor, we have:

$$\pi_t \circ (\psi_t/(\{0\} \times E(\theta)) = Y_\theta(t)$$

which means:

$$\psi_t/(\{0\} \times E(\theta)) = (\pi_t/\text{graph } U_\theta(t))^{-1} \circ Y_\theta(t). \tag{14}$$

But (5) implies that if $t \geqq 1$:

$$\|(\pi_t/\text{graph } U_\theta(t))^{-1}\| = (1 + \|U_\theta(t)\|^2)^{1/2} \leqq (1+A^2)^{1/2}. \tag{15}$$

Hence (13), (14) and (15) imply:

$$\|(\psi_t/G_\theta)^\wedge\| \leqq B^{-(n-1)}(1+A^2)^{1/2} \| Y_\theta(t)^\wedge\|.$$

Now the problem has been reduced to finding $A'>0$ satisfying

$$\| Y_\theta(t)^\wedge\| \leqq A' |\det Y_\theta(t)|$$

for every $\theta \in TM_1$ and $t \geqq 1$. Observe that $\| Y_\theta(t)^\wedge\|$ is the maximum of $|\det Y_\theta(t)/S|$ where S varies in the set of all the subspaces of $E(\theta)$. Choose S such that $\| Y_\theta(t)^\wedge\| = |\det Y_\theta(t)/S|$. With respect to the splittings $E(\theta) = S \oplus S^\perp$, $E(\varphi_t(\theta)) = Y_\theta(t) S \oplus (Y_\theta(t) S)^\perp$, $Y_\theta(t)$ can be written in the form

$$Y_\theta(t) = \left(\begin{array}{c|c} Y_1 & Y_3 \\ \hline 0 & Y_2 \end{array}\right).$$

Then:

$$|\det Y_1| = |\det Y_\theta(t)/S| = \| Y_\theta(t)^\wedge\|$$

and

$$|\det Y_\theta(t)| = |\det Y_1|\,|\det Y_2| = |\det Y_2|\, \| Y_\theta(t)^\wedge\|.$$

From Lemma I.3 it follows that:

$$\| Y_1 v + Y_3 w\|^2 + \| Y_2 w\|^2 \geqq B^2(\|v\|^2 + \|w\|^2)$$

for all $v \in S$, $w \in S^\perp$. Take $v = -Y_1^{-1} Y_3 w$. Then $Y_1 v + Y_3 w = 0$ and:

$$\| Y_2 w\|^2 \geqq B^2(\|v\|^2 + \|w\|^2) \geqq B^2 \|w\|^2.$$

Then the image by Y_2 of the unit ball of $S^{\perp}$ contains a ball of radius B. This means that:

$$|\det Y_2| \geqq B^k$$

where $k = \dim S$. Hence:

$$\| Y_\theta(t)^\wedge \| = |\det Y_2|^{-1} |\det Y_\theta(t)| \leqq B^{-k} |\det Y_\theta(t)|.$$

II. Proof of Theorem II

First recall the construction due to Green [4] of solutions of the Ricatti equation of M (Eq. (7) of the Introduction). If $\theta \in TM_1$ and $s \in \mathbb{R}$ let $Y_{\theta,s}(t)$ be the solution of the Jacobi equation (Eq. (1) of Sect. I) such that $Y_{\theta,s}(s) = 0$, $Y_{\theta,s}(0) = I$. In [4] Green proves that $\lim_{s \to -\infty} Y_{\theta,s}(t)$ exists for all $\theta \in TM_1$ and $t \in \mathbb{R}$ (see also Eberlein [3], Sect. 2). He also shows that defining:

$$Y_\theta^+(t) = \lim_{s \to -\infty} Y_{\theta,s}(t) \tag{1}$$

we obtain a solution of Jacobi equation such that $\det Y_\theta^+(t) \neq 0$. Moreover it is proved in [4] and [3] that $(DY_\theta^+/Dt)(t) = \lim_{s \to -\infty} (DY_{\theta,s}/Dt)(t)$. Then, if we define

$$U_s(\theta) = \frac{DY_{\theta,s}}{Dt}(0), \tag{2}$$

$$U^+(\theta) = \frac{DY_\theta^+}{Dt}(0) \tag{3}$$

we obtain

$$U^+(\theta) = \lim_{s \to -\infty} U_s(\theta). \tag{4}$$

Moreover it is easy to check that

$$Y_{\varphi_h(\theta), s-h}(t) = Y_{\theta,s}(t+h)\, Y_{\theta,s}^{-1}(h).$$

Then

$$Y_{\varphi_h(\theta)}^+(t) = Y_\theta^+(t+h)\, Y_\theta^{+\,-1}(h)$$

and

$$U^+(\varphi_h(\theta)) = \frac{DY_\theta^+}{Dt}(h)\, Y_\theta^{+\,-1}(h)$$

for all $h \in \mathbb{R}$. From this it follows that U^+ is a solution of the Ricatti equation of M. From (4) follows that it is measurable (because $U_s(\theta)$ is continuous on θ) and from Lemma 2 of [4] (or [3] Sect. 6) it is bounded. We shall prove that this solution satisfies Theorem II.

