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Support varieties for restricted Lie algebras

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Much progress has been made recently in studying the geometric properties of the cohomology of finite groups. In [7], the authors introduced analogous methods into the study of the cohomology of restricted Lie algebras. Although the formalism was established for arbitrary finite dimensional restricted Lie algebras, these results were by and large applicable only to p-unipotent Lie algebras (i.e., those with nilpotent restriction map). For such Lie algebras $\mathfrak{g}$, [7; 2.7] relates the cohomological support variety $|\mathfrak{g}|_M$ of a restricted $\mathfrak{g}$-module M with the rank variety, the subvariety of $\mathfrak{g}$ corresponding to those cyclic subalgebras for which M is not projective (thereby providing an extension of "Carlson's conjecture" [2] from elementary abelian p-groups to p-unipotent Lie algebras).

In this paper, we extend the results of our previous paper to arbitrary finite dimensional restricted Lie algebras and restricted modules. In particular, Corollary 1.4 presents a definitive generalization of Carlson's Conjecture for restricted Lie algebras. For a connected linear algebraic group G, Theorem 1.2 provides a description of the image inside the Lie algebra of the support variety of a rational G-module in terms of the corresponding variety of a Borel subgroup of G. In Theorem 1.3, we present a geometric cohomological criterion which identifies those cyclic, p-unipotent subalgebras of $\mathfrak{g}$ restricted to which a given restricted $\mathfrak{g}$-module is projective. Corollary 1.4 is then an easy consequence of Theorems 1.2 and 1.3 together with a recent theorem of Jantzen [10].

Section 2 presents various applications of these results. For example, Theorem 2.2 provides necessary and sufficient conditions for a subvariety of a finite dimensional restricted Lie algebra $\mathfrak{g}$ to be the image of the support variety of some restricted $\mathfrak{g}$-module. Proposition 2.4 indicates how knowledge of the G-orbit structure of the subvariety of nilpotent elements of $\mathfrak{g}=\mathrm{Lie}(G)$ for G simple can provide information concerning restricted $\mathfrak{g}$-modules. Section 3 concludes with a few problems worthy of further study.

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A first draft of this paper offered weaker results applicable only to Lie algebras of reductive algebraic groups with a restriction on the prime p and only to rational modules. A recent preprint [10] by J.C. Jantzen inspired us to rethink our arguments from a slightly different point of view, enabling us to obtain the present general results by eliminating any restriction on the size of p or on the module. We are most grateful to J.C. Jantzen for providing us with his manuscript [10].

1. Support varieties

Throughout this paper we consider a fixed algebraically closed field k of positive characteristic p and we consider only algebraic groups over k. Let G be an affine algebraic group defined over the prime field $\mathbf{F}_p$ with Frobenius morphism $\sigma: G \to G$. We denote by G_1 the (group-scheme theoretic) kernel of σ. If V is a rational G-module, we write $V^{(1)}$ for the corresponding "twisted" G-module, obtained by making G act on V though σ. Equivalently, $V^{(1)}$ has the same G-structure as V, but a new k-structure in which k acts on V through σ^{-1}. Given a rational G-module V such that G_1 is in the kernel of the action of G on V, there exists a rational G-module W such that $V \simeq W^{(1)}$, and we often write $V^{(-1)}$ for W. We refer the reader to [4; § 3] for a discussion of these matters.

As is well known, the category of rational G_1-modules is equivalent to the category of restricted $\mathfrak{g}$-modules with $\mathfrak{g} = \mathrm{Lie}(G_1)$. The latter is by definition the category of modules for the restricted enveloping algebra $V(\mathfrak{g})$ of the restricted Lie algebra $\mathfrak{g}$. This follows from the facts that a rational G_1-module is by definition a comodule for the coordinate ring $k[G_1]$ of G_1 and that $V(\mathfrak{g})$ is naturally isomorphic to the dual algebra of $k[G_1]$. Without further comment, we frequently view a restricted $\mathfrak{g}$-module as a rational G_1-module and cohomology of the algebra $V(\mathfrak{g})$ with coefficients in such a module as rational cohomology of G_1.

Let G be a connected linear algebraic group defined over $\mathbf{F}_p$ and let $B \subset G$ be a Borel subgroup over $\mathbf{F}_p$. For a rational B-module M, we denote by $H^0(G/B, M)$ the rational G-module obtained by inducing M from B to G (cf. [4; § 1]). The n^{th} right derived functor of $H^0(G/B, -)$ we denote by $H^n(G/B, -)$, the notation indicating the well-known interpretation of $H^*(G/B, M)$ as certain (sheaf) cohomology groups of the quotient variety G/B. In terms of rational cohomology, $H^*(G/B, M) \simeq H^*(B, M \otimes k[G])$, where the action of G on $H^*(B, M \otimes k[G])$ is defined by the action of G on the coordinate ring $k[G]$ of G given by $(g \cdot f)(x) = f(g^{-1}x)$ for $f \in k[G]$ and $x, g \in G$ (cf. [5; 2.9]).

We require the following spectral sequence first obtained for reductive algebraic groups by H.H. Andersen and J.C. Jantzen in [1; § 3]. For the reader's convenience, we sketch a particularly simple derivation.

(1.1) **Proposition.** *Let G be a connected linear algebraic group defined over $\mathbf{F}_p$ and let M be a rational G-module. There is a first quadrant spectral sequence*

$$E_2^{*,*}(M) = H^*(G/B, H^*(B_1, M)^{(-1)}) \Rightarrow H^*(G_1, M).$$

Proof. We have

$$H^*(G_1, M) \simeq H^*(G, M \otimes k[G]^{(1)}) \quad \text{by [4; 4.4]}$$
$$\simeq H^*(B, M \otimes k[G]^{(1)}) \quad \text{by [11; Theorem 2].}$$

The asserted spectral sequence is the Lyndon-Hochschild-Serre spectral sequence (cf. [4; 4.5]) with $E_2^{*,*}$-term

$$H^*(B/B_1, H^*(B_1, M \otimes k[G]^{(1)})) \simeq H^*(B, H^*(B_1, M)^{(-1)}) \otimes k[G]). \quad \square$$

If $\mathfrak{g}$ is a (finite dimensional) restricted Lie algebra, the cohomology variety $|\mathfrak{g}|$ of $\mathfrak{g}$ is the affine algebraic variety associated to the commutative k-algebra $H^{ev}(V(\mathfrak{g}), k)^{(-1)}$ (cf. [7; 1.4]). If M is a finite dimensional restricted $\mathfrak{g}$-module, the support variety $|\mathfrak{g}|_M$ of M is the support in $|\mathfrak{g}|$ of the $H^{ev}((V(\mathfrak{g}), k)^{(-1)}$-module $H^*(V(\mathfrak{g}), M \otimes M^*)^{(-1)}$. Since the action of $H^*(V(\mathfrak{g}), k)^{(-1)}$ on $H^*(V(\mathfrak{g}), M \otimes M^*)^{(-1)}$ is defined by the natural algebra homomorphism

$$H^*(V(\mathfrak{g}), k)^{(-1)} \simeq \mathrm{Ext}^*_{V(\mathfrak{g})}(k, k)^{(-1)}$$
$$\to \mathrm{Ext}^*_{V(\mathfrak{g})}(M, M)^{(-1)} \simeq H^*(V(\mathfrak{g}), M \otimes M^*)^{(-1)},$$

it follows that $|\mathfrak{g}|_M$ is the zero locus in $|\mathfrak{g}|$ of the homogeneous ideal $\ker\{H^{ev}(V(\mathfrak{g}), k)^{(-1)} \to H^*(V(\mathfrak{g}), M \otimes M^*)^{(-1)}\}$. Consequently, $|\mathfrak{g}|_M$ is a conical, closed subvariety of $|\mathfrak{g}|$.

As in [7; 2.1], we denote by $\Phi: |\mathfrak{g}| \to \mathfrak{g}$ the morphism of varieties associated to the natural map $\Phi^*: S^*(\mathfrak{g}^*) \to H^{ev}(V(\mathfrak{g}), k)^{(-1)}$, where $S^*(\mathfrak{g}^*)$ is the symmetric algebra on the dual of $\mathfrak{g}$ (or, equivalently, the algebra of polynomial functions on $\mathfrak{g}$). By [7; 2.4], we may identify $\Phi(|\mathfrak{g}|_M)$ for a finite dimensional restricted $\mathfrak{g}$-module M with the support of the graded $S^*(\mathfrak{g}^*)$-module $H^*(V(\mathfrak{g}), M \otimes M^*)^{(-1)}$. So defined, $\Phi(|\mathfrak{g}|_M)$ is a conical, closed subvariety of $\mathfrak{g}$ (cf. [7; 2.4]).

The map Φ^* is obtained by multiplicatively extending the natural map $\mathfrak{g}^* \to H^2(V(\mathfrak{g}), k)^{(-1)}$ constructed by Hochschild: for $\varphi \in \mathfrak{g}^{*(1)}$ and $H^2(V(\mathfrak{g}), K)$ identified with the group of isomorphism classes of restricted extensions of $\mathfrak{g}$ by k, φ is sent to the split Lie algebra $k \oplus \mathfrak{g}$ with p-th power given by $(\alpha, X)^{[p]} = (\varphi(X)^p, X^{[p]})$ (cf. [9; p. 575]). For $p > 2$, Hochschild's construction can be identified with the edge homomorphism of the spectral sequence

$$E_0^{2s,t} = S^s(\mathfrak{g}^*)^{(1)} \otimes \Lambda^t(\mathfrak{g}^*) \Rightarrow H^{2s+t}(V(\mathfrak{g}), k) \quad \text{(cf. [8; 1.1]).}$$

Namely, let $\mathfrak{g}_{ab}$ denote the underlying vector space of $\mathfrak{g}$ viewed as an abelian Lie algebra with trivial restriction. Then the associated graded complex of the filtered cobar resolution for $k[G_1]$ is the cobar resolution for $V(\mathfrak{g}_{ab})$, so that the E_0-term of the spectral sequence is $H^*(V(\mathfrak{g}_{ab}), k)$. The inclusion $E_0^{2,0} \to H^2(V(\mathfrak{g}_{ab}), k)$ is given by Hochschild's construction for $\mathfrak{g}_{ab}$. The edge homomorphism sends $\varphi \in \mathfrak{g}^{*(1)}$ viewed as a 2-cocycle for $\mathfrak{g}_{ab}$ to the same cochain for $\mathfrak{g}$, which we know to be a 2-cocycle for $\mathfrak{g}$. Identifying 2-cocycles with extensions and identifying the isomorphism $\mathfrak{g}^{*(1)} \simeq E_0^{2,0}$ with Hochschild's construction for $\mathfrak{g}_{ab}$, we identify the edge homomorphism with Hochschild's construction for $\mathfrak{g}$. For $p = 2$, Hochschild's map can be similarly related to