Given $\theta \in TM_1$ and $s \in \mathbb{R}$ define

$$V_{\theta,s} = \operatorname{graph} U_s(\theta) = \{(x, U_s(\theta)x) \mid x \in E(\theta)\}$$
$$V_\theta = \operatorname{graph} U^+(\theta).$$

From (4) it follows that:

$$V_\theta = \lim_{s\to -\infty} V_{\theta,s}.$$

We claim that:

$$\psi_{-s}(\{0\}\times E(\varphi_s(\theta))) = V_{\theta,s}. \tag{5}$$

To prove this take $v\in E(\varphi_s(\theta))$ and let $J(t)$ be a Jacobi perpendicular vector field along the geodesic $\gamma(t)$ with initial condition $(\gamma(0), \dot\gamma(0)) = \varphi_s(\theta)$ such that $J(0)=0$, $(DJ/Dt)(0)=v$. The by the definition of ψ:

$$\psi_{-s}(0,v) = (J(-s), (DJ/Dt)(-s)).$$

Consider the Jacobi vector field $\hat J(t) = J(t-s)$ along the geodesic $\hat\gamma(t) = \gamma(t-s)$. Then $\hat J(s)=0$ and $\hat J(0) = J(-s)$. Hence:

$$\hat J(t) = Y_{\theta,s}(t)\, J(-s)$$

or

$$J(t) = Y_{\theta,s}(t+s)\, J(-s).$$

Then

$$\left(\frac{DJ}{Dt}\right)(-s) = \frac{DY_{\theta,s}}{Dt}(0)\, J(-s) = U_s(\theta)\, J(-s)$$

which means by the definition of $V_{\theta,s}$ that:

$$\psi_{-s}(\{0\}\times E(\varphi_s(\theta))) \subset V_{\theta,s}.$$

The equality now follows from the fact that both members have the same dimension.

By Oseledec's theory there exists a total probability Borel set $\Lambda\subset M$ such that for every $x\in\Lambda$ there is a unique splitting $G_x = E^s(x)\oplus E^c(x)\oplus E^u(x)$ such that:

$$\lim_{t\to\pm\infty} \frac{1}{t}\log\|\psi_t v\| = 0 \qquad v\in E^c(x),$$

$$\lim_{t\to\pm\infty} \frac{1}{t}\log\|\psi_t v\| > 0 \qquad v\in E^u(x),$$

$$\lim_{t\to\pm\infty} \frac{1}{t}\log\|\psi_t v\| < 0 \qquad v\in E^s(x)$$

and for every subspace $E^u(x)\subset S\subset E^u(x)\oplus E^c(x)$ the limit

$$\chi(x) = \lim_{t\to+\infty} \frac{1}{t}\log(\det(\psi_t/S)) \tag{6}$$

exists and is independent of S.

We claim that for any φ-invariant probability μ on M the following properties hold for μ-a.e. $\theta\in\Lambda$:

$$E^u(\theta)\subset V_\theta\subset E^u(\theta)\oplus E^c(\theta), \tag{7}$$

$$\chi(\theta) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \operatorname{tr} U^+(\varphi_t(\theta))\, dt. \tag{8}$$

From (8) follows Theorem II because it is easy to prove, using the isomorphism $H: G \to F$ defined in Sect. I, that the function χ associated to the linear flow $\psi: \mathbb{R} \times G \to G$ coincides with that associated to the linear flow $D\varphi: \mathbb{R} \times T(TM_1) \to TM_1$.