$$H^*(V(\mathfrak{g}_{ab}), k) = (S^*(\mathfrak{g}^*), \partial) \quad \text{(cf. [8; 1.2])}$$

which plays the role of the above spectral sequence for $p>2$. Namely, Hochschild's map for $p=2$ and $\mathfrak{g}_{ab}$ can be identified with the diagonal map Δ: $\mathfrak{g}^{*(1)} \to S^2(\mathfrak{g}^*) \simeq H^2(V(\mathfrak{g}_{ab}), k)$. Hence, as argued for $p>2$, Hochschild's map for $p=2$ and $\mathfrak{g}$ sends $\varphi \in \mathfrak{g}^{*(1)}$ to the cocycle $\Delta(\varphi) \in (S^*(\mathfrak{g}^*), \partial)$.

The following theorem provides the key to extending the results of [7] for p-unipotent Lie algebras to general restricted Lie algebras.

(1.2) **Theorem.** *Let G be a connected linear algebraic group defined over $\mathbf{F}_p$, let $B=T\cdot U$ be a Borel subgroup also defined over $\mathbf{F}_p$, and let $\mathfrak{g}=\mathrm{Lie}(G)$, $\mathfrak{b}=\mathrm{Lie}(B)$. For any finite dimensional rational G-module M, we have*

$$\Phi(|\mathfrak{g}|_M) = G\cdot\Phi(|\mathfrak{b}|_M)$$

where $G\cdot\Phi(|\mathfrak{b}|_M)$ is the orbit of $\Phi(|\mathfrak{b}|_M)$ under the adjoint action of G. Consequently, if G is reductive and if p is greater than the Coxeter number of any simple quotient of G, then $|\mathfrak{g}|_M \simeq G\cdot|\mathfrak{b}|_M$.

Proof. We introduce the following ideals:

$$S^*(\mathfrak{b}^*) \supset I_M = \ker\{S^*(\mathfrak{b}^*) \to H^*(B_1, k)^{(-1)} \to H^*(B_1, M\otimes M^*)^{(-1)}\},$$
$$S^*(\mathfrak{g}^*) \supset J_M = \ker\{S^*(\mathfrak{g}^*) \to H^*(G_1, k)^{(-1)} \to H^*(G_1, M\otimes M^*)^{(-1)}\},$$
$$S^*(\mathfrak{g}^*) \supset K_M = \{f\in S^*(\mathfrak{g}^*)\colon \text{for all } x\in G,\ x\cdot f|_{\mathfrak{b}} \in I_M\},$$
$$S^*(\mathfrak{g}^*) \supset L_M = \{f\in S^*(\mathfrak{g}^*)\colon \text{for all } x\in G,\ x\cdot f|_{\mathfrak{b}} \in \sqrt{I_M}\}.$$

By definition, $\Phi(|\mathfrak{b}|_M) = Z(I_M) \subset \mathfrak{b}$, the zero locus of I_M. Similarly, $\Phi(|\mathfrak{g}|_M) = Z(J_M) \subset \mathfrak{g}$. Since the radical of I_M, $\sqrt{I_M}$, is the ideal of functions on $\mathfrak{b}$ vanishing on the closed subvariety $\Phi(|\mathfrak{b}|_M) \subset \mathfrak{b}$, L_M is the ideal of functions vanishing on $G\cdot\Phi(|\mathfrak{b}|_M)$. We first verify that $G\cdot\Phi(|\mathfrak{b}|_M)$ is a closed subvariety of $\mathfrak{g}$, so that to prove the theorem it will suffice to prove that $\sqrt{J_M} = L_M$. Namely, $\Phi(|\mathfrak{b}|_M) \subset \mathfrak{b}$ is stable with respect to the adjoint action of B on $\mathfrak{b}$ because M is a rational G-module. Since $R = \{(xB, Y)\colon x^{-1}\cdot Y \in \Phi(|\mathfrak{b}|_M)\}$ is clearly a closed subvariety of $G/B\times\mathfrak{g}$ and since the projection $G/B\times\mathfrak{g}\to\mathfrak{g}$ is a closed map, the image $G\cdot\Phi(|\mathfrak{b}|_M)$ of R via the projection is closed.

By naturality, $\Phi(|\mathfrak{b}|_M) \subset \Phi(|\mathfrak{g}|_M)$ (cf. [7; 2.4]). Since $\Phi(|\mathfrak{g}|_M)$ is G-stable, we conclude that $G\cdot\Phi(|\mathfrak{b}|_M) \subset \Phi(|\mathfrak{g}|_M)$ and thus $L_M \supset \sqrt{J_M}$. We claim that $\sqrt{K_M} \supset L_M$. Namely, let $f\in L_M$ and let I_f denote the ideal of $S^*(\mathfrak{g}^*)$ generated by the functions $x\cdot f$ as x ranges over the elements of G. Since $S^*(\mathfrak{g}^*)$ is Noetherian, I_f is finitely generated. Let $x_1\cdot f, \ldots, x_j\cdot f$ denote generators of I_f and let $A=p^B$ be a sufficiently large p-th power that each of $(x_1\cdot f^A)|_{\mathfrak{b}}, \ldots, (x_j\cdot f^A)|_{\mathfrak{b}}$ lie in I_M. (We implicity use the equality $x\cdot f^A = (x\cdot f)^A$.) For any $x\in G$, $x\cdot f = \sum \lambda_i (x_i\cdot f)$ for some choice of $\lambda_1, \ldots, \lambda_j \in S^*(\mathfrak{g}^*)$ depending on x; thus, $x\cdot f^A = \sum \lambda_i^A (x_i\cdot f^A)$ so that $(x\cdot f^A)|_{\mathfrak{b}} \in I_M$. We conclude that $f^A \in K_M$, so that $L_M \subset \sqrt{K_M}$ as claimed. Thus, to complete the proof of the equality $\Phi(|\mathfrak{g}|_M) = G\cdot\Phi(|\mathfrak{b}|_M)$, it will suffice to prove that $K_M \subset \sqrt{J_M}$.

The naturality of the map $\mathfrak{h}^* \to H^2(H_1, k)^{(-1)}$ for any restricted Lie algebra $\mathfrak{h}$ determines a commutative diagram

$$\begin{array}{ccccc} K_M & \to & S^*(\mathfrak{g}^*) & \to & H^*(G_1, k)^{(-1)} \\ \downarrow & & \downarrow & & \downarrow \\ I_M & \to & S^*(\mathfrak{b}^*) & \to & H^*(B_1, k)^{(-1)}. \end{array}$$

By Frobenius reciprocity, we obtain the following commutative diagram

$$\begin{array}{ccccc} K_M & \rightarrow & S^*(\mathfrak{g}^*) & \rightarrow & H^*(G_1, k)^{(-1)} \\ \downarrow & & \downarrow & & \downarrow \\ H^0(G/B, I_M) & \rightarrow & H^0(G/B, S^*(\mathfrak{b}^*)) & \rightarrow & H^0(G/B, H^*(B_1, k)^{(-1)}). \end{array}$$

Consider the composition of the following chain of maps

$$\begin{aligned} K_M \rightarrow S^*(\mathfrak{g}^*) \rightarrow H^*(G_1, k)^{(-1)} \rightarrow H^*(G_1, M \otimes M^*)^{(-1)} \\ \rightarrow H^0(G/B, H^*(B_1, M \otimes M^*)^{(-1)}) \end{aligned}$$

where the right-most map is the edge homomorphism of the spectral sequence $\{E_r^{*,*}(M \otimes M^*)\}$ of Proposition 1.1. Using the above commutative diagram and the naturality of the spectral sequence (and thereby of the edge homomorphism), we conclude that this composition equals the composition of the following chain of maps

$$\begin{aligned} K_M \rightarrow H^0(G/B, I_M) \rightarrow H^0(G/B, S^*(\mathfrak{b}^*)) \rightarrow H^0(G/B, H^*(B_1, k)^{(-1)}) \\ \rightarrow H^0(G/B, H^*(B_1, M \otimes M^*)^{(-1)}). \end{aligned}$$

Since the composition of the three right-most maps of this latter chain is induced by the chain of B-module maps

$$I_M \rightarrow S^*(\mathfrak{b}^*) \rightarrow H^*(B_1, k)^{(-1)} \rightarrow H^*(B_1, M \otimes M^*)^{(-1)}$$

and since this composition is zero by definition of I_M, we conclude that the composition of the original chain is likewise zero. This implies that the image of the composition

$$K_M \rightarrow S^*(\mathfrak{g}^*) \rightarrow H^*(G_1, k)^{(-1)} \rightarrow H^*(G_1, M \otimes M^*)^{(-1)}$$

has positive filtration with respect to the filtration on $H^*(G_1, M \otimes M^*)^{(-1)}$ associated to the spectral sequence $\{E_r^{*,*}(M \otimes M^*)\}$. Since $H^m(G/B, -)=0$ for $m > d = \dim(G/B)$, we conclude that $(K_M)^{d+1}$ maps to 0 in $H^*(G_1, M \otimes M^*)^{(-1)}$. Consequently, $K_M \subset \sqrt{J_M}$ as required.

Finally, if G is reductive and if p is greater than the Coxeter number of any simple quotient of G, then $\Phi\colon |\mathfrak{g}| \rightarrow \mathfrak{g}$ is injective with image the closed subvariety of nilpotent elements [7; 2.3] and $\Phi\colon |\mathfrak{b}| \rightarrow \mathfrak{b}$ is injective with image the closed subvariety $\mathfrak{u} \subset \mathfrak{b}$ (cf. [1; 2.3]). Consequently, the equality $\Phi(|\mathfrak{g}|_M) = G \cdot \Phi(|\mathfrak{b}|_M)$ of closed subvarieties of $\mathfrak{g}$ implies the isomorphism $|\mathfrak{g}|_M \simeq G \cdot |\mathfrak{b}|_M$. $\square$

We now consider an arbitrary finite dimensional restricted Lie algebra $\mathfrak{g}$ and an arbitrary finite dimensional restricted $\mathfrak{g}$-module M. We denote by $N \subset \mathfrak{g}$ the closed subvariety consisting of 0 together with those elements $X \in \mathfrak{g}$ for which there exists some non-negative integer $m(X)$ with $X^{[p^{m(X)}]} \neq 0$, $X^{[p^{m(X)+1}]} = 0$. For any $X \in \mathfrak{g}$, we denote by $\langle X \rangle$ the restricted Lie subalgebra of $\mathfrak{g}$ generated by X.