To prove the second inclusion in (7) observe that if $v \in E(\theta)$:

$$\psi_t(0, v) = \left(Y_\theta(t)v, \left(\frac{DY_\theta}{Dt}\right)(t)v\right),$$

Then Lemma I.3 implies that $(0, v) \notin E^s(\theta)$ if $v \neq 0$. Hence

$$E^s(\theta) \cap (\{0\} \times E(\theta)) = \{0\}.$$

Therefore for every $\theta \in \Lambda$ there exist a subspace $T(\theta) \subset E^u(\theta) \oplus E^c(\theta)$ and a linear map $L(\theta)$: $T(\theta) \to E^s(\theta)$ such that

$$\{0\} \times E(\theta) = \operatorname{graph} L(\theta).$$

Given $\varepsilon > 0$ take a compact subset $K \subset \Lambda$ with $\mu(K) \geqq 1 - \varepsilon$ and satisfying:

a) $\sup\{\|L(\theta)\| \,|\, \theta \in K\} < +\infty$.

b) There exists $T > 0$ and $0 < \xi < 1 < \tilde{\xi} < \xi^{-1}$ such that:

$$\|\psi_t / E^s(\theta)\| \leqq \xi^t,$$
$$\|\psi_t^{-1} / (E^u(\theta) \oplus E^c(\theta))\| \leqq \tilde{\xi}^t$$

for every $\theta \in K$, $t \geqq T$.

Then, if $\theta \in \Lambda$:

$$\begin{aligned} V_{\theta,s} &= \psi_s(\{0\} \times E(\varphi_{-s}(\theta))) = \psi_s(\operatorname{graph} L(\varphi_{-s}(\theta))) \\ &= \operatorname{graph}(\psi_s \circ L(\varphi_{-s}(\theta)) \circ (\psi_{-s} / L(\theta))). \end{aligned}$$

If $\theta \in K$ take a sequence $s_n \to -\infty$ such that $\varphi_{s_n}(\theta) \in K$. Then by property (a):

$$\|\psi_{s_n} \circ L(\varphi_{-s_n}(\theta)) \circ (\psi_{-s_n} / L(\theta))\| \leqq \xi^{s_n} \tilde{\xi}^{s_n} \sup_{\theta \in K} \|L(\theta)\|.$$

But $\xi^{s_n} \tilde{\xi}^{s_n} \to 0$ if $n \to +\infty$. Then:

$$V_\theta = \lim_{n \to +\infty} V_{\theta, s_n} = T(\theta) \subset E^u(\theta) \oplus E^c(\theta)$$

therefore the second inclusion in (7) is satisfied by any $\theta \in K$. Since $\mu(K) \geqq 1 - \varepsilon$ and ε is arbitrary this inclusion holds for a.e. $\theta \in \Lambda$.

To prove the first inclusion of (7) consider the splitting:

$$G_\theta = V_\theta + V_\theta^\perp$$

for $\theta \in \Lambda$. Since V_θ is ψ_t-invariant, with respect to this splitting ψ_t can be written in block form:

$$\psi_t = \left(\begin{array}{c|c} A_t(\theta) & C_t(\theta) \\ \hline 0 & B_t(\theta) \end{array}\right).$$

Now recall that if $J_1(t)$, $J_2(t)$ are perpendicular Jacobi vector fields of M it is known that:

$$\left\langle J_1(t), \frac{DJ_2}{Dt}(t)\right\rangle - \left\langle J_2(t), \frac{DJ_1}{Dt}(t)\right\rangle$$

is constant in t. This means that the vector bundle isomprphism $J: G \hookleftarrow$ covering the identity and acting on the fibers G_θ as:

$$J(v, w) = (-w, v)$$

satisfies:

$$\langle J\psi_t u, \psi_t v\rangle = \langle J u, v\rangle$$

for all $u, v \in G_\theta$, $t \in \mathbb{R}$, $\theta \in M$. This means that

$$\psi_t^* J \psi_t = J$$

for all t, or:

$$A_t(\theta) = B_t^{*-1}(\theta)$$

for all $\theta \in \Lambda$, $t \in \mathbb{R}$. Since $V_\theta \subset E^u(\theta) \oplus E^c(\theta)$:

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|B_t^{*-1}(\theta) w\| = \lim_{t\to\pm\infty} \frac{1}{t} \log \|A_t(\theta) w\| \geqq 0$$

for all $w \in V_\theta^\perp$. Then it is easy to see for all $w \in V_\theta^\perp$:

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|B_t^{-1}(\theta) w\| \geqq 0. \tag{9}$$

Now suppose that $E^u(\theta) \not\subset V_\theta$. Take $0 \neq v \in E^u(\theta)$, $v \notin V_\theta$. Then $v = w + z$, $0 \neq w \in V_\theta^\perp$, $z \in V_\theta$. Moreover:

$$\psi_{-t}(v) = B_t^{-1}(\theta)\, w + u_t$$

for some

$$u_t \in V_{\varphi_{-t}}(\theta).$$

Then:

$$\|\psi_{-t}(v)\| \geqq \|B_t^{-1}(\theta) w\|$$

and by (9):

$$\lim_{t\to+\infty} \frac{1}{t} \log \|\psi_{-t}(v)\| \geqq 0$$

contradicting $v \in E^u(\theta)$.