The following theorem interprets $\Phi(|\mathfrak{g}|_M)$ in module-theoretic terms involving the cyclic subalgebras $\langle X\rangle$ of $\mathfrak{g}$. Our proof uses an observation of Jantzen (cf. [10; 3.1]) concerning the usefulness of the natural map $\psi\colon \mathfrak{g}\to\mathfrak{gl}(M)$, where $GL(M)$ denotes the reductive algebraic group of k-linear automorphisms of M and $\mathfrak{gl}(M)=\mathrm{Lie}(GL(M))$. Both Theorem 1.3 and its Corollary 1.4 have an obvious reformulation for $|\mathfrak{g}|_M$ whenever $\Phi\colon |\mathfrak{g}|\to\mathfrak{g}$ is injective (e.g., $\mathfrak{g}=\mathrm{Lie}(G)$ with G reductive and p greater than the Coxeter number of any simple quotient of G).

(1.3) **Theorem.** *Let* $\mathfrak{g}$ *be a finite dimensional restricted Lie algebra and let* M *be a finite dimensional restricted* $\mathfrak{g}$*-module. There is an equality of subvarieties of* N*:*

$$\{X\in N\colon M \text{ is not projective as a restricted } \langle X\rangle\text{-module}\}\cup\{0\}$$
$$=\{X\in N\colon X^{[p^{m(X)}]}\in\Phi(|\mathfrak{g}|_M)\}\cup\{0\}.$$

Furthermore, $\Phi(|\mathfrak{g}|)\subset N$.

Proof. For notational convenience, let $R_{\mathfrak{g}}(M)\subset\mathfrak{g}$ denote the subvariety (the "rank subvariety" of M in $\mathfrak{g}$) on the left of the above equality and let $N_{\mathfrak{g}}(M)$ denote the subvariety on the right. Replacing M by $M\oplus P$ for P a faithful, projective, finite dimensional restricted $\mathfrak{g}$-module, we may assume that $\psi\colon \mathfrak{g}\to\mathfrak{gl}(M)$ is faithful. Then $R_{\mathfrak{g}}(M)\subset N_{\mathfrak{g}}(M)$: if M is not projective as a restricted $\langle X\rangle$-module, then $X^{[p^{m(X)}]}\in\Phi(\langle X\rangle_M)\subset\Phi(|\mathfrak{g}|_M)$ by [7; 1,6]. To prove that $R_{\mathfrak{g}}(M)\supset N_{\mathfrak{g}}(M)$, we observe that $\psi\colon \mathfrak{g}\to\mathfrak{gl}(M)$ clearly has the property that $\psi^{-1}(R_{\mathfrak{gl}(M)}(M))=R_{\mathfrak{g}}(M)$. Moreover, the naturality of Φ implies that $\psi^{-1}(N_{\mathfrak{gl}(M)}(M))\supset N_{\mathfrak{g}}(M)$. Consequently, it suffices to prove that $R_{\mathfrak{gl}(M)}(M)\supset N_{\mathfrak{gl}(M)}(M)$.

Let $X\in N_{\mathfrak{gl}(M)}(M)$. Choose a Borel subgroup $B\subset GL(M)$ as in Theorem 1.2 and let U be the unipotent radical of B with $\mathfrak{u}=\mathrm{Lie}(U)$. By Theorem 1.2, there exists some $g\in GL(M)$ such that $gX^{[p^{m(X)}]}\in\Phi(|\mathfrak{b}|_M)$. Since the restriction map $H^*(B_1, M\otimes M^*)\to H^*(U_1, M\otimes M^*)$ is injective, $\Phi(|\mathfrak{b}|_M)=\Phi(|\mathfrak{u}|_M)$ as subvarieties of $\mathfrak{b}$. Consequently, we may apply [7; 2.7] to $gX^{[p^{m(X)}]}\in\Phi(|\mathfrak{u}|_M)$ to conclude that M is not projective as a restricted $\langle g\cdot X\rangle$-module. Because M is a rational $GL(M)$-module, this implies that M is not projective as a restricted $\langle X\rangle$-module. Hence, $X\in R_{\mathfrak{gl}(M)}(M)$ as required.

To prove $\Phi(|\mathfrak{g}|)\subset N$, we consider an embedding $\mathfrak{g}\subset\mathfrak{gl}_n=\mathrm{Lie}(GL_n)$ of restricted Lie algebras as above and observe that it suffices by functorality to consider the case $\mathfrak{g}=\mathfrak{gl}_n$. If $B\subset GL_n$ denotes a Borel subgroup with unipotent radical U, then $\Phi(|\mathfrak{b}|)\subset\mathfrak{u}$, where $\mathfrak{b}=\mathrm{Lie}(B)$ and $\mathfrak{u}=\mathrm{Lie}(U)$. Thus, Theorem 1.2 implies that $\Phi(|\mathfrak{gl}_n|)\subset N$. □

We recall from [10] Jantzen's recent theorem that $\Phi(|\mathfrak{g}|)=\{X\in\mathfrak{g}\colon X^{[p]}=0\}$. Combining this theorem with Theorem 1.3, we obtain the following simple, module-theoretic characterization of $\Phi(|\mathfrak{g}|_M)$ proved by Jantzen in the special case of p equals 2 [10; 3.9].

(1.4) **Corollary.** *Let* $\mathfrak{g}$ *be a finite dimensional restricted Lie algebra and let* M *be a finite dimensional restricted* $\mathfrak{g}$*-module. Then* $\Phi(|\mathfrak{g}|_M)$ *equals*

$$\{X\in\mathfrak{g}\colon X^{[p]}=0 \text{ and } M \text{ is not projective as a restricted } \langle X\rangle\text{-module}\}\cup\{0\}.$$

Proof. Naturality implies that any $X \in \mathfrak{g}$ with $X^{[p]}=0$ and with M not projective as a restricted $\langle X\rangle$-module must lie in $\Phi(|\mathfrak{g}|_M)$. Conversely, if $X \neq 0$ is an element of $\Phi(|\mathfrak{g}|_M)$, then Jantzen's theorem implies that $X^{[p]}=0$ and Theorem 1.3 implies that M is not projective as a restricted $\langle X\rangle$-module. □

2. Applications

In this section, we present a few applications of the results of Sect. 1. Our first application presents certain natural properties of $\Phi(|\mathfrak{g}|_M)$ (cf. [7; 3.3]).

(2.1) **Proposition.** *Let* $\mathfrak{g}$ *be a finite dimensional restricted Lie algebra, let* $\mathfrak{h} \subset \mathfrak{g}$ *be a restricted Lie subalgebra, and let* M *and* Q *be finite dimensional restricted* $\mathfrak{g}$*-modules. Then*

a) $\Phi(|\mathfrak{h}|_M)=\Phi(|\mathfrak{g}|_M)\cap \mathfrak{h}$.

b) $\Phi(|\mathfrak{g}|_{M\oplus Q})=\Phi(|\mathfrak{g}|_M)\cup\Phi(|\mathfrak{g}|_Q)$

c) $\Phi(|\mathfrak{g}|_{M\otimes Q})=\Phi(|\mathfrak{g}|_M)\cap\Phi(|\mathfrak{g}|_Q)$.

Proof. The proof is the same as that given for [7; 3.3]. Parts a) and b) follow immediately from Corollary 1.4. Part c) follows from part a) by considering the diagonal map $\Delta\colon \mathfrak{g}\to\mathfrak{g}\times\mathfrak{g}$, since $|\mathfrak{g}\times\mathfrak{g}|_{M\otimes Q}\simeq|\mathfrak{g}|_M\times|\mathfrak{g}|_Q$. □

We next recall the restricted $\mathfrak{g}$-modules L_ζ first introduced by J. Carlson in the context of finite groups [3; § 2] and translated to the context of restricted Lie algebras by the authors in [7; § 4]. Namely, if $\zeta\in H^{2n}(V(\mathfrak{g}), k)$ is any non-trivial cohomology class, then ζ naturally determines a homomorphism of restricted $\mathfrak{g}$-modules $\Omega^{2n}(k)\to k$ whose kernel is defined to be L_ζ, where $\Omega^{2n}(k)$ is the $2n$-th syzygy module for the trivial module k (cf. [7; 3.1, 4.1]). As shown in [7; 4.1], L_ζ has the following property: for any $X\in\mathfrak{g}$ with $X^{[p]}=0$, L_ζ is not projective as a restricted $\langle X\rangle$-module if and only if the cohomology class $\zeta\in H^{2n}(V(\mathfrak{g}), k)$ restricts to zero in $H^{2n}(\langle X\rangle, k)$.

The existence of restricted $\mathfrak{g}$-modules L_ζ with this property together with Proposition 2.1 enables us to prove the following theorem, a generalization of [7; 4.3] from p-unipotent Lie algebras to arbitrary restricted Lie algebras. The sketch of the proof given below is merely a repetition of the proof of [7; 4.3] which in turn is based on arguments of Carlson [3]. The reader should observe that part a) of Theorem 2.2 is a complete characterization of those subvarieties of $\mathfrak{g}$ which can occur as images of support varieties of some restricted $\mathfrak{g}$-module in view of the fact that $\Phi(|\mathfrak{g}|_M)$ is always a closed, conical subvariety of $\mathfrak{g}$.

(2.2) **Theorem.** *Let* $\mathfrak{g}$ *be a finite dimensional restricted Lie algebra.*

a) *Any closed, conical subvariety of* $\Phi(|\mathfrak{g}|)$ *is of the form* $\Phi(|\mathfrak{g}|_M)$ *for some finite dimensional restricted* $\mathfrak{g}$*-module* M.

b) *If* M *is a finite dimensional, indecomposable restricted* $\mathfrak{g}$*-module, then the projective variety* $\operatorname{Proj}(\Phi(|\mathfrak{g}|_M))\subset\operatorname{Proj}(\mathfrak{g})\simeq P^{\dim(\mathfrak{g})-1}$ *is connected.*

Proof. Let $F\in S^n(\mathfrak{g}^*)$ be a homogeneous polynomial function on $\mathfrak{g}$ with non-trivial image $\Phi^*(F)=\zeta\in H^{2n}(V(\mathfrak{g}), k)$. By above (cf. the proof of [7; 4.2]),