Finally we shall prove (8). Let $\pi_\theta: V_\theta \to E(\theta)$ be the restriction to $V_\theta \subset E(\theta) \times E(\theta)$ of the canonical projection onto the first factor. Then:

$$\pi_\theta^{-1}(v) = (v, U_\theta^+(v)).$$

Therefore:

$$\sup_{\theta\in\Lambda} \|\pi_\theta^{-1}\| < +\infty. \tag{10}$$

Moreover it is easy to check that:

$$\psi_t/V_\theta = \pi^{-1}_{\varphi_t(\theta)} \circ Y_\theta^+(t) \circ \pi_\theta \tag{11}$$

for all $\theta \in \Lambda$, $t \in \mathbb{R}$. From (10, (11) and Lemma I.3 we conclude that:

$$\begin{aligned}\lim_{t\to+\infty} \frac{1}{t} \log\det(\psi_t/V_\theta) &= \lim_{t\to+\infty} \frac{1}{t} \log\det Y_\theta^+(t)\\ &= \lim_{t\to+\infty} \frac{1}{t} \int_0^t \operatorname{tr} Y_\theta^+(s) \frac{DY_\theta^+}{Dt}(s)\, ds = \lim_{t\to+\infty} \frac{1}{t} \int_0^t \operatorname{tr} U^+(\varphi_\theta(s))\, ds.\end{aligned}$$

This equality, together with (6) and (7) implies (8) for μ-a.e. $\theta \in V$.

Remark I. When M has negative sectional curvatures there exists $c > 0$ such that $\langle U^+(\theta)v, v\rangle \geqq c\|v\|^2$ for all $\theta \in TM_1$, $v \in E(\theta)$. To prove this set $A = \sup\{\|U^+(\theta)\| \mid \theta \in TM_1\}$ and take $\delta > 0$ such that for any perpendicular Jacobi vector field J with $\|J(0)\| = 1$ and $\|(DJ/Dt)(0)\| \leqq 2A$ then $\|J(t)\| \geqq \frac{1}{2}$ for $-\delta \leqq t \leqq 0$. Then, given $\theta \in TM_1$, it follows that

$$\|Y_\theta^+(t)v\| \geqq \tfrac{1}{2}\|v\|$$

for all $\theta \in TM_1$, $v \in E(\theta)$, $-\delta \leqq t \leqq 0$. Then (1) and (1) imply that given $\theta \in TM_1$ there exists $C > 0$ such that if $-s \geqq C$:

$$\|Y_{\theta,s}(t)v\| \geqq \tfrac{1}{2}\|v\|$$

for every $-\delta \leqq t \leqq 0$ and $v \in E(\theta)$. Fix any $v \in E(\theta)$ and define:

$$u(t) = \langle Y_{\theta,s}(t)v, Y_{\theta,s}(t)v\rangle.$$

Then:

$$u(s) = \dot u(s) = 0,$$

$$\dot u(0) = 2\langle U_s(\theta)v, v\rangle.$$

Hence:

$$\langle U_s(\theta)v, v\rangle = \tfrac{1}{2} \int_s^0 \ddot u(r)\, dr.$$

Choose $k > 0$ satisfying $\langle K(x)w, w\rangle \leqq -k\|w\|^2$ for all $x \in TM_1$ and $w \in E(x)$. It is easy to verify that:

$$\ddot u(r) \geqq k\, u(r).$$

Then, if $-s \geqq C$:

$$\begin{aligned}\langle U_s(\theta)v, v\rangle &\geqq \tfrac{1}{2}k \int_s^0 u(r)\, dr = \tfrac{1}{2}k \int_s^0 \langle Y_{\theta,s}(r)v, Y_{\theta,s}(r)v\rangle\, dr\\ &\geqq \tfrac{1}{2}k \int_{-\delta}^0 \langle Y_{\theta,s}(r)v, Y_{\theta,s}(r)v\rangle\, dr \geqq \tfrac{1}{8}\delta k\|v\|^2.\end{aligned}$$

Therefore $\langle U^+(\theta)v, v\rangle \geqq \lim_{s\to-\infty} \langle U_s(\theta)v, v\rangle \geqq 8^{-1}\delta k\|v\|^2$.

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