$\Phi(|\langle X\rangle|_M)\simeq|\langle X\rangle|_M=Z(F_{|\langle X\rangle})$ for any $X\in\mathfrak{g}$ with $X^{[p]}=0$ and for $M=L_\zeta$, where $Z(F_{|\langle X\rangle})$ is the locus of zeros in the space $\langle X\rangle$ of the function F. Consequently, part a) of Proposition 2.1 together with Corollary 1.4 implies for $M=L_\zeta$ that

$$\Phi(|\mathfrak{g}|_M)=\bigcup_{X\in\Phi(|\mathfrak{g}|)}\Phi(|\langle X\rangle|_M)=\bigcup_{X\in\Phi(|\mathfrak{g}|)}Z(F_{|\langle X\rangle})=Z(F)\cap\Phi(|\mathfrak{g}|)$$

where $Z(F)\subset\mathfrak{g}$ is the locus of zeros of the function F. Applying part a) of Proposition 2.1, we conclude that we can realize in the form of $\Phi(|\mathfrak{g}|_M)$ any subvariety of $\mathfrak{g}$ of the form $\Phi(|\mathfrak{g}|)$ intersected with the zeros of a homogeneous ideal of $S^*(\mathfrak{g}^*)$. Such subvarieties are precisely the closed, conical subvarieties of $\Phi(|\mathfrak{g}|)$, thereby proving a). As remarked in [7; 4.3], the proof of part b) is merely a translation of the proof of [3; Theorem 1] into our present context. $\square$

We obtain the following intriguing result as an immediate corollary of part b) of Theorem 2.2 and Jantzen's recent theorem that $\Phi(|\mathfrak{g}|)=\{X\in\mathfrak{g}\colon X^{[p]}=0\}$.

(2.3) **Corollary.** *For any finite dimensional restricted Lie algebra* $\mathfrak{g}$, *the projective variety* $\operatorname{Proj}(\{X\in\mathfrak{g}\colon X^{[p]}=0\})\subset\operatorname{Proj}(\mathfrak{g})\simeq P^{\dim(\mathfrak{g})-1}$ *is connected.* $\square$

Leonard Scott has shown us the following easy proof of Corollary 2.3 in the special case when $\mathfrak{g}=\operatorname{Lie}(G)$ with G a connected linear algebraic group. Let Y denote $\{X\in\mathfrak{g}\colon X^{[p]}=0\}$ and let x, y be non-zero elements of Y. Choose g, $h\in G$ such that $g\cdot x$, $h\cdot y$ belong to $\mathfrak{u}=\operatorname{Lie}(U)$, where $B=T\cdot U$ is a chosen Borel subgroup of G. Let z be a non-zero element in the center of $\mathfrak{u}$ with $z^{[p]}=0$, so that the span V_x (respectively, V_y) of z and $g\cdot x$ (resp., z and $h\cdot y$) lies in Y. Then $V_x\cap V_y\neq 0$ implies that $\operatorname{Proj}(V_x\cap V_y)$ lies in the connected component of $\operatorname{Proj}(Y)$ containing the image of $G\cdot x$ and that containing the image of $G\cdot y$, so that the images of x and y lie in the same component of $\operatorname{Proj}(Y)$.

We now restrict our attention to a simple algebraic group G defined and split over $\mathbf{F}_p$ and consider $\mathfrak{g}=\operatorname{Lie}(G)$. If p is greater than $h(G)$, the Coxeter number of G, then $|\mathfrak{g}|\simeq\Phi(|\mathfrak{g}|)=N$, the closed subvariety of $\mathfrak{g}$ consisting of nilpotent elements (cf. [7; 2.3]). Because the support varieties of rational G-modules are G-stable, knowledge of G-orbits of N can provide information about such modules.

Fix a set Π of simple roots for the root system of G relative to a maximal torus T of a split Borel subgroup B of G. For each root $\alpha\in\Pi$, let X_α be a non-zero α-weight vector in $\mathfrak{g}$; let $X_r=\sum X_\alpha$, with the sum indexed by $\alpha\in\Pi$.

(2.4) **Proposition.** *Let G be a simple algebraic group defined over* $\mathbf{F}_p$ *and let* $\mathfrak{g}=\operatorname{Lie}(G)$. *Let M and Q be finite dimensional, rational G-modules.*

a) *If $p>h(G)$, then $|\mathfrak{g}|_M=|\mathfrak{g}|$ if and only if M is not projective as a restricted $\langle X_r\rangle$-module.*

b) *If $p=2$, assume that G is not of type B, C, or F_4; if $p=3$, assume that G is not of type G_2. Then M is projective as a restricted* $\mathfrak{g}$*-module if and only if M is projective as a restricted $\langle X_\alpha\rangle$-module for some long root $\alpha\in\Pi$.*

c) *For p as in* b), *$M\otimes Q$ is projective as a restricted* $\mathfrak{g}$*-module if and only if either M or Q is projective as a restricted* $\mathfrak{g}$*-module.*

Proof. If $p>h(G)$, $X^{[p]}=0$ for all $X\in N$, whereas $G\cdot X_r$ is a dense subset of N ([14; 4.6]). Since $|\mathfrak{g}|_M\subset N=|\mathfrak{g}|$ is closed, part a) follows from Theorem 1.3. Using the usual commutator formules [13; p. 23] for $\mathfrak{g}$, it follows for p subject to the conditions of b) that $G\cdot X_\alpha$ is the unique non-trivial minimal orbit in N whenever α is a long root in Π. Consequently $G\cdot X_\alpha\cap\Phi(|\mathfrak{g}|_M)=0$ implies $\Phi(|\mathfrak{g}|_M)=0$ which implies by [7; 1.5, 2.4] that M is projective as a restricted $\mathfrak{g}$-module, so that part b) follows from Theorem 1.3. Finally, part c) follows from the uniqueness of a minimal non-trivial G-orbit in N as in part b) together with part c) of Proposition 2.1. □

The "Main Theorem" of [6] asserts that a finite dimensional rational TG_1-module M is projective as a restricted $\mathfrak{g}$-module if and only if M is projective as a restricted module for the restricted subalgebras $\langle X_\alpha\rangle$ for each root α. (This "Main Theorem" applies as well to the group schemes TG_r for $r>1$.) Thus, part b) of Proposition 2.4 gives an improved version of this result in the case when G has two root lengths provided that p satisfies the conditions of (2.4b) and M is a rational G-module. In fact, one can reprove this "Main Theorem" without restriction on p or M using Corollary 1.4: if M is a finite dimensional TG_1-module which is not projective, then $0\neq\Phi(|\mathfrak{g}|_M)\subset\mathfrak{g}$ is T-stable (cf. [7; 4.4]) and thereby must contain some non-zero weight vector X_α.

(2.5) *Remark.* It is likely that part a) of Proposition 2.4 has a suitable formulation for small primes. For example, the SL_n-orbits of the nilpotent variety $N\subset\mathfrak{sl}_n$ are in one-to-one order preserving correspondence with partitions of n. Write $n=pm+r$ with $0\leqq m$ and $0\leqq r<p$. If λ denotes the partition $(p, \ldots, p, r)$ of n, then the SL_n-orbit C_λ associated to λ is open and dense in $\Phi(|\mathfrak{sl}_n|)=\{X\in\mathfrak{sl}_n: X^{[p]}=0\}$. Consequently, if $X_\lambda\in C_\lambda$, then part a) of Proposition 2.4 can be generalized for type A without restriction on p as follows: a finite dimensional restricted $\mathfrak{sl}_n$-module M satisfies $|\mathfrak{sl}_n|_M=|\mathfrak{sl}_n|$ if and only if M is not projective as a restricted $\langle X_\lambda\rangle$-module. Using [12; 1.2.5], we may apply the same argument to the other classical types at least when $p\neq 2$.

When p is small, the statement of part c) of Proposition 2.4 is no longer valid. For example, in case G is of type C_2 and $p=2$, the 16 dimensional Steinberg module St factors as $St\simeq M\otimes Q$, where M and Q are the 4 dimensional irreducible modules with high weights corresponding to the two fundamental dominant weights. On the other hand, St is projective as a restricted $\mathfrak{g}$-module whereas neither M nor Q is. A similar factorization of the Steinberg module occurs for G of type F_4, $p=2$ and for G of type G_2, $p=3$.

3. Further questions

We conclude with a few problems to add to the list of those we presented in [7; § 5], having provided fairly satisfying answers to the problems [7; 5.2] and [7; 5.6].

(3.1) *Problem.* Let G be a simple algebraic group. Which G-stable, closed, conical subvarieties of $|\mathfrak{g}|$ can be realized as $|\mathfrak{g}|_M$ for some rational G-module M

(cf. [10; §4])? For small primes p (i.e., $p \leqq h(G)$), one would presumably first answer this question with $|\mathfrak{g}|$ replaced by $\Phi(|\mathfrak{g}|)$.

(3.2) *Problem.* Let G be a simple algebraic group with Coxeter number $h(G) < p$, and let B be a Borel subgroup of G with unipotent radical U. Then $H^*(G_1, k)$ is the coordinate ring of the nilpotent variety $N = G \cdot \mathfrak{u}$, where $\mathfrak{u} = \mathrm{Lie}(U)$. Can one generalize this in the following direction? Let P be a parabolic subgroup of G. Does there exist a natural G-algebra A_p such that $H^*(G_1, A_p)$ is the coordinate ring of $G \cdot \mathfrak{u}_p$, where $\mathfrak{u}_p$ is the Lie algebra of the unipotent radical of P? A somewhat easier problem is to find natural modules M_p whose support variety is $G \cdot \mathfrak{u}_p$.

(3.3) *Problem.* For simple algebraic groups G with $p > h(G)$ or G of classical type and $p \neq 2$, $\Phi(|\mathfrak{g}|)$ has a dense orbit (cf. Remark 2.5), so that $\mathrm{Proj}(\Phi(|\mathfrak{g}|))$ is irreducible where $\mathfrak{g} = \mathrm{Lie}(G)$. For which restricted Lie algebras $\mathfrak{g}$ is $\mathrm{Proj}(|\mathfrak{g}|)$ irreducible?

(3.4) *Problem.* Give a reasonably general necessary condition on a finite dimensional restricted $\mathfrak{g}$-module M which implies that $\mathrm{Proj}(\Phi(|\mathfrak{g}|_M))$ is irreducible (cf. Theorem 2.2, part b).

(3.5) *Problem.* By [7; 3.2], the dimension of $\Phi(|\mathfrak{g}|_M)$ can be interpreted module theoretically as the complexity of M. Give a module theoretic interpretation of the degree with multiplicities of the projective subvariety $\mathrm{Proj}(\Phi(|\mathfrak{g}|_M)) \subset \mathrm{Proj}(\mathfrak{g})$ in "good" situations (e.g., when $\mathrm{Proj}(\Phi(|\mathfrak{g}|_M))$ is irreducible).

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